Predictive Fault Tolerant Control for LPV systems using model reference

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Abstract: The present work proposes a Fault Tolerant Control (FTC) methodology for non-linear discrete-time systems that can be modeled as Linear Parameter Varying (LPV) systems. The proposed approach relies on the modeling of faults as additional scheduling parameters of the LPV model for the controlled system and it uses a triple loop architecture. The inner control loop is designed by means of the standard $\mathcal{H}_2/\mathcal{H}_\infty$ control methodologies based on Linear Matrix Inequalities (LMIs). The design takes into account a prespecified set of faults and the ranges of their magnitudes that are wanted to be tolerated and it assumes the availability of on-line fault estimations provided by a Fault Detection and Isolation (FDI) module. The resulting controller tries to compensate the system faults in order to maintain a satisfactory closed-loop dynamic performance, but it does not take into account possible system input and state constraints associated to actuator saturation and other physical limitations. Thus, an intermediate control loop determines the actual compensation feasibility using set invariance theory. And, when it is needed, it applies suitable additive predictive control actions that enlarge the invariant set, trying to assure that the current state remains inside the enlarged invariant set. Finally, an outer loop implements a model reference control that allows reference tracking. The use of the proposed FTC methodology is illustrated through its application to the well-known quadruple tank system benchmark.

Keywords: Linear Parameter Varying, $\mathcal{H}_2/\mathcal{H}_\infty$ performance, Fault Tolerant Control, Efficient Predictive Control, Set Invariance, Quadruple-Tank.

1. INTRODUCTION

Under certain undesired events, such as actuator malfunctions, sensor malfunctions or unexpected changes in the dynamics of the plant, conventional feedback control design could result in unexpected poor performance or even instability of the control loop system. For this reason, Fault Tolerant Control (FTC) arises with the aim of designing controllers capable of tolerating certain system malfunctions whilst still maintaining a desirable performance or an acceptable degradation. FTC has become an active and consolidated research topic during the last decades, see for instance Patton (1997), Blanke et al. (2003) or Zhang and Jiang (2008). Generally speaking, FTC systems can be categorized into two main groups: *active* and *passive*. On one hand, *passive FTC* systems (PFTC) are based on robust control laws that assure a satisfactory behavior not only in absence but also in presence of faults. However, this type of approach is restricted to faults of limited scope and, moreover, the obtained control tends to be conservative. On the other hand, the *active FTC* systems (AFTC) involve an adaptation of the control law by using the information provided on-line by a Fault Detection and Isolation (FDI) module. With this information, some automatic controller adjustments are done immediately after the fault appearance, trying to guarantee that the control objectives are still satisfied. In general, active FTC systems provide good performances due to their adaptation to the real system behavior, but the price to pay is that the overall system becomes more complicated and costly. The design of AFTC systems has been addresses in the literature by considering different control objectives and techniques, the interested reader is referred to the nice survey by Zhang and Jiang (2008). In recent years, several works in the literature have considered the problem of FTC for linear parameter varying (LPV) systems, see e.g. Montes de Oca et al. (2014). The main motivation is twofold. On one hand, LPV systems allows to represent non-linear dynamics. On the other hand, the powerful control design techniques for linear systems can be extended to such systems. It must be noticed that most of these works adapt

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the LPV framework to deal with particular types of faults, e.g. sensor or actuator faults. In contrast, in Montes de Oca et al. (2014) it is proposed an approach where faults are modeled as additional scheduling parameters. Under this general framework, it is possible to represent different types of faults. Moreover, it is also possible to specify, at design stage, the ranges of fault magnitudes that are wanted to be tolerated.

A known limitation of the LPV framework is that it does not allow to consider constraints. In particular, a suitable AFTC design methodology should consider that the compensation of faults could saturate the system actuators. To take this into account, in Witczak and Witczak (2013) it is proposed the use of a multi-loop control architecture that integrates Efficient Predictive Control (EPC) to deal with the constraints. The limitation is that the proposed formulation is restricted to a particular type of additive actuator faults.

This paper merges and extends the ideas proposed in Witczak and Witczak (2013) and Montes de Oca et al. (2014). The architecture and the techniques presented in the former work are extended to consider the general fault modeling framework proposed in the latter. Moreover, an additional model reference control loop is introduced to track step references. In summary, a novel FTC is proposed, able to deal with non-linear systems considering constraints, to achieve fault tolerance for prespecified faults and their size, and, finally, using model reference control for tracking purposes.

The rest of the paper is organized as follows. Section 2 introduces the LPV representation for faulty systems. In Section 3, the proposed control architecture is explained. In Sections 4 and 5, the technical details about $\mathcal{H}_2/\mathcal{H}_\infty$ control and Efficient Predictive Control design for LPV systems are provided, respectively. In Section 6, the results of applying our novel approach to the well-known quadruple-tank system are presented. Finally, in Section 7, the conclusions are presented and future work is proposed.

2. LPV MODELING

2.1 General framework: polytopic LPV systems

Let us consider the general space state representation for discrete-time LPV systems:

$$\mathbf{x}(k+1) = \mathbf{A}(\boldsymbol{\theta}(k))\,\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \qquad (1)$$

where $\mathbf{x}(k) \in \mathbb{R}^{n_x}$ is the state vector and $\mathbf{u}(k) \in \mathbb{R}^{n_u}$ is the input vector, both at time instant k. The vector $\boldsymbol{\theta}(k) \in \mathbb{R}^{n_p}$ is the vector of scheduling parameters, that are assumed to be available on-line. The system matrix $\mathbf{A}(\boldsymbol{\theta}(k))$ is considered time-varying while the input matrix $\mathbf{B}$ is assumed to be constant. If the matrix $\mathbf{A}(\boldsymbol{\theta}(k))$ is affine with respect to the scheduling parameters and those parameters are bounded, i.e. $\theta_i \in [\underline{\theta}_i, \bar{\theta}_i]$, then the parameter vector $\boldsymbol{\theta}(k)$ lies in a polytope $\boldsymbol{\Theta}$ whose vertexes are given by combinations of extreme values for the parameters. In this case, (1) can be rewritten in polytopic form (Apkarian et al. (1995)) as:

$$\mathbf{x}(k+1) = \sum_{i=1}^{q} h_i(k)\,\mathbf{A}_i\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \qquad (2)$$

where $\mathbf{A}_i$ is the matrix $\mathbf{A}(\boldsymbol{\theta}(k))$ evaluated at the vertex i, $q = 2^{n_p}$ is the number of vertexes, n_p is dimension of the vector of parameters and h_i are the scheduling functions that satisfy $h_i(k) \in [0,1]$, $\sum_{i=1}^{q} h_i(k) = 1$.

The initial interest on LPV systems relies on their ability to approximate non-linear dynamics. Moreover, powerful control design techniques for linear systems, such as $\mathcal{H}_2/\mathcal{H}_\infty$ control or pole placement, based on solving Linear Matrix Inequalities (LMIs), can be extended to polytopic LPV systems, see e.g. Apkarian et al. (1995), Chilali and Gahinet (1996) or Rotondo et al. (2014).

2.2 Modeling of the faults

In this work, it is considered that the scheduling parameters in the general representation (1) are associated to the system operating conditions and that they can be directly computed as functions of the state variables (quasi-LPV systems). On the other hand, following what is proposed in Montes de Oca et al. (2014), it is considered that additional scheduling parameters can be used to model the effects of faults in the system. Hence, the following system representation is proposed:

$$\mathbf{x}(k+1) = \mathbf{A}(\boldsymbol{\theta}_{\boldsymbol{x}}(k), \boldsymbol{\theta}_{\boldsymbol{f}}(k))\,\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k), \qquad (3)$$

where $\boldsymbol{\theta}_{\boldsymbol{x}}(k) \in \mathbb{R}^{n_x}$ represents the parameters associated to the state variables and $\boldsymbol{\theta}_{\boldsymbol{f}}(k) \in \mathbb{R}^{n_f}$ are the parameters associated to the faults, that are assumed to be estimated on-line by an FDI module. For this case, $n_p = n_x + n_f$ to compute q in (2).

The proposed formulation allows to model the system faults and to take them into account during the FTC design. In particular, it is considered that the FTC design has to cope with a set of faults whose magnitudes are assumed to be bounded, thus the parameters related to the faults are bounded as well which can be represented as $\theta_{\mathbf{f}} \in [\underline{\theta}_{\mathbf{f}}, \bar{\theta}_{\mathbf{f}}]$. The goal of the design is to obtain a satisfactory closed-loop behavior whenever the faults present in the system do not exceed their prespecified limits. On the other hand, if the state variables are also assumed to be bounded, i.e. $\mathbf{x} \in [\underline{\mathbf{x}}, \bar{\mathbf{x}}]$, then the proposed formulation casts into the general polytopic LPV framework and the associated powerful modeling and control design techniques can be directly applied.

3. PROPOSED CONTROL SCHEME

As the main control objective is to track references whilst guaranteeing fault tolerance, the mechanism for tracking purposes shown in Figure 1 is proposed, based on the model reference approach presented in Franklin et al. (1998). It is assumed that a good model of the plant to be controlled is available. Thus, through this model it is possible to compute the nominal input $\mathbf{u}_{nom}(k)$ to guarantee steady-state conditions for a given state reference $\mathbf{x}_{ref}(k)$. The value $\mathbf{u}_{nom}(k)$ is used to include a feed-forward loop in the control systems for guaranteeing

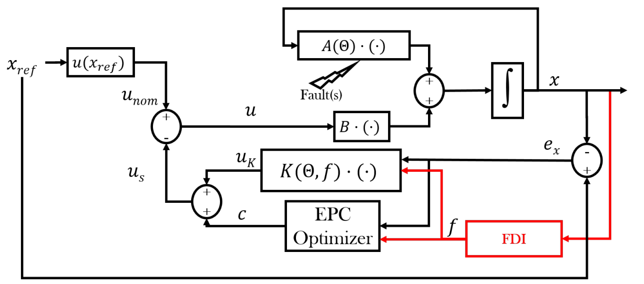

Fig. 1. Model reference control scheme for tracking purposes.

that once the system has reached the state reference it will remain there. Thus, the control problem is reduced to stabilize the tracking error.

To stabilize the tracking error, a feedback control law will be applied, defined as follows:

$$\mathbf{u}_s(k) = \mathbf{K}(\theta(k))\,\mathbf{e_x}(k) + \mathbf{c}(k) \tag{4}$$

$$= \sum_{i=1}^{q} h_i(k)\,\mathbf{K}_i\mathbf{e_x}(k) + \mathbf{c}(k), \tag{5}$$

where $\mathbf{e_x}(k)$ is the state error. $\mathbf{K}(\theta(k))$ is a scheduled control gain, where $\mathbf{K}_i$ are controllers that guarantees stability for each vertex (notice that $\mathbf{K}$ could be a robust unique controller valid for all vertices). Furthermore, it can be designed in order to guarantee closed-loop performance and robustness, but without taking into account the input constraints of the system. On the other hand, $\mathbf{c}(k)$ is an additional degree of freedom of the control actions based on the closed-loop paradigm explained in Rossiter (2013), that can be designed in order to satisfy the input constraints of the system whilst performance is enforced.

Considering that the references to be tracked are constant state references $\mathbf{x}_{ref}$, the tracking error dynamics for the closed-loop system (2)-(5) is given by the following expression:

$$\mathbf{e_x}(k+1) = \sum_{i=1}^{q} h_i(k)\,\mathbf{A}_i\mathbf{e_x}(k) + \mathbf{B}\mathbf{u}_s(k), \tag{6}$$

The proposed control architecture assumes that faults can be modeled as scheduling parameters and that a FDI module is able to provide precise fault estimations. For the results presented in this work, it is assumed that an ideal FDI module provides such estimations. For a more realistic fault estimation implementation, the methods proposed in Montes de Oca et al. (2014) can be used. However, the effect of fault estimation errors on the overall closed-loop behavior is still an open problem to be considered in future works.

Accordingly to what has been outlined, the control design problem can be divided into four stages. The first three stages are carried on off-line: the LPV modeling of the system and its faults; the computation of the stack of control gains $\mathbf{K}_i$ by using $\mathcal{H}_2/\mathcal{H}_\infty$ control techniques; and the computation of an invariant as proposed in Kouvaritakis et al. (2000), which is used to include a predictive mechanism into the control actions. By using the invariant sets computed off-line, the last and on-line stage is based on the solution of an optimization problem that allows to guarantee constraint handling whilst fault tolerance, control performance and robustness are preserved. The following sections are aimed at explaining this design procedure in more detail.

4. $\mathcal{H}_2/\mathcal{H}_\infty$ CONTROL DESIGN FOR LPV SYSTEMS

4.1 Control specifications

Consider the discrete-time system

$$\mathbf{e_x}(k+1) = \sum_{i=1}^{q} h_i(k)\,\mathbf{A}_i\mathbf{e_x}(k) + \mathbf{B_w}\mathbf{w}(k) + \mathbf{B}\mathbf{u}_s(k), \tag{7}$$

where $\mathbf{w}(k)$ is a vector of a bounded exogenous input disturbance and $\mathbf{B_w}$ is the disturbance distribution matrix. Then, we define two output performance functions to evaluate the closed loop system response as

$$\begin{aligned}&\mathbf{z}_\infty(k) = \mathbf{C}_{z_\infty}\mathbf{x}(k) + \mathbf{D}_{z_\infty w}\mathbf{w}(k) + \mathbf{D}_{z_\infty u}\mathbf{u}_s(k) \quad \text{and}\\ &\mathbf{z}_2(k) = \mathbf{C}_{z_2}\mathbf{x}(k) + \mathbf{D}_{z_2 u}\mathbf{u}_s(k),\end{aligned} \tag{8}$$

where the constant matrices $\mathbf{C}_*$ and $\mathbf{D}_*$ can be interpreted as weighting matrices chosen according to the control objectives. If (7) is under a scheduling state feedback control law $\mathbf{u}_s(k) = \sum_{i=1}^{q} h_i(k)\mathbf{K_i}\mathbf{e_x}(k)$, the resultant closed loop control system is

$$\mathbf{e_x}(k+1) = \sum_{i=1}^{q} h_i(k)\left[\mathbf{A}_i + \mathbf{B}\mathbf{K}_i\right]\mathbf{e_x}(k) + \mathbf{B}_w\mathbf{w}(k). \tag{9}$$

As proposed in Rotondo et al. (2014), we consider the following objectives for the closed loop system:

- Closed loop stability.
- $\mathcal{H}_\infty$ performance $\| \mathbf{T_{z_\infty w}(z)} \|_\infty < \gamma$, where $\mathbf{T_{z_\infty w}(z)}$ denotes the closed-loop transfer function from $\mathbf{w}$ to $\mathbf{z}_\infty$ and $\gamma > 0$ is the bound on the closed-loop $\mathcal{H}_\infty$ performance.
- $\mathcal{H}_2$ performance $\| \mathbf{T_{z_2w}(z)} \|_2 < \nu$, where $\mathbf{T_{z_2w}(z)}$ denotes the closed-loop transfer function from $\mathbf{w}$ to $\mathbf{z_2}$ and $\nu > 0$ is the bound on the closed-loop $\mathcal{H}_2$ performance.

4.2 Vertex controllers

In order to guarantee these objectives, the stack of gains for the scheduling control law $\mathbf{K}_i$ for $i = 1, 2, \ldots, q$ has to be found by solving the following LMI problems (Rotondo et al., 2014):

$$\begin{aligned}&\mathbf{X}_i = \mathbf{X}_i{}^T \succ 0, \quad \mathbf{Y} = \mathbf{Y}^T \quad \text{with} \quad trace(\mathbf{Y}) < \nu^2,\\ &\begin{bmatrix} \mathbf{X}_i & \mathbf{U}_i(\mathbf{X},\mathbf{\Gamma}) & \mathbf{B_w} & \mathbf{0}_{n_x\times n_x} \\ \mathbf{U}_i(\mathbf{X},\mathbf{\Gamma})^T & \mathbf{X}_i & \mathbf{0}_{n_x\times n_x} & \mathbf{V}_i(\mathbf{X},\mathbf{\Gamma})^T \\ \mathbf{B}_w{}^T & \mathbf{0}_{n_x\times n_x} & \mathbf{I}_{n_x\times n_x} & \mathbf{D}_{z_\infty w}{}^T \\ \mathbf{0}_{n_x\times n_x} & \mathbf{V}_i(\mathbf{X},\mathbf{\Gamma}) & \mathbf{D}_{z_\infty w} & \gamma^2\mathbf{I}_{n_x\times n_x}\end{bmatrix} \succ 0,\\ &\begin{bmatrix} -\mathbf{X}_i & \mathbf{A}_i\mathbf{Z} + \mathbf{B}_u\mathbf{\Gamma} \\ (\mathbf{A}_i\mathbf{Z} + \mathbf{B}_u\mathbf{\Gamma})^T & \mathbf{X}_i - \mathbf{Z} - \mathbf{Z}^T\end{bmatrix} \prec 0\\ &\text{and} \quad \begin{bmatrix} \mathbf{Y} & \mathbf{W}_i(\mathbf{X},\mathbf{\Gamma}) \\ \mathbf{W}_i(\mathbf{X},\mathbf{\Gamma})^T & \mathbf{Y}\end{bmatrix} \succ 0,\end{aligned} \tag{10}$$

where

$$\begin{aligned}\mathbf{U}_i(\mathbf{X},\mathbf{\Gamma}) &= \mathbf{A}_i\mathbf{X}_i + \mathbf{B}_u\mathbf{\Gamma},\\ \mathbf{V}_i(\mathbf{X},\mathbf{\Gamma}) &= \mathbf{C}_{z_\infty}\mathbf{X}_i + \mathbf{D}_{z_\infty u}\mathbf{\Gamma} \quad \text{and}\\ \mathbf{W}_i(\mathbf{X},\mathbf{\Gamma}) &= \mathbf{C}_{z_2}\mathbf{X}_i + \mathbf{D}_{z_2 u}\mathbf{\Gamma}.\end{aligned} \tag{11}$$

Once the solution (if any solution exists) to the LMI problems (10) with (11) is found, the stack of control gains for the scheduling control law is computed as $\mathbf{K}_i = \mathbf{\Gamma X}_i{}^{-1}$ for $i = 1, 2, \ldots, q$. Once the stack of matrices $\mathbf{K}_i$ for $i = 1, 2, \ldots, q$ has been obtained, the next step is to set up a scheduling mechanism to compute the control gain $\mathbf{K}(k)$ at each sampling instant. This is done according to (5).

5. EFFICIENT PREDICTIVE CONTROL FOR LPV SYSTEMS

As it was mentioned before, real systems used to have state and input constrains. Moreover it can be assumed that any initial condition $\mathbf{x}_0$ of the system belongs to a known set $\mathbf{X}_0$. Starting from this fact, the Efficient Robust Predictive Control (ERPC) formulated in Kouvaritakis et al. (2000), proposes to compute a matrix $\mathbf{Q}_x$ such that the ellipsoidal invariant set

$$E_x = \{\mathbf{x}(k) | \mathbf{x}(k)^T \mathbf{Q}_x{}^{-1} \mathbf{x}(k) \leq 1\} \qquad (12)$$

exists. Then, if for all $\mathbf{x}(k)$ belonging to the set $\mathbf{X}_0$ satisfy (12), the stability of the system is guaranteed.

Now, assuming that the stack of control gains $\mathbf{K}_i$ has been already computed according to the previous section, in order to make predictions that allows to compute $\mathbf{c}(k)$ optimally for the system (2), it is simulated n_c sampling instants, as proposed in Kouvaritakis et al. (2000) leading to the autonomous system

$$\mathbf{z}(k+1) = \sum_{i=1}^{q} h_i(k) \mathbf{\Psi}_i \mathbf{z}(k) \text{ where } \mathbf{z} \in \mathbb{R}^{n_x + n_u n_c},$$

$$\mathbf{z} = \begin{bmatrix} \mathbf{e_x}(k) \\ \mathbf{w}(k) \end{bmatrix}, \quad \mathbf{\Psi}_i = \begin{bmatrix} \mathbf{\Phi}_i & [\mathbf{B}\ \mathbf{0} \cdots \mathbf{0}] \\ \mathbf{0} & \mathbf{M} \end{bmatrix}, \qquad (13)$$

$$\mathbf{w}(k) = \begin{bmatrix} \mathbf{c}(k) \\ \mathbf{c}(k+1) \\ \vdots \\ \mathbf{c}(k+n_c-1) \end{bmatrix}, \mathbf{M} = \begin{bmatrix} \mathbf{0}_{n_u} & \mathbf{I}_{n_u} & \mathbf{0}_{n_u} & \cdots & \mathbf{0}_{n_u} \\ \mathbf{0}_{n_u} & \mathbf{0}_{n_u} & \mathbf{I}_{n_u} & \cdots & \mathbf{0}_{n_u} \\ \vdots & \vdots & \vdots & \ddots & \cdots \\ \mathbf{0}_{n_u} & \cdots & \cdots & \mathbf{0}_{n_u} & \mathbf{I}_{n_u} \\ \mathbf{0}_{n_u} & \mathbf{0}_{n_u} & \cdots & \cdots & \mathbf{0}_{n_u} \end{bmatrix},$$

where $\mathbf{\Phi}_i = \mathbf{A}_i + \mathbf{BK}_i$ for $i = 1, 2, \ldots, q$. Then, our problem is reduced to find the stack of matrices $\mathbf{Q}_z^i$, such that the ellipsoidal invariant set

$$E_z^i = \{\mathbf{z}(k) | \mathbf{z}(k)^T \mathbf{Q}_z^{i\,-1} \mathbf{z}(k) \leq 1\} \qquad (14)$$

exists for each vertex. Finding $\mathbf{Q}_z^i$ is equivalent to a Lyapunov stability test. Thus, in order to guarantee (14) for the system (2), the stack of positive-definite matrices $\mathbf{Q}_z^i$ must be found. This is done by finding a feasible solution for the matrix inequalities

$$\mathbf{\Psi}_i{}^T \mathbf{Q}_z^{i\,-1} \mathbf{\Psi}_i - \mathbf{Q}_z^{i\,-1} \prec 0 \text{ with } \mathbf{Q}_z^i \succ 0. \qquad (15)$$

Advantageously, according to the Schur complement (15) can be rewritten as the LMI system

$$\mathbf{Q}_z^i \succ 0 \text{ and } \begin{bmatrix} \mathbf{Q}_z^i & \mathbf{Q}_z^i \mathbf{\Psi}_j{}^T \\ \mathbf{\Psi}_j \mathbf{Q}_z^i & \mathbf{Q}_z^i \end{bmatrix} \succeq 0. \qquad (16)$$

Knowing that actual control actions are computed as (5), and the control limits are supposed to be constrained symmetrically as $|u_s^j| \leq d_j$ for $j = 1, 2, \ldots, n_u$, these constraints are equivalent to the inequality $\left[\left[\mathbf{K}_j^T\ \mathbf{e}_j^T\right] \mathbf{Q}_z^{1/2}\right] \leq d_j$ (Kouvaritakis et al., 2000), which squared and rearranged turns into

$$d_j^2 - \left[\mathbf{K}_j^T\ \mathbf{e}_j^T\right] \mathbf{Q}_z \left[\mathbf{K}_i^T\ \mathbf{e}_i^T\right]^T \succeq 0, \qquad (17)$$

Finally, (17) is equivalent to the following LMI

$$\begin{bmatrix} \mathbf{Q}_z^i & \mathbf{Q}_z^i \left[\mathbf{K}_j^T\ \mathbf{e}_j^T\right] \\ \left[\mathbf{K}_j^T\ \mathbf{e}_j^T\right]^T \mathbf{Q}_z^i & d_j^2 \end{bmatrix} \succeq 0, \qquad (18)$$

for $i = 1, 2, \ldots, n_u$ and $j = 1, 2, \ldots, q$, where $\mathbf{K}_i^T$ corresponds to the ith row of the matrix $\mathbf{K}$ and e_i is the ith column of the identity matrix $e \in \mathbb{R}^{n_c - n_x}$.

To illustrate graphically the objective of computing the invariant set $\mathbf{Q}_z^i$, first consider $\mathbf{T}_{xz}$ as a transformation matrix such that $\mathbf{x} = \mathbf{T}_{xz}\mathbf{z}$. Now consider the projection of the invariant set $\mathbf{Q}_z^i$ over the original state space represented by $\mathbf{Q}_{xz}^i = \mathbf{T}_{xz}\mathbf{Q}_z^i\mathbf{T}_{xz}^T$. To illustrate it, consider a simple example with $n_x = n_u = n_c = 1$, then the invariant sets E_x and E_{xz} are straight lines (see (Kouvaritakis et al., 2000) for more details). Then, to maximize the set invariance of $\mathbf{Q}_{xz}^i$, Q_z^i has to be optimized per each vertex of the polytope where the system is defined according to the objective function

$$\max_{\mathbf{Q}_z^j} \log\det\left(\mathbf{T}_{xz}\left(\mathbf{Q}_z^j\right)^{-1}\mathbf{T}_{xz}^T\right) \qquad (19)$$

subject to the LMIs (16) and (18). The advantage is that, $\mathbf{Q}_z^i$ can be computed offline in order to guarantee that (12) is satisfied for any $\mathbf{x}_0 \in \mathbf{X}_0$.

Once the stack of ellipsoidal invariant sets $\mathbf{Q}_z^i$ has been computed, the next step is to use it on-line. During the on-line stage, at each sampling instant the optimisation problem

$$\min_{w} \mathbf{w}(k)^T \mathbf{w}(k) \quad \text{subject to} \quad \mathbf{z}(k)^T \mathbf{Q}_z^{i\,-1} \mathbf{z}(k) \leq 1 \quad (20)$$

for all $i = 1, 2, \ldots, q$ must be solved. Once $\mathbf{w}(k)$ is optimized its first element $\mathbf{c}(k)$ is implemented to the system (2) under the guarantee of stability, performance given by $\mathbf{K}_i$ and constraints awareness.

6. APPLICATION TO THE QUADRUPLE-TANK SYSTEM

6.1 Description

To test the proposed FTC approach the well-known four tank case study is proposed. The physical equations describing this system are the following

$$\begin{aligned} A_t\frac{dh_1}{dt} &= -a_1\sqrt{2gh_1} + a_3\sqrt{2gh_3} + \gamma_a\frac{q_a}{3600}, \\ A_t\frac{dh_2}{dt} &= -a_2\sqrt{2gh_2} + a_4\sqrt{2gh_4} + \gamma_b\frac{q_b}{3600}, \\ A_t\frac{dh_3}{dt} &= -a_3\sqrt{2gh_3} + (1-\gamma_b)\frac{q_b}{3600} \quad \text{and} \\ A_t\frac{dh_4}{dt} &= -a_4\sqrt{2gh_4} + (1-\gamma_a)\frac{q_a}{3600}. \end{aligned} \qquad (21)$$

where: $q_{a,b}$ represents the flow pumped by each one of the pumps, in m^3/s; h_1, h_2, h_3, and h_4 represents the water levels of the tanks, in m; $\gamma_{a,b}$ represents the percentage of pumped water that goes to the respective bottom tanks (its complementary corresponds to the remaining percentage of pumped water that goes to the upper tanks); $a_1, \ldots, a_4$ are the areas of the holes at the bottom the tanks, with $a_1 = 1.3104\ cm^2$, $a_2 = 1.5074\ cm^2$, $a_3 = 0.92673\ cm^2$, and $a_4 = 0.88164\ cm^2$; $A_t = 0.3\ m^2$ is the area of the transverse section of the tanks; and finally,

g represents the gravity constant, with value 9.81 m/s^2. Other important values for the considered system are the operational limits: $h_i \in [0.2m,\ 1.2m]\ \forall i \in [1,4]$ and $q_s \in [0m^3/s,\ 2.5m^3/h]$ for $s = a, b$. The values for the opening of the valves are $\gamma_a = 0.3$ and $\gamma_b = 0.4$. Finally, the sampling time of the control system is $T_s = 5s$.

6.2 LPV model including faults

In order to obtain an LPV model of the quadruple-tank system, firstly the non-linearities of the system are embedded into parameters of the form $\theta_i(h_i(t)) = 1/\sqrt{h_i(t)}$, generating the vector of parameters

$$\boldsymbol{\theta}_{\mathbf{x}}(t) = [\theta_1(h_1(t))\ \theta_2(h_2(t))\ \theta_3(h_3(t))\ \theta_4(h_4(t))]^T. \quad (22)$$

Secondly, it is supposed the existence of leakages at the bottom of the tanks during the modeling of the system. This kind of fault affects directly the discharge holes at the bottom of the tanks, i.e., the parameters a_i. Thus, the parameters related to the leakages at the bottom of the tanks are of the form $\theta_{f,i}(a_{f,i}(t), h_i(t)) = a_{f,i}/\sqrt{h_i(t)}$, where $a_{f,i} \in [\underline{a}_{f,i}, \bar{a}_{f,i}]$ are the admissible leakages areas for which the control scheme will be designed. The vector of parameters related to the faults becomes

$$\begin{aligned}\boldsymbol{\theta}_{\mathbf{f}}(t) = [&\theta_1(a_{f,1}(t), h_1(t))\quad \theta_2(a_{f,2}(t), h_2(t))\\ &\theta_3(a_{f,3}(t), h_3(t))\ \theta_4(a_{f,4}(t), h_4(t))]^T. \quad (23)\end{aligned}$$

Finally, given the vector $\boldsymbol{\theta}(t) = [\boldsymbol{\theta}_{\mathbf{x}}(t)\ \boldsymbol{\theta}_{\mathbf{f}}(t)]^T$, the model (21) is rewritten as LPV model as

$$\mathbf{A}(\boldsymbol{\theta}(t)) = \frac{\sqrt{2g}}{A_t}\begin{bmatrix} a_{11}(t) & 0 & a_{13}(t) & 0\\ 0 & a_{22}(t) & 0 & a_{24}(t)\\ 0 & 0 & a_{33}(t) & 0\\ 0 & 0 & 0 & a_{44}(t)\end{bmatrix}, \quad (24)$$

where

$$\begin{aligned} a_{11}(t) &= -\left(a_1\theta_1(t) + \theta_{f,1}\right), \quad a_{13}(t) = a_3\theta_3(t) + \theta_{f,3},\\ a_{22}(t) &= -\left(a_2\theta_2(t) + \theta_{f,2}\right), \quad a_{24}(t) = a_4\theta_4(t) + \theta_{f,4},\\ a_{33}(t) &= -\left(a_3\theta_3(t) + \theta_{f,3}\right) \text{ and}\\ a_{44}(t) &= -\left(a_4\theta_4(t) + \theta_{f,4}\right). \end{aligned}$$

On the other hand the matrix $\mathbf{B}$ follows directly from (21) being independent of the operating point.

Given that all the parameters in (24) are known to be bounded, it is possible to define the polytope where the LPV model is defined and after evaluating it at all the vertices of the polytope and discretizing the model at each vertex, the model (21)could be rewritten as

$$\dot{\mathbf{h}}(k) = \sum_{i=1}^{256} A_i(\mathbf{h}_j(k), \mathbf{f}_k(k))\mathbf{h}(k) + \mathbf{B}\mathbf{q}(k). \quad (25)$$

6.3 FTC design

In the predictive control approach introduced in this work, the input constraints should be symmetric around zero, such that $-u_{max} \le u_i(k) \le u_{max}$ for all $i = 1, 2, \ldots, n_u$ where n_u is the number of inputs. The input bounds of the four-tank system are both equal to $2.5m^3/h$. But considering the control scheme shown in Figure 1, the inputs constraints that must be considered for computing $\mathbf{u}_s$ (represented by $\mathbf{q}_s$ for the quadruple-tank) are obtained below. Consider $\mathbf{q}_{\mathbf{K}}(k) = \mathbf{K}(k)\mathbf{e}_h(k-1)$, given that $\mathbf{q}(k) = \mathbf{q}_{nom}(k) - \mathbf{q}_s(k)$, where $\mathbf{q}_s(k) = \mathbf{q}_{\mathbf{K}}(k) + \mathbf{c}(k)$, then $\mathbf{q}_s(k) = \mathbf{q}_{nom}(k) - \mathbf{q}(k)$, where $\mathbf{q}_i(k)$ must have a maximum value of $2.5m^3/h$ and a minimum of zero. Then, since $\mathbf{q}_{i,nom}(k) \in [0, 2.5m^3/h]$, we have

$$\begin{aligned}\mathbf{q}_{s,i,min}(k) &= \mathbf{q}_{nom}(k) - 2.5m^3/h\\ \mathbf{q}_{s,i,max}(k) &= \mathbf{q}_{nom}(k) - 0m^3/h.\end{aligned} \quad (26)$$

The obtained bounds for $\mathbf{q}_i(k)$, except for one case, are non symmetric to zero. The present work has not considered yet the case of non-symmetric constraints. For illustrative purposes the experiment presented considers $-1.25m^3/h \le \mathbf{q}_{s,i}(k) \le 1.25m^3/h$. The objectives of the control scheme are to guarantee good reference tracking whereas achieving constraints awareness and fault tolerance. Those references are obtained finding the equillibrium of the system that minimizes the cost function presented in the competition organised by CEA IFAC (2014):

$$J(\mathbf{h}, \mathbf{q}, c, p) = \left(q_a^2 + cq_b^2\right) + p\frac{V_{min}}{A_T(h_1 + h_2)}, \quad (27)$$

i.e., the volume of water in tanks 1 and 2 must be maximized whereas the value of the inputs must be minimized, with a preference established by the cost parameters c and p, $V_{min} = 0.012m^3$. Notice that when a fault appears, the dynamics of the system changes and the optimal references must be recomputed according to the estimated faults.

6.4 Results

Given that not only a change in the tracking references modifies the nominal input $\mathbf{q}_{nom}$, but the appearance of a fault also does, it can be seen from Table 1, that when a fault appears, the nominal input is modified as well. Table 1 is aimed at showing the sequence of tracking references used for the experiments presented hereby and the fault magnitude with its occurrence time. In Figures 2-3 and Table 1, it is possible to see how the FTC scheme has a good performance, in the case of a leakage in the tank 1 and considering that the fault is perfectly identified. It can also be noticed that after a change of the references or after the change in the fault size the nominal input changes because the dynamics of the system changes. Then, the optimal operation according to the cost function (27) as well changes. When the references change, the feedback loop makes an effort to reach the given references satisfying input constraints due to the compensation provided by the extra degrees of freedom introduced by $\mathbf{c}(k)$, Notice that sometimes the constraints are a slightly violated because of the assumption of symmetric constraints. Finally, in order to show the advantages in the performance, it is shown in Figure 4 how the architecture using the EPC overperforms an identical control architecture where the predictive control action is not used, the index of performance corresponds to

$$IP(t) = \int_0^t \left(J\left(\mathbf{h}(\tau), \mathbf{q}(\tau), c(\tau), p(\tau)\right) - J^*(c(\tau), p(\tau))\right) \quad (28)$$

where $J^*(c(\tau), p(\tau))$ corresponds to the optimal cost at time τ (CEA IFAC, 2014).

Time (s)	Costs	Time (s)	$a_{f,1}$ (m^2)
0 to 1200	$c = 1.5$, $p = 20$	0 to 1800	0
1201 to 2400	$c = 1$, $p = 40$	1801 to 3000	$1e-05$
2401 to 3600	$c = 0.5$, $p = 10$	3001 to 4200	$2.5e-05$
3601 to 4800	$c = 1$, $p = 20$	4201 to 4800	$5e-05$

Table 1. Economic costs and fault size.

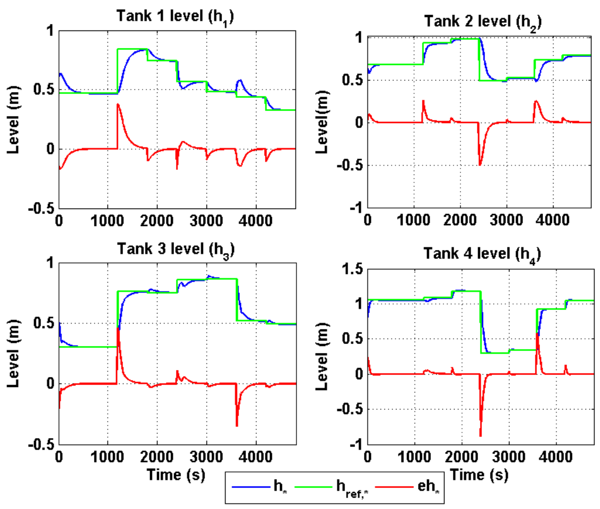

Fig. 2. States evolution for the closed loop system.

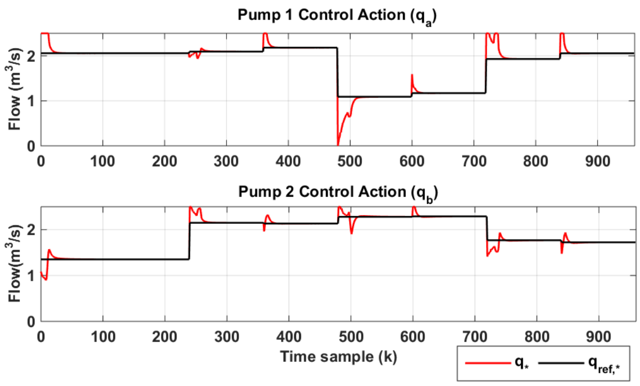

Fig. 3. Control contributions of the closed system.

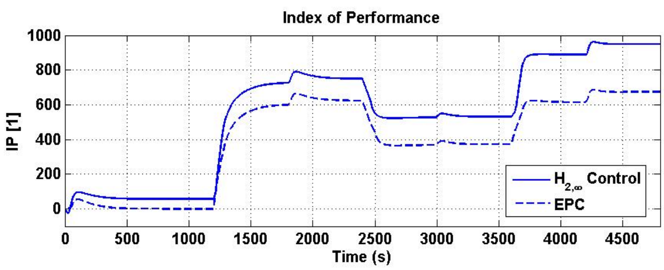

Fig. 4. Index of performance evolution.

7. CONCLUSIONS

In this work, a FTC methodology for non-linear discrete-time systems that can be modeled as LPV systems has been proposed. The proposed approach relies on the modeling of faults as additional scheduling parameters of the LPV model for the controlled system and it uses a triple loop architecture. The inner control loop is designed by means of the standard $\mathcal{H}_2/\mathcal{H}_\infty$ control methodologies based on LMIs. The design takes into account a prespecified set of faults and the ranges of their magnitudes that are wanted to be tolerated and it assumes the availability of on-line fault estimations provided by a FDI module. The resulting controller tries to compensate the system faults in order to maintain a satisfactory closed-loop dynamic performance, but it does not take into account possible system input and state constraints associated to actuator saturation and other physical limitations. And, when it is needed, it applies suitable additive predictive control actions that enlarge the invariant set, trying to assure that the current state remains inside the enlarged invariant set. Finally, an outer loop implements a model reference control that allows reference tracking. The use of the proposed FTC methodology has been illustrated through its application to the well-known quadruple tank system benchmark.

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