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Title: The Isoperimetric Problem in Johnson Graphs

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UNIVERSITAT POLITÈCNICA DE CATALUNYA

Master Thesis

The Isoperimetric Problem in Johnson Graphs

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ABSTRACT. It has been recently proved that the connectivity of distance regular graphs is the degree of the graph. We study the Johnson graphs $J(n, m)$, which are not only distance regular but distance transitive, with the aim to analyze deeper connectivity properties in this class.

The vertex k-connectivity of a graph G is the minimum number of vertices that have to be removed in order to separate the graph into two sets of at least k vertices in each one. The isoperimetric function $\mu_G(k)$ of a graph G is the minimum boundary among all subsets of vertices of fixed cardinality k. We give the value of the isoperimetric function of the Johnson graph $J(n, m)$ for values of k of the form $\binom{t}{m}$, and provide lower and upper bounds for this function for a wide range of its parameter. The computation of the isoperimetric function is used to study the k-connectivity of the Johnson graphs as well. We will see that the k-connectivity grows very fast with k, providing much sensible information about the robustness of these graphs than just the ordinary connectivity.

In order to study the isoperimetric function of Johnson graphs we use combinatorial and spectral tools. The combinatorial tools are based on compression techniques, which allow us to transform sets of vertices without increasing their boundary. In the compression process we will show that sets of vertices that induce Johnson subgraphs are optimal with respect to the isoperimetric problem. Upper bounds are obtained by displaying nested families of sets which interpolate optimal ones. The spectral tools are used to obtain lower bounds for the isoperimetric function. These tools allow us to display completely the isoperimetric function for Johnson graphs $J(n, 3)$.

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CHAPTER 1

Introduction

The study of connectivity properties of graphs is a wide subject in graph theory. The survey [**3**] gives a panoramic view of the many results in the area in recent years. The classical notion of connectivity of a graph is defined as the minimum number of vertices whose deletion disconnects the graph. Stronger measures of connectivity have been proposed in the literature, see e.g. [**2**]. In this work we will consider two of these stronger measures, the k–connectivity and the isoperimetric function.

Let $G = (V, E)$ be a graph, usually considered to be simple (no loops nor multiple edges) and undirected. Given a set $X \subset V$ of vertices, we denote by

$$\partial X = \{y \in V \setminus X : d(X, y) = 1\},$$

the *boundary* of X, and by

$$B(X) = \{y \in V : d(X, y) \leq 1\} = X \cup \partial X,$$

the *ball* of X. we write ∂_G and B_G when the reference to G has to be made explicit.

The k–connectivity of G is defined to be

$$\kappa_k(G) = \min\{|\partial X| : k \leq |X| \leq |B(X)| \leq |V| - k\}.$$

In other words, the k–connectivity of G is the minimum cardinality of a set of vertices which separate the graph into two sets, each with cardinality at least k. This notion was introduced by Hamidoune (see e.g. [**5**]) and is close to the notion of extraconnectivity [**2**], where the additional requirement that the separator leaves connected components of size at least k. For $k = 1$, $\kappa_1(G)$ is just the ordinary connectivity of the graph.

The *isoperimetric function* of G is defined as

$$\mu(k) = \min\{|\partial X| : X \subset V,\ |X| = k\},$$

that is, $\mu(k)$ is the size of the smallest boundary among sets of vertices with cardinality precisely k. It is clear that

$$\kappa_k(G) = \min\{\mu(t) : k \leq t \leq |V| - \mu(t) - k\}.$$

Hence, the knowledge of the isoperimetric function allows us to obtain the k–connectivity of the graph.

The isoperimetric function is known only for some rather specific classes of graphs. One of the seminal results is the exact determination of the isoperimetric function for the n–cube obtained by Harper [**13**] in 1966. Analogous results were obtained for cartesian products of chains by Lindstrom [**7**], cartesian products of even cycles by Riordan [**9**], and some other cartesian products by Bezrukov and Serra [**34**]. The survey on discrete isoperimetric problems by Bezrukov [**6**] gives a good panoramic view of the isoperimetric problem in discrete spaces.

The above generalizations arise from the fact that the n–cube can be seen as the cartesian product of edges. However the n–cube is also one of the most important examples of distance regular graphs. A graph G is said to be *distance–regular* if, for any two vertices v and w, the number of vertices at distance j from v and at distance k from w depends only upon j, k, and the distance $i = d(v, w)$ between v and w. An analogous definition is the following one. Let x_0 be a distinguished vertex in the graph and denote by

$$S_i = \{y \in V : d(x_0, y) = i\},$$

the sphere of radius i centered at x_0. The graph is distance regular if, for each i and every vertex $y \in S_i$, the number of neighbours of y in S_{i-1} on S_i and on S_{i+1} do not depend on y. These numbers are usually denoted by

$$c_i = |\partial y \cap S_{i-1}|,\ a_i = |\partial y \cap S_i|,\ b_i = |\partial y \cap S_{i+1}|,$$

with the convention that $a_0 = b_0 = 0$. Of course distance regular graphs are regular, and the degree is $b_0 = k$, while $a_i + b_i + c_i = k$ for all suitable i. The monograph by Brouwer and Cohen [**14**] is an excellent reference for the topic of distance regular graphs.

Distance regular graphs enjoy combinatorial regularity properties which make them a very interesting family of graphs for which detailed properties should be particularly accessible. In spite of this, their connectivity properties have been an open problem for long time. The first result in the study of connectivity properties of distance regular graphs was obtained by Cioaba, Kim and Koolen [**1**], who proved that the connectivity of strongly regular graphs, the class of distance regular graphs of diameter two, equals the degree. They actually prove more, that the only separating sets with cardinality k, the degree, isolate a single vertex. In terms of the k–connectivity this is equivalent to the statement that $\kappa_2(G) > k$. Some years later the same result was extended by Brouwer and Koolen [**4**] to the general class of distance regular graphs.

In view of the much larger values of the isoperimetric function in the particular class of n–cubes, it is likely that stronger bounds on the k–connectivity can be obtained in the general class of distance regular graphs. The purpose of this work is to explore the situation in the case of the Johnson graphs defined as follows.

DEFINITION 1.1. Given $n, m \in \mathbb{N}$ with $m \leq \frac{n}{2}$, the Johnson graphs $J(n, m)$ is the graph defined by:

(1) The vertex set is the set of all subsets of $[n]$ with cardinality exactly m.
(2) Two vertices are adjacent if and only if the symmetric difference of the corresponding sets has cardinality two.

Johnson graphs arise from the so–called Johnson association schemes and are objects with a particular high degree of symmetry. However the study of their connectivity properties, beyond the ones provided by the fact that they are distance regular graphs, is quite open. In the next section the general properties of Johnson graphs will be discussed. It will be shown that, in fact, the Johnson graphs are not only distance regular but also distance–transitive. A graph is distance–transitive if for any two pairs of vertices x, y and u, v such that $d(x, y) = d(u, v)$, there is an automorphism of the graph which send one pair to the other one. Distance-transitivity is a very strong property and it is known that there is only a finite number of distance transitive graphs of given degree.

The plan of the work is as follows. In Chapter 2 we set the notation we are going to use and discuss the basic properties of Johnson graphs. In Chapter 3 we introduce and analyze compression operators which will be our main combinatorial tool in the analysis of the isoperimetric function for Johnson graphs. The main result in the chapter says that optimal sets with respect to the isoperimetric problem can be found among stable sets under some given classes of compression operators. This result is used in Chapter 4 to determine the values of the isoperimetric function $\mu(k)$ in the Johnson graph $J(n, m)$ for values of k of the form $k = \binom{t}{m}$. Moreover we will identify optimal sets for these values as sets of vertiecs inducing subgraphs isomorphic to Johnson graphs. In Chapter 5 we will show how to interpolate between two consecutive optimal sets to obtain an upper bound for the isoperimetric function in a dense set of values of its argument. Lower bounds for the isoperimetric function will be obtained via spectral techniques. There is a well–known connection between isoperimetric properties and spectral analysis of graphs whch allows us to obtain good lower bounds, particularly for cardinalities of sets which are not too big nor too small. Finally, in Chapter 6 we analyze in particular the cases of the Johnson graphs $J(n, 2)$ and $J(n, 3)$ were more specific results can be given.

CHAPTER 2

Johnson graphs

In this chapter we introduce the Johnson graphs and discuss some of their basic properties. As it has been mentioned in the introduction, Johnson graphs arise from the association schemes with the same name. They are defined as follows.

Definition 2.1. Given $n, m \in \mathbb{N}$ with $m \leq \frac{n}{2}$, the Johnson graph $J(n,m)$ is defined by:

(1) The vertex set is the set of all subsets of $[n]$ with cardinality exactly m.
(2) Two vertices are adjacent if and only if the symmetric difference of the corresponding sets is two.

It follows from the definition that, for $m = 1$, the Johsnon graph $J(n,1)$ is the complete graph K_n. For $m = 2$ the Johnson graph $J(n,2)$ is the line graph of teh complete graph on n vertices, also known as the triangular graph $T(n)$. Thus, for instance, $J(5,2)$ is the complement of the Petersen graph, displayed in Figure 1, and, in general, $J(n,2)$ is the complement of the Kneser graph $K(n,2)$.

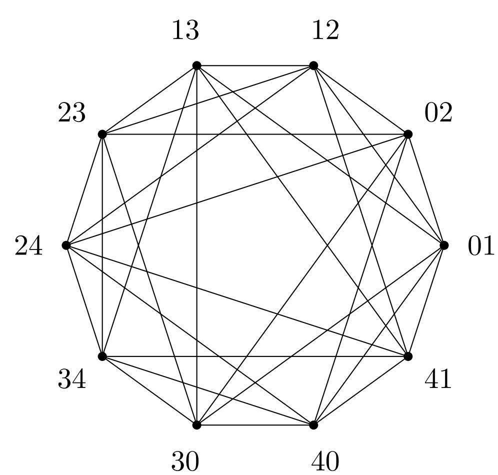


Figure 1. The Johnson graph $J(5,2)$.

Throughout this work we will identify subsets of $[n]$ by its characteristic vectors, which will be denoted by lower boldface characters. Thus the subset $\{1,2,3\} \subset \{1,2,3,4,5\}$ is denoted by

$\mathbf{x} = (1, 1, 1, 0, 0)$. We usually write $\mathbf{x} = (x_1, \ldots, x_n)$ so x_i denotes the i–th coordinate of $\mathbf{x}$, which is 1 if i is in the corresponding set and 0 otherwise. We denote the support of $\mathbf{x}$ as

$$supp(\mathbf{x}) = \{i : x_i = 1\}.$$

We use the XOR operation

$$\mathbf{x} \oplus \mathbf{y}$$

of binary vectors defined as $(\mathbf{x} \oplus \mathbf{y})_i = x_i + y_i \pmod 2$. It corresponds to the symmetric difference between the corresponding sets. Finally we write

$$|\mathbf{x}| = \sum_i x_i$$

for the weight (or ℓ_1 norm) of a vector. It is the cardinality of its support. With this notation, the set of vertices of the Johnson graph are all vectors with weigth m, and we refer to their support as teh set corresponding to the vector. We also denote by $\mathbf{e}_1, \ldots, \mathbf{e}_n$ the unit vectors

$$\mathbf{e}_i = (0, \ldots, 0, \overset{i)}{1}, 0, \ldots, 0),\ i = 1, \ldots, n.$$

Thus, the neighbors of $\mathbf{x}$ in $J(n, m)$ are the vectors

$$\mathbf{x} \oplus \mathbf{e}_i \oplus \mathbf{e}_j,$$

for each pair i, j such that $x_i + x_j = 1$.

Some elementary properties of the Johnson graph are summarized in the next proposition.

PROPOSITION 2.1. *Given the graph $J = J(n, m)$ with $m \leq n/2$*

(1) $|V(J)| = \binom{n}{m}$
(2) *J is a regular graph with degree $m \cdot (n - m)$.*
(3) $E(J) = \frac{1}{2} m(n - m)\binom{n}{m}$
(4) *Given $\mathbf{x}, \mathbf{y} \in V(J)$, then $d(\mathbf{x}, \mathbf{y}) = \frac{1}{2}|\mathbf{x} \oplus \mathbf{y}|$*
(5) *The diameter of J is m.*

PROOF. The two first properties come directly from the definition. The third one can be derived from (1) and (2).

Given two vertices $\mathbf{x}$ and $\mathbf{y}$, a minimum length path from $\mathbf{x}$ to $\mathbf{y}$ requires exchanging the elements in the symmetric difference of the sets corresponding to $\mathbf{x}$ and $\mathbf{y}$. Each step in the path decreases the symmetric difference by two and therefore, the distance $d(\mathbf{x}, \mathbf{y})$ is just $\frac{1}{2}|\mathbf{x} \oplus \mathbf{y}|$. In particular, since $m \leq n/2$, the maximum distance between two vertices is $m = diam(J)$. □

One of the key properties of the Johnson graph is that they have a high degree of symmetry. It is clear that every permutation $\sigma \in Sym(n)$ in the symmetric group induces in a natural way a permutation of the vertices of the graph:

$$\sigma(x_1, \ldots, x_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)}),$$

which is an automorphism of the graph. By using these automorphisms one can see that $J(n, m)$ is distance transitive.

PROPOSITION 2.2. *The Johnson graph $J(n, m)$ is distance transitive.*

PROOF. Let $\mathbf{x}_1, \mathbf{x}_2$ and $\mathbf{y}_1, \mathbf{y}_2$ two pairs of vertices of $J(n, m)$ at mutual distance d,

$$d(\mathbf{x}_1, \mathbf{x}_2) = d(\mathbf{y}_1, \mathbf{y}_2) = d.$$

We consider three disjoint subsets of the set $[n]$. Let be $A_x = supp(\mathbf{x}_1) \cap supp(\mathbf{x}_2)$, $B_x = supp(\mathbf{x}_1) \setminus supp(\mathbf{x}_2)$ and $C_x = supp(\mathbf{x}_2) \setminus supp(\mathbf{x}_1)$. Let us define in the same way the sets A_y, B_y and C_y for the vertices $\mathbf{y}_1$ and $\mathbf{y}_2$.

Since $d(\mathbf{x}_1, \mathbf{x}_2) = d(\mathbf{y}_1, \mathbf{y}_2)$, we have $|A_x| = |A_y|, |B_x| = |B_y|$ and $|C_x| = |C_y|$ and, by definition, the two collections of three sets are pairwise distinct. Therefore, there is a permutation $\sigma \in Sym(n)$ whith

$$\sigma(A_x) = A_y,\ \sigma(B_x) = B_y, \ \text{ and } \sigma(C_x) = C_y.$$

As we have already mentioned, this permutation induces an automorphism of the graph $J(n, m)$ with $\sigma(\mathbf{x}_i) = \mathbf{y}_i$, $i = 1, 2$. □

Being distance transitive, the Johnson graph $J(n, m)$ is in particular vertex and arc–transitive and distance regular. It follows that the subgraph induced by two vertices $\mathbf{x}, \mathbf{y}$ at distance d and all vertices lying in minimal paths connecting $\mathbf{x}$ and $\mathbf{y}$ are isomorphic. For $d = 2$ the resulting graph is the octahedron, isomorphic to $J(4, 2)$ as displayed in Figure 2. For general d we call such graph a d–octahedron, which is isomorphic to $J(2d, d)$.

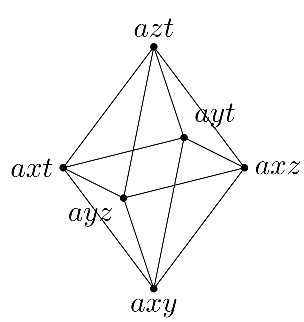


FIGURE 2. The octahedron induced by two vertices at distance two.

CHAPTER 3

Reducing the neighbourhood of a subset of the Johnson graph

1. The transformations T_{ij} and T^*_{ij}

Shifting techniques are one of the key ingredients in the study of set systems. They were initially introduced in the original proof of the Erdős–Ko–Rado theorem [**12**] and have been particularly used in the solution of the isoperimetric problem for hypercubes due to Frankl, see e.g. [**11, 8**].

We shall use the shifting transformations T_{ij} and T^*_{ij} defined below. The main purpose of this chapter is to show that, for each set S f vertices, the boundary of a transformed set $T_{ij}(S)$ in the Johnson graph $J(n,m)$ is not larger than the boundary of S.

We recall that subsets of $[1,n]$ are identified by their characteristic vectors $\mathbf{x} \in \{0,1\}^n$, and that $\mathbf{e}_i$ denotes the n–vector with all entries equal to zero but the i–th one which equals one.

Definition 3.1. Let S be a family of sets in $2^{[n]}$ and let $i, j \in [n]$ with $i \neq j$. For each $\mathbf{x} \in \{0,1\}^n$ we define

$$T^*_{ij}(\mathbf{x}) = \begin{cases} \mathbf{x} \oplus \mathbf{e}_i \oplus \mathbf{e}_j & \text{if } x_i = 0 \text{ and } x_j = 1 \\ \mathbf{x} & \text{otherwise} \end{cases}$$

We also define

$$T_{ij}(S) = \{T_{ij}(S, \mathbf{x}) : \mathbf{x} \in S\},$$

where, for $\mathbf{x} \in S$,

$$T_{ij}(S, \mathbf{x}) = \begin{cases} T^*_{ij}(\mathbf{x}) & \text{if } T^*_{ij}(\mathbf{x}) \notin S \\ \mathbf{x} & \text{otherwise} \end{cases}$$

We next prove some basic properties of the transformations T^*_{ij} and T_{ij} which will be useful later on.

Lemma 3.1. *The following statements hold for the Johnson graph $J(n,m)$ and each pair $i, j \in [1,n]$ of distinct elements.*

(i) T^*_{ij} *is a contraction: for every pair of vertices $\mathbf{x}, \mathbf{y} \in V(J(n,m))$ we have*

$$d(T^*_{ij}(\mathbf{x}), T^*_{ij}(\mathbf{y})) \leq d(\mathbf{x}, \mathbf{y}).$$

In particular, for each subset S containing $\mathbf{x}$ and $\mathbf{y}$, we also have

$$d(T_{ij}(S,\mathbf{x}), T_{ij}(S,\mathbf{y})) \leq d(\mathbf{x},\mathbf{y}).$$

(ii) *For each subset $S \subset V(J(n,m))$ we have $|T(S)| = |S|$.*

PROOF. (i) Recall that the distance in the Johnson graph is given by

$$d(\mathbf{x},\mathbf{y}) = \frac{1}{2}|\mathbf{x} \oplus \mathbf{y}|.$$

If the transformation T^*_{ij} either leaves both vertices $\mathbf{x}$ and $\mathbf{y}$ invariant or changes both of them then $|\mathbf{x} \oplus \mathbf{y}| = |T^*_i(\mathbf{x}) \oplus T^*_{ij}(\mathbf{y})|$ and their distance is preserved by T^*_{ij}. On the other hand, if $T^*_{ij}(\mathbf{x}) = \mathbf{x} \oplus \mathbf{e}_i \oplus \mathbf{e}_j$ and $T^*_{ij}(\mathbf{y}) = \mathbf{y}$ then the transformed vectors agree in the coordinate j where $\mathbf{x}$ and $\mathbf{y}$ disagree and its distance is reduced by one unit.

The same argument proves that $d(T_{ij}(S,\mathbf{x}), T_{ij}(S,\mathbf{y})) \leq d(\mathbf{x},\mathbf{y})$.

(ii) This follows directly from the definition of T_{ij}.

□

We now can prove that the switching transformation T_{ij} on a set S of vertices of the Johnson graph $J(n,m)$ does not enlarge its neighbourhood.

THEOREM 3.2. *Let S be a set of vertices in the Johnson graph $J(n,m)$. For each $1 \leq j < i \leq n$ we have*

$$B(T_{ij}(S)) \subseteq T_{ij}(B(S)). \tag{1}$$

In particular,

$$|\partial(T_{ij}(S))| \leq |\partial(S)|. \tag{2}$$

PROOF. We will show that, for each $\mathbf{y} \in T_{ij}(S)$, we have $B(\mathbf{y}) \subseteq T_{ij}(B(S))$. This will prove (1). We observe that then (2) follows since

$$|\partial(T_{ij}(S))| = |B(T_{ij}(S))| - |T_{ij}(S)| \leq |T_{ij}(B(S))| - |T_{ij}(S)| = |B(S)| - |S| = |\partial S|.$$

Let $\mathbf{x}$ be the element in S such that $\mathbf{y} = T_{ij}(S,\mathbf{x})$. We consider two cases.

Case 1. $(x_i, x_j) \neq (0,1)$. In this case we certainly have $\mathbf{y} = \mathbf{x}$. Moreover, for each $\mathbf{z} \in N(\mathbf{x})$ such that $z_i = 0$ and $z_j = 1$, we have $\mathbf{z} \oplus \mathbf{e}_i \oplus \mathbf{e}_j \in B(\mathbf{x})$. Therefore, since $B(\mathbf{x}) \subseteq B(S)$, the transformation $T_{ij}(B(S), \cdot)$ leaves all vectors in $B(\mathbf{x})$ invariant. Hence,

$$B(\mathbf{y}) = B(\mathbf{x}) = T_{ij}(B(S), B(\mathbf{x})) \subseteq T_{ij}(B(S)).$$

Case 2. Suppose now that $x_i = 0$ and $x_j = 1$. Then $\mathbf{z} = \mathbf{x} \oplus \mathbf{e}_i \oplus \mathbf{e}_j$ is the only neighbour of $\mathbf{x}$ with $z_i = 1$ and $z_j = 0$.

Case 2.1 If $\mathbf{y} = \mathbf{x}$ then, by the definition of $T_{ij}(S, \cdot)$, we have $\mathbf{z} \in S$ and $T_{ij}(S, \mathbf{z}) = \mathbf{z}$. Observe that every neighbour $\mathbf{z}'$ of $\mathbf{x}$ is left invariant by $T_{ij}(B(S), \cdot)$. This is clearly the case if $(z'_i, z'_j) \neq (0, 1)$ and, if $(z_i, z_j) = (0, 1)$, because we then have $\mathbf{z}'' = \mathbf{z}' \oplus \mathbf{e}_i \oplus \mathbf{e}_j \in B(\mathbf{z}) \subset B(S)$. Hence

$$B(\mathbf{y}) = B(\mathbf{x}) \subseteq T_{ij}(B(S), B(\mathbf{x})) \subseteq T_{ij}(B(S)).$$

Case 2.2 Suppose that $\mathbf{y} \neq \mathbf{x}$. Then $\mathbf{y} \notin S$ but $\mathbf{y} \in B(\mathbf{x}) \subseteq B(S)$. Each neighbour $\mathbf{z}$ of $\mathbf{y}$ distinct from $\mathbf{x}$ is of the form $\mathbf{z} = \mathbf{z}' \oplus \mathbf{e}_i \oplus \mathbf{e}_j$ for some neighbour $\mathbf{z}'$ of $\mathbf{x}$ and therefore it belongs to $T_{ij}(B(S))$. For $\mathbf{x}$ itself we have $T_{ij}(B(S), \mathbf{x}) = \mathbf{x}$ because $\mathbf{y} = T^*_{ij}(\mathbf{x}) \in B(S)$. Thus we again have $B(\mathbf{y}) \subset T_{ij}(B(S)$. This completes the proof of (1). $\square$

Example 3.1. Let us consider $J(6, 2)$ and the set $S = \{\{1, 2\}, \{1, 3\}, \{3, 4\}\}$. The boundary of this set is

$$\partial S = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 5\}, \{3, 6\}, \{4, 5\}, \{4, 6\}\},$$
$$|\partial S| = 11.$$

Choose $x = 4$ and $y = 2$ for the transformation, and apply T_{42} (see Fig. 3.1). We have $T_{42}S = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$; and

$$\partial T_{42}S = \{\{1, 4\}, \{1, 5\}, \{1, 6\}, \{2, 4\}, \{2, 5\}, \{2, 6\}, \{3, 4\}, \{3, 5\}, \{3, 6\}\}$$
$$|\partial S| = 9.$$

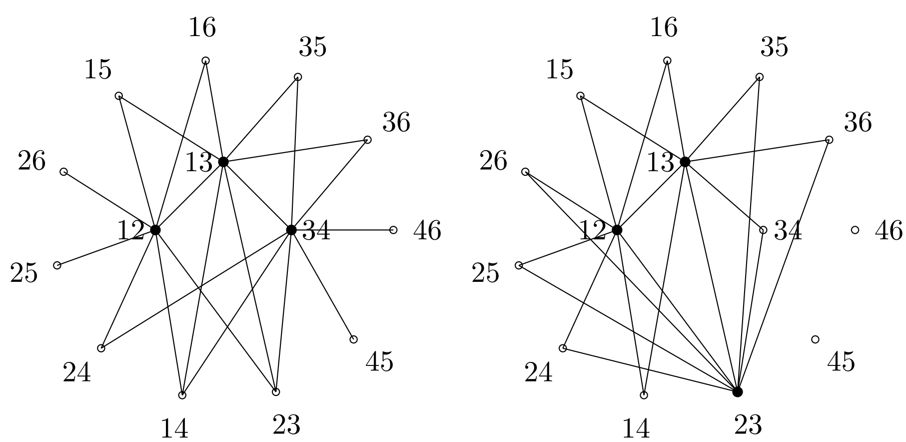


Figure 1. The effect of compression T_{42} on $S = \{12, 13, 34\}$ on the left hand side of the picture.

We can see how the vertex $\{3, 4\}$ belongs to the boundary of $T_{42}S$ but can not be the transformed by T^*_{42} of any vertex of ∂S. This is the ghost neighbour, which replaces the vertex $\{2, 3\}$. They are just the vertices transformed by T_{42}. The other two vertices that the boundary lose are the vertices touching the element 4, the vertices $\{4, 5\}$ and $\{4, 6\}$.

Now we know how the transformation T_{ij} works, we are going to apply a sequence of these transformations to a given set S. Consider the set of transformations

$$\mathcal{T} = \{T_{ij} : 1 \leq j < i \leq n\}.$$

By the condition $j < i$, we eventually reach a set S' such that $T_{ij}(S') = S'$ for each $T_{ij} \in \mathcal{T}$. This kind of sets are called $\mathcal{T}$–**stable sets**. It follows from Theorem 3.2 that, for each given cardinality, we will find a set of vertices of $J(n, m)$ with smallest boundary among stable sets. We conclude this chapter by stating the main result which motivates the use of compression techniques.

Theorem 3.3. *For every cardinality k there is an optimal set in the Johnson graph which is $\mathcal{T}$–stable for $\mathcal{T} = \{T_{ij} : 1 \leq j < i \leq n\}$.*

CHAPTER 4

Critical cardinals

In this chapter we determine optimal sets of the Johsnon graph $J(n,m)$ for cardinalities k of the form $k = \binom{t}{m}$, $m \le t \le n$. We first observe that sets of vertices inducing a subgraph isomorphic to a Johnson graph must be (isomorphic to) $\mathcal{T}$–stable sets.

LEMMA 4.1. *If the graph induced by $S \subset V(J(n,m))$ is the Johnson graph $J(t,m)$ on $\{i_1,\dots,i_t\} \subset [1,n]$ for some t with $m \le t \le n$ on , then S is isomorphic to a $\mathcal{T}$–stable set.*

PROOF. We may assume that $\{i_1,\dots,i_t\} = [1,t]$. Then, for every $\mathbf{x} \in S$ and each pair $1 \le j < i \le n$ with $x_i = 1$ and $x_j = 0$ we have $\mathbf{x} \oplus \mathbf{e}_i \oplus \mathbf{e}_j \in S$ and thus $T_{ij}(S,\mathbf{x}) = \mathbf{x}$. □

It turns out that these stable sets are actually optimal sets. The next Theorem is the main result in this chapter.

THEOREM 4.2. *Let $J(n,m)$ be a Johnson graph and let t be an integer with $m \le t \le n$. The set $S \subset V(J(n,m))$ consisiting of all m–subsets of $[t]$ is an optimal set of $J(n,m)$ with cardinality $\binom{t}{m}$.*

The proof of Theorem 4.2 will follow from the more general statement below.

LEMMA 4.3. *Let k be a positive integer with $\binom{t-1}{m} < k \le \binom{t}{m}$. There is an optimal set S with cardinality k such that*

$$\binom{[t-1]}{m} \subset S \subseteq \binom{[t]}{m}.$$

PROOF. Choose a $\mathcal{T}$–stable optimal set S which maximizes the number $|S \cap \binom{[t]}{m}|$ of elements in $\binom{[t]}{m}$. Suppose that $S \not\subseteq \binom{[t]}{m}$ and we will obtain a contradiction. Denote by

$$S_0 = S \cap \binom{[t]}{m} \ \text{ and } \ S_1 = S \setminus S_0.$$

We observe that, if either $i,j \le t$ or $i \le t < j$, then $S' = T_{ij}(S)$ satisfies

$$|S'_0| = |S' \cap \binom{[t]}{m}| \ge |S_0| \ \text{ and } \ |\partial S'| \le |\partial S|.$$

The first part of the sentence comes from the conditions on i, j, and the second part by Theorem 3.2.

Let $\mathbf{x} \in \binom{[t]}{m} \setminus S_0$ be a vertex in $\binom{[t]}{m}$ and not in S, and let $\mathbf{y} \in S_1$ a vertex whose support has largest intersection with the support of $\mathbf{x}$ among the vertices in S_1. Let

$$A = (supp(\mathbf{y}) \setminus supp(\mathbf{x})) \cap [t], \ B = supp(\mathbf{y}) \setminus [t], \text{ and } C = supp(\mathbf{x}) \setminus supp(\mathbf{y}).$$

By our choice of $\mathbf{y}$, the cardinality of A is smallest for all choices of elements in S. If $A \neq \emptyset$, choose $i \in A$ and $j \in C$ and consider the transformation T_{ij}. We note that $T_{ij}(\mathbf{y}, S) = \mathbf{y} \oplus \mathbf{e}_i \oplus \mathbf{e}_j$ by the minimality of $|A|$ (otherwise there is an element in S whose support has larger intersection with the support of $\mathbf{x}$.)

Denote by $S' = T_{ij}(S)$ and define similarly $S'_0 = S' \cap \binom{[t]}{m}$ and $S'_1 = S' \setminus S'_0$. Choose again an element $\mathbf{y}' \in S'_1$ whose support has largest intersection with the support of $\mathbf{x}$. By defining $|A'| = (supp(\mathbf{y}') \setminus supp(\mathbf{x})) \cap [t]$ we clearly have $|A'| < |A|$, because $\mathbf{y} \oplus \mathbf{e}_i \oplus \mathbf{e}_j$ belongs to S'. By iterating this procedure we eventually reach a set S' which contains an element $\mathbf{y}'$ whose support has a largest intersection with the suport of $\mathbf{x}$ and with

$$supp(\mathbf{y}') \cap [t] \subseteq supp(\mathbf{x}).$$

If $B' = supp(\mathbf{y}') \setminus [t] = \emptyset$ then $\mathbf{y}' = \mathbf{x}$ and $|S'_0| > |S_0|$, contradicting our choice of S that it maximizes its intesection with $\binom{[t]}{m}$. Otherwise we consider transformations T_{ij} with $i \in B' = supp(\mathbf{y}') \setminus [t]$ and $j \in C' = supp(\mathbf{x}) \setminus supp(\mathbf{y})$ which again increase the intersection with $\mathbf{x}$. At the end of the process we eventually reach a set with one more element in $\binom{[t]}{m}$ without increasing its boundary, so the transformed set is still optimal. This contradicts our initial choice of S.

Since in the above argument the choice of $\mathbf{x}$ is arbitrary, we can select $\mathbf{x} \in \binom{[t-1]}{m} \setminus S$ as long as there are elements in this set. Therefore we can also construct an optimal set S' with $\binom{[t-1]}{m} \subset S'$. This completes the proof. $\square$

Once the optimal sets with cardinality $\binom{[t]}{m}$ have been identified, the isoperimetric function for these values can be easily determined.

PROPOSITION 4.4. *The isoperimetric function of the Johnson graph $J(n, m)$ satisfies*

$$\mu \binom{t}{m} = \binom{t}{m-1}(n-t), \ t = m, m+1, \ldots, n.$$

PROOF. Let $S = \binom{[t]}{m}$. Given a vertex in ∂S, its support has $m-1$ elements in common with any vertex of S and exactly one element of the support lies in $[n] \setminus [t]$. $\square$

CHAPTER 5

Boundary of optimal subsets with non critical cardinality

In this chapter we determine the isoperimetric function of the Johnson graph $J(n,m)$ for cardinalities different from $\binom{t}{m}$. For this we will use the fact, proved in Lemma 4.3, that, if $\binom{t}{m} < k < \binom{t+1}{m}$, then there is an optimal set S satisfying

$$\binom{[t-1]}{m} \subset S \subset \binom{[t]}{m}. \tag{3}$$

The above equation gives general upper and lower bounds for the isoperimetric function.

PROPOSITION 5.1. *If* $\binom{t-1}{m} < k < \binom{t}{m}$ *then*

$$\binom{t-1}{m-1}(n-t) + \binom{t}{m} - k \le \mu(k) \le \binom{t}{m-1}(n-t) + \binom{t}{m} - k.$$

PROOF. Let S be an optimal set with cardinality k satisfying (3). Let $S_0 = \binom{[t]}{m}$. Since $S \subset S_0$ we have $\partial S \subset \partial S_0 \cup (S_0 \setminus S)$. This gives the upper bound.

For the lower bound, let $S_0' = \binom{[t-1]}{m}$. Since $S_0' \subset S \subset S_0$, we have $\partial S_0' \setminus S_0 \subset \partial S$. Moreover, since $S_0 \subset \partial S_0'$ we have $\partial S \supset (\partial S_0' \setminus S_0) \cup (S_0 \setminus S)$. The cardinality of $\partial S_0' \setminus S_0$ is $\binom{t-1}{m-1}(n-t)$ and the lower bound follows. $\square$

In what follows we will see that the upper bound can be tight. We will give a tight improvement of the lower bound for some values of k.

1. Taking out vertices of a critical set

We first determine the isoperimetric function of $J(n,m)$ for cardinalities below but close to $\binom{t}{m}$.

PROPOSITION 5.2. *Let* $k < \binom{t}{m}$*. If* $\binom{t}{m} - k \le t - m$ *then*

$$\mu(k) = \mu\left(\binom{t}{m}\right) + \binom{t}{m} - k = \binom{t}{m-1}(n-t) + \binom{t}{m} - k. \tag{4}$$

PROOF. Let S be an optimal set with cardinality k satisfying (3). Let $S_0 = \binom{[t]}{m}$. We observe that every vertex $\mathbf{w}$ in ∂S_0 has precisely $t-(m-1)$ neighbours in S_0 (one for each choice of a coordinate in $[t] \setminus supp(\mathbf{x})$.) Since $|S_0 \setminus S| \leq t-m$, such a vertex $\mathbf{w}$ still belongs to ∂S. On the other hand, since S contains $\binom{[t-1]}{m}$ every vertex $\mathbf{u} \in S_0 \setminus S$ belongs to ∂S. It follows that $\partial S = \partial S_0 \cup (S_0 \setminus S)$. $\square$

The above proposition shows that the isoperimetric function of the Johnson graph is not monotone.

2. Adding vertices to a critical set

We next give the value of the isoperimetric function for values of k above $\binom{t-1}{m}$ but close to it.

PROPOSITION 5.3. *If* $x = k - \binom{t-1}{m} \leq t-m+1$ *then*

$$\mu(k) = \binom{t-1}{m-1}(n-t) + \binom{t}{m} - k + (x(m-2)+1)(n-t). \tag{5}$$

PROOF. Choose an optimal set S satisfying (3). Let $S_0 = \binom{[t]}{m}$, $S_0' = \binom{[t-1]}{m}$ and set $U = S \setminus S_0'$. As shown in the proof of Proposition 5.1, we have

$$(\partial S_0' \setminus S_0) \cup (S_0 \setminus S) \subset \partial S.$$

Let us denote by $W = (\partial S_0' \setminus S_0) \cup (S_0 \setminus S)$ this subset of ∂S. Each vertex $\mathbf{w}$ in U contributes to the boundary of S in $(m-1)(n-t)$ vertices which do not belong to W, namely the vertices which contain t in its support. They can be found by choosing $m-2$ elements in $supp(\mathbf{w}) \cap [t-1]$ and adding one element in $[n] \setminus [t]$. This shows that $|\partial S|$ is at least $|W| + (m-1)(n-t)$, and the equality holds for $x = 1$.

We observe that every two vertices $\mathbf{w}, \mathbf{w}'$ in $S \setminus S_0'$ have at most $n-t$ common vertices in $\partial S \setminus W$, because the support of every vertex in this last set contains precisely $m-2$ elements in $[t-1]$ in addition to the element t. This shows that

$$|\partial S \setminus W| \geq x(m-1)(n-t) - (x-1)(n-t) = (x(m-2)+1)(n-t), \tag{6}$$

and therefore the righthand side of (5) is a lower bound for $\mu(k)$.

However we can find up to $m-t+1$ vertices in $S \setminus S_0$ with this same number of common neighbours in $S \setminus W$, for instance the vertices $\mathbf{w}_1, \ldots, \mathbf{w}_{m-t+1}$ with support

$$supp(\mathbf{w}_i) = \{1, 2, \ldots, m-2\} \cup \{i+m-2\} \cup \{t\},\ 1 \leq i < t-m+1.$$

By choosing U to contain x of those vertices, we have equality in (6) and therefore equality in (5) as claimed. $\square$

3. Some optimal subsets without critical cardinality

Let consider a optimal subset of vertices $S = \binom{[t]}{m}$ with $n > t > m$ and consider a vertex v of its boundary. This vertex v has exactly $m-1$ elements of $[1:t]$, lets call B_v to this block of $m-1$ elements. All the other vertices of the boundary which contains B are adjacent to the same subset of vertices of S.

In other words, there exist a partition of the vertices of ∂S such that all the vertices in the same class are adjacent to exactly the same vertices of S. It means that if we remove the vertices of S that we need for delete a vertex of the boundary, automatically, we will delete all the vertices of the class.

All this vertices of the same class contain the same block in common; and for delete this class of the boundary of S, we have to remove all the vertices of S that contains this block. We can see this like a bijection between the quotient set of the boundary and $J(t, m-1)$. We will study this bijection in the future.

For delete the first class of vertices, we just have to remove all the vertices that contain a block B_1. There are exactly $t-(m-1)$ vertices to remove.

For delete the second class of vertices of ∂S we have to remove all the vertices that contains another block B_2. Totally, there are $t-(m-1)$ vertices, but we can use some of the vertices that we removed for the first class of vertices. Given two different blocks, there are at most one vertex that contains both blocks, and this happen if and only if this two blocks have $m-2$ elements in common. Thus, for delete the second class of vertices, we just have to remove $t-(m-1)-1$ vertices.

Taking the bijection between the blocks and $J(t, m-1)$, we have that two blocks have a vertex that contains them if and only if their images are adjacent in $J(t, m-1)$. This is a prove for the following property:

PROPERTY 5.4. *The optimal way of chose the classes of vertices of the boundary for delete, is to maximize the number of edges in the induced subgraph of $J(t, m-1)$.*

In particular, the complete subgraphs correspond to optimal classes of vertices for delete. For use this particular case, we have to know that the maximum complete subgraph of $J(t, m-1)$ has size $t-(m-2)$. It means that we can remove $t-m+2$ classes of vertices knowing that for all the possible pair of blocks there exist a vertex containing both of them.

We remove $t-m+1$ for the first class, $t-m$ for the second, $\ldots t-m+2-i$ for the i class, $\ldots$. The last one of this set, for $i = t-m+2$, we have to remove 0 vertices, it means that if we delete $t-m+1$ classes, the $t-m+2$ class is deleted for free.

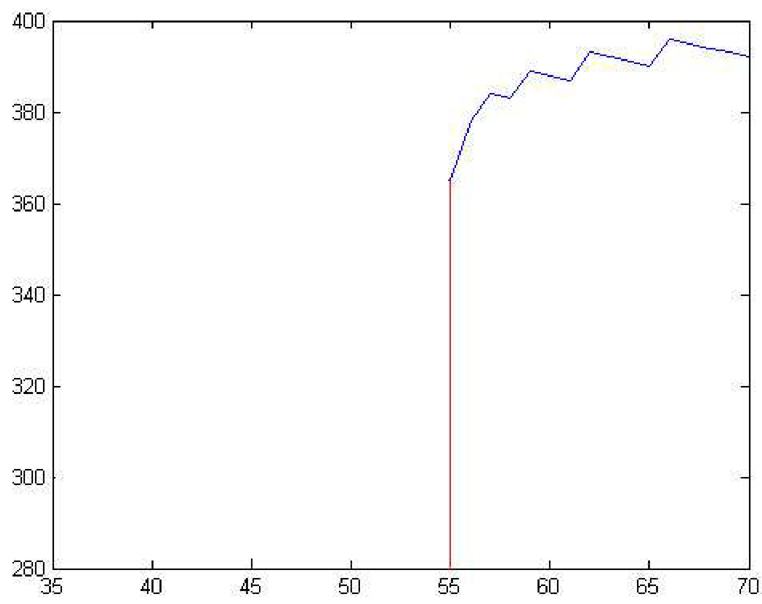
After this, we will have removed

$$\sum_{i=1}^{t-m+2} t-m+2-i = \sum_{i=0}^{t-m+1} i = \frac{(t-m+1)\cdot(t-m+2)}{2}.$$

It means that we can draw exactly the function of optimal boundary for the intervals of size $\frac{(t-m+1)\cdot(t-m+2)}{2}$ before the critical cardinal $\binom{t}{m}$. We now only need to calculate how many vertices are in each class of the boundary.

This classes has all the possible vertices that contains a given block of $m-1$ elements, and we can chose the last element between all the other $n-t$ elements. Thus there are $n-t$ vertices in each class.

Let see an example of the function for the graph $J(15,4)$ and the parameter $t=8$.



The initial critical set has size $\binom{8}{4} = 70$ and the expression $\frac{(t-m+1)\cdot(t-m+2)}{2}$ takes the value 15. Then, using this, we can know the function in the interval $[55, 70]$. We can appreciate how the graphic has steps of constant high, separated each one much more when we walk near the critical set of size 70. And also, we can appreciate that the size of the last step is twice the size of the others. This is because the last block that we have to remove, has size 0; it means, the last one are two steps together.

For the first interval, between the critical cardinals $\binom{m}{m} = 1$ and $\binom{m+1}{m} = m+1$, we need that $\frac{(t-m+1)\cdot(t-m+2)}{2} \geq m-1$ for $t = m+1$. It means, $m \leq 4$. It means, we can draw exactly the first interval between the critical cardinals for some values of m.

We can say more. For $m = 3$ the interval of the function that we know to the left of the critical cardinals cover completely the distance to the previous critical cardinal. Given a cardinal $\binom{t+1}{3}$, the number of vertices that we know how to take out is

$$\frac{(t-m+2)\cdot(t-m+3)}{2} = \binom{t}{2}$$

and we have that

$$\binom{t+1}{3} - \binom{t}{3} = \binom{t}{2}.$$

Let suppose now that we have a graph $J(n,4)$ and we try to add vertices to an optimal set of cardinality $\binom{t}{4}$. The vertices that we have to add belong to $J(t,4) \setminus J(t+1,4)$. Thus this vertices has to contain the element $t+1$. Suppose t large enough, we may assume that the first vertex to add is

$$v_1 = \{1, 2, 3, t+1\}.$$

The new vertices of the boundary, the adjacent to v_1 but not to $J(t,4)$ have to have the element $t+1$, two elements of the subset 1, 2 or 3 and another element larger than $t+1$. Then the number of vertices that we are adding to the boundary is proportional to the number of pairs that appear in the first part of the vertices added to the optimal set.

For example, if $v_1 = \{1, 2, 3, t+1\}$ and $v_2 = \{1, 4, 5, t+1\}$, the pairs are

$$\{1,2\}, \{1,3\}, \{2,3\}, \{1,4\}, \{1,5\}, \{4,5\}.$$

Choosing the second vertex at this way, the number of pairs is 6. But we can do it in a better way. We can use the pairs added with v_1. Let define as

$$v_2 = \{1, 2, 4, t+1\}.$$

At this way, v_2 only add the pairs $\{1,4\}$ and $\{2,4\}$. The choice of this second vertex can looks evident, but for the third one, we can follow a lexicographic or colexicographic order. Let see what happen adding as v_3 the vertices $\{1, 2, 5, t+1\}$ or $\{1, 3, 4, t+1\}$.

With $\{1, 2, 5, t+1\}$ there appear the pairs $\{1,5\}$ and $\{2,5\}$. Two new pairs. In fact, we can see how if we keep adding vertices at this way, we always are going to add two new pairs.

With $\{1, 3, 4, t+1\}$ there only appear the pair $\{3, 4\}$. And keeping with the colexicographic order, the fourth vertex will be $\{2, 3, 4, t+1\}$, which doesn't add any new pair.

This order is equivalent to say that we want to "finish" all the possible vertices of $J(t, 3)$ ($m = 3$ because we know that the element $t+1$ always belong to the vertices that we add and the vertices that appear in the boundary) with the first three elements, then with the first four elements. We want to keep low the number of elements that we use, for keep low the number of possible pairs.

We can use the proof of the lemma 4.3 for justificate that. Given $k = |S|$ such that $\binom{t}{m}+\binom{t'}{m-1} \leq k \leq \binom{t+1}{m}$ for some $m \leq t' < t$, we know that S will contain $\binom{[t]}{m}$. Now for the vertices in $S_1 = S \setminus \binom{[t]}{m}$ we can apply the same process as in the proof of this theorem, and compact S_1 as much as possible into the set $\binom{[t'] \cup \{t\}}{m-1}$, having that

$$\binom{[t-1]}{m} \bigcup \binom{[t'] \cup \{t\}}{m-1} \subseteq S \subseteq \binom{[t]}{m}.$$

For each value of t' we obtain a second family of critical cardinals.

At this way we maybe can not know which vertex choose to add at each moment. But we always know that after 1 vertex we have to have added the subgraph $J(3, 3)$, after 4 vertices the subgraph $J(4, 3)$, after 10 vertices the subgraph $J(5, 3)$, and this keep growing up with intervals of size the triangular numbers. Let see why the size of the intervals are the triangular numbers.

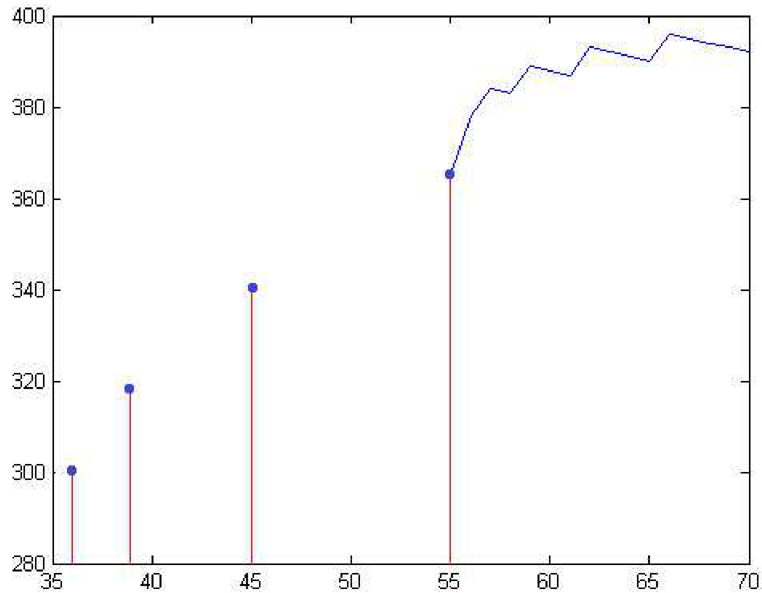
Suppose that we have added all the subgraph $J(t', 3)$. The vertices that we have to add for arrive to the $J(t'+1, 3)$ have to have the element $t'+1$ and the other two elements have to be less or equal to t'. This is the cause for there are $\binom{t'}{2} = T_{t'-1}$ possible vertices that we have to add.

Now we can calculate the cardinality of the neighbourhood in each point that we have seen. Let suppose that we have added the subgraph $J(t', 3)$. The possible neighbours to this vertices have to have the element $t+1$ and two elements of the interval $[1 : t']$. There are

$$\binom{t'}{2}(n - t - 1)$$

new neighbours.

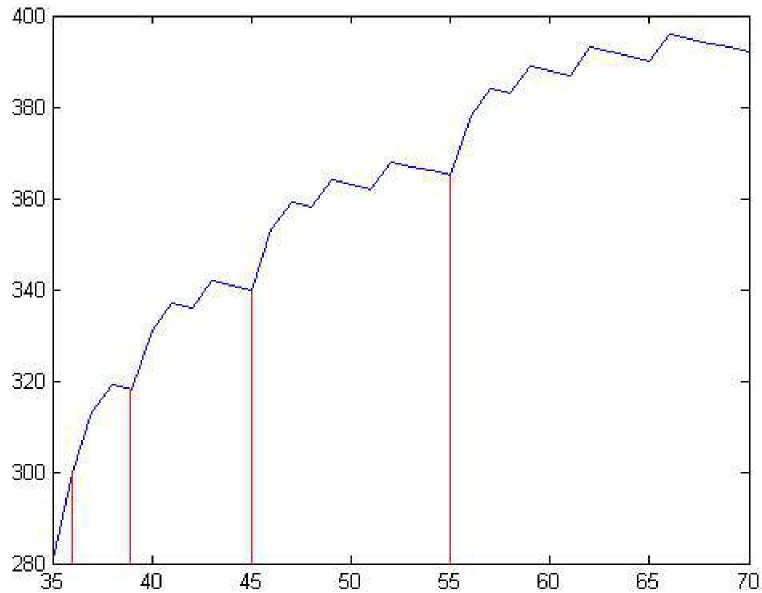
Lets complete the graphic with this second critical cardinals:



From each one of this second critical cardinals we can use the process of the blocks explained some pages before. Remember that we could draw the function $\binom{t-m+3}{2} = T_{t-m+2}$ points before the cardinal $\binom{t+1}{m}$. In this case, if we want to take out the vertices before the cardinal $\binom{t}{m}+\binom{t'}{m-1}$ we already know that we have to take out the vertices which contain the pair $\{t', t+1\}$ and the vertices with the pairs $\{x, t+1\}$ are already taken out for any $x > t'$. In sense of the Johnson graph $J(t, m-1)$ from which we have to chose the maximum clique, it means that we already have taken some vertices from every clique.

Then, for $t' = t-1$ we can draw the function T_{t-m+1} points, for $t' = t-2$ we can draw the function T_{t-m} points,... (note that for $t' = t$, $\binom{t}{m} + \binom{t'}{m-1} = \binom{t+1}{m}$). This decreasing sequence of triangular numbers is exactly the sequence of size of the intervals in which this second cardinals are spread.

And finally, complete the intervals with the process of the blocks:



The function has slope -1 in the most part of the points. The only moments when if grows up is when a block of vertices is taken out, when the function grows a step of size 7 in this case ($n - t - 1$ in general); and when the complete clique is taken out, in whose case the jump has double size. As we know, this double jumps match with the second critical cardinals.

Note Remember that every vertex that we add to an optimal set, is a vertex of its boundary. This is the cause for the slope is -1 when there is no jump, and the jumps in the graphic has size 6 and 13 and not $n - t - 1 = 7$ and $2 \cdot (n - t - 1) = 14$ respectively.

CHAPTER 6

Two particular cases

The results in the above chapter allow us to completely describe the isoperimetric function of the Johnson graphs $J(n,m)$ for $m = 2, 3$.

1. The case $J(n,2)$

In this case we have $\binom{t}{2} - \binom{t-1}{2} = t - 1$. It follows from Proposition 5.2 that the values of $\mu(k)$ are determined if $\binom{t}{2} - k \leq t - 1$. Thus we know the complete values of the isoperimetric function in this case.

PROPOSITION 6.1. *The isoperimetric function $\mu(k)$ of the Johnson graph $J(n,2)$ satisfies*

$$\mu\left(\binom{t}{2} - i\right) = t(n-t) + i, \ i = 0, 1, \ldots, t-1.$$

Figure 1 displays the isoperimetric function of the complement of the Petersen graph $J(5,2)$ displayed in Figure 2.

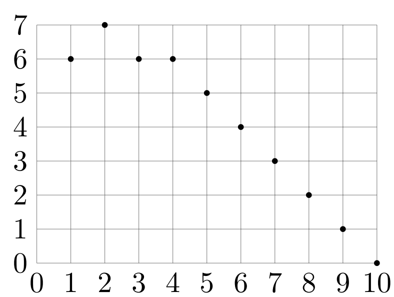


FIGURE 1. The isoperimetric function of $J(5,2)$.

2. Case $J(n,3)$

Let consider a Johnson graph $J(n,3)$ and a set S of vertices, a vertex in the boundary has to have two elements of $[1:n]$ in common with a vertex in the set S. There are two kind of vertices in the boundary. Let $v \in \partial S$. As we said, two elements of v are contained in a vertex of S. The first kind of vertices, has the third element in $\cup S$; and the second one, has the last element outside of $\cup S$.

For minimize the number of vertices of the first and the second kind, we have to get S inside of a $J(t,3)$, for the smallest integer t that we can use. Let see.

Let call a *pair* to a pair of elements of $[1:n]$ such that both of them belong to a vertex of S. Let consider $\mathcal{P} = \{p_1, p_2, \ldots, p_k\}$ the family of all the pairs in S. The number of vertices of the second kind, is exactly

$$(n-t)|\mathcal{P}|.$$

Looking only for this number, we should make as small as possible the number of pairs of S. For do that, we should overlap the vertices as much as possible. It mean that the vertices should lie in the smallest subset of $[1:n]$ as possible. Also, we have to note that the expression $(n-t)|\mathcal{P}|$ grows when t gets small, but when we the t grows in 1 element, the number of possible pairs grows up quadratically.

The number of vertices of the first kind, is exactly

$$\binom{t}{3} - |S|.$$

We can see that the number of vertices of this kind, also decrease when t decrease.

CHAPTER 7

Spectral approach

In this chapter we use spectral techniques to obtain lower bounds for the isoperimetric function of the Johnson graph.

1. Preliminaries

The isoperimetric number $i(G)$ of a graph G is defined as (see e.g. [**33**])

$$i(G) := \min_S \frac{|e(S)|}{|S|},$$

where $e(S)$ denotes the set of edges with exactly one end point in S and the minimum is taken over all proper subsets of $V(G)$. Since $e(S) = e(V \setminus S)$, there is no loss of generality in taking the minimum over all subsets $S \subset V(G)$ with $|S| \leq |V(G)|/2$.

Thus, the isoperimetric number of a graph determines the minimum ratio between the edges going out of a subset of vertices and the cardinality of this subset.

Let us see some simple examples:

Example 7.1. (1) $i(G) = 0$ if and only if G is disconnected.
(2) $i(G) \leq \delta$, where δ denotes the minimal degree of G.

We are going to use the isoperimetric number to obtain lower bounds for the (vertex) isoperimetric function of the Johnson graph $J(n, m)$. Using that $J(n, m)$ is r-regular, we can write

$$r \cdot |\partial S| \geq |e(S)|$$

and, by the definition of $i(G)$,

$$|e(S)| \geq i(G) \cdot |S|$$

for any $S \subset V(J(n, m))$. Then, we have that

$$|\partial S| \geq \frac{i(G)}{r}|S|. \tag{7}$$

Either r and $i(G)$ are fixed values on $J(n, m)$. Then the cardinality of the vertex boundary of a subset S is lower bounded by a linear function on its cardinality $|S|$.

Let us define $i_k(G)$ as

$$i_k(G) := \min_{|S|=k} \frac{|e(S)|}{|S|}$$

for each k with $1 \leq k < n$. In particular, we have that

$$i(G) = \min_k i_k(G).$$

We next recall general bounds for the isoperimetric number in terms of the Laplacian eigenvalues of the graph $G = J(n,m)$, see e.g. [**33**].

PROPOSITION 7.1. *Let G a graph with N vertices and $S \subset V = V(G)$. Then we have*

$$\mu_2 \frac{|S|(N-|S|)}{N} \leq e(S) \leq \mu_N \frac{|S|(N-|S|)}{N},$$

where μ_2 and μ_N are the second smaller and the largest eigenvalues of the Laplacian matrix of G respectively.

PROOF. Let $S \subset V$ and $v = (v_1, \dots, v_n) \in \mathbb{R}^n$ the characteristic vector of S, it means $v_i = 1$ if $i \in S$ and $v_i = 0$ otherwise. If we write

$$\sum_{i \in V} \sum_{j \in V} (v_i - v_j)^2$$

we are counting, twice the total number of pairs of vertices that one of them belongs to S and not the other. Then we have

$$\sum_{i \in V} \sum_{j \in V} (v_i - v_j)^2 = 2|S|(n - |S|).$$

But if we want to obtain $e(S, V \setminus S)$, we have to sum only the pairs of vertices which induces an edge:

$$\sum_{i \in V} \sum_{j \in V} A_{ij}(v_i - v_j)^2 = 2e(S, n - |S|)$$

where the matrix A is the adjacent matrix of G. And by the previous lemma and assuming that $S \neq \emptyset, V$, we have that

$$\mu_2 \leq 2n \cdot \frac{e(S, V \setminus S)}{2|S|(n-|S|)} \leq \mu_n.$$

$\square$

It follows from the above proposition and the fact that $e(S) = e(V \setminus S)$ that, for every $1 \leq k < N$, we have

$$i_k(G) \geq \frac{(N-k)\mu_2}{N},$$

and

$$i(G) \geq \frac{\mu_2}{2}.$$

Combining with (7), we obtain

$$|\partial S| \geq \frac{\mu_2}{2r}|S|. \tag{8}$$

for every $S \subset V(J(n,m))$.

2. Eigenvalues of $J(n,m)$

The eigenvalues of the Johnson graph $J(n,m)$ are known to be (see e.g. [**18**]):

$$(m-i)(n-m-i)-i,$$

with multiplicity

$$\binom{n}{i} - \binom{n}{i-1}$$

for each i with $0 \leq i \leq m$.

Using that $J(n,m)$ is a regular graph, of order $m(n-m)$, we can deduce that the eigenvalues of the Laplacian matrix of the graph, are

$$m(n-m)-(m-i)(n-m-i)-i,\ 0 \leq i \leq m.$$

In particular, for $i=0$, we can check that the first Laplacian eigenvalue is 0. Taking $i=1$, we have that

$$\mu_2 = m(n-m)-(m-1)(n-m-1)-1 = n-2.$$

We are going to use this value to complete the lower bound of the isoperimetric number and then that of $|\partial S|$.

Plugging in this value of μ_2 in the inequality (8) we obtain

$$|\partial S| \geq \frac{n-2}{r}|S|.$$

We can significantly improve this lower bound by using Lemma 4.3. According to this Lemma, if $\binom{t-1}{m} < k \leq \binom{t}{m}$ we can find an optimal set S of cardinality k such that

$$\binom{[t-1]}{m} \subset S \subseteq \binom{[t]}{m}.$$

It follows that every vertex in ∂S is adjacent from at most $t-m+1$ vertices in S. Therefore, we can replace r in the lower bound (7) with the much smaller value $t-m+1$. This gives, for $\binom{t-1}{m} < k \leq \binom{t}{m}$ and $|S| = k$,

$$|\partial S| \geq \frac{n-2}{t-m+1}|S|. \tag{9}$$

We know from Proposition 5.1 that, if $\binom{t-1}{m} < k < \binom{t}{m}$ then

$$\mu(k) \geq \binom{t-1}{m-1}(n-t) + \binom{t}{m} - k.$$

Our goal is to use spectral bounds for $i(G)$ which improve this lower bound for $\mu(k)$ for some values of k where the exact value of μ is not known.

Figure 1 shows the plots of the lower bounds for the isoperimetric function given by Proposition 5.1 and the spectral bound (9). As it is illustrated in this figure, the spectral bound gives better estimations of the isoperimetric function for values of k close to half the total number of vertices.

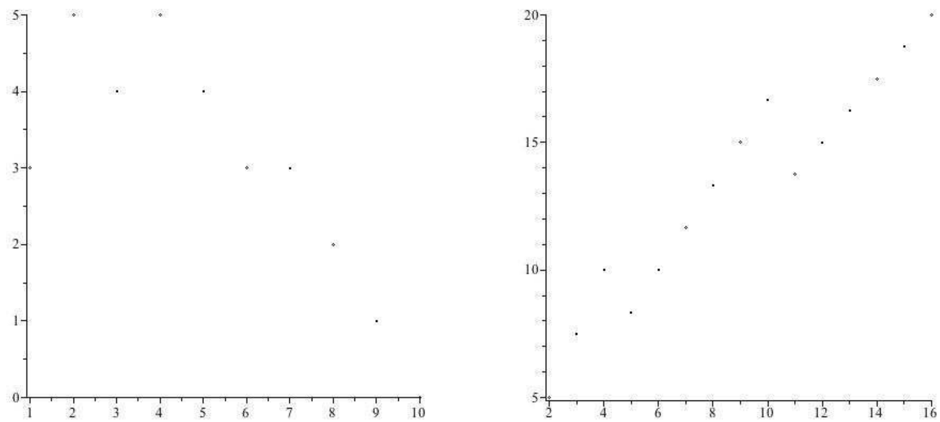


FIGURE 1. Comparing the lower bounds for the isoperimetric function, from Proposition 5.1 on the left and from the spectral inequality (9) on the right.

CHAPTER 8

Connectivity in Johnson graph

Let define the k-connectivity of a graph as:

Definition 8.1. Let G a graph and let k a positive integer. The k-connectivity of G is the minimum number of vertices that we have to remove of G for break the graph in two connected components with at least k vertices in each one of this two connected components.

It means, let consider all the sets X of vertices such that $G \setminus X$ has at least two different connected components with at least k vertices in each one.

We will denote it by $\kappa_k(G)$.

If there exist a set of vertices U of a graph G such that $G \setminus U$ has at least two connected components with at least k vertices in each one, we will say that G is k-separable, for any value of k. Otherwise, we say that G is not k-separable.

Clearly, if $k \geq \dfrac{|V|}{2}$, G is not k-separable.

For use the properties and results studied in the other sections, we write this lemma:

Lemma 8.1. *Given S an optimal set of vertices in the Johnson graph $J(n,m)$, the subgraph*

$$J(n,m) \setminus (S \cup \partial S)$$

is a connected subgraph.

This is an important lemma, because we are going to use the boundary of optimal subsets of vertices as cut sets; and we want that in each connected component has at least k vertices. But if we break one of the connected components, it may has not satisfy this property.

Proof. Let consider a Johnson graph and an optimal set S. And let be $u, v \in V(J(n,m))$ two different vertices such that $u, v \notin S \cup \partial S$. If $|S| \leq \binom{t+1}{m}$, we may suppose that $u, v \cap [1:t+1]$ is at most $m-2$; because $S \cup \partial S \subseteq J(t+1,m) \cup \partial J(t+1,m)$.

Let $C_u = u \cap [1:t+1]$ and $C_v = v \cap [1:t+1]$. If $|C_u| \neq |C_v|$, suppose without lose of generality $|C_u| + k = |C_v| \leq m-2$ for some positive integer k. Take $\{y_1, \ldots, y_k\}$ subset of $C_v \setminus C_u$ and apply the transformations T^*_{xy} to the vertex u with

$$x \in u \setminus [1 : t+1]$$

and

$$y \in \{y_1, \ldots, y_k\}$$

Each one of this transformations give us a new vertex at distance 1 of the previous vertex. And each vertex has at most $m-2$ elements of $[1 : t+1]$, so every vertex belong to $J(n,m)\setminus(S \cup \partial S)$. Let call $\hat{u}$ the last vertex of this sequence of transformations.

Now, we have $\hat{u}$ and v with $|\hat{u} \cap [1 : t+1]| = |v \cap [1 : t+1]|$. We need to transform now only the elements of

$$(\hat{u} \setminus v) \cap [1 : t+1]$$

on the elements of

$$(v \setminus \hat{u}) \cap [1 : t+1].$$

And the elements in

$$(\hat{u} \setminus v) \cap [t+2 : n]$$

on the elements of

$$(v \setminus \hat{u}) \cap [t+2 : n].$$

We have a path

$$u \sim \ldots \sim \hat{u} \sim \ldots \sim v$$

with all the vertices at most $m-2$ elements of $[1 : t+1]$. Finally, if any of the vertex u or v belong to $J(t+1,m) \cup \partial J(t+1,m)$, then for sure exist a vertex adjacent to each one of them out of $J(t+1,m) \cup \partial J(t+1,m)$. Let call $\bar{u}$ and $\bar{v}$ respectively. There exist, in this case the following path:

$$u \sim \bar{u} \sim \ldots \sim \hat{\bar{u}} \sim \ldots \sim \bar{v} \sim v$$

with all the vertices but the first and the last at most $m-2$ elements of $[1 : t+1]$. And the first and the last one, we know that don't belong to $S \cup \partial S$. $\square$

Using the results studied in the past sections, we can write some easy properties.

Property 8.2. *Given a positive integer k, and S an optimal subset of vertices of a graph $J(n,m)$ with $|S| = k$. Then, the k connectivity is at most $|\partial S|$.*

Proof. Let k and S an optimal subset of vertices such that $|S| = k$ satisfying that

$$2k + |\partial S| \leq |V|.$$

Thus, if we remove the set ∂S we have two connected components with at least k vertices in each one. $\square$

As we saw in the previous sections, the boundary of a set of vertices in a Johnson (and in general in the most of the graphs) graph don't grow monotonically with the cardinality of the set. This implies that sometimes, the optimal cut set is not just the boundary of an optimal set of length k.

Example 8.1. Let the graph $J(15,4)$, and $k=14$. We can check that

$$\binom{5}{4} < k < \binom{6}{4}.$$

Then, an optimal set S of length k will satisfy

$$J(5,4) \nsubseteq S \nsubseteq J(6,4).$$

We know that $|\partial J(6,4)| = \binom{6}{3}(15-6) = 180$. Given a vertex in $\partial J(6,4)$, it has $t-m+2=3$ neighbours in $J(6,4)$ If we remember the results of section 6.3, we have to delete 3 vertices for delete the block of $J(,)$ associated to the vertex in the boundary and $\binom{6}{4}-14=1$. It means that there is not decreasing of the boundary. Then

$$|\partial S| = |\partial J(6,4)| + 1.$$

The boundary of S is just the boundary of $J(6,4)$ and the vertex which is in $J(6,4)$ but not in S. Even if $k=13$, as $t-m+2=3$ which is larger than the difference between $\binom{6}{4}$ and k, we have

$$|\partial S| = |\partial J(6,4)| + 2.$$

If we keep going down in k and we fix $k=12$, the distance between $\binom{6}{4}$ and k is already 3 which is equal to the value $t-m+2$. It means that there exist a vertex in $\partial J(6,4)$ which is not any more in ∂S (in fact, there are $15-6=9$ vertices which share neighbour in $J(6,4)$ that have been removed), but there are 3 vertices in $J(6,4)$ which are not in S, then

$$|\partial S| = |\partial J(6,4)| - 6.$$

Obviously, the set $\binom{[6]}{4}$ of cardinality 15 has a better boundary than the sets of cardinality 14 and 13, but not than the set of 12 vertices. The 13-connectivity, the 14-connectivity and the 15-connectivity has the same value: 180. The 12-connectivity of $J(15,4)$ is 174.

The graph $J(15,4)$ has 1365 vertices. And for break it in to parts of only 13 vertices in each one (parts of the order of less of the 1% of the total graph), we have to remove 180 vertices, 15 times the length of each part.

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