Master in Artificial Intelligence (UPC-URV-UB)

Master of Science Thesis

Application of Fuzzy Techniques to Biomedical Images

Gabriel Mattioli Aramburu

Advisors: Dr. Eduard Montseny, Dr. Jordi Recasens, Dr. Maite López

21/06/2012

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Acknowledgment

The Master Thesis work I provide in this pages is the result of two years of effort. It must be said that the path that has taken me to present this work and to have derived these results has gone along with the one of several people that have helped me on my way. In these lines I would like to thank them all for their molehill.

First of all, I would like to thank project number TIN2009-07235 for funding me.

Secondly I would like to thank very specially my three Master Thesis Tutors: Eduard Montseny, Maite Sànchez and Jordi Recasens, for their work and dedication in some parts of the work; as well as for their supervision in the final project I present. I would like to thank as well Department of Pathology and Anatomical Sciences Ellis Fischel Cancer Center of the University of Missouri-Columbia for providing me the images used further in this work.

Thirdly, I want to have a word for the large amount of people that have been there these two years, specially my friends and family. Despite sometimes they did not understand very much my technical work, they have been there hearing me, trying to understand and most important of all, always being there to back me up.

And finally thank all the libraries, rooms, classrooms, municipal and entities' sites where the Master Thesis that borns today has been conceived and created.

1 Introduction

60 years after, we can state that sunrising of computers in the second half of the 20th century has constituted an integral revolution for Humanity. The fast emergence of this new technology has implied multiple changes in almost everything, and these changes are here to stay.

Human interaction and communication, information and services exchange, work, behavior, science, notion of time distance...

At a deeper level, even one of the most important assumptions human beings do about ourselves has earthquaked. As a specie, its been thousands of years since we stated that we are intelligent. In less that one century the emergence of computers has somehow reeled this dogma.

Science Fiction hit the nail on the head in many of their predictions, but it definitely missed when it ran into intelligence. Surprisingly, it has been easier to get to the Moon than to understand our thousands-years old statement: What is intelligence?

Despite this question has been approached by many people in this last century, it can definitely be said that there is no consensuated answer at all. Philosophical attempts usually crash into the Wall of ontological questions while engineer-like ones are totally insufficient as they use to stay in the problem-solving field without really walking at the Knowledge level.

However, the situation is not so dramatic. Some check points have been defined, for instance that a system able to "'behave"' like a human being can be considered "'intelligent"', according to the fundamental statement that we are actually intelligent. This idea can be already distilled from the first assays of Alan Turing [43] on the connection between naive Intelligence and Artificial Intelligence. However, to the opinion of the author of this work, this proposition has a deep lack as it only identifies intelligent agents when they are indistinguishable from other agents from whom we priorly assume intelligence . Hence, it does not focus at all on the previous question of what is intelligence, neither on what conditions should an agent verify to be considered intelligent in a non-comparative context.

Closing this initial discussion, it is definitely out of the scope of this

Master's Thesis trying to attack this problem and to provide a new answer or proposal of it. However, it is the opinion of the author that this nuclear question should be in every discussion around the topic of all fields related to intelligence, such as AI.

After all, it is the question that drives us [45]. And without being able to explain its own title, how can a scientifical discipline bear to assert its own existence?

1.1 Aim of the work and Motivation

The main goal of this work is to bring together Fuzzy Logic techniques and Biomedical images analysis, linking this two fields through an image segmentation problem approached by a Fuzzy c-Means algorithm.

This work has two main cores. The first one is a formal structural analysis of some Fuzzy Logic concepts within the theory of Indistinguishabilitty Operators. The main actors of this part will be indistinguishability operators, sets of extensional fuzzy subsets, upper approximation operators and lower approximation operators. All these concepts will be explained in depth further in this work. The second core is a practical application of the Fuzzy c-Means algorithm to segmentate biomedical images of mammographies and bone marrow microscopical captures. This two approaches will be brought together in the chapter regarding future lines of research.

Fuzzy Logic has proved to have things to say on how images should be segmentated [6], but a quick look at the main methods used shows that in many cases very low-level Fuzzy Logic has been used in practical contexts. This statement is not to underestimate the existing methods, but to recall that only the ideas of fuzzy sets and fuzzy rules have been used to construct algorithms for fuzzy clustering and classification and image analysis; whereas theoretical Fuzzy Logic has gone far beyond.

This is an interesting point because it is the main Wall several researchers who have given up up FL point about this theory: "'Fuzzy Logic is a nice and interesting theory, but of questionable usefulness when applied to practical contexts"' [23].

In this sense this project can be seen as an attempt to show how high-level Fuzzy Logic can prove to be not just useful but efficient in order to attack biomedical image segmentation problems, and encourage its use in other fields as well.

Finally, it is a must to recall that this is not a complete project but an intermediate stage of a whole PhD work. For that reason the strong connection aimed to be done between theoretical Fuzzy Logic and Biomedical Engineering image segmentation will not be completely found on this work. This work will build two strong columns around these two topics, finding further new results and finally pointing the future lines of research with which it is expected to build one complete house.

1.2 Structure of the Work

This work is structured as follows:

Chapter 2 will be devoted to a quick review of the State of the Art of the fields this work lays upon. Firstly a quick review of the Biomedical Image Segmentation topic will be done, as well as the main existing methods to solve this problem.

Secondly the work will recall the definition and main results of the concepts within the Indistinguishability Operators theory that will be used in further chapters. A special strength will be put on Indistinguishability Operators, Sets of Extensional Fuzzy Subsets and Upper and Lower Approximation Operators by Extensional Fuzzy Subsets. Natural means and powers of t-norms will be explained as well in this Section.

Finally the Fuzzy C-Means algorithm will be deeply explained as it will play a central role in the practical application in Chapter 4 of this work.

Chapter 3 will focus on Indistinguishablity Operators theory and several new results will be found. First of all it will be shown how the concepts remarked above are mathematical lattices and that these lattices are isomorphic with respect to the natural operations of a lattice. Further it will be shown that, counterintuitively, this isomorphism cannot be completely extended to more complex Aggregation Operators such as natural weighted means. Finally, a study of how powers can be defined among indistinguishability operators and related concepts will be performed and several technical results will emerge. This results will open some very interesting and promising lines of research that will be explained in Chapter 5.

In Chapter 4 a practical image segmentation problem will be faced. After an histogram-based preprocessing of the image a Fuzzy c-Means algorithm will be used to segment mammographies and bone marrow images. A brief study of the performance of this algorithm will be done and a discussion of the results obtained will be provided.

At this point we can say that the two columns or backbones of the whole project will have been strongly stated. On one side we will have FL theory and on the other Biomedical Image Segmentation, as well as the Fuzzy C-Means algorithm which will give us a weak provisional beam to join them.

As said, this Master Thesis will not construct the whole building, but Chapter 5 can be seen as a map on how to build walls and roofs, as it will be devoted to the future lines of research this work suggests to the author. Several possible continuations will be pointed which will enhance the existing FCM beam or suggest other possible approaches or attempts.

Finally, Chapter 6 will close the work with the Concluding Remarks where the main results obtained will be summarized as well as the future lines of the project will be outlined.

2 State of the Art

2.1 Biomedical Image Segmentation

This section will focus on realising a quick review of the state of the art of Biomedical Engineering and specially of Biomedical Image Segmentation.

The first section will provide a quick historical review of Biomedical Engineering and will try to point out how engineering approaches and medicine have walked aside for centuries providing mutual profit in a simbiotical way. Finally, though biomedical engineering is an ancient field we will focus on biomedical image segmentation which is much more recent and the basis of one of the columns this work lays upon.

In the second section the main methods of image segmentation will be briefly recalled and explained. It is a must to say here that the amount of techniques and algorithms that have been used in this field is so vast that it is almost impossible to actually be exhaustive in this list. After all, almost all techniques of clustering and machine learning can be adapted to segmentate an image. Nevertheless, the list provided will explain the most representative and used methodologies in image segmentation.

2.1.1 History of Biomedical Engineering and Biomedical Image Segmentation

According to some researchers [18], the first proof of engineering applications in a medical context was found in an Egyptian tomb of 3000 years ago, where a protesis of a foot thumb was found. It has been proved as well that Egyptians also used canes and empty pipes to watch and listen what happened inside the human body.

Other researchers [47] claims that we can find the first biomedical engineering assays in the work of Leonardo Da Vinci on the perfect anthropometric measures and his works on levers and its interpretation in the human body.

Further in time, the physicist Renne Laennec used a newspaper to canalize the sound of the heart and his idea would finally derive in the sthetoscope. By that time we can recall as well the first assays on electrical conductivity in life beings by Galvani and Lord Kelvin.

The previously stated situations are just a short list to illustrate how it has been since the birth of medicine (which can be maybe identified with the birth of humanity) that engineering approaches have been used to better understand, comprehend, diagnose and heal human diseases and injuries. After all, if we listen to those that, modestly, assert that the human body is the perfect machine, then it is an immediate must to consider medicine as an engineering specification.

However, despite several works can be invoked before, it is between 1890 and 1930 that Biomedical Engineering took off and has not stopped. Röntgen [35] discovered X-rays in 1895 and its first clinical application was just a week after. Siemens and General Electric were already selling industrial X-rays systems just one year after in 1896. In 1887, Waller [46] started recording human heart beats and in 1924 Berger [3] was already processing electroencefalographies.

Since 1930 it is already impossible to try to track all the applications engineering has developed for human medicine (and vice versa).

However, the discipline of biomedical image segmentation does not appear until the decade of 1970. In this years the first papers on automatic image segmentation are published by Brice & Fennema [8], Pavlidis [32] and Rosenfeld & Kak (1976) [36].

It can be stated that the birth of biomedical image segmentation was at simultaneous to the one of its mother discipline image segmentation. Why it is important to use this techniques of automatic processing of images in a biomedical context seems immediate, however a good compilation of these reasons can be found in [37]

From then on, the number of papers published on this topic has grown exponentially up to the point that now it is impossible to follow the waterfall of publications that include the keyword "'image segmentation"' that appear every day.

2.1.2 Biomedical Image Segmentation Techniques

According to Gonzalez and Woods [19] the segmentation on an image is the subdivision of the mother image into its constituent regions or objects. In plain words given an input image, the goal of an image segmentation process is to identify and discriminate the different regions, classes or objects in the image.

As said in the previous lines, there is an extremely vast variety of techniques that are used nowadays in the field of biomedical image segmentation. Without any aim of being exhaustive, below it is provided a list of the main families of techniques and most used algorithms for this purpose:

- Clustering Methods: Almost all clustering methods can be adapted in order to segmentate an image [6] [19]. An image can be seen as a sample of pixels and a segmentation process is in this approach nothing but a clustering of the sample according to a certain distance. Hence, with a suitable data representation almost any clustering algorithm (hierarchical clustering, k-means, spectral clustering...)can be used to segmentate an image. The Fuzzy c-Means algorithm that will be explained in Section 2.3 and used in Chapter 4 can be included in this family.

- Histogram-based Methods: The idea underlying these methods is not to work over the raw image, but on the histogram defined by the gray-level of the pixels in the image. This methods are used mainly in B/W images, because the histogram is unidimensional, while colored images have 3-dimensional histograms. The assumption behind histogram processing algorithms is that the main information to discriminate objects in an image is encoded in the color or gray-level of the pixels. This methods aim at identifying the main peaks and valleys in the histogram of the image and in general are very efficient. However a further problem is that it may be very difficult to identify these peaks in the image, and hence they are usually combined with other techniques of this list. In this work a histogram-based combined with a watershed method will be explained in depth and used in Section 4.2.

- Edge Detection: Edge detection is focused on finding the confines of each region. After all, every compact region is defined by the closed curve related to its border [16]. The output of these methods are not in general compact curves, but families of disconnected segments of them. The problem to be solved next in order to have a proper image segmentation is how to connect these segments. Lindeberg & Li propose a general method in order to perform this task [22].

- Region Growing Methods: In this family we can find many different algorithms [19] based on the same basic idea, which is to choose an initial family of seeds or regions and let them grow by comparing the regions with the neighbor pixels until we get a full partition of the whole image. The main methods of this family are seeded region growing methods [19] and λ-connected segmentation [9].

- Split-and-Merge Methods: As its name says, these methods are based on an iterative application of splitting and merging the image. The algorithms begin with the whole image and if its found non-homogeneous its splitted into n components. Inversely, if at any time of the process two connected components are found homogeneous they can be merged into one [19]. When $n = 4$ and the components are squared then its called quadtree segmentation.

- Partial Differential Equations (PDE) Methods: These methods are based on the idea of numerically solving PDEs in order to evolve an initial curve into the lowest potential of a cost function [39]. Constraints dealing with smoothness have to be considered in order to get plausible solutions.

- Watershed Methods: A Watershed transformation considers the gradient magnitude of an image as a topographic surface. Symbolically, the idea is to overflow a representation of the image with water and study the behavior of the flow as the water is drained [19]. In Section 4.2 a Watershed method will be explained in depth as it will be used together with a histogram-based method.

- Fourier-Transform-based Methods: The idea of this methods is to use the Fourier (or other kind of transformations) Transformation of the image and work over the image within the space of coefficients of the Fourier Transformed [16].

- Graph Partitioning Methods: The image is seen as a weighted undirected graph where each node is a pixel, each edge a neighboring relation and the weight of each edge as the degree of (dis-)similarity between nodes. A segmentation of the image corresponds in this representation with a clustering over the graph [16]. All the techniques of clustering among graphs can be used then with very good results in the final segmentation.

As said, the previous list is not exhaustive and many other methods can be used like Neural Networks, Multi-Scale Segmentation, Model-based Segmentation, Rule-based Systems...

Finally, it is important to recall that the usage of one technique or another one depends on the kind of image, the choice of the researcher and very specially on the task to be performed. Mainly we can find two different approaches to image segmentation: Those that want to determine the number of regions within an image (Watershed methods for instance), or to determine these regions known the amount of them (the Fuzzy c-Means algorithm used in this work for instance); and those that aim at thresholding the image in order to determine Regions of Interest (ROIs) to clean the image (erasing the background, the image pollution and noise in general) in order to work it better.

2.2 Fuzzy Logic Preliminaries

In this section we will recall the main definitions and results that background the work that will be developed further in Chapter 3.

Historically, the story of the development and evolution of the main concepts involved in this work can be summarized as follows.

It is well known that Fuzzy Logic was firstly introduced by Lofti Zadeh in 1965 [49]. However, back in 1951 Menger already worked on what he called probabilistic relations [30], which were nothing but fuzzy equivalence relations (or indistinguishabilities, as they will be recalled in this work) with respect to the Product t-norm. Menger's results were taken further by Ovchinnikov, who studied in depth probabilistic relations in the 80s [31].

Indistinguishabilities with respect to the Łukasiewicz t-norm were studied by Ruspini in the late 60s who called them likeness relations and used them in the field of Reasoning by Analogy [38]; which was an approach to define a similarity relation between possible worlds, in a very close aim to modal logic's one.

Indistinguishabilities with respect to the Minimum t-norm were studied by Zadeh [48] in 1971 who called them similarities. It was in this paper either where it was first stated that this results could be extended by choosing a t-norm in general. However, it was not until 1981 that Trillas reformulated the theory taking an arbitrary t-norm [40], coining then the term "'indistinguishability operators"' with which fuzzy equivalence relations have been recalled since then.

This general definition was found to be of extreme importance following the research carried by Trillas & Valverde in the following years [41] [42] [44] and since then several research groups all aroung the globe have focused on trying to comprehend and enhance this formal approach to intuitive "equality".

This section will be structured as follows.

First of all the concept of t-norm will be recalled together with the main results around it that are important for this work.

After that powers of t-norms will be constructed. This idea will prove to be semantically interesting in Chapter 3.

Then, the main definitions and results of Indistinguishability Operators theory will be provided. Extensional sets, upper approximations and lower approximations will be defined as well as they will play a central role in this work.

Finally, the concept of quasi-arithmetic mean will be recalled.

2.2.1 Fuzzy Logic general concepts

It is well known that the classic conjunction $\wedge$ of classical bivaluated logic is generalized into the multivalued logic field by the concept of t-norm. The formal definition of t-norms is provided below.

Definition 2.1. *Let T be a map $T : [0,1]^2 \to [0,1]$. We will say that T is a t-norm if and only if T verifies the following properties $\forall a, b, c, d \in [0,1]$:*

a) $T(a,b) = T(b,a)$ *(Commutativity)*

b) $a \leq c,\ b \leq d \Rightarrow T(a,b) \leq T(c,d)$ *(Monotonicity)*

c) $T(a, T(b,c)) = T(T(a,b),c)$ *(Associativity)*

d) $T(a,1) = a$ *(Identity Element)*

Example 2.2. *The following maps T are t-norms:*

- $T(x,y) = \min\{x,y\}$ *(Minimum t-norm)*
- $T(x,y) = x \cdot y$ *(Product t-norm)*
- $T(x,y) = \max\{0, x+y-1\}$ *(Łukasiewicz t-norm)*

Once a T-norm is chosen, the implication and biimplication connectives are subsequently defined. To define a multivalued disjunction given a t-norm T it is necessary to have a (strong) negation.

Definition 2.3. *Let T be a t-norm.*

- *The residuation $\overrightarrow{T}$ of T is defined for all $x, y \in [0,1]$ by*

$$\overrightarrow{T}(x|y) = \sup\{\alpha \in [0,1] | T(\alpha, x) \leq y\}.$$

- *The birresiduation $\overleftrightarrow{T}$ of T is defined for all $x, y \in [0,1]$ by*

$$\overleftrightarrow{T}(x,y) = \min\{\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)\} = T(\overrightarrow{T}(x|y), \overrightarrow{T}(y|x)).$$

T-norms can fulfill several properties such as continuity, idempotency... A very interesting property a t-norm can verify is Archimedeanity. We will say that T is Archimedean if 0 is the only nilpotent element. Archimedean t-norms are characterized in the following Theorem in terms of additive generators, which will prove to be operationally useful further.

Theorem 2.4. *[21] A continuous t-norm T is Archimedean if and only if there exists a continuous and strictly decreasing function $t : [0,1] \to [0,\infty)$ with $t(1) = 0$ such that:*

$$T(x,y) = t^{[-1]}(t(x) + t(y))$$

where $t^{[-1]}$ is the pseudo inverse of t defined by:

$$t^{[-1]}(x) = \begin{cases} 1 & \text{if } x \leq 0 \\ t^{-1}(x) & \text{if } 0 \leq x \leq t(0) \\ 0 & \text{if } t(0) \leq x \end{cases}$$

The function t will be called the additive generator of the t-norm.

Additive generators allow to transform expressions including t-norms and its logical derivatives. This is very useful, as relational equations can be rewritten as numerical ones which are more handly and easier to compute.

Proposition 2.5. *Let T be a t-norm generated by an additive generator t. Then:*

- $T(x,y) = t^{[-1]}(t(x) + t(y))$
- $\overrightarrow{T}(x|y) = t^{[-1]}(t(y) - t(x))$
- $\overleftrightarrow{T}(x|y) = t^{[-1]}|t(y) - t(x)|$

2.2.2 Powers of t-norms

One of the most active lines of research nowadays within Fuzzy Logic is to modellize the inacuraccy and intrinsecal vagueness of language [29]. A very interesting topic within this field is how to modellize linguistic reinforcement. For instance how the predicate "very near to four" can be defined from the predicate "near to four".

Lofti Zadeh itself pointed this problem [52] [53] [54], and worked in the problem of how to modellize linguistic hedges.

Since t-norms were established as the standard to model conjunction in Fuzzy Logic this problem can be solved as follows. Let $p(x)$ stand for a fuzzy set measuring how near x is from four, then $T(p(x), p(x))$ can be seen as a measure of how much x is very near to four. Mathematically, we can generalize this idea by means of powers of t-norms.

Let us define first the n'th power of an object with respect to a t-norm.

Definition 2.6. *Let T be a t-norm and n a natural number. We will call the n'th power of T to:*

$$T^n(x) = T(\overbrace{x, x, ..., x}^{n})$$

To simplify notation sometimes we will denote $T^n(x) = x^n$.

It is possible to extend this definition to all positive rational numbers as follows.

Definition 2.7. *Let T be a t-norm and n a natural number. We will define T to the power of $1/n$ as:*

$$T^{1/n}(x) = \sup_{z\in[0,1]} T^n(z) \leq x$$

Naturally, $x^{p/q}(x) = (x^{1/q})^p$.

Finally, passing to the limit it is possible to define $T^r(x)$ for all $r \in \mathbb{R}^+$.

The following result is very useful, as it allows to calculate these powers we have just defined by using an additive generator t of T, which makes operations more manageable.

Proposition 2.8. *Let T be an Archimedean t-norm with additive generator t and $r \in \mathbb{R}^+$. Then:*

$$x^r = t^{[-1]}(r \cdot t(x))$$

2.2.3 Indistinguishability Operators Theory Preliminaries

Let us define now the concept of indistinguishability operator. This object will be the main actor of Chapter 3, and many operators and concepts will be defined after indistinguishability operators. Mathematically, they are nothing but the fuzzyfication of classical mathematical equivalence relations, and in the literature have also been called similarities, fuzzy equalities or fuzzy equivalence relations.

The skill of identifying objects is deeply rooted in human intelligence and reasoning. However, when it turns to modellize this ability things become tough. Indistinguishability operators provide a good approach for this purpose, as they measure a "'degree"' of similarity between objects softer than classical equality (things may be similar according to a certain feature and not globally equal) and smoother than classical equivalence relations (as objects may have degrees of similarity between 0 and 1, not being fit to be totally similar or dissimilar)

Definition 2.9. *Let T be a t-norm. A fuzzy relation E on a set X is a T-indistinguishability operator if and only if for all $x, y, z \in X$*

a) $E(x, x) = 1$ *(Reflexivity)*

b) $E(x, y) = E(y, x)$ *(Symmetry)*

c) $T(E(x, y), E(y, z)) \leq E(x, z)$ *(T-transitivity).*

Example 2.10. *Let $X = \{x_1, x_2, x_3\}$ be a finite set of 3 elements and T the Łukasiewicz t-norm. Then the relation E defined by the following matrix*

$$E = \begin{pmatrix} 1 & 0.6 & 0.7 \\ 0.6 & 1 & 0.8 \\ 0.7 & 0.8 & 1 \end{pmatrix}$$

is a T-indistinguishability operator.

A very interesting fact about indistinguishability operators is that they are dual with respect to distances. This is that given an indsitinguishability E and a strong negation ϕ (for instance $\phi(x) = 1 - x$) then $1 - E$ defines a distance. Even more, all distances can be seen as the dual with respect to a strong negation of an indistinguishability operator.

This fact is of extreme relevance because distances define metric spaces, which are spaces where things can be measured. As all sciences use to need some kind of metrics, without loss of generality we can consider that we have indistinguishabilities operating on sets and defining these metric spaces. This way indistinguishability operators become the key to understand the intuitive and very human capability of "'measure"'.

Following from Section 2.2.2, we can define as well powers of T-indistinguishability operators with respect to the t-norm T.

Definition 2.11. *Let E be a T-indistinguishability operator and $r \in \mathbb{R}^+$. We define the r'th power of E as:*

$$E^r(x, y) = T^r(E(x, y))$$

It can be proved that the power E^r of a T-indistinguishability operator E is a T-indistinguishability operator as well.

Theorem 2.12. *Let E be a T-indistinguishability operator and $r \in \mathbb{R}^+$. Then E^r is a T-indistinguishability operator.*

Indistinguishability operators can be generated in multiple ways. One of the most simple is the indistinguishability operator related to a fuzzy subset.

Proposition 2.13. *Let X be a set, T a t-norm and μ a fuzzy subset of X. The fuzzy relation E on X defined by*

$$E_\mu(x, y) = \overleftrightarrow{T}(\mu(x), \mu(y))$$

for all $x, y \in X$ is a T-indistinguishability operator on X.

Whereas indistinguishability operators are the direct fuzzification of equivalence relations, the set of extensional fuzzy sets related to an indistinguishability operator defined below plays the role of the set of fuzzy equivalence classes, with its unions and intersections.

Definition 2.14. *Let X be a set and E a T-indistinguishability operator on X. A fuzzy subset μ of X is called extensional if and only if:*

$$\forall x, y \in X \quad T(E(x, y), \mu(y)) \leq \mu(x).$$

We will denote H_E the set of all extensional fuzzy subsets of X with respect to E.

The following proposition provides a useful result to study whether a fuzzy subset is extensional.

Proposition 2.15. *Let X be a set, E a T-indistinguishability operator on X and μ a fuzzy subset of X. Then:*

$$\mu \in H_E \Leftrightarrow E_\mu \geq E$$

One of the main results of the theory of indistinguishability operators is the Representation Theorem. This theorem proves that any family of fuzzy subsets on a set X generates a T-indistinguishability operator E on X and, reciprocally, that every T-indistinguishability operator on X can be generated by a family of fuzzy subsets.

Theorem 2.16. [44] *Representation Theorem. Let R be a fuzzy relation on a set X and T a continuous t-norm. Then R is an T-indistinguishability operator if and only if there exists a family $(\mu_i)_{i\in I}$ of fuzzy subsets of X such that for all $x, y \in X$*

$$R(x,y) = \inf_{i\in I} E_{\mu_i}(x,y).$$

It is well known that the set H_E of extensional sets with respect to an indistinguishability operator E coincides with the set of generators in the sense of the Representation Theorem.

It has been proved that the set H_E of extensional fuzzy subsets related to an indistinguishability operator E can be fully characterized by he following family of properties.

Proposition 2.17. *[10] Let E be a T-indistinguishability operator and H_E its set of extensional fuzzy sets. Then, $\forall \mu \in H_E, (\mu_i)_{i\in I}$ a family of extensional sets and $\forall \alpha \in [0,1]$ the following properties hold:*

1. $\bigvee_{i\in I} \mu_i \in H_E$
2. $\bigwedge_{i\in I} \mu_i \in H_E$
3. $T(\alpha, \mu) \in H_E$
4. $\overrightarrow{T}(\mu|\alpha) \in H_E$
5. $\overrightarrow{T}(\alpha|\mu) \in H_E$

Theorem 2.18. *[10] Let H be a subset of $[0,1]^X$ satisfying the properties of Proposition 2.17. Then there exists a unique T-indistinguishability operator E such that $H = H_E$.*

The set of all sets of extensional sets on X will be denoted by $\mathcal{H}$.

Following we define two operators ϕ_E and ψ_E that, given a fuzzy subset μ, provide its best upper and lower approximation by extensional sets of E respectively.

Definition 2.19. *Let X be a set and E a T-indistinguishability operator on X. The maps ϕ_E: $[0,1]^X \to [0,1]^X$ and ψ_E: $[0,1]^X \to [0,1]^X$ are defined $\forall x \in X$ by:*

$$\phi_E(\mu)(x) = \sup_{y\in X} T(E(x,y), \mu(y)).$$

$$\psi_E(\mu)(x) = \inf_{y\in X} \overrightarrow{T}(E(x,y)|\mu(y)).$$

$\phi_E(\mu)$ is the smallest extensional fuzzy subset greater than or equal to μ; hence it is its best upper approximation by extensional sets. Analogously, $\psi_E(\mu)$ provide the best approximation by extensional fuzzy subsets less than or equal to μ. These operators can be seen as a topological closure and interior operator respectively, and appear as well in a natural way in the field of fuzzy rough sets and modal logic where they stand for a fuzzy possibility and necessity respectively [33].

An important property of upper and lower approximations is that they are fix over extensional sets. This is that $\mu \in H_E \Rightarrow \phi(\mu) = \psi(\mu) = \mu$.

These operators can be completely characterized by the following families of properties.

Theorem 2.20. *[13] Given a set X and an operator ϕ: $[0,1]^X \to [0,1]^X$, ϕ is the upper approximation of a certain indistinguishability if and only if the following properties are fulfilled.*

1. $\mu \leq \mu' \Rightarrow \phi_E(\mu) \leq \phi_E(\mu')$
2. $\mu \leq \phi_E(\mu)$
3. $\phi_E(\mu \vee \mu') = \phi_E(\mu) \vee \phi_E(\mu')$
4. $\phi_E(\phi_E(\mu)) = \phi_E(\mu)$
5. $\phi_E(\{x\})(y) = \phi_E(\{y\})(x)$
6. $\phi_E(T(\alpha, \mu)) = T(\alpha, \phi_E(\mu))$

Theorem 2.21. *[13] Given a set X and an operator ψ: $[0,1]^X \to [0,1]^X$, ϕ is the lower approximation of a certain indistinguishability if and only if the following properties are fulfilled.*

1. $\mu \leq \mu' \Rightarrow \psi_E(\mu) \leq \psi_E(\mu')$

2. $\psi_E(\mu) \leq \mu$

3. $\psi_E(\mu \wedge \mu') = \psi_E(\mu) \wedge \psi_E(\mu')$

4. $\psi_E(\psi_E(\mu)) = \psi_E(\mu)$

5. $\psi_E(\overrightarrow{T}(x|\alpha)(y) = \psi_E(\overrightarrow{T}(y|\alpha)(x)$

6. $\psi_E(\overrightarrow{T}(\alpha|\mu)) = \overrightarrow{T}(\alpha, \psi_E(\mu))$

2.2.4 Aggregation Theory Preliminaries

A transversal problem to almost all scientifical discipline is how to aggregate information when there is multiple different data to be considered at the same time. A natural way to face this problem is considering means. Below we introduce the concept of quasi-arithmetic means, which are a slight generaralization of classical arithmetic means.

Definition 2.22. *Let $t : [0,1] \to [-\infty, \infty]$ be a strict monotonic map and $x, y \in [0,1]$. The quasi-arithmetic mean m_t of x and y is defined as*

$$m_t(x, y) = t^{[-1]}(\frac{t(x) + t(y)}{2})$$

m_t is continuous if and only if $\{-\infty, \infty\} \nsubseteq Ran(t)$.

The quasi-arithmetic mean of two indistinguishability operators E, F can be defined as the quasi-arithmetic mean of E and F taking the additive generator of the t-norm as the monotonic map that defines the mean.

Proposition 2.23. *Let T be an Archimedean t-norm with additive generator t and E, F two T-indistinguishability operators on a set X. Then the natural mean of E and F defined as:*

$$m_t(E, F)(x, y) = t^{[-1]}(\frac{t(E(x, y)) + t(F(x, y))}{2})$$

is a T indistinguishability operator on X.

For the rest of the work, quasi-aritmetic means taking the additive generator of the (Archimedean) t-norm T as the monotonic map will be called natural means. The term natural comes from the fact that chosen the Archimedean t-norm T the mean is implicitelly fixed.

2.3 Fuzzy c-Means Algorithm

In this section the Fuzzy c-Means (from now on FCM) clustering algorithm will be explained in depth. This algorithm will be used in Chapter 4 in order to segmentate biomedical images and its performance will be compared with a watershed histogram-based segmentation technique.

FCM is an iterative algorithm that was developed by Dunn in 1973 [17] and improved by Bezdek in 1981 [4]. The aim of FCM is to cluster a given sample in to a given number C of fuzzy clusters.

Formally, the algorithm takes as input a p-dimensional sample $X = \{x_i\}_{i=1,\ldots,N}$ and an integer number C of clusters to be found in the sample and provides an output matrix U of dimension NxC such that $\sum_{j=1}^{C} u_{ij} = 1$ that represents the degree of membership of the i'th element of the sample to the j'th fuzzy set (cluster).

Besides, the FCM algorithm is parametrized by two parameters. The first one is called the fuzzifier and it will be represented by m and the second is a distance function d. The fuzzifier m must be in the range $(1, \infty)$.

The algorithm tries to minimize the following objective function:

$$J_m = \sum_{i=1}^{N} \sum_{j=1}^{C} u_{ij}^m \cdot d(x_i, c_j)$$

where x_i corresponds are the elements of the sample X, c_j the centroid (center of masses) of the j'th cluster and d the distance used.

In order to minimize this objective function the algorithm follows the scheme below:

1. Initialize U, U_0
2. At k step calculate the centers c_j with U_k

$$c_j = \frac{\sum_{i=1}^{N} u_{ij}^m \cdot x_i}{\sum_{i=1}^{N} u_{ij}^m}$$

3. Update U_k, U_{k+1}

$$u_{ij} = \frac{1}{\sum_{k=1}^{C} \left(\frac{d(x_i,c_j)}{d(x_i,c_k)}\right)^{\frac{2}{m-1}}}$$

4. If $\left\|U^{(k+1)} - U^{(k)}\right\| \leq \epsilon$ STOP; Else return to 2

The second step of the algorithm updates the centroids by calculating the center of masses of the elements of the clusters.

The third step calculates the membership degree of each element to each cluster according to the distance to the centroids.

In [17] it is proved that this iterative process converges toward a matrix U_{ij} that minimizes the objective function J_m.

It is a must to remark that though this algorithm has evolved up to being one of the cornerstones of the applications of Fuzzy Logic, it is a computationally costly algorithm. This is because many operations, assignations and iterations are usually necessary in order to obtain a good solution. One of the problems is the fact of beginning with a random matrix U_{ij} which implies a random family of centers $\{c_j\}_{j=1,\ldots,C}$ at the first iteration.

In order to overcome this problem a strong effort has been done in the research of heuristics and methods to improve the performance of the algorithm. A review of the enhancements proposed in the literature can be found at [5].

However, the exponential evolution of the processing power of computers in the last decade has left aside this efficiency problem in general, as for few dimensions and normal sizes of samples, FCM provides a good solution in a very little amount of time.

Finally, it must be recalled that FCM needs a priori the number of clusters to be found. For that reason, some kind of previous method must be used in order to determine the number C of clusters to be found. In Chapter 4, a watershed method will be used to state the number of clusters to be found in a histogram.

3 Indistinguishabilty operators theory

This chapter will be devoted to study in depth the concepts introduced in Section 2.2 of indistinguishability operators, sets of fuzzy extensional subsets and upper and lower approximation operators.

For the rest of the work, given a t-norm T, the set of all T-indistinguishability operators will be recalled $\mathcal{E}$, the set of sets of extensional fuzzy subsets related to T-indistinguishability operators will be named $\mathcal{H}$, and the sets of upper and lower approximations with the T t-norm will be recalled $\mathcal{U}$ and $\mathcal{L}$ respectively.

In the first section it will be showed how the sets $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ can be given a lattice structure. Further it will be explicitly proved that these lattices are isomorphic.

The second section will deal with more complicated aggregation operators such as natural means, and it will be studied how some results of the previous section can be extended while others fail. Counterexamples will be provided to illustrate the tricky points. This will be studied either in a finite and non-finite case.

The third section will derive some properties of applying powers of t-norms to $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$. These properties open some interesting lines of research, either theoretical and practical, that will be explained in depth in Chapter 5.

3.1 Structural Analysis of $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$

3.1.1 The Lattice Structure of $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$

In this section we will show how the sets $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ can be given a lattice structure. In order to do so it will be necessary to define some previous concepts to assure that both the union and intersection operations are internal of the set.

The effort of finding the algebraic structure underlying these sets is necessary to understand in depth how tight is the internal relation of the sets, as well as the relations between them.

First of all, let us recall the definition of transitive closure of a reflexive and symmetric fuzzy relation (a proximity) which provides the smallest indistinguishability operator containing it.

Definition 3.1. *Let T be a t-norm and R a fuzzy proximity (a reflexive and symmetric fuzzy relation) on a set X. The transitive closure $\overline{R}$ of R is the T-indistinguishability operator on X defined by*

$$\overline{R} = \bigcap_{E \in A} E$$

where A is the set of T-indistinguishability operators on X greater than or equal to R.

The transitive closure of a reflexive and symmetric fuzzy relation is transitive and thus an indistinguishability operator. Moreover, given a family of indistinguishability operators, its intersection is also an indistinguishability operator. However the union $E \cup F$ does not preserve transitivity and hence is not in general an indistinguishability operator. In order to close the union under transitivity we can consider the operation $\overline{E \cup F}$.

This way, the set $\mathcal{E}$ becomes a lattice with the natural ordering $E \leq F$ and the operations $E \cap F$ and $\overline{E \cup F}$.

A similar construction can be done with the set $\mathcal{H}$. The intersection of sets of extensional fuzzy subsets H_E and H_F preserves extensionality, but the union does not. It becomes necessary then to define a closure for the union.

Definition 3.2. *Let $J \subseteq [0,1]^X$. Its extensional closure $\overline{J}$ is defined by:*

$$\overline{J} = \bigcap_{H \in A, J \subseteq H} H$$

where A is the set of subsets of $[0,1]^X$ satisfying the properties of Proposition 2.17.

Now the set $\mathcal{H}$ is a lattice with the ordering $H_E \subseteq H_F$ and the operations $H_E \cap H_F$ and $\overline{H_E \cup H_F}$.

An analogous construction can be done in the set of upper approximations $\mathcal{U}$. Below we define the concept of ϕ-closure of a map.

Definition 3.3. *Let f be a map $f : [0,1]^X \to [0,1]^X$. The ϕ-closure $\overline{f}$ of f is defined as:*

$$\overline{f} = \bigwedge_{f \leq \phi, \phi \in U} \phi$$

where U is the set of maps $\phi : [0,1]^X \to [0,1]^X$ satisfying the properties of Theorem 2.20.

Proposition 3.4. *Let f be a map $f : [0,1]^X \to [0,1]^X$. Then $\overline{f} \in \mathcal{U}$.*

Proof. It is straightforward to check that $\overline{f}$ fullfills the properties of Theorem 2.20. □

It is straightforward to see that $\mathcal{U}$ is a lattice with the ordering $\phi_E \leq \phi_F$ and the operations $\phi_E \wedge \phi_F$ and $\overline{\phi_E \vee \phi_F}$.

Finally, the case of lower approximations is dual to the construction above. This is because lower approximations are in fact dual with respect to upper approximations, as one is a closure operator and the other an interior one.

Definition 3.5. *Let f be a map $f : [0,1]^X \to [0,1]^X$. The ψ-interior $\underline{f}$ of f is defined as:*

$$\underline{f} = \bigvee_{\psi \in L, f \leq \psi} \psi$$

where L is the set of maps $\psi : [0,1]^X \to [0,1]^X$ satisfying the properties of Theorem 2.21.

Proposition 3.6. *Let f be a map $f : [0,1]^X \to [0,1]^X$. Then $\underline{f} \in \mathcal{L}$.*

Proof. It is straightforward to check that $\underline{f}$ fullfills the properties of Theorem 2.20. □

Considering the ordering $\psi_E \leq \psi_F$ and the operations $\underline{\psi_E \wedge \psi_F}$ and $\psi_E \vee \psi_F$, $\mathcal{L}$ is a lattice.

3.1.2 Isomorphisms between the Lattices $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$

In this section it will be shown that $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ are isomorphic lattices. Firstly, the bijections between them will be explicitly stated and it will be shown how these bijections are compatible with the structure and the operations constructed in Section 2.

In Theorem 2.18 it is proved that $\mathcal{E}$ and $\mathcal{H}$ are in bijection. Following let us state the bijections for upper and lower approximations.

Proposition 3.7. *[13] Let $\phi : [0,1]^X \to [0,1]^X$ be a map satisfying the properties of Proposition 2.20. Then the fuzzy relation E_ϕ defined by:*

$$E_\phi(x,y) = \phi(\{x\})(y)$$

is a T-indistinguishability operator on X.

Proposition 3.8. *[13] Let $\psi : [0,1]^X \to [0,1]^X$ be a map satisfying the properties of Proposition 2.21. Then the fuzzy relation E_ψ defined by*

$$E_\psi(x,y) = \inf_{\alpha \in [0,1]} \overrightarrow{T}(\psi(\overrightarrow{T}(\{x\}|\alpha)(y)|\alpha)).$$

is a T-indistinguishability operator on X.

It can be seen that the maps assigning $E \to \phi_E$ and $\phi \to E_\phi$ are inverse one from the other and therefore $\mathcal{E}$ and $\mathcal{U}$ are in bijection. A detailed proof of this fact can be found in [13]. The same situation is given with lower approximations.

The following result proves how the ordering between the lattices are completely correlated through the bijections.

Proposition 3.9. *Let E, F be two indistinguishability operators on X. Then:*

1. $E \leq F \Leftrightarrow H_F \subseteq H_E$
2. $E \leq F \Leftrightarrow \phi_E \leq \phi_F$
3. $E \leq F \Leftrightarrow \psi_F \leq \psi_E$

Proof.

1. Let $\mu \in H_F$. Then $\forall x, y\ \ \mu(x) \geq T(F(x,y), \mu(y)) \geq T(E(x,y), \mu(y))$. Hence $\mu \in H_E$.

2. Straightforward because T is increasing with respect to the first variable.

3. Straightforward because $\overrightarrow{T}$ is decreasing with respect to the first variable.

$\square$

Finally, the following theorem shows how the structure given by the operations is preserved through the bijections. Furthermore, it is proved how the operations are transformed. As a corollary of this key theorem we will have that $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ are isomorphic lattices.

Theorem 3.10. *Let E,F be two T-indistinguishability operators on a set X. Then:*

1. $H_{\overline{E \cup F}} = H_E \cap H_F$
2. $H_{E \cap F} = \overline{H_E \cup H_F}$
3. $\phi_{\overline{E \cup F}} = \overline{\phi_E \vee \phi_F}$
4. $\phi_{E \cap F} = \phi_E \wedge \phi_F$
5. $\psi_{\overline{E \cup F}} = \underline{\psi_E \wedge \psi_F}$
6. $\psi_{E \cap F} = \psi_E \vee \psi_F$

Proof.

1. $\mu \in H_E \cap H_F \Leftrightarrow \mu \in H_E$ and $\mu \in H_F$. This is equivalent to $E_\mu \geq E$ and $E_\mu \geq F$ or $E_\mu \geq E \cup F$ and by the definition of the extensional closure, this is equivalent to $E_\mu \geq \overline{E \cup F} \Leftrightarrow \mu \in H_{\overline{E \cup F}}$

2. $\supseteq$)
   $E \cap F \leq E \Rightarrow H_E \subseteq H_{E \cap F}$.
   $E \cap F \leq F \Rightarrow H_F \subseteq H_{E \cap F}$.
   Then $H_E \cup H_F \subseteq H_{E \cap F} \Rightarrow \overline{H_E \cup H_F} \subseteq H_{E \cap F}$.

   $\subseteq$)

Let $\mu \in H_{E\cap F}$. Let us suppose $\mu \notin \overline{H_E \cup H_F}$. Then there exists an extensional set H_G with indistinguishability operator G such that $H_G \supseteq H_E \cup H_F$ and $\mu \notin H_G \Rightarrow \exists (x,y)$ such that $E_\mu(x,y) < G(x,y)$. Without loss of generality we can assume $E(x,y) \leq F(x,y)$. As $H_G \supseteq H_E \cup H_F$, $H_E \subseteq H_G$ and $H_F \subseteq H_G \Rightarrow G \leq E$ and $G \leq F$. So

$E_\mu(x,y) < G(x,y) \leq E(x,y) \leq F(x,y)$. Hence

$E_\mu(x,y) < (E \cap F)(x,y)$.

Therefore, $\mu \notin H_{E\cap F}$ which contradicts our hypothesis.
So $\mu \in \overline{H_E \cup H_F}$.

3. Let us recall $\overline{\phi_E \vee \phi_F} = \phi_G$. We have to see that $\phi_G = \phi_{\overline{E\cup F}}$.

$\leq)$
$E \leq \overline{E \cup F} \Rightarrow \phi_E \leq \phi_{\overline{E\cup F}}$
$F \leq \overline{E \cup F} \Rightarrow \phi_F \leq \phi_{\overline{E\cup F}}$
Hence, $\overline{\phi_E \vee \phi_F} = \phi_G \leq \phi_{\overline{E\cup F}}$.

$\geq)$
$\phi_G \geq \phi_E \Rightarrow G \geq E$
$\phi_G \geq \phi_F \Rightarrow G \geq F$
Hence, $G \geq \overline{E \cup F} \Rightarrow \phi_G \geq \psi_{\overline{E\cup F}}$.

4. Let us recall $\phi_E \wedge \phi_F = \phi_G$. We have to see that $\phi_G = \phi_{\overline{E\cup F}}$.

$\geq)$
$E \cap F \leq E \Rightarrow \phi_{E\cap F} \leq E$
$E \cap F \leq F \Rightarrow \phi_{E\cap F} \leq F$
Hence, $\phi_E \wedge \phi_F = \phi_G \geq \phi_{E\cap F}$.

$\leq)$
$\phi_G \leq \phi_E \Rightarrow G \leq E$
$\phi_G \leq \phi_F \Rightarrow G \leq F$
Hence, $G \leq E \cap F \Rightarrow \phi_G \leq \phi_{E\cap F}$.

5. Let us recall $\underline{\psi_E \wedge \psi_F} = \psi_G$. We have to see that $\psi_G = \psi_{\overline{E\cup F}}$.

$\geq$)
$E \leq \overline{E \cup F} \Rightarrow \psi_E \geq \psi_{\overline{E \cup F}}$
$F \leq \overline{E \cup F} \Rightarrow \psi_F \geq \psi_{\overline{E \cup F}}$
Hence, $\underline{\psi_E \wedge \psi_F} \geq \psi_{\overline{E \cup F}} \Rightarrow \psi_G \geq \psi_{\overline{E \cup F}}$.

$\leq$)
$\psi_G \leq \psi_E \Rightarrow G \geq E$
$\psi_G \leq \psi_F \Rightarrow G \geq F$
Hence, $G \geq \overline{E \cup F} \Rightarrow \psi_G \leq \psi_{\overline{E \cup F}}$.

6. Let us recall $\psi_E \vee \psi_F = \psi_G$. We have to see that $\psi_G = \psi_{\overline{E \cup F}}$.

   $\leq$)
   $E \cap F \leq E \Rightarrow \psi_{E \cap F} \geq E$
   $E \cap F \leq F \Rightarrow \psi_{E \cap F} \geq F$
   Hence, $\psi_E \vee \psi_F = \psi_G \leq \psi_{E \cap F}$.

   $\geq$)
   $\psi_G \geq \psi_E \Rightarrow G \leq E$
   $\psi_G \geq \psi_F \Rightarrow G \leq F$
   Hence, $G \leq E \cap F \Rightarrow \psi_G \geq \psi_{E \cap F}$.

□

Corollary 3.11. $\mathcal{E} \cong \mathcal{H} \cong \mathcal{U} \cong \mathcal{L}$

This result has a deep importance either from a theoretical and practical viewpoint.

At a theoretical level it has just been proved that $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ are isomorphic lattices, and hence despite its different semantic approach at a structural level there is no difference between them.

From a practical viewpoint, these isomorphisms provide a dictionary to translate information between the different lattices. This can turn to be useful in situations where different attributes that have to be considered come from different sources and are formally codified in different lattices. Knowing that these lattices are equivalent and knowing how to move information from one to the other allows us to overcome the problem of having data of different nature to be considered at a time.

The results of Theorem 3.10 can be summarized in the following diagram:

$$\begin{array}{ccccccc}
\mathcal{L} & \approx & \mathcal{U} & \approx & \mathcal{E} & \approx & \mathcal{H} \\
\psi_E \vee \psi_F & \leftrightarrow & \phi_E \wedge \phi_F & \leftrightarrow & E \cap F & \leftrightarrow & \overline{H_E \cup H_F} \\
\underline{\psi_E \wedge \psi_F} & \leftrightarrow & \overline{\phi_E \vee \phi_F} & \leftrightarrow & \overline{E \cup F} & \leftrightarrow & H_E \cap H_F
\end{array}$$

3.2 Natural Means over $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$

3.2.1 The Finite Case

In this section natural weighted mean operators on $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ will be studied. Specifically we will focus on analyzing if similar equalities to the ones found in Theorem 3.10 hold with natural weighted mean operators. The answer will be positive between $\mathcal{E}$ and $\mathcal{H}$ and negative with upper and lower approximations. A counterexample will be provided at the end to illustrate where equality fails.

In order to be able to define weighted natural means, we will assume in this section that the t-norm is continuous Archimedean.

In sake of simplicity we have only considered quasi-arithmetic means of two indistinguishability operators, though the generalization to a finite number of them is straightforward. In Section 7 the possibility of aggregating a non-finite number of them will be studied.

First of all, let us recall the concept of quasi-arithmetic mean.

Definition 3.12. *[1] Let $t : [0, 1] \to [-\infty, \infty]$ be a strict monotonic map and $x, y \in [0, 1]$. The quasi-arithmetic mean m_t of x and y is defined as:*

$$m_t(x, y) = t^{-1}(\frac{t(x) + t(y)}{2})$$

m_t is continuous if and only if $\{-\infty, \infty\} \nsubseteq Ran(t)$.

If we take as strict monotonic map the additive generator of a continuous Archimedean t-norm T, we will talk of the natural mean with respect to T. The term natural comes from the fact that chosen the t-norm, the mean is fixed by this choice as there is a bijection between continuous Archimedean t-norms and continuous quasi-arithmetic means.

Proposition 3.13. *[11] The map assigning every continuous Archimedean t-norm with additive generator t the quasi-arithmetic mean m_t is a bijection between continuous Archimedean t-norms and continuous quasi-arithmetic means.*

Quasi arithmetic weighted means are defined below:

Definition 3.14. *Let $t : [0,1] \to [-\infty, \infty]$ be a strict monotonic map, $x, y \in [0,1]$and $r \in [0,1]$. The weighted quasi-arithmetic mean (with weight r) m_t^r of x and y is defined as:*

$$m_t^r(x, y) = t^{-1}(r \cdot t(x) + (1 - r) \cdot t(y)).$$

We can observe that the parameter r defines an homotopy between x and y which recovers the definition of quasi-arithmetic mean for $r = \frac{1}{2}$.

It can be proved that the natural weighted means of T-indistinguishability operators is a T-indistinguishability operator as well. This way, natural means can be seen as operators on $\mathcal{E}$.

Proposition 3.15. *[11] Let T be a continuous Archimedean t-norm with additive generator t, E, F two T-indistinguishability operators on a set X and $r \in [0,1]$. Then the natural weighted mean of E and F defined as:*

$$m_t^r(E, F)(x, y) = t^{[-1]}(r \cdot t(E(x, y)) + (1 - r) \cdot t(F(x, y)))$$

is a T indistinguishability operator on X.

Natural weighted means can be defined as well over $\mathcal{H}, \mathcal{U}$ and $\mathcal{L}$ as follows:

Definition 3.16. *Let T be a continuous Archimedean t-norm with additive generator t, E, F two T-indistinguishability operators on a set X and $r \in [0,1]$. Then the set $\mu \in m_t^r(H_E, H_F) \in [0,1]^X$ and the operators $m_t^r(\phi_E, \phi_F), m_t^r(\psi_E, \psi_F) : [0,1]^X \to [0,1]^X$ are defined in the following way:*

- $\mu \in m_t^r(H_E, H_F) \Leftrightarrow \exists \nu \in H_E, \rho \in H_F \quad \mu(x) = t^{[-1]}(r \cdot t(\nu(x)) + (1 - r) \cdot t(\rho(x)))$
- $m_t^r(\phi_E, \phi_F)(\mu)(x) = t^{-1}(r \cdot t(\phi_E(\mu)(x)) + (1 - r) \cdot t(\phi_F(\mu)(x)))$
- $m_t^r(\psi_E, \psi_F)(\mu)(x) = t^{-1}(r \cdot t(\psi_E(\mu)(x)) + (1 - r) \cdot t(\psi_F(\mu)(x)))$

We will prove that $\overline{m_t(H_E, H_F)} = H_{m_t(E,F)}$. The following definition and lemmas are previous results to simplify the proof of the previous equality.

Definition 3.17. *Let E be an indistinguishability operator on a set X. The columns μ_x of E are defined as:*

$$\mu_x(y) = E(x, y) \quad \forall x, y \in X.$$

Lemma 3.18. *Let μ_x be a column of $m_t^r(E, F)$. Then $\mu_x \in m_t^r(H_E, H_F)$.*

Proof. μ_x is a column of $m_t^r(E, F)$ if and only if $\mu_x(y) = m_t^r(E, F)(x, y)$.

Then $\mu_x = m_t^r(\nu_x, \rho_x)$ with ν_x and ρ_x the corresponding columns of E and F respectively.

Indeed $\mu_x(y) = m_t^r(E, F)(x, y) = m_t^r(E(x, y), F(x, y)) = m_t^r(\nu_x(y), \rho_x(y))$ □

Lemma 3.19. *[10] Let A, B be two families of fuzzy subsets of X. Then:*

$$E_A = E_B \Leftrightarrow \overline{A} = \overline{B}$$

where E_A and E_B are the T-indistinguishability operators generated by the families A and B according to the Representation Theorem 2.16 respectively.

Theorem 3.20. *Let T be a continuous Archimedean t-norm with additive generator t, E and F two T-indistinguishability operators on a set X with associated sets of extensional fuzzy subsets H_E and H_F respectively and $r \in [0, 1]$. Then:*

$$\overline{m_t^r(H_E, H_F)} = H_{m_t^r(E,F)}$$

Proof. $\leq$)

We will prove that $m_t^r(H_E, H_F) \leq H_{m_t^r(E,F)}$. This will prove the inequality, as $\overline{m_t^r(H_E, H_F)}$ is the smallest set of extensional fuzzy subsets greater than or equal to $m_t^r(H_E, H_F)$.

Let $\mu \in H_E$ and $\nu \in H_F$. We have to see that

$$E_{m_t^r(\mu,\nu)} \geq m_t^r(E, F)$$

and this is equivalent to prove that

$$T(m_t^r(E, F)(x, y), m_t^r(\mu(y), \nu(y))) \leq m_t^r(\mu(x), \nu(x)).$$

Expanding, this is analogous to prove that

$$t^{[-1]}(t(t^{-1}(r\cdot t(E(x,y))+(1-r)\cdot t(F(x,y))))+t(t^{-1}(r\cdot t(\mu(y))+(1-r)\cdot t(\nu(y)))))$$
$$\leq t^{-1}(r\cdot t(\mu(x))+(1-r)\cdot t(\nu(x))).$$

Simplifying,

$$t^{-1}(r\cdot t(E(x,y))+(1-r)\cdot t(F(x,y)))+r\cdot t(\mu(y))+(1-r)\cdot t(\nu(y))))$$
$$\leq t^{-1}(r\cdot t(\mu(x))+(1-r)\cdot t(\nu(x))).$$

Which is equivalent to:

$$r\cdot t(E(x,y))+(1-r)\cdot t(F(x,y))+r\cdot t(\mu(y))+(1-r)\cdot t(\nu(y))$$
$$\geq r\cdot t(\mu(x))+(1-r)\cdot t(\nu(x))$$

And this is true because $\mu\in H_E$ and $\nu\in H_F$.

$\underline{\geq)}$

$\overline{m_t^r(H_E,H_F)}\geq H_{m_t^r(E,F)}\Leftrightarrow E_{m_t^r(H_E,H_F)}\leq m_t^r(E,F)$. We will prove this last inequality.

Let μ_x be a column of $m_t^r(E,F)$. Then thanks to Lemma 3.18 $\mu\in m_t^r(H_E,H_F)$ and

$$E_{m_t^r(H_E,H_F)}(y,z)\leq E_{\mu_x}(y,z)=\overleftrightarrow{T}(m_t^r(E,F)(x,y),m_t^r(E,F)(x,z))$$
$$\leq m_t^r(E,F)(y,z)$$

as $\overleftrightarrow{T}$ is T-transitive.

Hence $E_{m_t^r(H_E,H_F)}\leq m_t^r(E,F)$ or, equivalently, $\overline{m_t^r(H_E,H_F)}\geq H_{m_t^r(E,F)}$.

□

This Theorem answers positively the question at the beginning of the section. We have just proved constructively that it is possible to extend the isomorphism found in the previous chapter to the field of natural weighted means in the case of $\mathcal{E}$ and $\mathcal{H}$. It would be expectable, given the (dual) symmetry that has emerged along all the work between $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ that the answer was positive either in the last two lattices. Counterintuitively, this is not true and symmetry is broken at this point.

Below we will prove that half of the analogy is kept, as one inequality is preserved when we consider the effect of natural means, either in $\mathcal{U}$ and $\mathcal{L}$. However, the other inequality fails as we will illustrate with a counterexample that will show how we cannot overcome this fact.

Proposition 3.21. *Let T be a continuous Archimedean t-norm with additive generator t, E and F two T-indistinguishability operators on a set X with associated upper approximations ϕ_E and ϕ_F respectively and $r \in [0,1]$. Then:*

$$m_t^r(\phi_E, \phi_F) \geq \phi_{m_t^r(E,F)}.$$

Proof. Let $\mu \in [0,1]^X$ be a fuzzy subset and $x \in X$. We have to see that

$$m_t^r(\phi_E, \phi_F)(\mu(x)) \geq \phi_{m_t^r(E,F)}(\mu(x)).$$

Expanding the expression we have to see that

$$t^{-1}(r \cdot t(\sup_{y\in X} t^{[-1]}(t(E(x,y)) + t(\mu(y))) + (1-r) \cdot t(\sup_{y\in X} t^{[-1]}(t(F(x,y)) + t((\mu(y))))$$
$$\geq \sup_{y\in X} t^{[-1]}(t(t^{-1}(r \cdot t(E(x,y)) + (1-r) \cdot t(F(x,y)) + t(\mu(y))).$$

This is equivalent to

$$t^{-1}(r \cdot \inf_{y\in X}(t(E(x,y)) + t(\mu(y))) + (1-r) \cdot \inf_{y\in X}(t(F(x,y)) + t(\mu(y))))$$
$$\geq t^{-1}(\inf_{y\in X}(r \cdot t(E(x,y)) + (1-r) \cdot t(F(x,y)) + t(\mu(y))))$$

or, equivalently,

$$r \cdot \inf_{y\in X}(t(E(x,y)) + t(\mu(y))) + (1-r) \cdot \inf_{y\in X}(t(F(x,y)) + t(\mu(y)))$$
$$\leq \inf_{y\in X}(r \cdot t(E(x,y)) + (1-r) \cdot t(F(x,y)) + t(\mu(y))),$$

which is true because the addition of infima is smaller than or equal to the infimum of the addition. □

Proposition 3.22. *Let T be a continuous Archimedean t-norm with additive generator t, E and F two T-indistinguishability operators on a set X with associated lower approximations ψ_E and ψ_F respectively and $r \in [0,1]$. Then:* $m_t^r(\psi_E, \psi_F) \leq \psi_{m_t^r(E,F)}$

Proof. Let $\mu \in [0,1]^X$ be a fuzzy subset and $x \in X$.
Rewriting this inequality terms of t, it has to be proved that

$$t^{-1}(r \cdot t(\inf_{y\in X}(t^{[-1]}(t(\mu(y)) - t(E(x,y))) + (1-r) \cdot t(\inf_{y\in X}(t^{[-1]}(t(\mu(y)) - t(F(x,y))))$$
$$\leq \inf_{y\in X} t^{[-1]}(t(\mu(y)) - t(t^{-1}(r \cdot t(E(x,y)) + (1-r) \cdot t(F(x,y)))))$$

Simplifying we have the equivalent expression,

$$t^{-1}(r \cdot \sup_{y\in X}(t(\mu(y)) - t((E(x,y))) + (1-r)\cdot \sup_{y\in X}(t(\mu(y)) - t((F(x,y))))$$
$$\leq \inf_{y\in X} t^{-1}(t(\mu(y)) - r\cdot t(E(x,y)) - (1-r)\cdot t(F(x,y))),$$

which is equivalent to

$$r\cdot \sup_{y\in X}(t(\mu(y)) - t(E(x,y))) + (1-r)\cdot \sup_{y\in X}(t(\mu(y)) - t(F(x,y)))$$
$$\geq \sup_{y\in X}(t(\mu(y)) - r\cdot t(E(x,y)) - (1-r)\cdot t(F(x,y))).$$

And this is true because the addition of suprema is greater than or equal to the supremum of the addition. □

As it was discussed before, the reciprocal inequality does not hold and hence it is not possible to reach equality as the following counterexample shows.

Example 3.23. *Let X be a finite set of cardinality 3. Let us consider the following T-indistinguishability operators E and F with T the Łukasiewicz t-norm.*

$$E = \begin{pmatrix} 1 & 0.6 & 0.7 \\ 0.6 & 1 & 0.8 \\ 0.7 & 0.8 & 1 \end{pmatrix} \quad F = \begin{pmatrix} 1 & 0.8 & 0.7 \\ 0.8 & 1 & 0.6 \\ 0.7 & 0.6 & 1 \end{pmatrix}$$

The natural mean $m_t(E,F)$ of E and F is:

$$m_t(E,F) = \begin{pmatrix} 1 & 0.7 & 0.7 \\ 0.7 & 1 & 0.7 \\ 0.7 & 0.7 & 1 \end{pmatrix}$$

Let us consider the fuzzy subset $\mu = \begin{pmatrix} 1 & 1 & 0.7 \end{pmatrix}$

$\mu \in H_{m_t(E,F)}$ since $E_\mu \geq m_t(E,F)$ (Proposition 2.15).

$\mu \in H_{m_t(E,F)}$ and $\mu \in H_F$ but $\mu \notin H_E$.

Extensional fuzzy subsets are fixed points with respect to upper and lower approximations, so:

$\phi_{m_t(E,F)}(\mu) = \phi_F(\mu) = \mu$, $\psi_{m_t(E,F)}(\mu) = \psi_F(\mu) = \mu$.

But $\mu \notin H_E \Rightarrow \phi_E(\mu) > \mu$ and $\psi_E(\mu) < \mu$.

So $\phi_{m_t(E,F)}(\mu) < m_t(\phi_E, \phi_F)$ and $\psi_{m_t(E,F)}(\mu) > m_t(\psi_E, \psi_F)$, and hence equality is not reached.

The differential fact between $\mathcal{H}$ and $\mathcal{U}$, $\mathcal{L}$ that makes the difference which breaks the symmetry is the following: Without considering closures it is straightforward to prove the following inequalities:

- $m_t^r(H_E, H_F) \leq H_{m_t^r(E,F)}$
- $m_t^r(\phi_E, \phi_F) \geq \phi_{m_t^r(E,F)}$
- $m_t^r(\psi_E, \psi_F) \leq \psi_{m_t^r(E,F)}$

If we take the closures defined in Section 4 in the previous expressions we have that

- $m_t^r(H_E, H_F) \leq \overline{m_t^r(H_E, H_F)} \leq H_{m_t^r(E,F)}$
- $\overline{m_t^r(\phi_E, \phi_F)} \geq m_t^r(\phi_E, \phi_F) \geq \phi_{m_t^r(E,F)}$
- $\underline{m_t^r(\psi_E, \psi_F)} \leq m_t^r(\psi_E, \psi_F) \leq \psi_{m_t^r(E,F)}$

And as it has been proved, taking the closure is the key to reach equality in the first expression, but brings no new information to overcome the inequality in the second and third case.

Finally, let us prove some final results around natural means applied on $\mathcal{U}$ and $\mathcal{L}$

Theorem 3.24. *Let F, G be two T-indistinguishability operators on a set X with associated upper and lower approximations ϕ_F, ψ_F, ϕ_G and ψ_G respectively and $r \in [0,1]$. Then:*

- $\phi_{E_{m_t^r(H_F,H_G)}} = \phi_{m_t^r(F,G)}$
- $\psi_{E_{m_t^r(H_F,H_G)}} = \psi_{m_t^r(F,G)}$

Proof. From Lemma 3.19 $E_{m_t^r(H_F,H_G)} = E_{\overline{m_t^r(H_F,H_G)}}$.

From Theorem 3.20 $m_t^r(H_F, H_G) = H_{m_t^r(F,G)}$.

Joining results, $E_{m_t^r(H_F,H_G)} = E_{H_{m_t^r(F,G)}} = m_t^r(F,G) \Rightarrow \phi_{E_{m_t^r(H_F,H_G)}} = \phi_{m_t^r(F,G)}$ and $\psi_{E_{m_t^r(H_F,H_G)}} = \psi_{m_t^r(F,G)}$.

$\square$

3.2.2 The Non-Finite Case

In some cases we have to aggregate a non-finite number of relations. For example, if we need to calculate the similarity between fuzzy subsets. This section will generalize the natural mean of T-indistinguishability operators to a non-finite family of them.

Suppose that we have a family of T-indistinguishability operators $(E_i)_{i\in[a,b]}$ on a set X with the indices in the interval $[a,b]$ of the real line and that for every couple (x,y) of X the map $f_{(x,y)} : [a,b] \to \mathbb{R}$ defined by $f_{(x,y)}(i) = E_i(x,y)$ is integrable in some sense.

Definition 3.25. *Let T be a continuous Archimedean t-norm with additive generator t and $(E_i)_{i\in[a,b]}$ a family of T-indistinguishability operators on a set X. The mean aggregation of the family $(E_i)_{i\in[a,b]}$ is the T-indistinguishability operator E defined for all $x,y \in X$ by*

$$E(x,y) = t^{-1}\Big(\frac{1}{b-a}\int_a^b t(E_i(x,y))di\Big).$$

Proposition 3.26. *This definition is independent of the generator of T thanks to the linearity of integration.*

Proof. If $t' = \alpha t$ is another additive generator of T, then

$$\begin{aligned} & t'^{-1}\Big(\frac{1}{b-a}\int_a^b t'(E_i(x,y))di\Big) \\ &= t^{-1}\Big(\frac{1}{b-a}\frac{\int_a^b \alpha t(E_i(x,y))di}{\alpha}\Big) \\ &= t^{-1}\Big(\frac{1}{b-a}\int_a^b t(E_i(x,y))di\Big). \end{aligned}$$

□

Proposition 3.27. *The fuzzy relation E obtained in Definition 3.25 is a T-indistinguishability operator on X.*

Proof. It is trivial to prove that E is a reflexive and symmetric fuzzy relation.

Let us prove that it is T-transitive.

Let $x, y, z \in X$.

$$\begin{aligned}
& T(E(x,y), E(y,z) \\
= \;& t^{[-1]}(t \circ t^{-1}(\frac{1}{b-a}\int_a^b t(E_i(x,y))di) + t \circ t^{-1}(\frac{1}{b-a}\int_a^b t(E_i(y,z))di)) \\
= \;& t^{[-1]}(\frac{1}{b-a}\int_a^b t(E_i(x,y))di + \frac{1}{b-a}\int_a^b t(E_i(y,z))di) \\
= \;& t^{[-1]}(\frac{1}{b-a}\int_a^b (t(E_i(x,y)) + t(E_i(y,z)))dx).
\end{aligned}$$

Since every E_i is T-transitive,

$$t(E_i(x,y)) + t(E_i(y,z)) \geq t(E_i(y,z)) \; \forall i \in [a,b]$$

and therefore

$$\int_a^b (t(E_i(x,y)) + t(E_i(y,z)))di \geq \int_a^b t(E_i(x,z))di$$

Since $t^{[-1]}$ is a non increasing map, we get

$$T(E_i(x,y), E_i(y,z) \leq E_i(x,z).$$

□

The most important necessity of aggregating a non-finite family of T-indistinguishability operators is when we need to calculate the degree of similarity between two fuzzy subsets μ and ν of a set X.

Probably the most natural way is comparing $\mu(x)$ and $\nu(x)$ for all $x \in X$ using $\overleftrightarrow{T}$ and then taking the infimum of all the results.

Definition 3.28. *Let μ, ν be two fuzzy subsets of a set X and T a t-norm. The degree of similarity $E_T(\mu, \nu)$ between μ and ν is defined by*

$$E_T(\mu, \nu) = \inf_{x \in X} \overleftrightarrow{T}(\mu(x), \nu(x)).$$

However this definition suffers from the drastic effect of the infimum. For example, if we have two fuzzy subsets μ, ν of a set X with $\mu(x) = \nu(x)$ for all $x \in X$ except for a value x_0 for which $\mu(x_0) = 1$ and $\nu(x_0) = 0$, then $E_T(\mu, \nu) = 0$ which means that both subsets are considered completely different or dissimilar. An average of the values obtained for every $x \in X$ seems a suitable alternative.

Definition 3.29. *Let μ, ν be two integrable fuzzy subsets of an interval $[a, b]$ of the real line. The averaging degree of similarity or indistinguishability $E_T^A(\mu, \nu)$ between μ and ν with respect to a continuous Archimedean t-norm T with additive generator t is defined by*

$$E_T^A(\mu, \nu) = t^{[-1]} \left(\frac{1}{b-a} \int_a^b t(\overleftrightarrow{T}(\mu(x), \nu(x)))dx \right).$$

With this definition, the degree of indistinguishability of the two fuzzy subsets considered after Definition 3.28 is 1, which is a very intuitive result.

Example 3.30. *Let us consider the two fuzzy subsets μ and ν of the interval $[0, 2]$ defined by $\mu(x) = 1/2$ and $\nu(x) = x/2$ $\forall x \in [0, 2]$. Let T_α be the Yager family of t-norms ($T_\alpha(x, y) = 1 - \min(1, (1-x)^\alpha + (1-y)^\alpha)^{\frac{1}{\alpha}}$ and $t_\alpha(x) = (1-x)^\alpha$ a generator of T_α with $\alpha \in (0, \infty)$).*

$$\overleftrightarrow{T_\alpha}(x, y) = 1 - |(1-y)^\alpha - (1-x)^\alpha|^{\frac{1}{\alpha}}$$

which implies that

$$\begin{aligned} \overleftrightarrow{T_\alpha}(\mu(x), \nu(x)) &= 1 - |(1-\nu(x))^\alpha - (1-\mu(x))^\alpha|^{\frac{1}{\alpha}} \\ &= \begin{cases} 1 - ((\frac{1}{2})^\alpha - 1 - \frac{x}{2})^\alpha)^{\frac{1}{\alpha}} & \text{if } x > 1 \\ 1 - ((1 - \frac{x}{2})^\alpha - (\frac{1}{2})^\alpha)^{\frac{1}{\alpha}} & \text{if } x \leq 1. \end{cases} \end{aligned}$$

and therefore

$$\begin{aligned} E_{T_\alpha}^A(\mu, \nu) &= t^{[-1]}(\frac{1}{2}(\int_0^1 ((1 - \frac{x}{2})^\alpha - (\frac{1}{2})^\alpha)dx) + \int_1^2 ((\frac{1}{2})^\alpha - (1 - \frac{x}{2})^\alpha)dx)) \\ &= t^{-1}(\frac{1 - (\frac{1}{2})^\alpha}{\alpha + 1}) = 1 - (\frac{1 - (\frac{1}{2})^\alpha}{\alpha + 1})^{\frac{1}{\alpha}}. \end{aligned}$$

If $\alpha = 1$, then T_α is the Łukasiewicz t-norm T and in this case the previous formula gives $E_T^A(\mu, \nu) = 3/4$ whereas using the infimum to aggregate we obtain $E_T(\mu, \nu) = 1/2$.

Following the lines of research of this work let us now relate the sets H_{E_i} of extensional fuzzy subsets of the family $(E_i)_{i \in [a,b]}$ with H_E.

Proposition 3.31. *Let T be a continuous Archimedean t-norm with additive generator t, $(E_i)_{i\in[a,b]}$ a family of T-indistinguishability operators on a set X and $(\mu_i)_{i\in[a,b]}$ a family of fuzzy subsets of X with $\mu_i \in H_{E_i}$ for all $i \in [a, b]$ and μ the fuzzy subset of X defined by*

$$\mu(x) = t^{-1}(\frac{1}{b-a}\int_a^b t(\mu_i(x))di).$$

Then $\mu \in H_E$, where E is the T-indistinguishability operator mean aggregation of the family $(E_i)_{i\in[a,b]}$.

Proof. Since μ_i is extensional with respect to E_i,

$$\mu_i(x) \leq t^{[-1]}(t(E_i(x, y)) + t(\mu_i(y)))$$

or

$$t(\mu_i(x)) \geq t(E_i(x, y)) + t(\mu_i(y)).$$

$$\begin{aligned} t(\mu(x) &= \frac{1}{b-a}\int_a^b t(\mu_i(x))di \\ &\geq \frac{1}{b-a}\int_a^b t(E_i(x,y))di + \frac{1}{b-a}\int_a^b t(\mu_i(y))di \\ &= t(E(x, y) + t(\mu(y)). \end{aligned}$$

□

As happened in the previous section, the relationship between upper and lower approximations lead to inequalities.

Proposition 3.32. *Let T be a continuous Archimedean t-norm with additive generator t, $(E_i)_{i\in[a,b]}$ a family of T-indistinguishability operators on a set X and $(\phi_{E_i})_{i\in[a,b]}$ the family of corresponding upper approximations. Then* *the upper approximation ϕ_E of the T-indistinguishability operator E, mean aggregation of the family $(E_i)_{i\in[a,b]}$ satisfies*

$$\phi_E(\mu(x)) \leq t^{-1}(\frac{1}{b-a}\int_a^b t(\phi_i(\mu(x)))di)$$

for all fuzzy subsets μ of X and $x \in X$.

Proof.

$$\begin{aligned}\phi_E(\mu(x)) &= \sup_{y\in X} T(E(x,y),\mu(y)) = \sup_{y\in X} t^{[-1]}(t(E(x,y)) + t(\mu(y))) \\ &= \sup_{y\in X} t^{[-1]}(\frac{1}{b-a}\int_a^b t(E_i(x,y))di + t(\mu(y))) \\ &\leq t^{-1}(\frac{1}{b-a}\int_a^b t(\phi_i(\mu(x)))di).\end{aligned}$$

□

Similarly the following result can be proved.

Proposition 3.33. *Let T be a continuous Archimedean t-norm with additive generator t, $(E_i)_{i\in[a,b]}$ a family of T-indistinguishability operators on a set X and $(\psi_{E_i})_{i\in[a,b]}$ the family of corresponding lower approximations. Then the upper approximation ϕ_E of the T-indistinguishability operator E, mean aggregation of the family $(E_i)_{i\in[a,b]}$ satisfies*

$$\psi_E(\mu(x)) \geq t^{-1}(\frac{1}{b-a}\int_a^b t(\psi_i(\mu(x)))di)$$

for all fuzzy subsets μ of X and $x \in X$.

3.3 Study of $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ under the Effect of an Isomorphism of t-norms

In this section it will be studied how all the results developed in the previous sections are affected if the t-norm is isomorphically changed and hence the indistinguishability operator is changed as well. As it would be expectable, it is proved that extensional fuzzy sets, upper approximations and lower approximations can be directly translated through this isomorphism from one indistinguishability operator to the other one, so all the results shown in this work are preserved.

First of all let us recall the definition of isomorphism of t-norms and of indistinguishability operators.

Definition 3.34. *[21] Two continuous t-norms T, T' are isomorphic if and only if there exists a bijective map $f : [0,1] \to [0,1]$ such that $f \circ T = T' \circ (f \times f)$.*

Definition 3.35. *[12] Given two t-norms T, T', a T-indistinguishability operator E on a set X and T'-indistinguishability E' on X', a morphism φ between E and E' is a pair of maps $\varphi = (h, f)$ such that the following diagram is commutative*

$$\begin{array}{ccc} X \times X & \xrightarrow{E} & [0,1] \\ \Big\downarrow{\scriptstyle h \times h} & & \Big\downarrow{\scriptstyle f} \\ X' \times X' & \xrightarrow{E'} & [0,1] \end{array}$$

(i.e. $f(E(x,y)) = E'(h(x), h(y))$ for all $x, y \in X$).
When h and f are bijective maps, φ is called an isomorphism.

In sake of simplicity we will take from now on $h = Id_X$ and $X' = X$. This way, isomorphisms will be only characterized by the map f.

An interesting result is that isomorphism between t-norms implies directly isomorphism between indistinguishability operators. The formal statement of this fact is given below:

Proposition 3.36. *Let $f : [0,1] \to [0,1]$ be an isomorphism between two t-norms T and T' and E a T-indistinguishability operator on a set X. Then E and $f \circ E$ are isomorphic indistinguishability operators.*

Proof. Straightforward. $\square$

Lemma 3.37. *[12] Let T, T' be isomorphic t-norms. Then $\overrightarrow{T}$ and $\overrightarrow{T'}$ are isomorphic.*

The following proposition is the key result of this section. It states that under an isomorphism of t-norms defined by a map f either extensional fuzzy subsets, upper approximations and lower approximations can be directly translated through this same map f.

Proposition 3.38. *Let $f : [0,1] \to [0,1]$ be a map defining an isomorphism between two t-norms T and T'. Let E be a T-indistinguishability operator on a set X and $f \circ E$ a T'-indistinguishability operator on X isomorphic to E. Then:*

- $\mu \in H_E \Leftrightarrow f \circ \mu \in H_{f \circ E}$
- $f \circ \phi_E = \phi_{f \circ E}$
- $f \circ \psi_E = \psi_{f \circ E}$

Proof. Straightforward. □

As sit was remarked in the discussion that opened this section, all the structural analysis and relationship done between $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ was conditioned by a previous choice of a certain t-norm. This final result is very important because it overcomes the dependence on this choice as is states that everything suits properly under isomorphism of t-norms. Hence, it has been show that all the relations found between $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ are robust with respect to the choice of the t-norm.

3.4 Powers of $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$

3.4.1 Powers of Indistinguishability Operators

The following two subsections will change the scope of interest of the previous subsections and will focus on studying the effect of the applications of powers (as introduced in Section 2.2.1) on indistinguishability operators and its related concepts.

It will be shown how the ordering on the real line derives an ordering on the lattices $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$ and some very promising results will emerge.

First of all, the following result is very useful, as it allows to calculate powers as they were defined in 2.2.1 by using an additive generator t of T, which makes operations more manageable.

Proposition 3.39. *Let T be an Archimedean t-norm with additive generator t and $r \in \mathbb{R}^+$. Then:*

$$x^r = t^{[-1]}(r \cdot t(x))$$

We have defined powers with objects x. In particular, we can consider the power of a T-indistinguishability operator E.

Definition 3.40. *Let E be a T-indistinguishability operator and $r \in \mathbb{R}^+$. We define the r'th power of E as:*

$$E^r(x,y) = T^r(E(x,y))$$

It can be proved that the power E^r of a T-indistinguishability operator E is a T-indistinguishability operator as well.

Theorem 3.41. *Let E be a T-indistinguishability operator and $r \in \mathbb{R}^+$. Then E^r is a T-indistinguishability operator.*

It is remarkable that the order we have on the real line is preserved order-reversed when we consider the associated powers of indistinguishability operators. Moreover, it is straightforward to see that this application is continuous and hence we can understand the increasing of the power as an homotopy or smooth transformation from one indistinguishability to a sharper or softer one.

From a linguistic viewpoint, this fact is trivial. Greater powers were related with strengthening the predicate, hence if we consider E^r and allow r to grow, it is natural that the result of the power gets sharper.

Proposition 3.42. *Let E be a T-indistinguishability operator and $r, s \in \mathbb{R}^+$. Then:*

$$r \leq s \Rightarrow E^r \geq E^s$$

Proof. $r \leq s \Rightarrow r \cdot t(E) \leq s \cdot t(E)$

And as t^{-1} is a monotone decreasing function:

$t^{-1}(r \cdot t(E)) \geq t^{-1}(s \cdot t(E)) \Rightarrow E^r \geq E^s$ □

In order to have continuity on the limits, we will define $E^0(x,y) = 1 \;\; \forall x, y \in X$ and $E^\infty(x,y) = Id_E(x,y)$.

This way we can see powers as homotopy operators that given any T-indistinguishability operator E provide us a gradation from the universal indistinguishability operator to the most sharp one passing through E when $r = 1$.

As it follows from this discussion, there is a tight relation between the ordering of powers and the ordering of indistinguishability operator. The next result proves that this relation remains as well when we consider the effect of natural means.

Theorem 3.43. *Let E be a T-indistinguishabilty operator on a set X and $r, s \in \mathbb{R}^+$. Then:*

$$E = m_t(E^r, E^s) \Leftrightarrow r + s = 2$$

Proof. Let $x, y \in X$. Then:

$m_t(E^r(x,y), E^s(x,y) = t^{-1}(\frac{t(E^r(x,y))+t(E^s(x,y))}{2})$

By Proposition 3.39 this is equal to:

$t^{-1}(\frac{t(t^{-1}(r \cdot t(E(x,y))))+t(t^{-1}(s \cdot t(E(x,y))))}{2})$

Simplifying we have:

$t^{-1}(\frac{r \cdot t(E(x,y))+s \cdot t(E(x,y))}{2})$

Which coincides with $E(x,y)$ if and only if $r + s = 2$. □

In particular we have that $E = m_t(\mathbf{1}, E^2))$ where $\mathbf{1}(x,y) = E^0(x,y) = 1 \ \forall x, y$. An interesting question that does not arise from the previous result is whether given an indistinguishability operator E there exists an indistinguishability F such that $E = m_t(F, Id)$.

3.4.2 Powers over $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$

This section will extend the application of powers to the lattices $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$.

First of all, let us prove the following lemma which will be useful further as it allows to understand the condition of extensionality in terms of additive generators.

Lemma 3.44. *Let E be a T-indistinguishability operator on a set X. $\mu \in H_E$ if and only if $\forall x, y \in X$:*

$$t(E(x,y)) + t(\mu(y)) \geq t(\mu(x))$$

Proof. $\mu \in H_E \Leftrightarrow T(E(x,y), \mu(y)) \leq \mu(x) \Leftrightarrow t^{-1}(t(E(x,y)) + t(\mu(y))) \leq \mu(x)$.

And as t is a monotone decreasing function this is equivalent to $t(E(x,y)) + t(\mu(y)) \geq t(\mu(x))$ □

The following result proves how tight is the relation between the set of extensionals H_E of an indistinguishability operator E and the one of its powers E^r.

Proposition 3.45. *Let E be a T-indistinguishability operator, H_E its set of extensional fuzzy subsets and $r \in \mathbb{R}^+$. Then*

$$\mu \in H_E \Leftrightarrow \mu^r \in H_{E^r}$$

Proof. By Lemma 3.44 $\mu^r \in H_{E^r} \Leftrightarrow t(E^r(x,y)) + t(\mu^r(y)) \geq t(\mu^r(x))$.

Expanding by Proposition 3.39 this is equivalent to:
$t(t^{-1}(r \cdot t(E(x,y)))) + t(t^{-1}(r \cdot t(\mu(y)))) \geq t(t^{-1}(r \cdot t(\mu(x))))$
Simplifying:
$r \cdot t(E(x,y)) + r \cdot t(\mu(y)) \geq r \cdot t(\mu(x))$
Which is equivalent to:
$t(E(x,y)) + t(\mu(y)) \geq t(\mu(x)) \Leftrightarrow \mu \in H_E$. □

The following lemma for extensional sets is analogous to the one for indistinguishability operators proved in Theorem 3.43 and is key to prove the Theorem below.

Lemma 3.46. *Let E be a T-indistinguishability operator over a set X, $\mu \in [0,1]^X$ a fuzzy subset and $r, s \in \mathbb{R}^+$ such that $r + s = 2$. Then:*

$$\mu = m_t(\mu^r, \mu^s)$$

Proof. Let $x \in X$. Then:
$m_t(\mu^r(x), \mu^s(x)) = t^{-1}(\frac{r \cdot t(\mu(x)) + s \cdot t(\mu(x))}{2})$
Which is equivalent to $\mu(x)$ because $r + s = 2$. □

The next result shows how H_E, ϕ_E and ψ_E can be obtained by considering means of them under the effect of powers. This result can be useful in cases where the effective computation of these concepts may be very costly but it is easy to compute them with powers E^r and E^s of E such that $r + s = 2$.

Theorem 3.47. *Let E be a T-indistinguishability operator over a set X, $r, s \in \mathbb{R}^+$ such that $r + s = 2$. Then:*

- $H_E = m_t(H_{E^r}, H_{E^s})$

- $\phi_E = m_t(\phi_{E^r}, \phi_{E^s})$
- $\psi_E = m_t(\psi_{E^r}, \psi_{E^s})$

Proof. We will prove explicitly the equality in the case of sets of extensional fuzzy subsets. With upper and lower approximations the proof is totally analogous to the one of Lemma 3.46.

Let $\mu \in H_E$. By Proposition 3.45 we have $\mu^r \in H_{E^r}$ and $\mu^s \in H_{E^s}$.

Besides, by Lemma 3.46 $\mu = m_t(\mu^r, \mu^s)$ and hence $H_E = m_t(H_{E^r}, H_{E^s})$. □

Finally, the following Theorem proves how the ordering $r \leq s$ we have over the positive real line derives an ordering too between the sets of extensionals, upper and lower approximations related to powers of indistinguishability operators.

This fact takes us to a similar situation to the one with indistinguishability operators where the use of powers provides homotopies between universal extensional sets, upper and lower approximations and sharp crisp ones. From a geometric and topologic viewpoint this turns out to be very interesting because it allows to see the degradation of fuzzy equivalence classes by just varying the power of E.

Theorem 3.48. *Let E be a T-indistinguishability operator over a set X and $r, s \in \mathbb{R}^+$. Then:*

- $r \leq s \Rightarrow H_{E^r} \subseteq H_{E^s}$
- $r \leq s \Rightarrow \phi_{E^r} \geq \phi_{E^s}$
- $r \leq s \Rightarrow \psi_{E^r} \leq \psi_{E^s}$

Proof. Straightforward from Lemma 3.42 and the Galois Connections between the lattices of T-indistinguishability operators, sets of extensional fuzzy subsets, upper and lower approximations shown in [24]. □

These last results open a very interesting line of research.

An indistinguishability operator E can be understood as a filter on the perception of reality that identifies objects. In this approach, extensional sets turn to be nothing but the observable sets in X. This means that under the effect of E not every set is perceivable but only the ones that can be discriminated by E.

For instance, given that our visual perception is limited, given a glass of wine what we see is only a dark-red liquid flow. However, wine is composed by many particles and different substances but to our vision these are indistinguishable. For that reason, all these different objects are captured into the same observable (extensional) set.

This example shows the importance of the approximation of fuzzy subsets by extensional ones. In this paper we have stated and worked with two operators that provided approximations: ϕ_E and ψ_E. These operators provide the best upper and lower approximations of a fuzzy subset by extensionals. However, it can be the case that despite being the best approximations containing or being contained (respectively) in μ, there is an extensional set α in the middle which is a much better approximation of μ in absolute terms.

This future line of research will be explained further in Chapter 5, as well as other possible ideas that emerge from the theoretical work done on the application of powers to $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$.

4 Image Segmentation with Fuzzy c-Means

Whereas the previous chapter had a pure theoretical scope, this one will face a practical biomedical segmentation problem. The challenge that emerges from bringing together these two scientifical viewpoints of solving problems will be treated in the next chapter.

As it was said in the introduction, in this chapter we will run a Fuzzy c-Means algorithm to segment six real patients' images. Four of these images are microscopical captures of Bone Marrow and the last two come from mammographies. The FCM algorithm will be ran after a previous preprocessing of the data through a watershed histogram-based method.

In the literature, the output of a segmentation process on an image is the same image labeled to discriminate the different objects. In this work the output aimed is to obtain as many images as objects identified on the initial one, and each image showing us one different object.

This chapter is structured as follows:

The first section will explain in depth the aim and interest of segmenting Bone Marrow and Mammography images. The images to be segmentated further will be introduced here.

The second one will show the resolution of the problem using the FCM algorithm beginning from a histogram analysis. The adjustment of parameters will be explained carefully and the final segmentation proposed by this method will be shown.

4.1 Bone Marrow and Mammography Segmentation Problem

For this problem we are going to work with six biomedical images. Figures 1, 2, 3 and 4 correspond to a bone marrow image where we are interested in studying the White Blood Cells (from now on WBC) in the image. Theses images have been provided by the Department of Pathology and Anatomical Sciences Ellis Fischel Cancer Center of the University of Missouri-Columbia, and correspond to microscopical captures of real patients. Figures 5 and 6

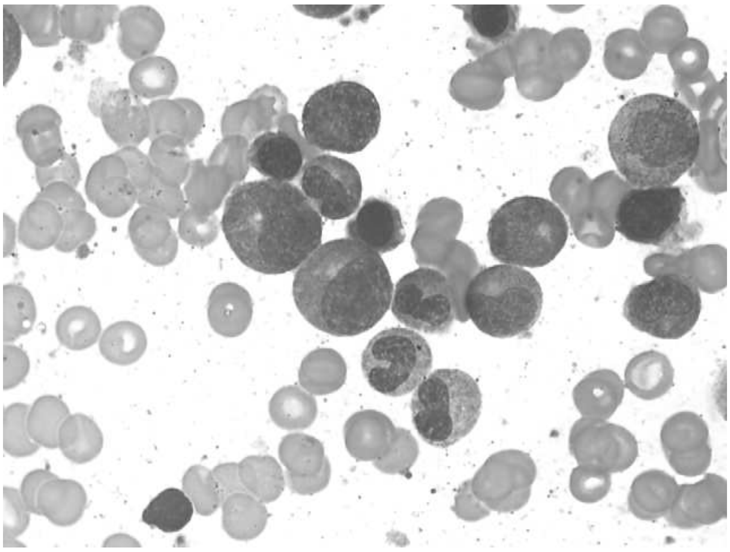


Figure 1: Bone marrow image 1

are mammographies and we are interested in segmenting them in order to find calcifications.

Despite the differences between these images, in order to segment them they have been treated in the same way. This is because in spite of their different origin and nature, at a technical level they are similar as both of them are gray-scale images, and precisely gray-level is the feature that will be used to perform clustering and image segmentation.

It is important to recall that the variability, unaccuracy and complexity that this kind of biomedical images have makes it very difficult in general to perform an efficient analysis of the images.

Segmenting WBC is useful in what is medically called Differential Recount. There are 5 different types of WBC: Neutrophils, Basophils, Eosinophils, Lymphocytes and Monophytes. They can be distinguished studying th morphology and geometry of the cores of the WBC, as it can be seen in Figure 7. Differential Recount test counts how many WBC of each class there are in a certain sample.

These 5 types are normally within a certain proportion, however some blood pathologies break the equilibrium between these 5 types and this is very clearly seen in a Differential Recount. For instance, when a patient has

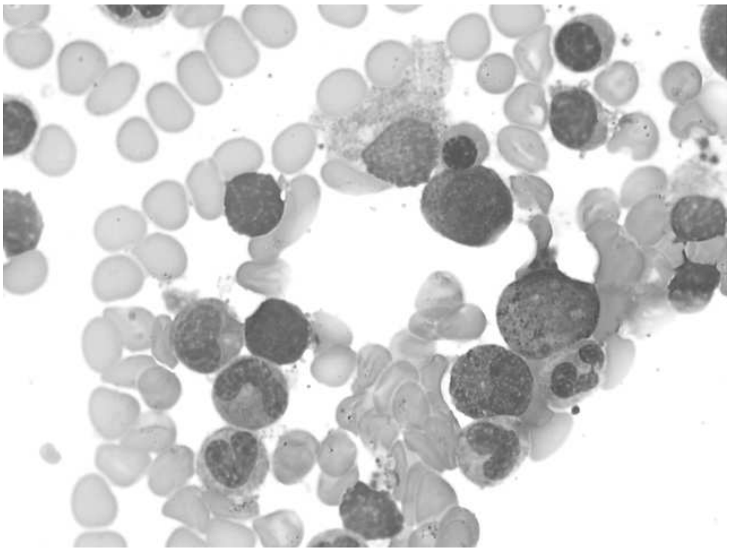


Figure 2: Bone marrow image 2

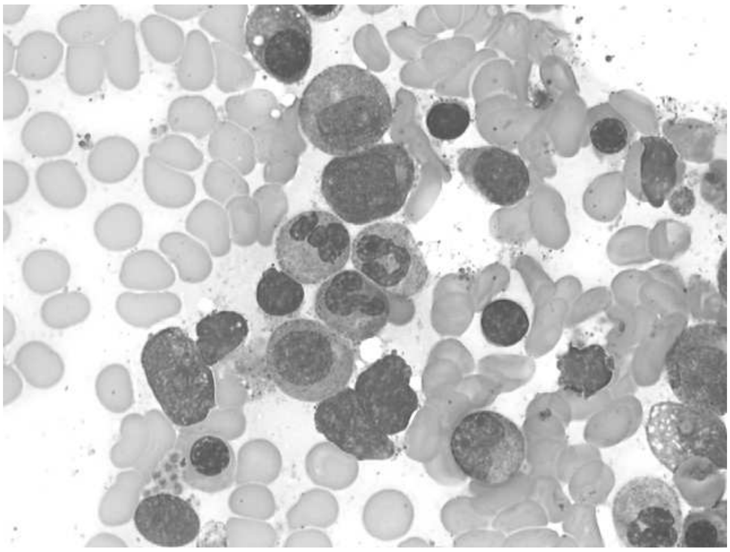


Figure 3: Bone marrow image 3

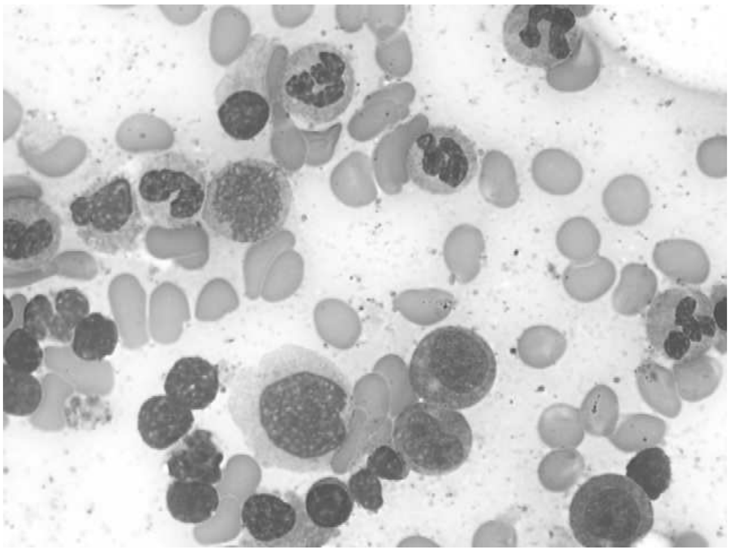


Figure 4: Bone marrow image 4

leukemia cancer this proportions are totally distorted, hence a Differential Recount test is of great use for onchlogysts in order to diagnose this kind of cancer.

On the other hand, the segmentation of mammographies aims at early detection of mama tumors (or calcifications). Young women's breasts have many fibers and this fact complicates segmenting calcifications while older women loose fiber and calcifications are more distinguishable. In Figures 9 and 10 the calcifications we are looking for are highlighted.

Hence, either in WBC images and mammographies we are not actually interested in all the different objects identified on the image, but only on those that show the pathology (in the case of mammographies) or those that define the regions to study (WBC images).

Finally, the motivation on automatic segmentation methods for these problems is to overcome subjectivity and non-controllable error. Let us illustrate this statement with an example:

In the case of WBC Differential Recount, a manual analysis of an image depends as much as 30% on the thresholding of the protocol used by each hospital. This means that depending on where the study is done the resulting proportions can vary up to a 30%. Automatic segmentation has a "'protocol"'

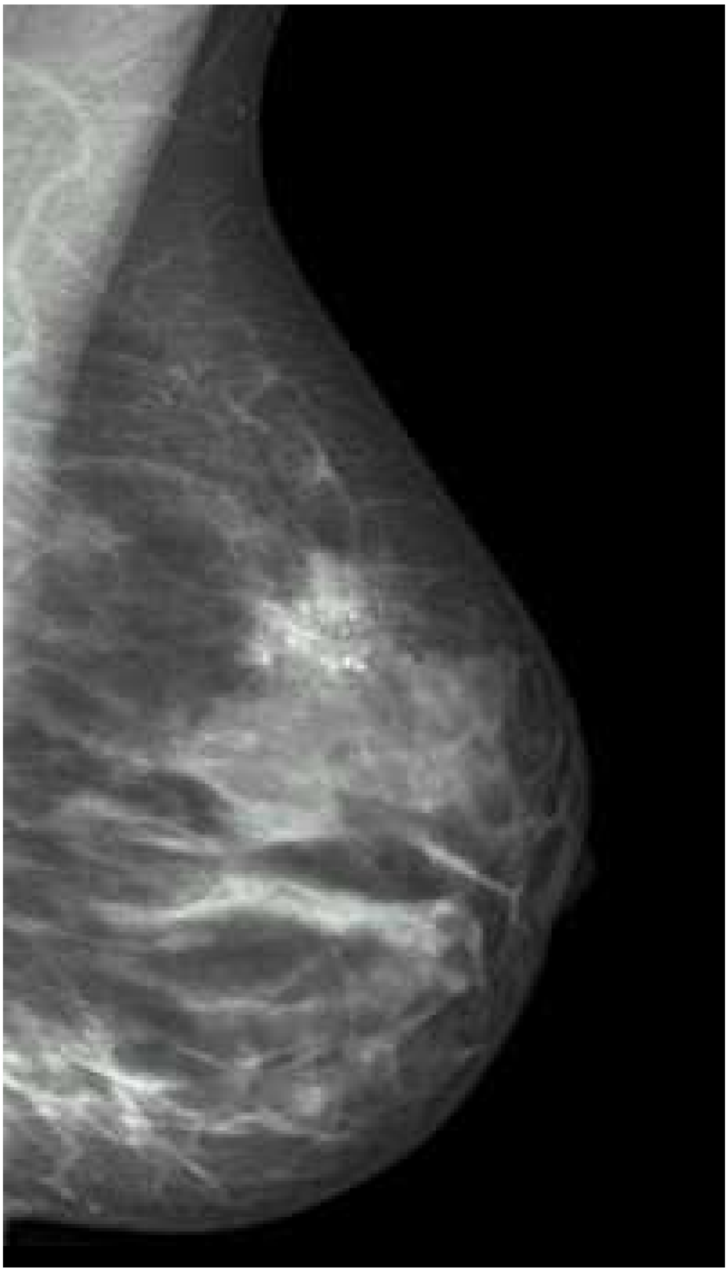


Figure 5: Mammography 1

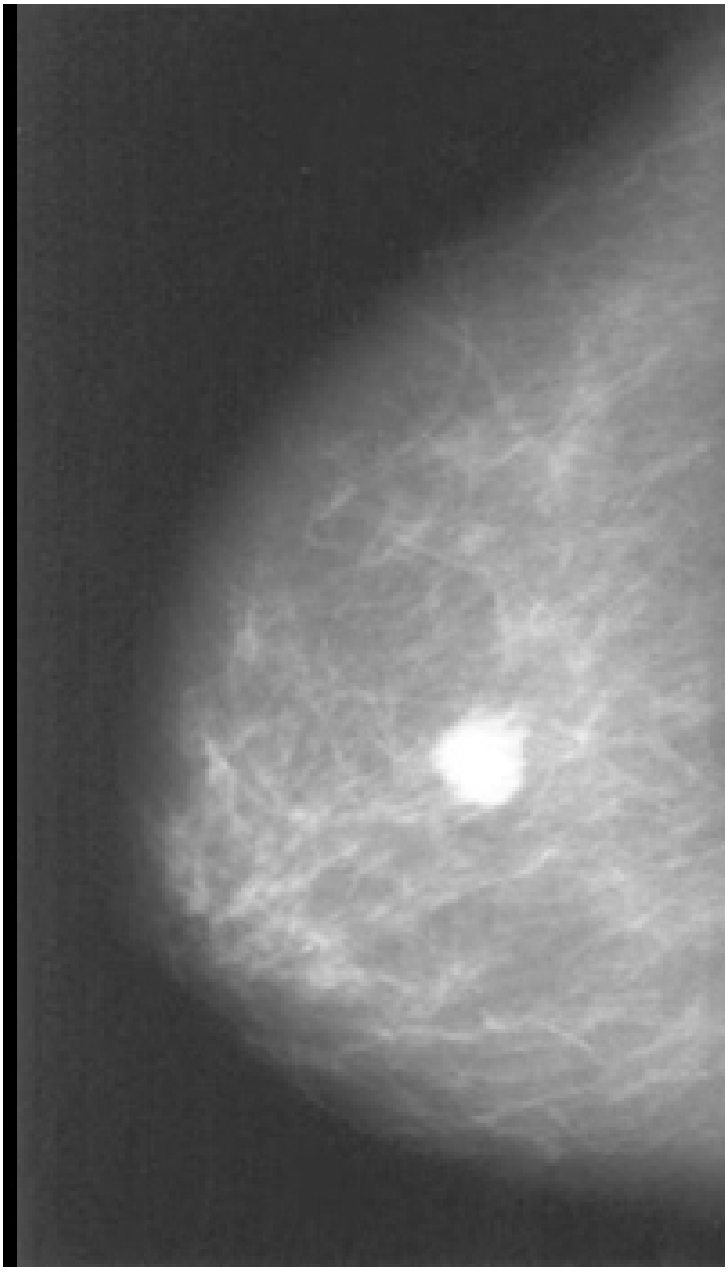


Figure 6: Mammography 2

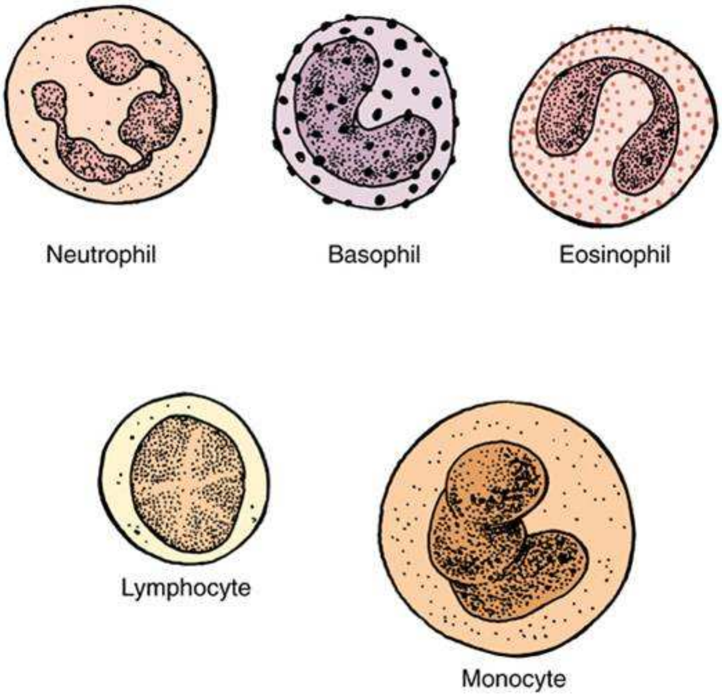


Figure 7: Classification of WBC into 5 groups

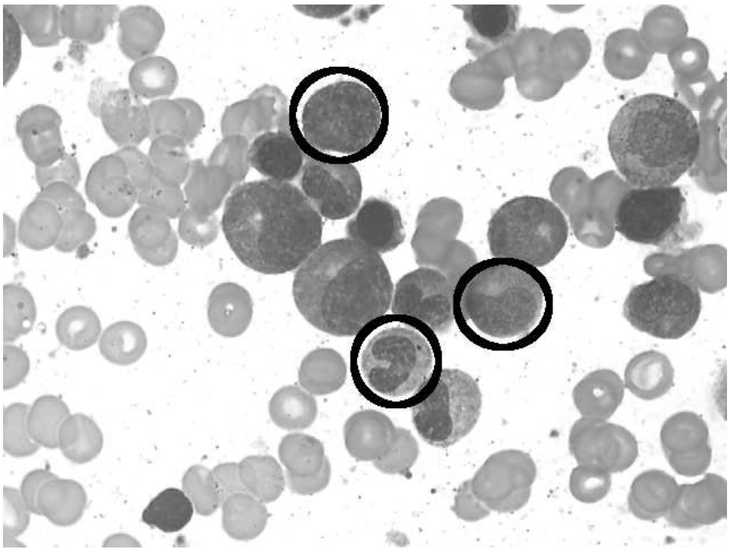


Figure 8: Capture from Bone marrow image 1 where different types of WBC have been highlighted. The difference between them can be found reagrding at the cores of the cells.

as well, but at least this one is non-subjective and intrinsically hardcoded in the algorithm used. Hence, applied on different hospitals will provide no differences because of this variability.

Further, a part from this 30% of variance among protocols, human error has to be considered as well. It is well-known that a human valorative classification depends on the current state of the diagnoser and that this error is non-controlable, as it depends on too many random factors. An automatic segmentation method will provide errors as well, but the errors commited are not subjected to so many variable factors. This fact makes it easier to keep under control the error of the program.

4.2 Fuzzy c-Means Approach

Now that the problem is correctly stated, let us explain the method used to solve it.

As said before, the FCM algorithm has been used after a histogram-based watershed preanalysis of data. This preprocessing will be explained in the

Figure 9: Caption of Mamography 1 which corresponds to a young breast. Circled we find the calcification

Figure 10: Caption of Mamography 1 which corresponds to an old breast. Circled we find the calcification

first subsection.

In the following subsection it will be explained in detail how the FCM algorithm has been used.

The third subsection will deal with the choice of the parameters in the algorithm and some studies will be done to improve the results.

Finally, the output segmentation will be given either with bone marrow images and mammographies, together with a discussion of the results obtained.

4.2.1 Histogram preanalysis

The choice of using a histogram-based watershed method to preprocess the data before using FCM has been done due to two reasons.

The first one is because this analysis proposes an amount of relevant peaks on the histogram which give a clue of the number of clusters FCM has to find. As said in Section 2.3, FCM algorithm needs as input the amount of fuzzy subsets that has to find.

The second reason, related partially with the first one, has to do with computational effort. As said in Section 2.3, FCM is a costly algorithm and hence running it from raw data makes it quite inefficient. Plotting and analyzing the histogram first simplifies and structure data, and so FCM can work on data much more efficiently. Besides, the analysis of the histogram provides as well a proposal of centers that are used by FCM as seeds to find the best centroids. It will be shown further that there in these problems this choice implies a slight advantage with respect with random initial seeding.

This watershed method runs in the following way:

Firstly, the medical image to be segmentated is loaded through a script and transformed into a histogram. Figure 11 shows the histogram related to Figure 1

Then the watershed method is applied in order to identify the relevant peaks in the histogram. The number of relevant peaks is found within the algorithm as well.

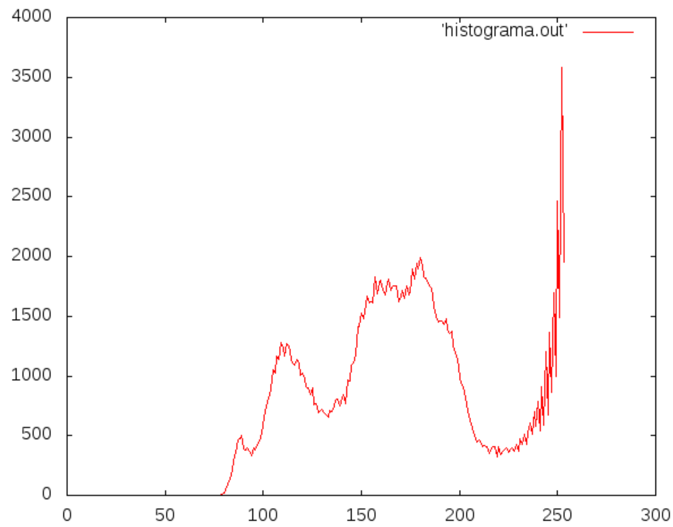


Figure 11: Histogram related to Bone marrow image 1

A peak is meant to be relevant if the area defined under the curve and a cut parallel to the x-axis is above a certain threshold. This is parametrized by some parameters of the algorithm and can be changed in order to make the algorithm more sensitive to finding more clusters or not. However, in the case of WBC it is known that the number of clusters to be found in the final segmentation is 4 and in the case of mammographies we will let the amount of clusters vary and compare results. Thus, the parameters controlling the sensitiveness of the program to finding more clusters have no further relevance on this work and we will not get into details of them.

The watershed algorithm follows the following scheme:

1. fix k such that the whole histogram is under the waterlevel line $x = k$

2. while (peaks_limit_are_disjoint)

3. reduce waterlevel k

4. evaluate intersections $\{x_i\}$ between waterlevel and histogram

5. update peaks_limits

6. if (peak_was_relevant) peak_is_relevant

7. if (peak_was_not_relevant) Reevaluate relevance of the peak

8. end_while

Figure 12 illustrates the process defined by the watershed algorithm.

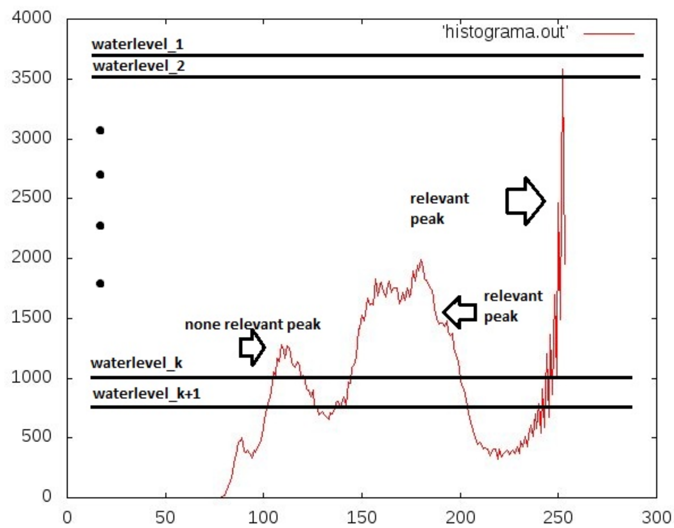


Figure 12: Watershed algorithm. The waterlevel decreases iteratively and the islands are identified as relevant or none relevant peaks according to their are in the current step

Finally, given a raw medical image this algorithm provides an amount C of relevant peaks and the maximums $\{c_j\}_{j=1,\dots,C}$ of each peak that will be used in the following step by the FCM algorithm as the number of fuzzy clusters to be found and their initial centroids.

4.2.2 FCM method

After the histogram analysis, we have the histogram related to the initial image and a partition of it in n relevant peaks.

As said in Section 2.3, the FCM algorithm follows the following scheme:

1. Initialize U, U_0
2. At k step calculate the centers c_j with U_k

$$c_j = \frac{\sum_{i=1}^{N} u_{ij}^m \cdot x_i}{\sum_{i=1}^{N} u_{ij}^m}$$

3. Update U_0, $U_(k+1)$

$$u_{ij} = \frac{1}{\sum_{k=1}^{C} \left(\frac{d(x_i,c_j)}{d(x_i,c_k)}\right)^{\frac{2}{m-1}}}$$

4. If $\left\| U^{(k+1)} - U^{(k)} \right\| \leq \epsilon$ STOP; Else return to 2

Let us recall the notation used: $\{x_i\}_{i=1,\ldots,N}$ stand for the data sample, $\{c_j\}_{j=1,\ldots,C}$ stand for the centroids of the clusters and U_{ij} is a NxC matrix that codifies the degree of membership of x_i to cluster j.

In our problem, the data sample x_i correspond to gray-levels, and hence $N = 256$. As explained in Section 3.2.3 we will initially consider C to be equal to the number of relevant peaks given by the histogram analysis. However, as it will be shown below this parameter can be changed depending on what we are looking for on the image.

Finally, as gray-level is a unidimensional space, the distance considered has been the euclidean (which is the same as the Manhattan one in 1D spaces). This choice has been done because it seems natural from the nature of the problem. However, other distances could be considered as well and we will regard this possibility in Section 5.1.

	Bone Marrow image 1	Mamography 1
Random seeding	22.8	29
Watershed seeding	22.1	28.4

Table 1: Difference on the number of iterations between watershed and random seeding

4.2.3 Adjustment of parameters

In order to start the FCM algorithm we have taken as initial centroid seeds the output maximums of the histogram watershed. On the contrary, classical FCM usually starts with random centers.

Table 1 shows the differences between random and watershed seeding. The table shows the average number of iterations taken by 10 runs of the algorithm to converge using each method.

It must be recalled that the centers found in these 20 runs of the algorithm have been always the same, for that reason the comparison may be done taking into account only the number of iterations. The fact that the algorithm always converges to the same solution shows that the problem is well conditioned, and the solutions found are global optima.

As it can be observed, there is a slight improvement using watershed seeding over random seeding. However, a further study should be done to an analyze if this difference is statistically significant.

The other parameter to be adjusted id the fuzzifier m. Tables 2 and 3 show the output fuzzy clusters of the algorithm varying the value of the parameter m. As it can be seen, for values over $m = 5$ the algorithm begins to fall off.

Finally, the last parameter to be fixed is the number C of clusters to be found. In the case of WBC, we will fix $C = 4$, as we know that there are

	Center 1	Border 1	Center2	Border 2	Center 3	Border 3	Center 4
m=1.1	111.89	134	156.99	173	190.86	221	252.98
m=1.3	111.64	134	156.84	173	190.84	221	253.03
m=1.7	111.24	134	157.23	174	191.13	222	253.27
m=2.5	111.17	138	157.97	176	190.72	228	253.94
m=3.5	111.39	145	157.94	178	189.32	236	254.58
m=5	112.65	154	159.59	182	187.23	245	254.83
m=7	113.14	162	164.88	195	197.72	250	254.99

Table 2: Centers and borders between clusters varying m found on Bone Marrow image 1

	Center 1	Border 1	Center2	Border 2	Center 3	Border 3	Center 4
m=1.1	0.48	30	60.88	80	100.65	130	160.47
m=1.3	0.54	30	61.05	81	100.97	131	161.02
m=1.7	0.59	31	61.99	81	101.67	132	162.09
m=2.5	0.43	39	65.30	85	102.11	137	162.49
m=3.5	0.32	54	70.90	93	104.79	145	162.75
m=5	0.03	67	73.99	100	105.91	152	161.11
m=7	0.00	63	66.12	88	91.15	152	157.04

Table 3: Centers and borders between clusters varying m found on Mammography 1

4 classes to be found on the image: Background, Red Blood Cells, White Blood Cells Cytoplasm and White Blood Cells cores. In the case of mammographies, the aim is different. We want to segmentate the image in order to see if the algorithm finds the calcifications shown in Figures 9 and 10, hence we will run the algorithm letting the number C of clusters to be found vary from 4 to 6.

4.2.4 Resulting segmentation of images using FCM

Finally let us show the final segmentation of the target images. As said in Section 4.2.3, the fuzzifier m of the algorithm has been fixed to $m = 1.3$ and we have used watershed seeding.

Figure 13: Segmentation of Bone Marrow Image 1 (Figure 1) in 4 clusters

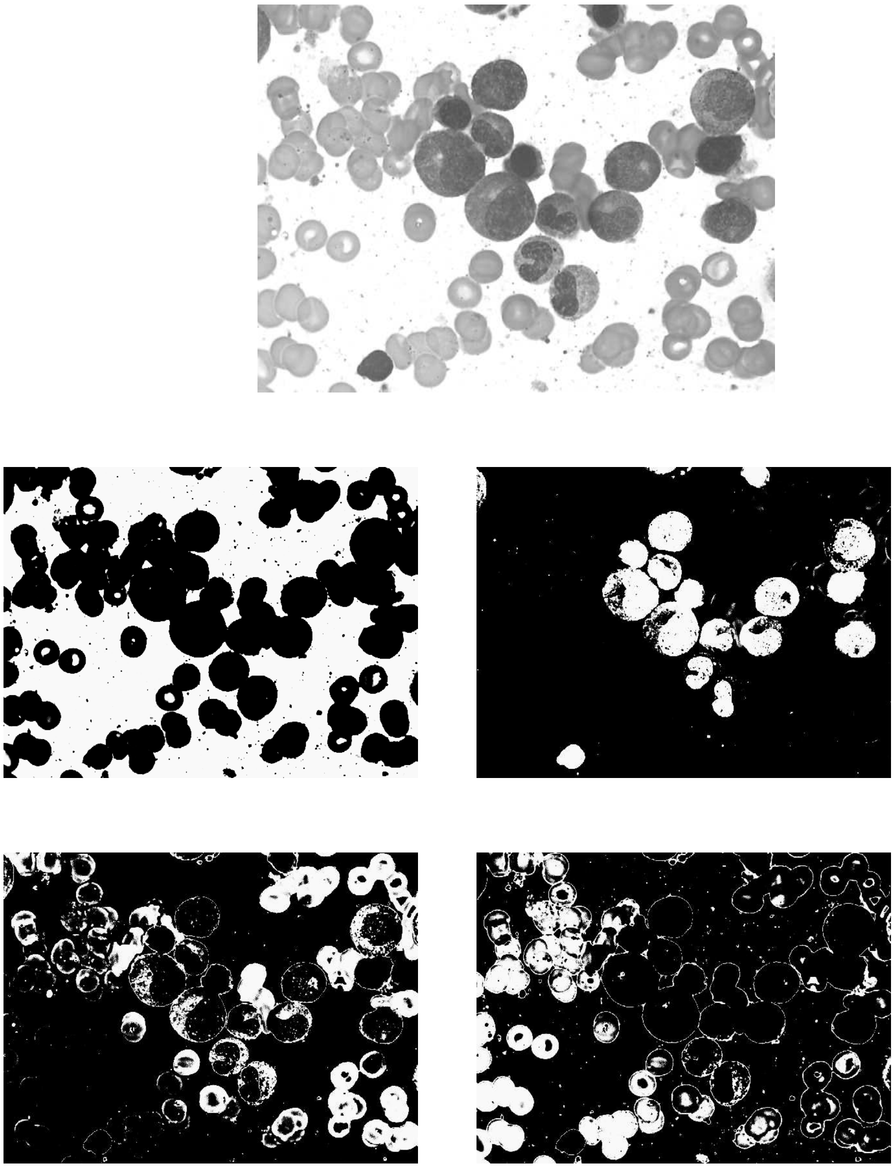


Figure 14: Segmentation of Bone Marrow Image 2 (Figure 2) in 4 clusters

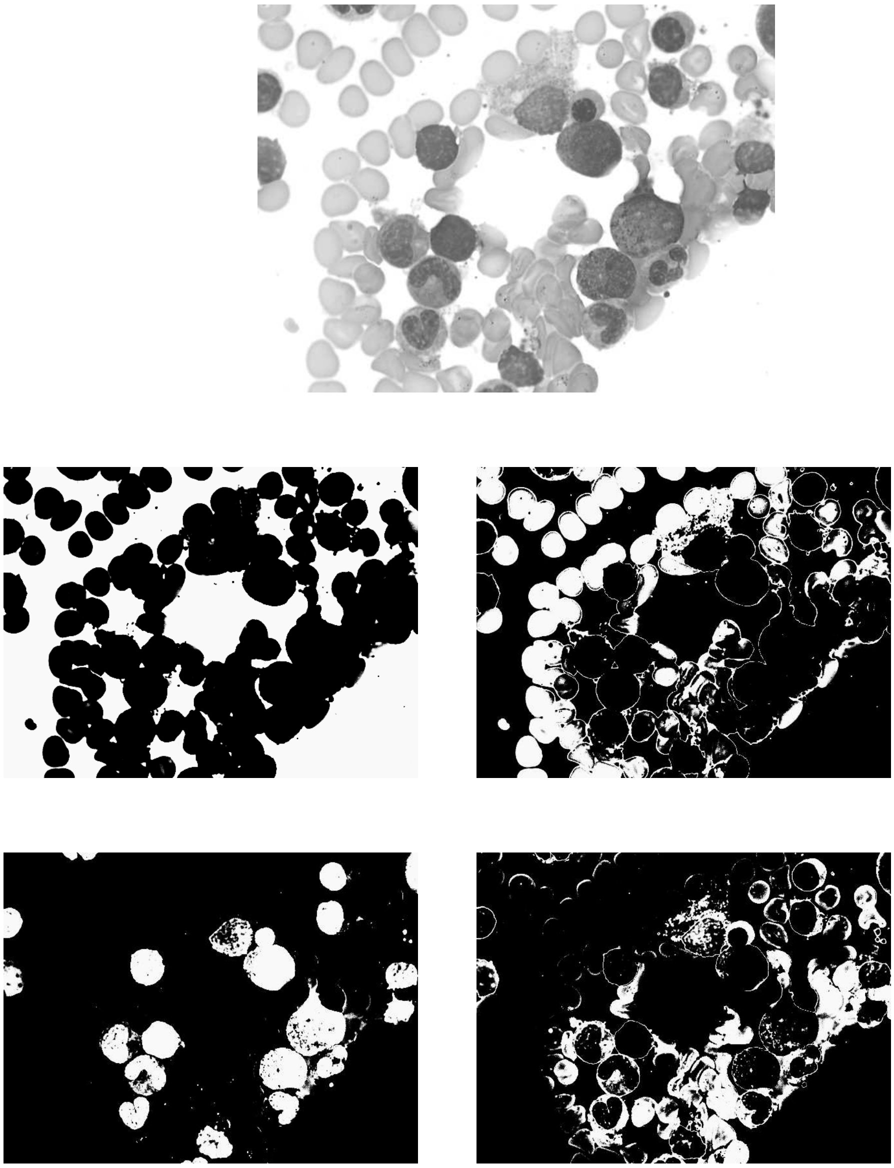


Figure 15: Segmentation of Bone Marrow Image 3 (Figure 3) in 4 clusters

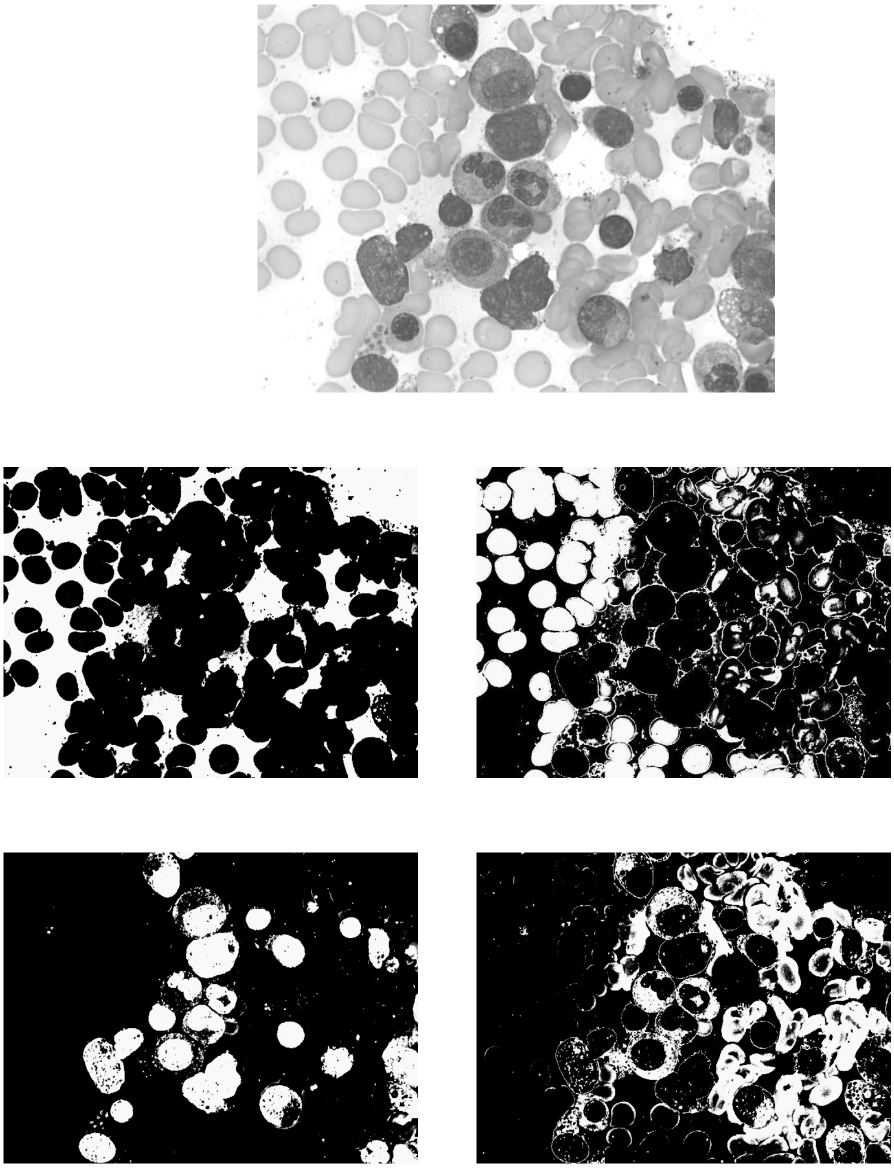


Figure 16: Segmentation of Bone Marrow Image 4 (Figure 4) in 4 clusters

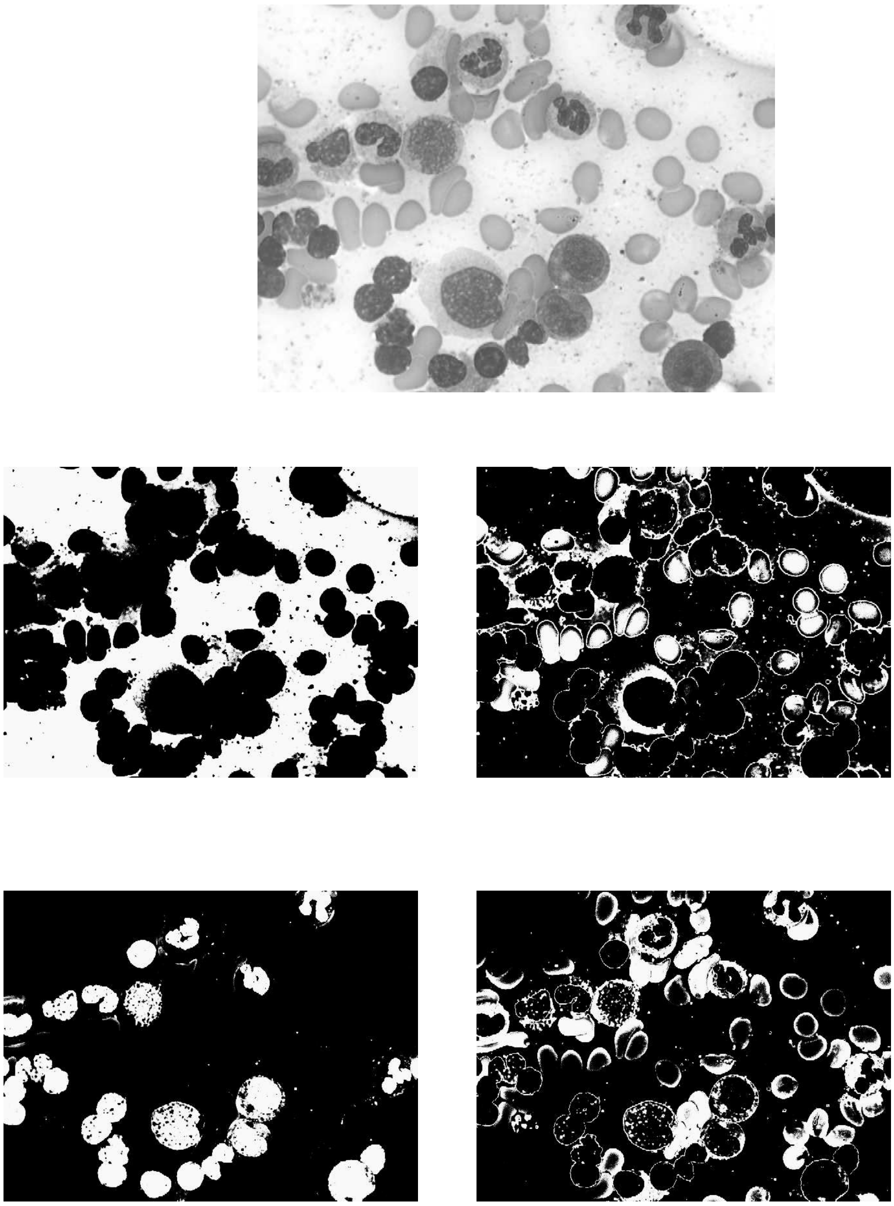


Figure 17: Segmentation of Mamography 1 (Figure 5) in 4 clusters

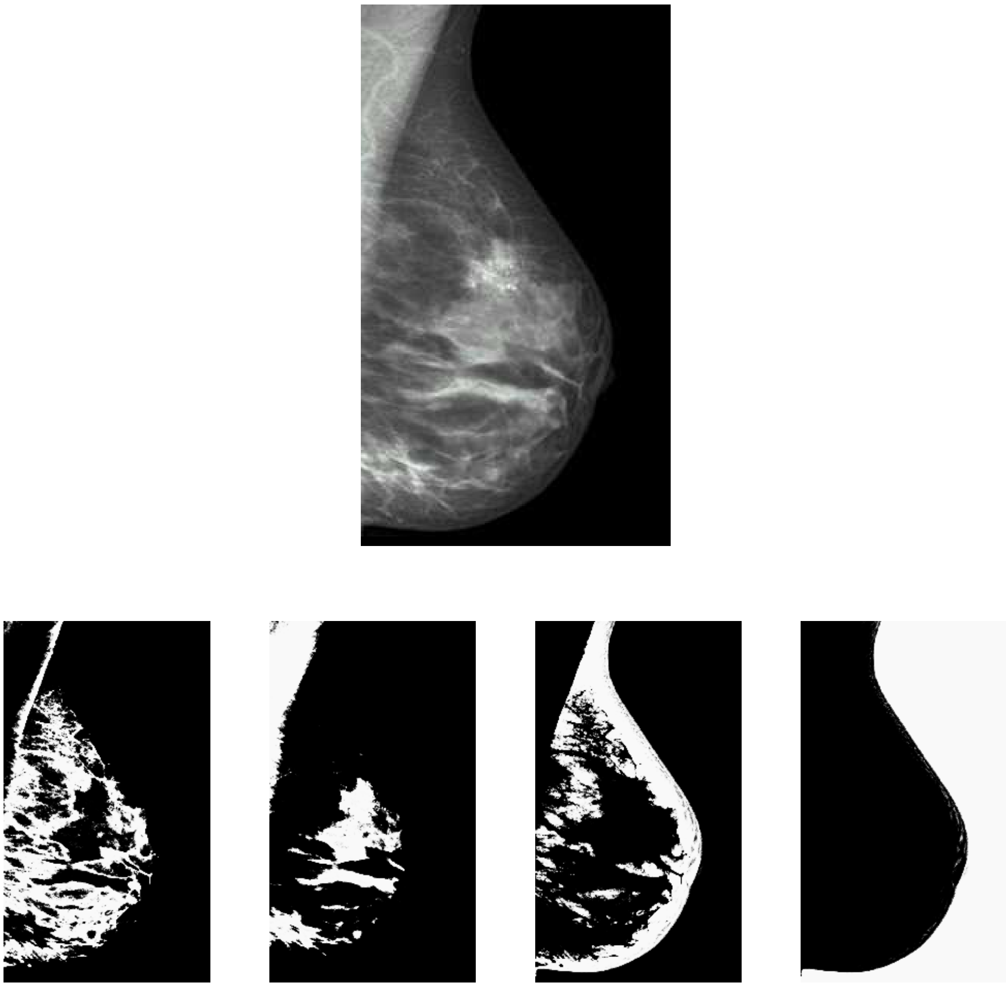


Figure 18: Segmentation of Mamography 1 (Figure 5) in 5 clusters

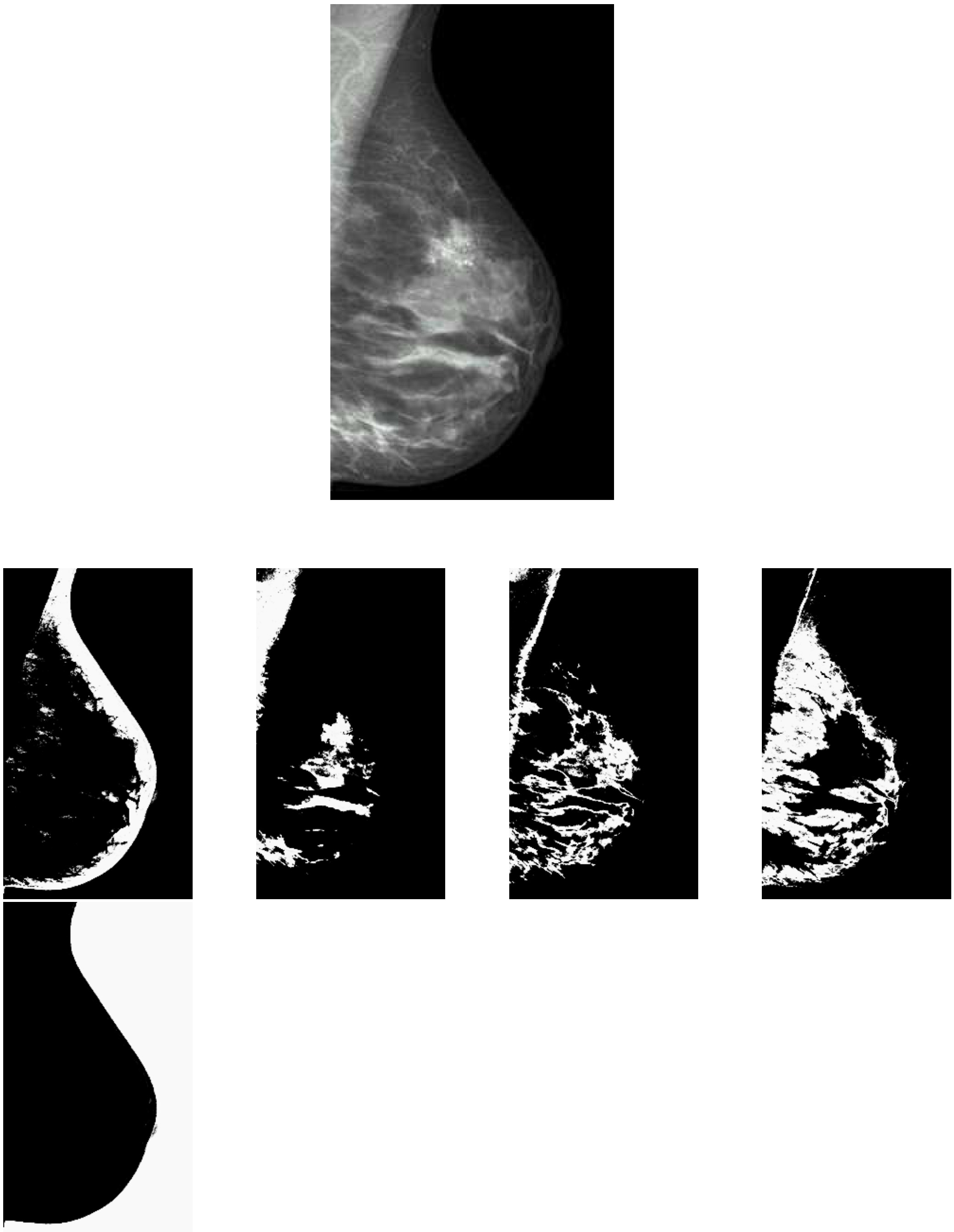


Figure 19: Segmentation of Mamography 1 (Figure 5) in 6 clusters

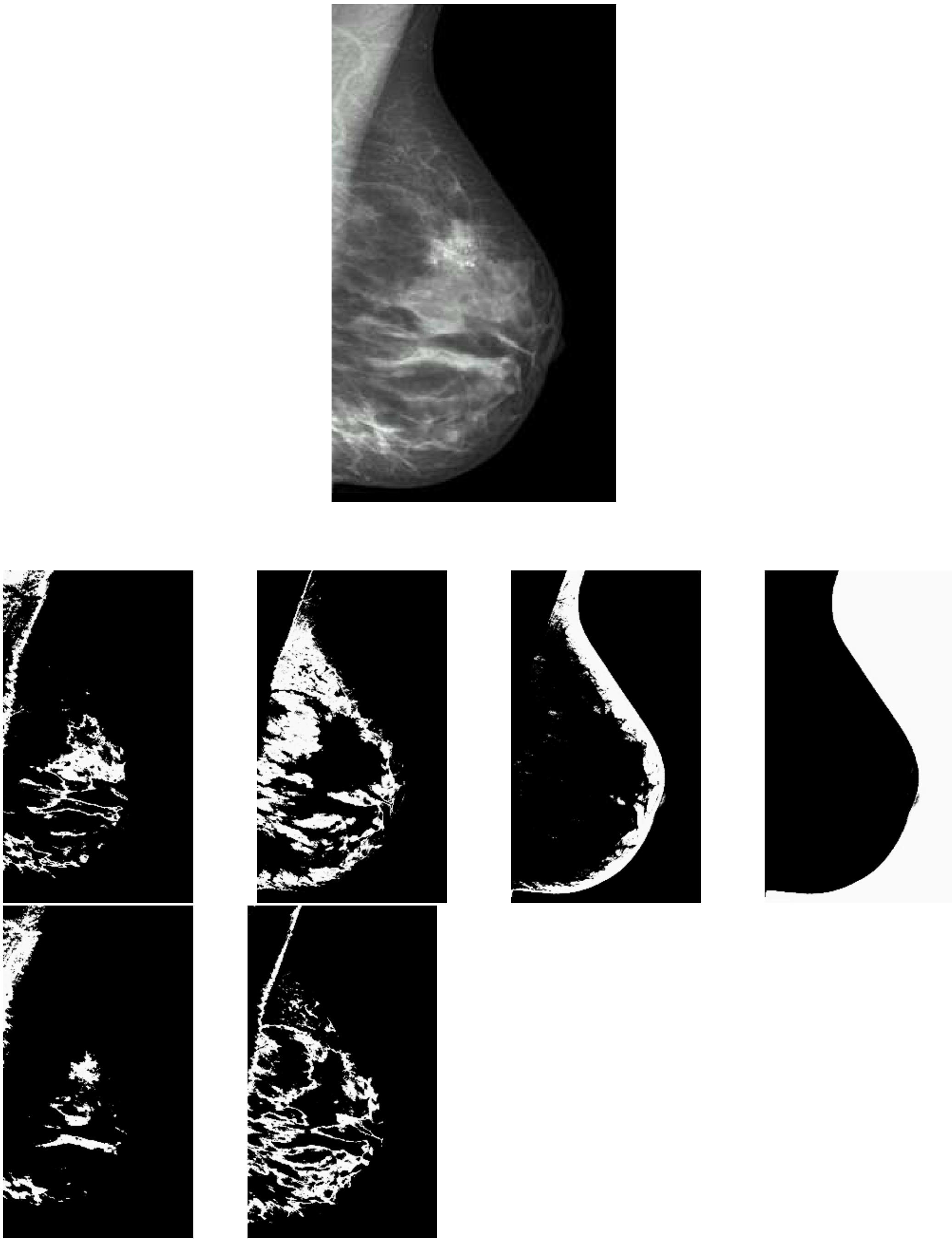


Figure 20: Segmentation of Mamography 2 (Figure 6) in 4 clusters

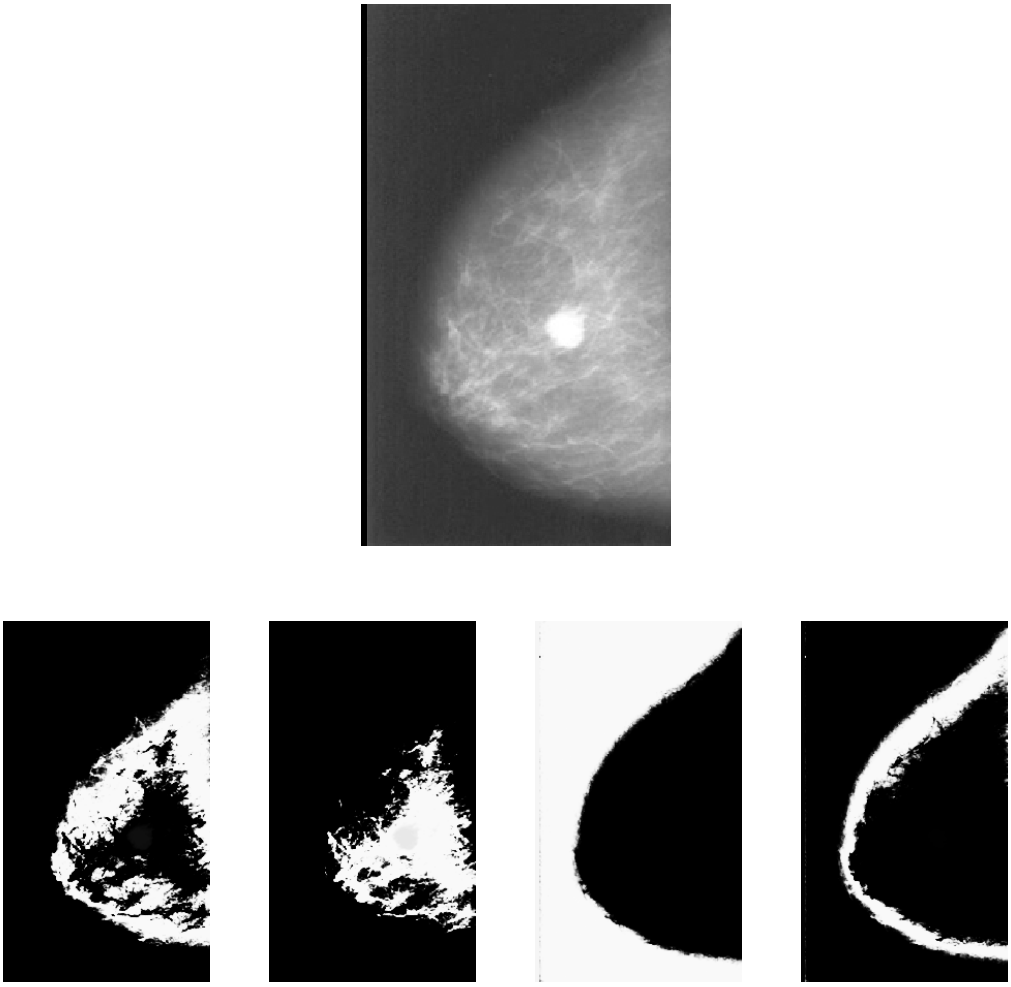


Figure 21: Segmentation of Mamography 2 (Figure 6) in 5 clusters

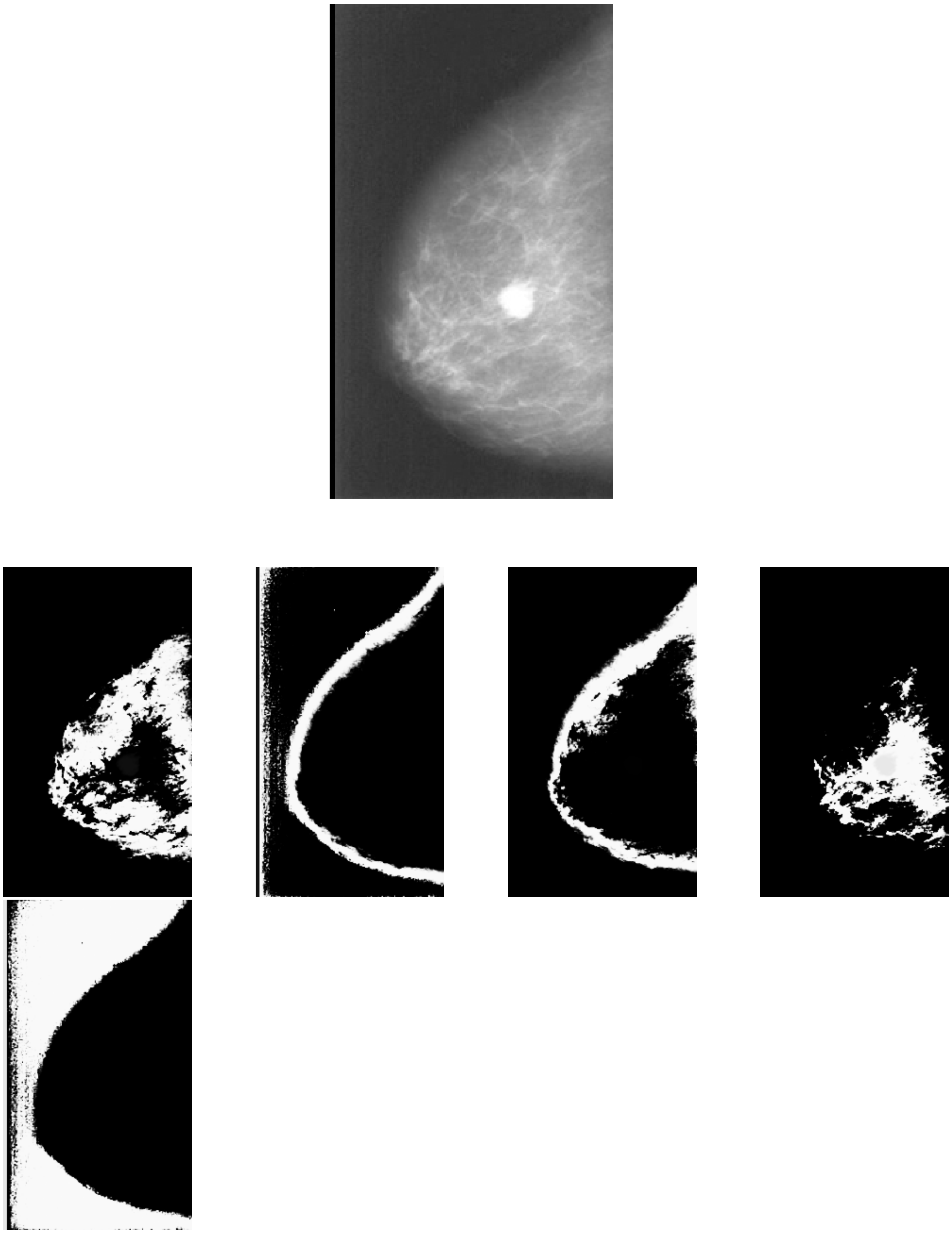
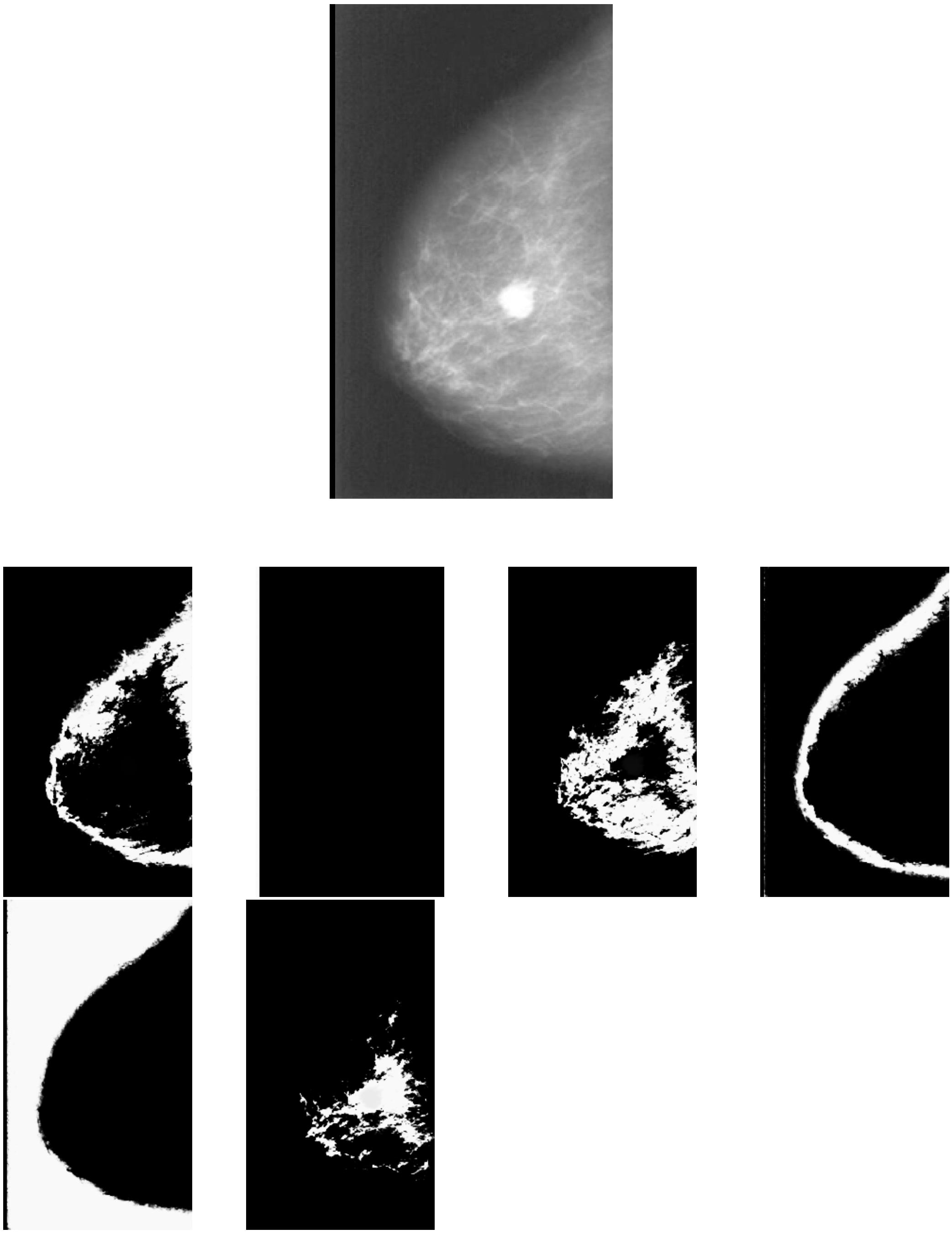


Figure 22: Segmentation of Mamography 2 (Figure 6) in 6 clusters



4.2.5 Discussion of the results obtained

Let us analyze now the results shown in Section 4.2.4.

The aim of segmenting Bone Marrow images was to develop an automatic process to distinguish the White Blood Cells cores from the rest of the image in order to perform a Differential Recount. As it can be seen in Figures 13, 14, 15 and 16 the segmentation of the cores is correctly achieved in the four images.

A slight distortion can be seen in the segmentation of these images when it comes to cluster Red Blood Cells. Several phenomena distort the expectable output segmentation. The first of all, which can be seen very clearly in Figure 13, is that, due to external causes of how the sample has been prepared in order to be microscopically captured, the gray level of the same object can be different in the left and right side of the image. In the segmentation of Figure 13 it can be observed that the RBC in the right side of the image are clustered together with the cytoplasm of WBC, while RBC in the left side of the image are clustered differently.

Another phenomena that brings difficulties is overlapping of RBC, which makes the overlapped region darker and may bring errors to the segmentation. This can be seen in Figure 14 and Figure 16.

However, despite these unavoidable complications, the segmentation obtained can be stated to be quite correct. Furthermore, restricting to the areas of interest of the image (cores of WBC), the segmentation obtained is clear and suitable for a further Differential Recount.

On the other hand, the goal of segmenting mammographies was to analyze if an automatic FCM segmentation could cluster the calcifications shown in Figures 8 and 9.

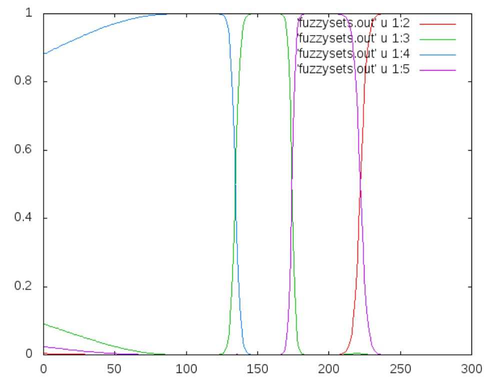
As it can be seen in Figures 17, 18, 19, 20, 21 and 22 in none of these segmentations a specific cluster has been found for the calcifications. The results have not been shown in this work but this was not achieved either clustering into 7 and 8 classes. However, these segmentations actually provide good Regions of Interest (ROIs), where a further image processing can be done in order to automatically detect mama tumors. In Figure 19 a good

ROI is obtained containing the calcifications of Figure 10.

An interesting phenomena can be observed in Figures 20, 21 and 22. A deep look at the final segmentation shows that the calcification appears in a soft gray in some of the clusters. Though the tumor is too little, and hence the number of pixels within the tumor provide neither a relevant peak in the histogram nor a cluster in the further FCM process, the phenomena that makes the tumor actually distinguishable is a side effect of the FCM algorithm. Let us make this point clear.

The FCM algorithm does not work over the raw image, but on the 1D histogram of frequencies of gray-levels. The images shown in Section 4.2.4 are the translation to images of the clustering performed by FCM.

The real output of for instance clustering into 4 classes Mamography 2 (Figure 6) are the following fuzzy sets:



As it can be observed in Figure 23 there is a strange distortion in the queue of the fuzzy sets at low intensities of gray-level. This is a theoretically strange fact following the philosophy of the algorithm, but it provides good results in our case. Because of this slight increasing (or decreasing depending on the cluster) on the membership degree of these pixels to the cluster, it comes that the tumor can be actually distinguished from the rest of the cluster.

Finally, it can be stated that the method developed is robust (it does not depend much on the choice of parameters), consistent (as it provides unique solutions) and computationally quite fast. It has proved to be useful to achieve the goals established at the beginning of this Chapter and could be used in real diagnosing situations, at least to determine ROIs. However, in order to do so, a much more strict and exhaustive analysis and test of the method should be done in order to assure its convergence and efficiency in more situations and images.

5 Future lines of research

As it was explained in the introduction of this work, up to now two disjoint works have been done.

On one side, Chapter 3 devoted a strong effort on understanding in depth formal fuzzy relations and fuzzy set theory. In this context four main concepts were introduced: Indistinguishability Operators, Sets of Fuzzy Extensional Sets and Upper and Lower Approximation Operators. It has been shown that there exists a isomorphic relation between them at a lattice-level and that this result is not kept when we consider further aggregation operators like natural weighted means.

In Chapter 3 it was also studied how taking powers of the fixed t-norm T can be seen as considering homotopies in the lattices $\mathcal{E}$, $\mathcal{H}$, $\mathcal{U}$ and $\mathcal{L}$.

On the other hand, Chapter 4 focused on solving the practical problem of segmenting medical images. The fuzziness of the approach was introduced here by the classical Fuzzy c-Means Clustering Algorithm, and this method was compared with an Histogram-based one.

As it can be seen, up to now we still haven't overcome the Wall fuzzy logic researchers use to run into, as we are constructing theory and attacking practice in a disjoint way. However, it is a must to be recalled here that this Master Thesis is an intermediate stage of a full PhD project. In this sense, the research done up to now and explained in the previous chapters is regarded by the author as a technical reinforcement of theory (Chapter 3) and a fuzzy approach to practice (Chapter 4) in order to provide a suitable context to bring it all together.

The aim of this section is to point the main lines of research the author considers that are promising to build these bridges. Different ideas will be considered in different subsections.

After all building a house is a tough task. First you must design the maps, then prepare the field and afterward build the skeletic structure. Once here, finishing the work is just a matter of filling the gaps. With the field prepared (studied), and the skeletical structure built (chapters 3 and 4), this chapter will provide the map for the next stages of the project./

5.1 Enhancing the FCM algorithm

One possible line of research is to try to enhance the FCM algorithm. This can be done in several ways.

First of all, as it was explained in Section 2.3, one of the problems of this algorithm is its computational cost. Since its proposal in 1981 [4], several researchers have worked on this problem and proposed improvements in order to reduce its computational time cost. A quick review of this improved FCM algorithms can be found in [5]. The problem faced in this work has been treated with the classical old FCM and due to the unidimensionality of the B/W images time has not been a problem. However, colored images could be considered as well (either RGB or HSI which is theoretically more suitable as it is more orthogonal than RGB) and this would take the problem into a 3-D space where possibly this improvements should have to be considered.

Besides, FCM is an algorithm for general clustering. In the local problem we are working on with images, the spatial location is an important variable that should be considered. The author is already taking into account models where the distance used by the algorithm is not only the euclidean between the gray-level, but a linear combination of gray-level and euclidean distance between pixels in the image.

$$d(p1, p_2) = \lambda \cdot d_{gray-level}(p_1, p2) + (1 - \lambda) \cdot d_{euc}(p_1, p_2)$$

This way, the algorithm could not just identify in the same cluster the same class of objects but to discriminate all the objects in the image.

The problem that emerges in this approach is how to decide a priori the number of clusters to be found by the algorithm, as this is one of the inputs required to run it.

5.2 Indistinguishability-based FCM

Another possible idea to bring together practical FCM and theoretical indistinguishability operators theory can also be derived from the idea of modifying the distance of the FCM algorithm. As it was explained in Chapter 2, there is a direct relation between distance operators and indistinguishabilities, as they are reverse one from the other. Hence, instead of taking an arbitrary distance in the FCM algorithm, we could choose this distance as the reverse of an indistinguishability.

The main gain of this approach with respect to the classical one is that this indistinguishability operator can be built taking into account prior knowledge on the field and thus it will be suitable and adaptaded to the specific problem. On the contrary, classical FCM just takes an arbitrary distance, usually euclidean or Manhattan, without any prior evidence that this choice suits properly to the specific problem.

For instance, knowing that in Bone Marrow images we have 4 classes to find and having an approximate estimation of the gray level associated to each cluster we can define an indistinguishability (and hence a distance) that takes into account this fact and its more sensitive around these gray levels and less in the cues.

5.3 Taking the output fuzzy sets of the FCM algorithm as generators of an Indistinguishability Operator

Another line of research that emerges from the usage of the FCM algorithm on this kind of problems of image segmentation can be seen on the output of the algorithm.

FCM takes as input the sample of data and the number n of clusters to be found and provides an adjusted family of n fuzzy sets that segmentate the biomedical image. It can be wondered then if the output fuzzy sets of the algorithm are extensional (observable) with respect to the indistinguishability operator defined by the reverse of the distance chosen for the algorithm. The answer to this question is negative in general.

What can be done, though, is to take this fuzzy sets as the generators (in the sense of the Representation Theorem) of an indistinguishability operator. This way, not only the fuzzy subsets become extensional which can be methodologically interesting, but the FCM algorithm can be understood to provide a model of how to understand "‘similarity"’ in this particular problem, as well as a particular solution of clusters (fuzzy extensional sets now) that segmentate the image.

This idea is interesting, as it takes profit of one of the main strong points of Fuzzy Logic: the plausibility and understandability of the models built. Other methods and techniques (with statistical methods on top of this list) can be very powerful and accurate, but the solutions found are the result of a complicated operational numerical manipulation and cannot be comprehended "'why"' this solution is correct and makes sense. On the other hand, Fuzzy Logic techniques use to simplify the operational calculus (by means of fuzzifying crisp concepts and introducing a new problem of interpretation here) and at the same time provide a semantical meaning to the solution.

This previous discussion can be seen in this problem as follows: A classical FCM that fuzzy subsets provides a model of "'what objects can be seen by an eye"' on a particular medical image. This semantically enhancement of FMC by constructing the indistinguishability operator given by taking the output fuzzy subsets as generators provides not only a model of "'what objects can be seen by an eye"', but a model (the indistinguishability) of "'how the eye sees"'. Hence this holistic view is giving us a complete map of the act of vision as we are modeling both the object and the subject, and we are not at the plain problem solving level but transcending to the problem comprehension level

5.4 Constructing a plausible model by means of iterating the previous ideas

After the previous two lines of research, an idea that comes out here is to iterate this process.

Given a particular image, a FCM algorithm can be ran in order to find the clusters that segmentate this image. As said before, this fuzzy sets can be taken as the generators of an indistinguishability operator that explains how the similarity relation works on this particular image.

Then this indistinguishability operator can be reversed to define a new distance that can be used to run a new FCM and iterate the process.

Algorithmically, this process would be the following:

1. Define the number of clusters and the distance d to be used by FCM.

2. Run FCM algorithm with the given distance d.
3. Compute the indistinguishability operator E generated from the output fuzzy sets.
4. Reverse the indistinguishability $d = 1 - E$.
5. if no improvement STOP; else Go to 2.

This algorithm can be seen as an iterative adjusting of the method in order to fit the problem, at the same time that the indistinguishability operator that defines the similarity model is enhanced as well.

Several further and deeper considerations should be made here. First of all, in order to run this iterative process its convergence should be proved first. Intuitively, this seems trivial as we are simply adjusting an algorithm that in its first iteration provides a good solution to the problem. However, in order to run this proposed method, this convergence should be shored up.

Another fact to take into account is overfitting problems. At the level of a pure image segmentation problem overfitting is a harmless problem (because we will not generalize the model, so overiterations will just converge to a better image segmentation), but if we tried to extent this method as it can be done to more fields within Machine Learning a diagnosis of the risk and limits of overfitting with this method should be carried up.

5.5 A reverse approach toward diagnosing

Up to this point, all the ideas proposed have followed the main path defined by applying FCM to biomedical images and enhancing this algorithm and method. The last two ideas proposed change drastically this paradigm and leave aside the FCM algorithm in order to face diagnosing problems from other viewpoints.

The first of this ideas comes from analyzing the whole chain of diagnosing. Up to this point, this process has had two steps as it is shown in the diagram of Figure **??**.

- Firstly an image segmentation is performed. In this work this step has been done with a FCM algorithm, but as said in Section 2.1 several methods can be used here.

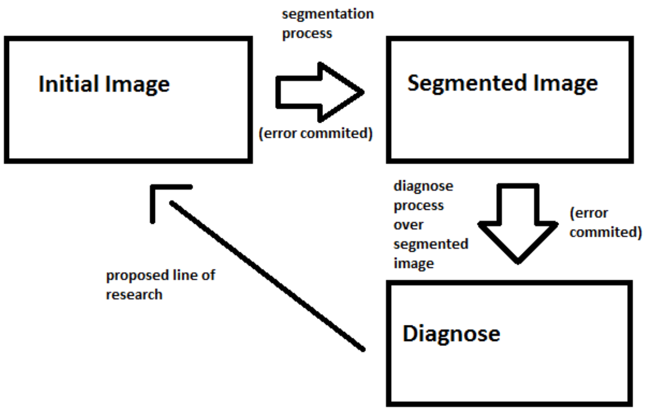


Figure 13: Current chain of automatic diagnosis. From raw image to final diagnose

- The second step is to diagnose over the segmentated image. This can be done either by an expert or automatically.

However, it can be seen that a certain error is committed in each of this steps. This error is accumulative and more important: it is very difficult to measure it accurately.

The idea that can be explored is to reverse this process. Instead of going from the image to the diagnose, think Bayesian and go from the different expected diagnoses to the suitability of these in the image. For instance in the mammography images instead of delimitating the possible tumor, one could modellize the different kinds of breast cancers and evaluate the degree of similarity between the model and the input image.

This approach brings up several technical problems. On one side we have that it can become computationally unaffordable to face a little model of a few pixels with the whole image exhaustively (that is with all the connex windows of this few pixels that can be found in the image). However, with the use of some heuristics this problem could possibly be overcome.

The second problem is that our model is fix, whereas our tumor even if its equal to the model can be rotated or bigger than our initial model. Thus, we should analyze the degree of indistinguishability between our model and the image windows modulo affine transformations.

A deeper study of how indistinguishability operators are affected by affine and other kind of topological transformations can be found in [34].

5.6 Focusing and unfocusing images

You can't see the wood for the trees!

The last and most interesting possible future line of research to the viewpoint of the author is the idea of using powers of indistinguishability operators to model the concept of focusing and unfocusing an image.

Fixed a given image there are optical methods to focus and unfocus it. The gain of this capability is that unfocused images loose details but allow to see clearly which are the big structures. Because its the loss of mental focus on the trees what allows to see the woods.

On the contrary, if it is possible to focus an image then all the small details can be zoomed and taken into account.

Being able to adjust the parameter of image focusing facilitates segmentation algorithms to find more easily whatever you want to find. Because in some cases when you look for big structures details become noise.

This discussion motivates why at a practical level this is relevant. Theoretically this can be modeled with the powers of t-norms defined in Section 3.4.1 and 3.4.2. In this sections it was showed how letting the power vary provides an homotopy between the universal indistinguishability operator (which models unfocused images where everything is "'equal"') and the drastic one (focused images, where every detail makes the difference).

Besides and recalling lines of research previously explained, taking the indistinguishability operator that models the "'similarity"' in the image and letting the exponent to vary gives a complete model of the subjacent similarity relation robust to focus and unfocusness.

6 Concluding remarks

The aim of this work was to develop a framework to bring together high-level Fuzzy Logic with Biomedical Engineering. In order to do so to main cores have been developed.

In Chapter 3 several results within the theory of Indistinguishability Operators have been derived. First of all a structural analysis of the lattice of indistinguishability operators $\mathcal{E}$, the one of sets of extensional fuzzy subsets $\mathcal{H}$ and the ones of upper and lower approximation operators by extensional fuzzy sets ($\mathcal{U}$ and $\mathcal{L}$ respectively) has shown that they are isomorphic and it has been discussed why this result has a semantically deep impact. Further natural weighted mean aggregation operators have been defined (either in finite and non finite cases) and it has been studied and shown how the previous results could not be extended to this new operation.

Furthermore, it has been studied how powers can be defined over t-norms, and the results found following these definition have yield to a very interesting and promising line of research explained in Section 5.6.

Besides, a biomedical image segmentation problem has been faced in this work. Two different kind of images have been treated: Bone marrow microscopical captures, where the cores of WBC were to be segmented in order to perform a Differential Recount, and Mammographies, where the aim was to find the tumors in the image.

In order to solve these problems, a double stage histogram watershed method and Fuzzy c-Means has been used. The results found are promising and a deeper study about the validity and accuracy of the method could be done to strengthen the assertion that the method proposed works.

The work presented in this Master Thesis is an itermediate stage of a whole PhD project. For that reason maybe the most intersting ideas of this work can be distilled from the last chapter gathering future lines of research. It is the work on these future lines what will provide the whole structure that is expected to prove that high level Fuzzy Logic is useful for practical contexts.

In this last chapter several different possible continuations to this work have been presented. All of them may bring new interesting problems and challenges that the future research will have to find, face and solve. It is the intention of the author to get deeper into the wild of Unknown following these lines in the following years and complete an original PhD with proofs of how FL has very much to say in pur comprehension of the world and derive new results, methodologies and check points in the sake of Science.

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