Escola d'Enginyeria de Telecomunicació i Aeroespacial de Castelldefels

UNIVERSITAT POLITÈCNICA DE CATALUNYA

TREBALL DE FI DE CARRERA

TÍTOL DEL TFC: The degree-number of vertices problem in Manhattan networks

TITULACIÓ: Enginyeria Tècnica Aeronàutica, especialitat Aeronavegació

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DATA: 19 de juliol de 2012

Títol: El problema grau-nombre de vèrtexs en les xarxes Manhattan

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Resum

En termes generals, l'objectiu d'aquest treball és estudiar el problema (Δ, N) (o problema grau-nombre de vèrtexs) per al cas del digraf Manhattan.
Un digraf és una xarxa constituïda per vèrtexs i per arestes dirigides anomenades arcs (en el cas de grafs, les arestes no tenen direcció). El diàmetre d'un graf és la mínima distància possible que hi ha entre dos dels vèrtexs més allunyats entre si. En el diàmetre d'un digraf hem de tenir en compte que els arcs tenen direcció.
Un digraf de doble pas consta de N vèrtexs i un conjunt d'arcs de la forma $(i, i + a)$ i $(i, i + b)$, amb a i b enters positius anomenats "passos", és a dir, que existeixen enllaços des del vèrtex i cap els vèrtexs $i + a$ i $i + b$ (les operacions s'han d'entendre sempre mòdul N). Aquest digraf es denota $G(N; a, b)$. Un graf de doble pas $G(N; \pm a, \pm b)$ també consta de N vèrtexs, però les arestes són de la forma $(i, i \pm a)$ i $(i, i \pm b)$, amb passos a i b (enters positius), per tant, existeixen enllaços des del vèrtex i cap els vèrtexs $i \pm a$ i $i \pm b$ (mod N) .
En un digraf Manhattan els arcs tenen les direccions com les dels carrers i les avingudes de Manhattan (o de l'Eixample de Barcelona), és a dir, si un arc va cap a la dreta, el "següent" va cap a l'esquerra i si un arc va cap a dalt, el "següent" va cap a baix.
El problema (Δ, N) consisteix a trobar el diàmetre mínim d'un graf o digraf fixats el nombre de vèrtexs N i el grau Δ. Com que aquest problema ha estat resolt per al cas de grafs de doble pas $G(N; \pm a, \pm b)$, hem expandit aquests grafs transformant cada vèrtex en un cicle dirigit de 4 vèrtexs i cada aresta en dos arcs de sentits oposats, de manera que obtenim un digraf Manhattan M.
En aquest treball trobem la relació entre els passos del graf de doble pas $G(N; \pm a, \pm b)$ i els del digraf Manhattan M. A més, hem fet un programa que calcula el diàmetre del digraf anomenat New Amsterdam NA, que està relacionat amb el Manhattan M, a partir dels paràmetres del graf original $G(N; \pm a, \pm b)$.

Title: The degree-number of vertices problem in Manhattan networks

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Overview

Generally speaking, the aim of this work is to study the (Δ, N) problem (or the degree-number of vertices problem) for the case of a Manhattan digraph.
A digraph is a network formed by vertices and directed edges called arcs (in the case of graphs the edges have no direction). The diameter of a graph is the minimum distance that exists between two of the farthest vertices. In the diameter of a digraph we must take into account that arcs have direction.
A double-step digraph consists of N vertices and a set of arcs of the form $(i, i+a)$ and $(i, i+b)$, with a and b positive integers called 'steps', that is, there are connections from vertex i to vertices $i+a$ and $i+b$ (operations are modulo *N*). This digraph is denoted by $G(N; a, b)$. A double-step graph $G(N; \pm a, \pm b)$ consists of N vertices, but the edges are of the form $(i, i \pm a)$ and $(i, i \pm b)$, with steps a and b (positive integers), therefore, there are connections from vertex i to vertices $i \pm a$ and $i \pm b$ (mod N).
In a Manhattan digraph, the arcs have directions like the ones of the streets and avenues of Manhattan (or *l'Eixample* in Barcelona), that is, if an arc goes to the right, the 'next one' goes to the left and if an arc goes upwards, the 'next one' goes downwards.
The (Δ, N) problem consists in finding the minimum diameter of a graph or digraph given the number of vertices N and the degree Δ. As this problem has been solved for the case of double-step graphs $G(N; \pm a, \pm b)$, we expand these graphs transforming every vertex into a directed cycle of order 4 and every edge into two arcs in opposite directions, so that we obtain a Manhattan digraph M.
In this work we find the relation between the steps of the double-step graph $G(N; \pm a, \pm b)$ and the ones of the Manhattan digraph M. Moreover, we made a program that calculates the diameter of the so-called New Amsterdam digraph NA, related to the Manhattan digraph M, from the parameters of the original graph $G(N; \pm a, \pm b)$.

Acknowledgements

Our most sincere thanks to Cristina Dalfó Simó for her valuable collaboration and the time she devoted to us contributing with her knowledge and experience in the development of this work, in a time where finding support from teachers was almost impossible. For that, we consider her cooperation very worthy, since this work could have not been done without her.

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INTRODUCTION

The aim of this work is to study the (Δ, N) problem (or the degree-number of vertices problem) for the case of a Manhattan digraph. In order to do this, we introduce some concepts of graph theory in Chapter 1, where we explain its origin and the huge variety of applications it has. We also introduce the concepts of double-step graphs and digraphs, which we need along this work. In Chapter 2, we solve the degree-number of vertices problem by means of a program we did with the free compiler Free Basic. In Chapter 3, we present the theoretical results that we obtained with their corresponding proofs. Finally, in Chapter 4, we expose our conclusions and the results obtained from the problems studied.

It is important to emphasize that the solution of the problems considered in this work can have theoretical applications, although they are no immediate.

CHAPTER 1. GRAPH THEORY

1.1. Introduction

Graph theory is an area of discrete mathematics that has experienced a huge development. It allows modelling any system in which there is a binary relation between certain objects, and that is why its scope is very broad and covers areas such as mathematics (topology, probability or numerical analysis), electrical engineering, telecommunications and computer science, operations research, sociology or even linguistics (see Comellas, Fàbrega, Sànchez, and Serra [8]).

The origin of graph theory dates back to the eighteenth century, when the Swiss mathematician Leonhard Euler (1707-1783) solved the 'Seven Bridges of Königsberg' problem. The method he used to solve this problem is considered by many to be the birth of graph theory.

1.2. The 'Seven Bridges of Königsberg' Problem

Euler wrote a short paper addressing an amusing problem originated in Königsberg, a town not too far from Euler's home in St. Petersburg. Königsberg (now known as Kaliningrad) was located on the Pregel River in the ancient Prussia. The river divided the city into four separate landmasses, which were linked by seven bridges. Most of these connected the island Kneiphof, which was caught between the two branches of the Pregel, with other parts of the city. Two additional bridges crossed both branches of the river (see **Fig.1.1**).

The people of Königsberg thought about the following mind puzzle: 'Can one walk across the seven bridges and never cross the same one twice?' No one found such a path until a new bridge was built in 1875.

Almost 150 years before the new bridge, in 1736, Euler offered a rigorous mathematical proof stating that with the seven bridges such a path does not exist. He not only solved the Königsberg problem but in his brief paper unintentionally started a huge branch of mathematics known as graph theory. Today graph theory is the basis for our thinking about networks.

Euler's great insight lay in viewing Königsberg's bridges as a graph, a collection of nodes connected by links. For this he used nodes to represent each of the four land areas separated by the river, distinguishing them with letters *A*, *B*, *C* and *D*. Next he called the bridges the links and connected with lines those pieces of land that had a bridge between them. He, thus, obtained a graph whose nodes were the pieces of land and the links were the bridges.

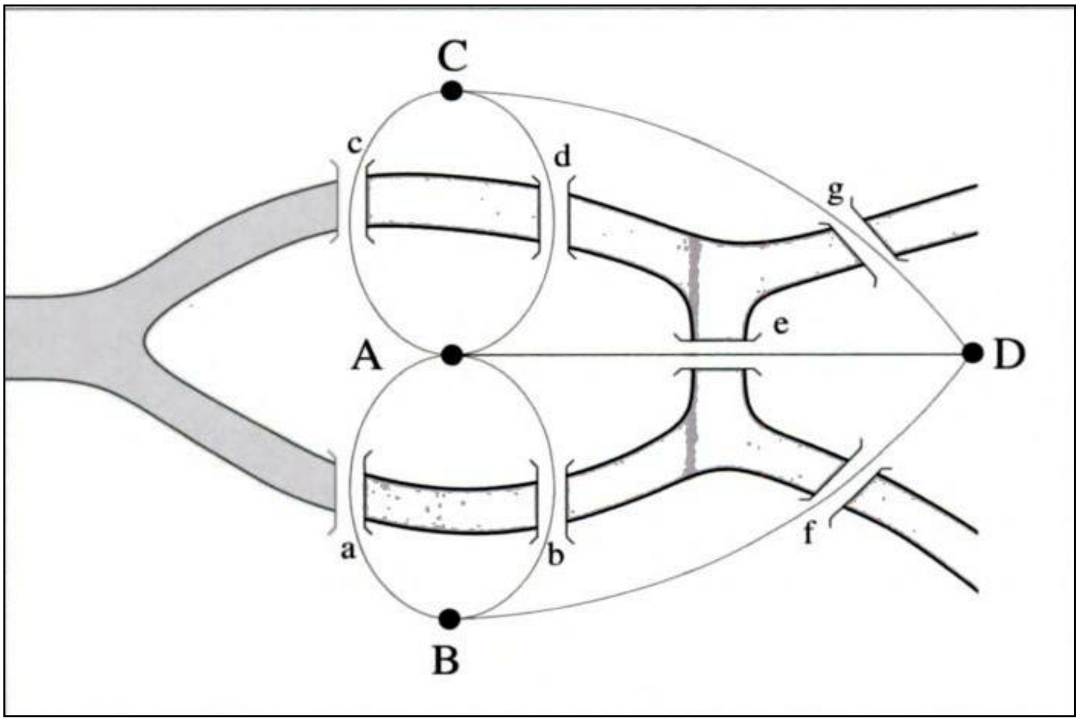


Fig.1.1 The layout of Königsberg before 1875, with Kneiphof island (*A*) and the land area *D* caught between the two branches of the Pregel River. Solving the Königsberg problem meant proving or disproving if there was a route around the city that would require a person to cross each bridge only once. In 1736, Euler gave birth to graph theory by replacing each of the four land areas with nodes (*A* to *D*) and each bridge with a link (*a* to *g*), obtaining a graph with four nodes and seven links. He then proved that on the Königsberg graph, a route crossing each link only once does not exist (Barabási [2]).

Euler's proof that in Königsberg there is no path crossing all seven bridges only once was based on a simple observation. Nodes with an odd number of links must be either the starting or the end point of the journey. A continuous path that goes through all bridges can have only one starting and one end point. Thus, such a path cannot exist on a graph that has more than two nodes with an odd number of links. As the Königsberg graph had four of such nodes, one could not find the desired path (Barabási [2]).

1.3. Graphs

In order to formalize our discussion on graph theory, we need to introduce some terminology.

A *graph* $G = (V, E)$ is a combinatorial structure consisting of a set $V = V(G)$ of elements called *vertices* and a set $E = E(G)$ of *edges*, which consist of unordered pairs of different vertices. If there is an edge $e = \{u, v\} = uv$, then it is said that u and v are *adjacent vertices*. If there is no edge between vertices u and v, then u and v are *independent vertices*. Two *edges* are *independent* if they have no vertices in common. Vertices are represented by points and edges are represented by lines that connect pairs of vertices.

The *order* of a graph G is the number of vertices of G and the *size* of G is the number of its edges. The *degree* of a vertex is the number of adjacent vertices it has. If all the vertices of a graph have the same degree, the graph is called *regular*.

A graph is *connected* if there is a path between any pair of vertices. If a graph is connected, its diameter is the minimum distance between two of the farthest vertices. As an example of a graph, we illustrate the Petersen graph in **Fig.1.2**, which has order 10, size 15, diameter 2 and it is regular of degree 3.

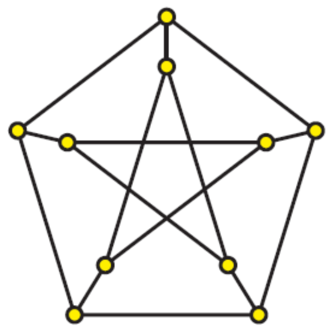


Fig.1.2 Petersen graph

In many applications, there are graphs that have neither loops nor multiple copies of the same edge (these are known as *parallel edges*). Such graphs are called *simple graphs*.

All in all, there are two types of graphs we must distinguish (see **Fig.1.3**):

1. Multigraphs (no restrictions on loops and parallel edges).

2. Simple graphs (may not have loops or parallel edges).

For more details about graphs, see Comellas, Fàbrega, Sànchez, and Serra [8].

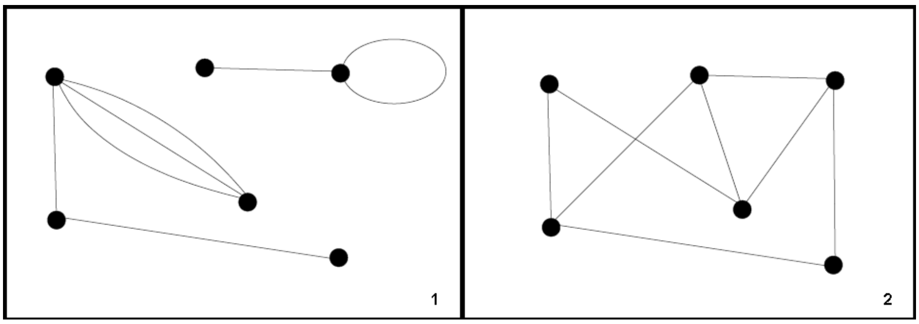


Fig.1.3 Two types of graphs: 1. Multigraph, 2. Simple graph (Fields [10]).

1.4. Applications

Graphs can be used to model many types of relations and process dynamics in physical, biological and social systems. Many problems can be represented by graphs.

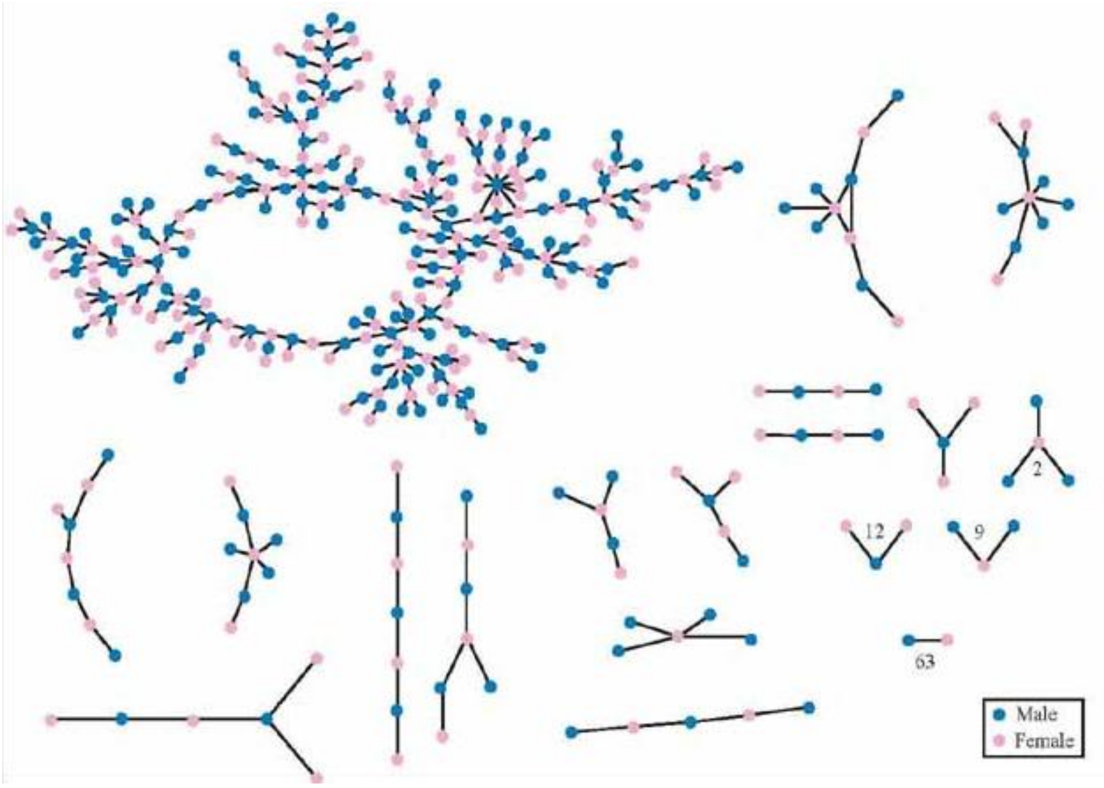


Fig.1.4 The structure of romantic and sexual relations at 'Jefferson High School' in Los Angeles, USA. Each circle represents a student and lines connecting students represent romantic relations within the 6 months preceding the interview. Numbers under the figure count the number of times that pattern was observed (Bearman, Moody and Stovel [3]).

Graph theory is used in biology where a vertex represents regions where certain species exist and the edges represent migration paths or movement between the regions. This information is important when looking at reproduction patterns, tracking the spread of diseases or parasites and also studying the impact of migration that affects other species.

In chemistry, graphs are used to model chemical compounds. In computational biochemistry some sequences of cell samples have to be excluded to resolve the conflicts between two sequences. This is modelled in the form of a graph where the vertices represent the sequences in the sample. An edge will be drawn between two vertices if and only if there is a conflict between the corresponding sequences.

Graphs play a very important role in computer science. They are used to model systems or some aspects of systems. In fact, computer science has strongly contributed to the development of graph theory. Computer systems may be extremely complex and it is very difficult to model all the details in such a way that one can keep an overview of the system as a whole. One accepted model of a computer system is called *transition system.* With a *transition system* we can determine the possible transitions from a state to a possible next state.

Moreover, graphs are used to represent networks of communication, data organization, computational devices, the flow of computation, etc. One practical example: The link structure of a website could be represented by a directed graph. The vertices are the web pages available at the website and a directed edge from page *A* to page *B* exists if and only if *A* contains a link to *B*.

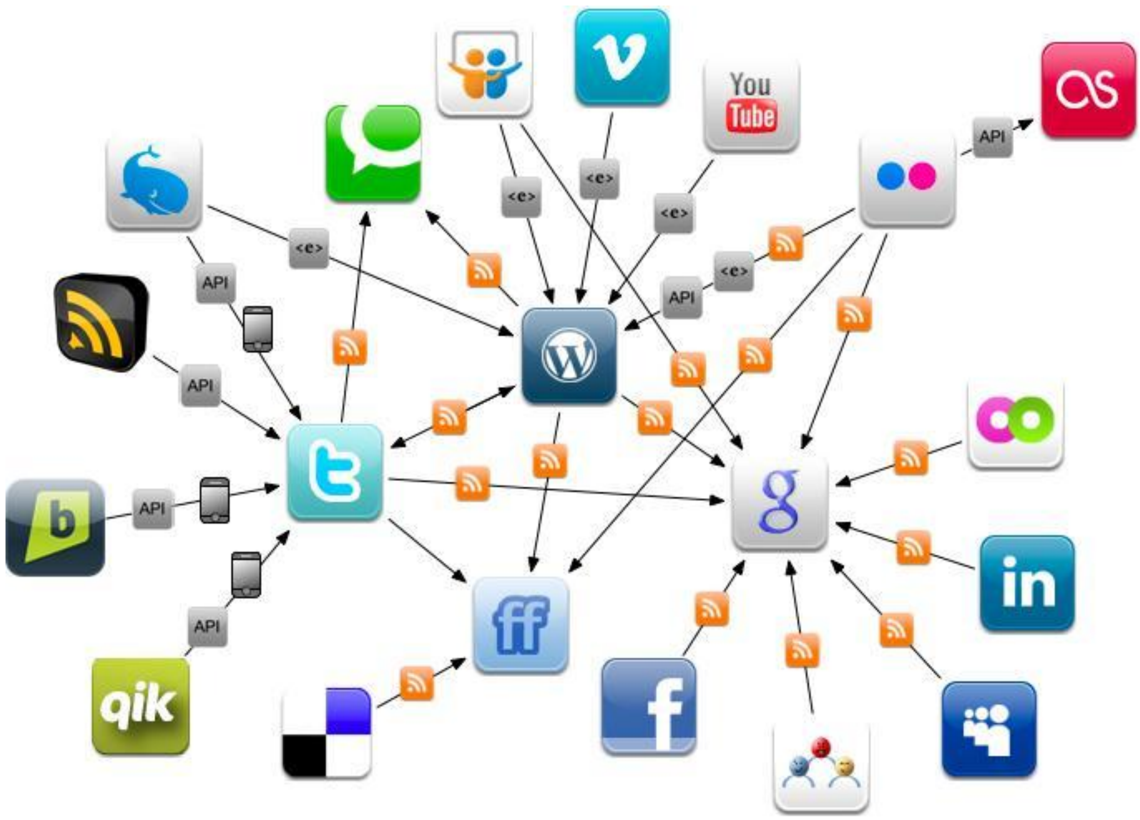


Fig.1.5 Relation between different online social networks (Helmond [13]).

Graph theoretical concepts are widely used in operations research. For example, the Travelling Salesman Problem (also known as TSP, see **Fig.1.6**), that is, to find the shortest spanning tree in a weighted graph, obtaining an optimal match of jobs and people and locating the shortest path between two vertices in a graph.

Graph theory is also used in modelling transport networks, activity networks and theory of games, which is applied in engineering, economics and science to find an optimal way to perform certain tasks in competitive environments.

One of the most popular and successful applications of networks is the planning and scheduling of large and complicated projects. For further information of graph applications, see Shirinivas, Vetrivel, and Elango [15].

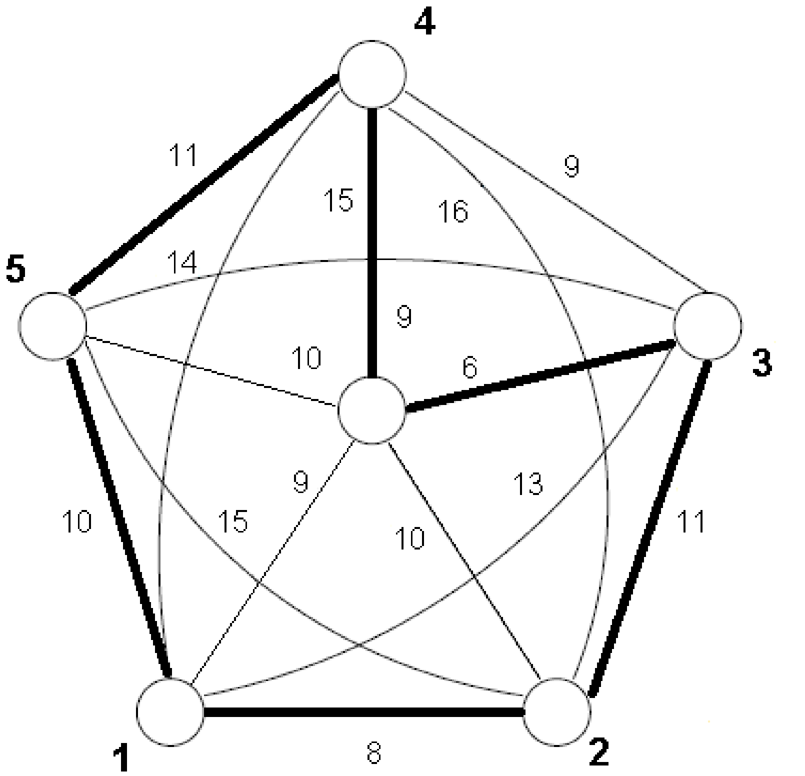


Fig.1.6 Travelling Salesman Problem. The number of each edge is called the weight of the edge. For example, if every vertex is a city, the weight of every edge could represent the distance between cities.

1.5. Double-step graphs

A *double-step graph* $G(N;\pm a,\pm b)$ consists of N vertices, which we denote by $0,1,\ldots,N-1$ and a set of edges of the form $(i,i\pm a)$ and $(i,i\pm b)$, with a and b positive integers called 'steps', so that $1\leq a,b\leq \lfloor N/2\rfloor$, that is, vertex i is adjacent to vertices $i\pm a$ and $i\pm b$ (operations are modulo N). When two numbers x and y are congruent modulo z, it means that $\frac{x-y}{z}$ is an integer and it is written $x\equiv y \pmod z$.

A necessary condition for a double-step graph to be connected is the following: $G(N;\pm a,\pm b)$ is connected if and only if $\gcd(N,a,b)=1$, that is, if N,a,b are relatively prime (Fiol, Yebra, Alegre, and Valero [11]). We represent the steps of a double-step graph and, as an example, the double-step graph $G(9;\pm 2,\pm 3)$ in **Fig.1.7**.

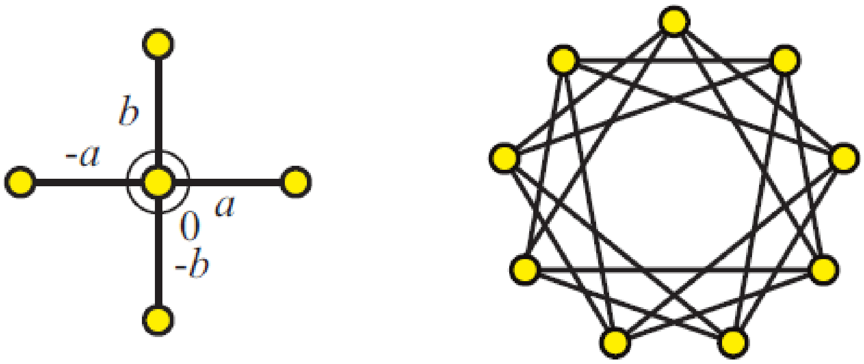


Fig.1.7 Steps of a double-step graph $G(N;\pm a,\pm b)$ and the double-step graph $G(9;\pm 2,\pm 3)$.

We can study the diameter of $G(N;\pm a,\pm b)$ from any arbitrary vertex. For convenience, we do it from the vertex labelled '0'. At distance one from this vertex, there are four vertices $\pm a$ and $\pm b \pmod N$; at distance two, there are eight vertices $\pm 2a, \pm a \pm b, \pm 2b \pmod N$; at distance three, there are twelve vertices $\pm 3a, \pm 2a \pm b, \pm a \pm 2b, \pm 3b \pmod N$, and so on. Therefore, the maximum order of a double-step graph with diameter k is

$$N_k = 1 + \sum_{q=1}^{k} 4q = 2k^2 + 2k + 1.$$

The vertices reached from vertex 0 can be arranged in a planar pattern as shown in **Fig.1.8**. This leads to the following two remarks, which are the base of our study:

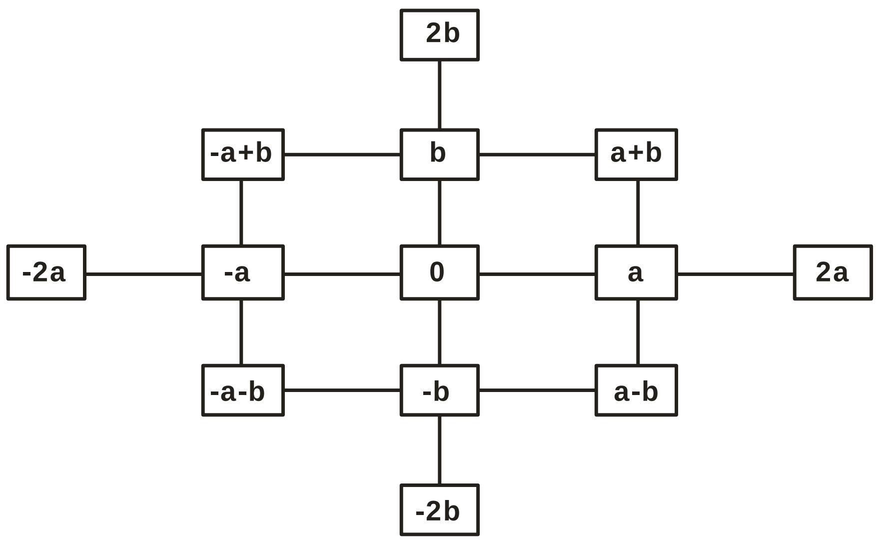


Fig.1.8 Planar pattern of the double-step graph $G(N;\pm a,\pm b)$.

1 – Periodicity: Let the infinite plane be divided in equal squares that we number as in **Fig.1.8** starting with 0 in an arbitrary one. Every square contains a number from 0 to $N-1$ and the distribution of these numbers in the plane repeats itself periodically. This fact is illustrated in **Fig.1.9** for $N = 13$, $a = 2$ and $b = 3$.

2 – Tessellation: Assume $\gcd(N, a, b) = 1$ and form a tile with the squares numbered from 0 to $N-1$. This tile tessellates periodically the plane, taking into account that the 0 square must be placed at the centre of every tile (see again **Fig.1.9**).

For further information about double-step graphs, see Yebra, Fiol, Morillo, and Alegre [16].

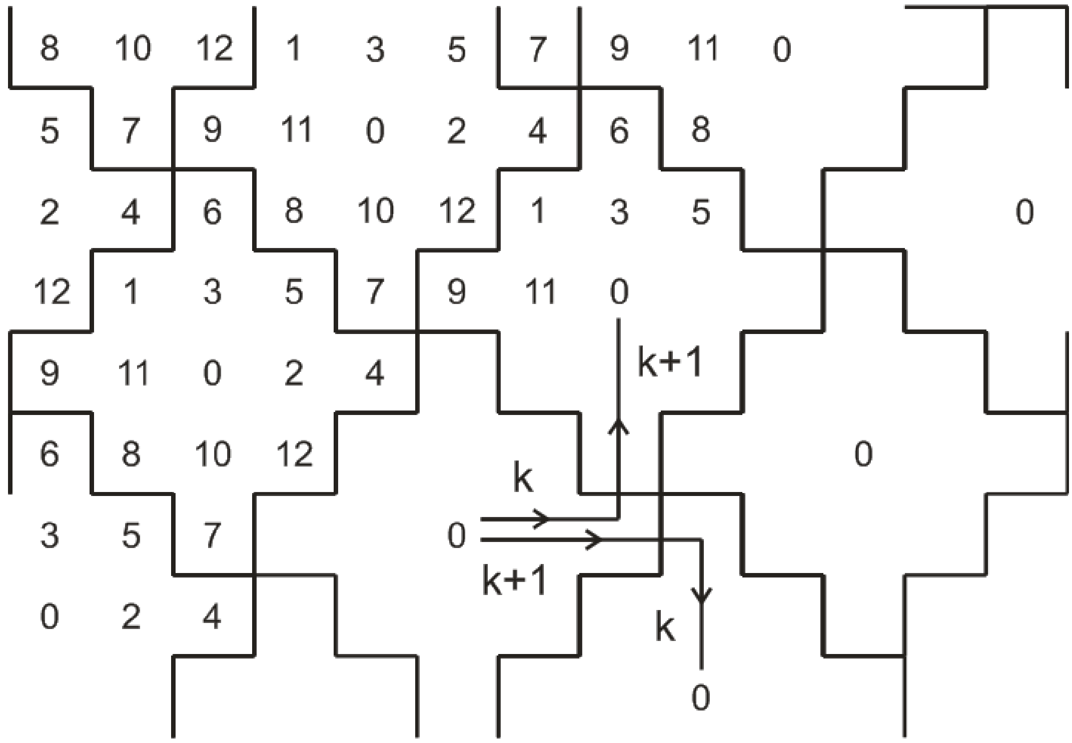


Fig.1.9 Tessellation of the double-step graph $G(13; \pm 2, \pm 3)$ (Yebra, Fiol, Morillo, and Alegre [16]).

1.6. Digraphs

A *digraph* $G = (A, V)$ is a combinatorial structure consisting of a set of *vertices* $V = V(G)$ and a set of *arcs* $A = A(G)$, which consist of ordered pairs of different vertices. If there is an arc $a = (u, v)$, then it is said that vertex u is *adjacent to* v or that v is *adjacent from* u.

As in the case of graphs, the *order* of a digraph G is the number of its vertices, and the *size* of G is the number of its arcs. Vertices are represented by points and arcs are represented by lines with direction that connect pairs of vertices. The diameter of a digraph is the minimum distance between two of the farthest vertices following the directions of arcs. In a digraph, we must distinguish between the outdegree of a vertex u (number of adjacencies from u) and the indegree (number of adjacencies to u).

As an example of a digraph, see the directed cycle C_5 in **Fig.1.10**, which has order 5, size 5, diameter 4 and it is regular with outdegree and indegree 1.

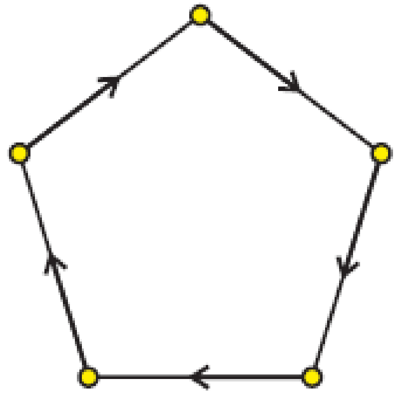


Fig.1.10 A directed cycle C_5.

As in the case of graphs, we can distinguish two types of digraphs (see **Fig.1.11**):

3. Multidigraphs (no restrictions).

4. Simple digraphs (no loops, no parallel edges).

Note that, in a simple digraph, edges that go between the same vertices but in opposite directions are not considered parallel. For more details about digraphs, see Comellas, Fàbrega, Sànchez, and Serra [8].

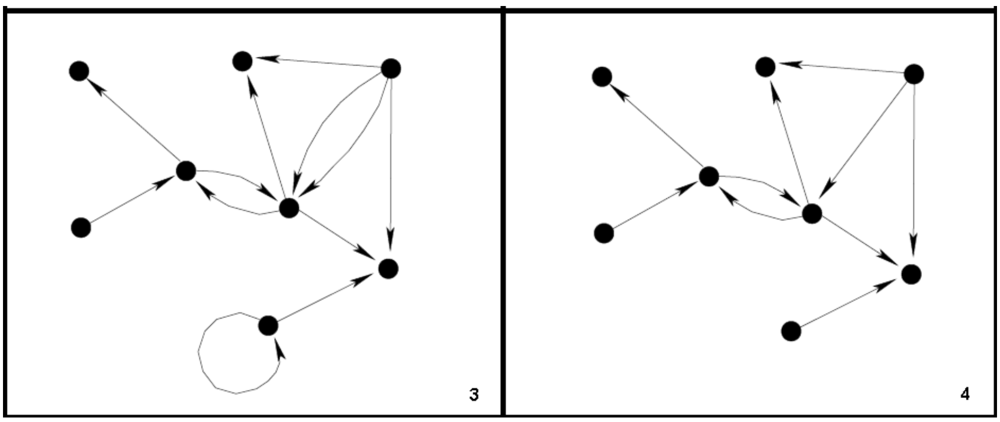


Fig.1.11 Two types of digraphs: 3. Multidigraph, 4. Simple digraph (Fields [10]).

1.7. Double-step digraphs

A *double-step digraph* $G(N; a, b)$ consists of N vertices, which we denote by $0, 1, \ldots, N-1$ and a set of unidirectional links or arcs of the form $(i, i+a)$ and $(i, i+b)$, with a and b positive integers called 'steps', that is, vertex i is adjacent to vertices $i+a$ and $i+b$ (operations are modulo N).

A digraph is strongly connected if there is a directed path (that is, taking the directions of arcs into account) between any pair of vertices. More precisely, $G(N; a, b)$ is strongly connected if and only if $\gcd(N, a, b) = 1$, that is, if N, a, b are relatively prime, which was proved by Fiol, Yebra, Alegre, and Valero [11]. In **Fig.1.12** we represent the steps of a double-step digraph $G(N; a, b)$ and, as an example, the double-step digraph $G(8; 1{,}3)$.

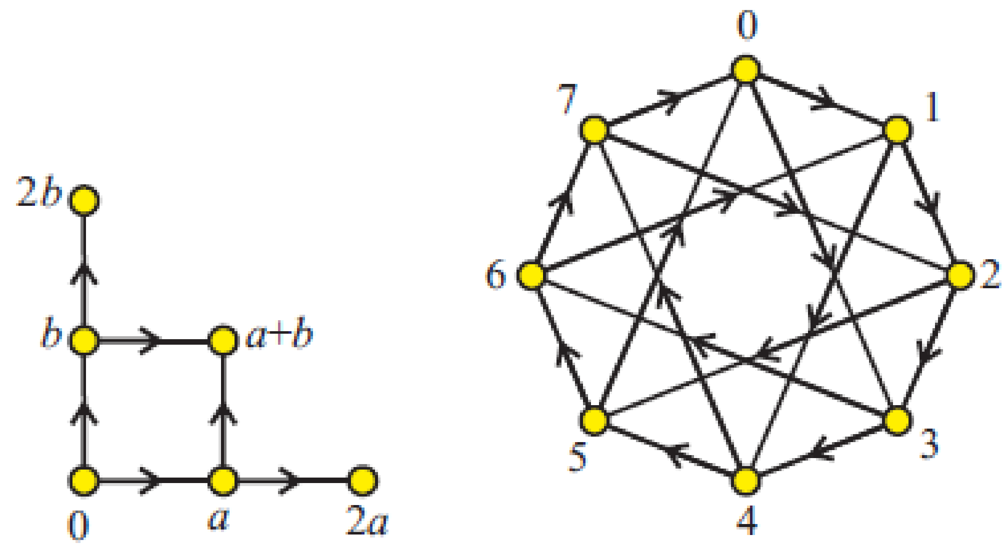


Fig.1.12 Steps in a double-step digraph $G(N; a, b)$ and the double-step digraph $G(8; 1{,}3)$.

1.8. From a digraph to an L-shaped form

We are going to see that every digraph $G(N; a, b)$ with $\gcd(N, a, b) = 1$ has an L-shaped form associated characterized by parameters l, h, w, y, so that $\gcd(l, h, w, y) = 1$ (see **Fig.1.13** (left)).

We consider the digraph $G(N; a, b)$ which is supposed to be strongly connected, that is, $\gcd(N, a, b) = 1$. We take the plane divided by unitary squares (centred in the integer coordinate points that form a reticle). From a square – or reticular point – labelled with zero, we add $a \pmod N$ when we move horizontally to the right to the next square, and $b \pmod N$ when we move vertically upwards. Then, the plane is covered by integers modulo N, as it is shown in **Fig.1.13** (right) with the example of a tessellation of the double-step graph $G(8;1,3)$. We observe that, in this way, every vertex of a digraph $G(N;a,b)$ is associated to a point of the reticle.

Now we choose an initial vertex, for example the 0 circled in **Fig.1.13** (right), and we label all the vertices from 1 to $N-1$ that are at a minimum distance from 0 in the corresponding digraph. The numbering can be done by following a simple algorithm that considers the successive diagonals, just as it is shown again in **Fig.1.13** (right). With this method, we obtain a tile which tessellates (that is, it covers without overlappings) the plane periodically.

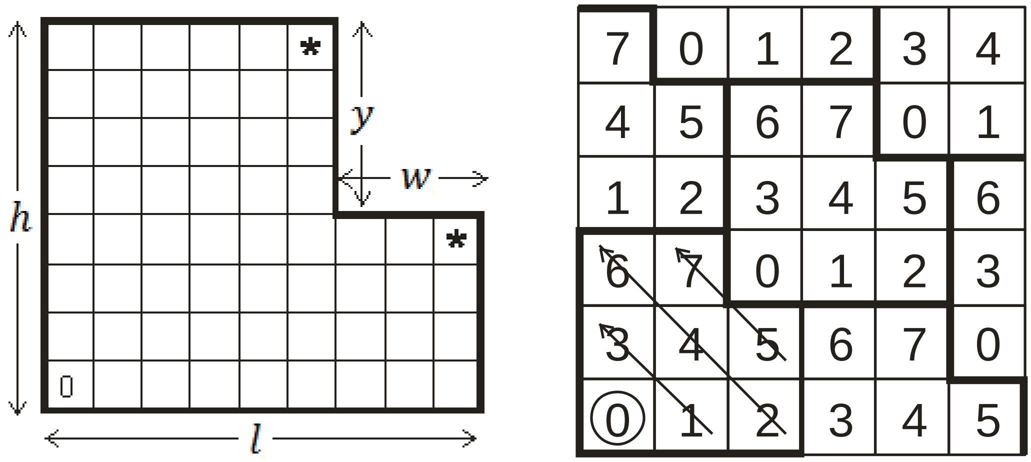


Fig.1.13 A generic L-shaped form and a tessellation of the double-step graph $G(8;1,3)$.

Brawer and Shokley [6] proved that such tiles are always L-shaped forms, that is, tiles with the shape of an L. The tile obtained in our example is also shown in the same figure. From the digraph symmetry, the L-shaped form obtained does not depend on the initial vertex and, therefore, it represents the digraph univocally. Then, an L-shaped form is characterized by its dimensions (l,h,w,y), with $l,h \geq 1$, $0 \leq w \leq l$, $1 \leq y \leq h$, as it is shown in **Fig.1.13** (left) and, as said before, it represents a digraph if $\gcd(l,h,w,y)=1$. We observe that, then, the diameter D of the digraph correspond to the minimum distances from 0 to the vertices labelled with an asterisk, that is,

$$D = \max\{l - w + h - 2, h - y + l - 2\}.$$

1.9. From an L-shaped form to a digraph

From an L-shaped form of dimensions (l,h,w,y), with $\gcd(l,h,w,y)=1$, and area $N=lh-wy$, steps a and b of the corresponding digraph of N vertices can be obtained as follows: From the distances between the zeros of different L-shaped forms we find (see **Fig.1.14**):

$$la - yb = \alpha N\ \big(\equiv 0\ (\mathrm{mod}\ N)\big),$$

$$-wa + hb = \beta N\ \big(\equiv 0\ (\mathrm{mod}\ N)\big).$$

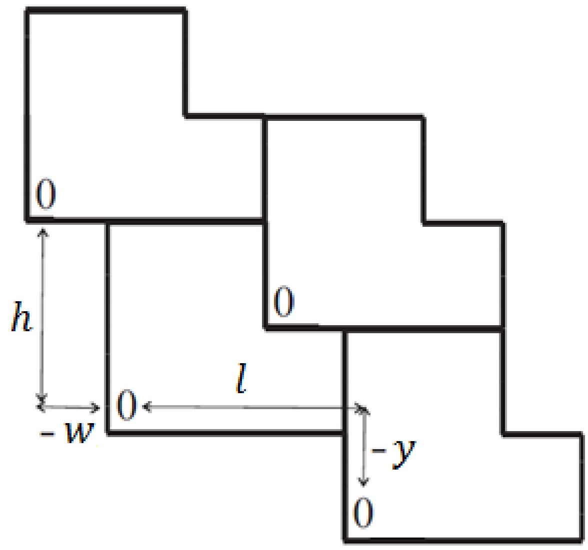


Fig.1.14 Distances between zeros of different L-shaped forms.

In a matricial form, we have

$$\begin{pmatrix} l & -y \\ -w & h \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} \alpha N \\ \beta N \end{pmatrix},$$

$$M = \begin{pmatrix} l & -y \\ -w & h \end{pmatrix},$$

$$\det M = lh - wy = N,$$

$$M^{-1} = \frac{1}{N} \begin{pmatrix} h & y \\ w & l \end{pmatrix},$$

$$\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} h & y \\ w & l \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix}.$$

Therefore, the steps are

$$a = \alpha h + \beta y,$$

$$b = \alpha w + \beta l.$$

We observe that the values of steps a and b are not unique, they depend on α and β, which must be chosen so that the condition $\gcd(N, a, b) = 1$ is satisfied. If this is not possible, steps a and b do not exist and, therefore, the digraph does not exist.

1.10. The (Δ, D) and (Δ, N) problems

The (Δ, D) problem in double-step digraphs $G(N; a, b)$ consists in finding the maximum number of vertices given a diameter D and a degree $\Delta = 2$, that is, finding which are the two steps of a double-step digraph that maximize the number of vertices. The (Δ, N) problem consists in finding the minimum diameter in double-step digraphs given a number of vertices N and a degree $\Delta = 2$, that is, finding which are the two steps of a double-step digraph that minimize the diameter.

1.11. Manhattan and New Amsterdam networks

The structure of a Manhattan network is as shown in **Fig.1.15**, and corresponds to a standard pattern for the allowed traffic directions in some neighbourhoods of our modern cities, like New York or Barcelona, with their system of straight orthogonal streets.

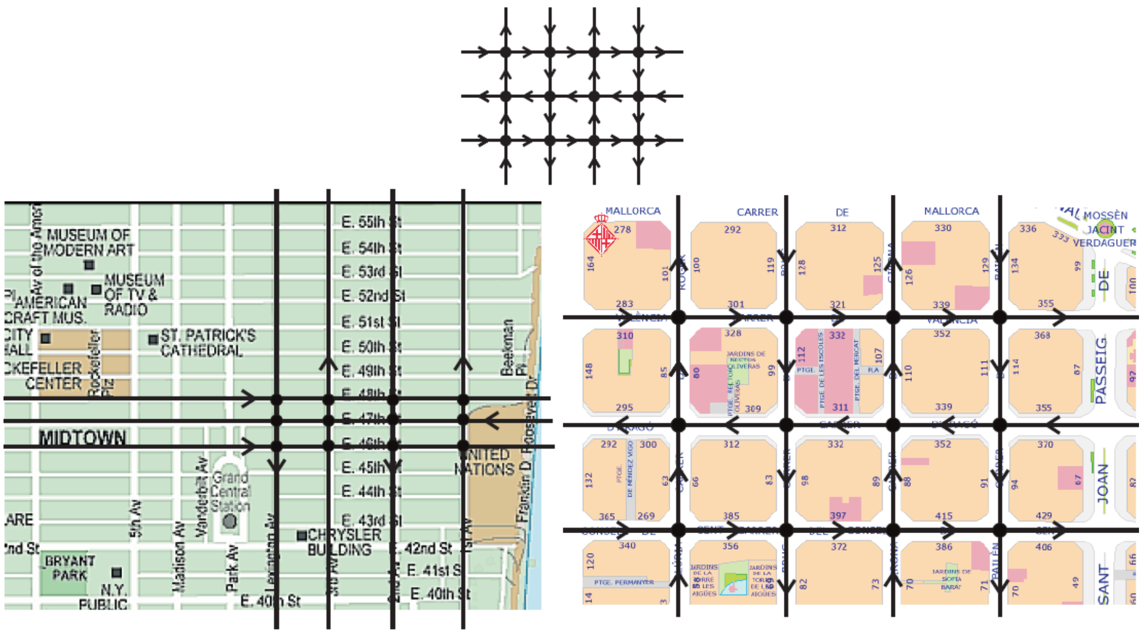


Fig.1.15 The local pattern of a Manhattan network and two real-life examples: Orthogonal streets of Manhattan and *l'Eixample* in Barcelona (Comellas, Dalfó, and Fiol [7]).

The definition of a toroidal Manhattan digraph M is the following (for more details, see Comellas, Dalfó, and Fiol [7]): Given two positive integer numbers N_1, N_2, the toroidal *Manhattan* digraph $M = M(N_1, N_2)$ has a set of vertices $V(M) = V(N_1) \times V(N_2)$. Each of its vertices is represented by a 2-dimension vector (u_1, u_2). The set of arcs $A(M)$ is defined by the following adjacencies:

$$(u_1, u_2) \rightarrow (u_1 + (-1)^{u_2}, u_2),$$

$$(u_1, u_2) \rightarrow (u_1, u_2 + (-1)^{u_1}).$$

Therefore, M is a 2-regular digraph with $N = N_1 N_2$ vertices.

Some of the properties of the toroidal Manhattan digraph are the following ones:

(a) M is a bipartite and 4-partite digraph.
(b) M is a vertex-symmetric digraph.
(c) For $N_1, N_2 > 4$, the diameter of $M = M(N_1, N_2)$ is

$$D(M_2) = \begin{cases} \frac{N_1}{2} + \frac{N_2}{2} + 1 & \text{if } N_1, N_2 \equiv 0 \pmod 4, \\ \frac{N_1}{2} + \frac{N_2}{2} & \text{otherwise.} \end{cases} \qquad (\mathbf{1.1})$$

In **Fig.1.16** there is an example of the shortest paths from any vertex to vertex $(0,0)$ in $M(10,8)$.

A more general way to define a Manhattan digraph M is by considering as follows (Morillo, Fiol, and Fàbrega [14]): A *Manhattan* digraph M has a set of vertices (with a multiple of 4 order) $V = V_0 \cup V_1 \cup V_2 \cup V_3$ with $V_j = i$, $0 \leq i \leq N - 1$, $i \equiv j \pmod 4$, and every vertex i is adjacent to vertices $i + a_j$ and $i + b_j \pmod N$, where $a_j \equiv 3$, $b_j \equiv 1 \pmod N$ and

$$a_0 + a_1 + a_2 + a_3 \equiv 0 \pmod N,$$

$$b_0 + b_1 + b_2 + b_3 \equiv 0 \pmod N,$$

$$a_0 + a_2 \equiv b_0 + b_2 \pmod N.$$

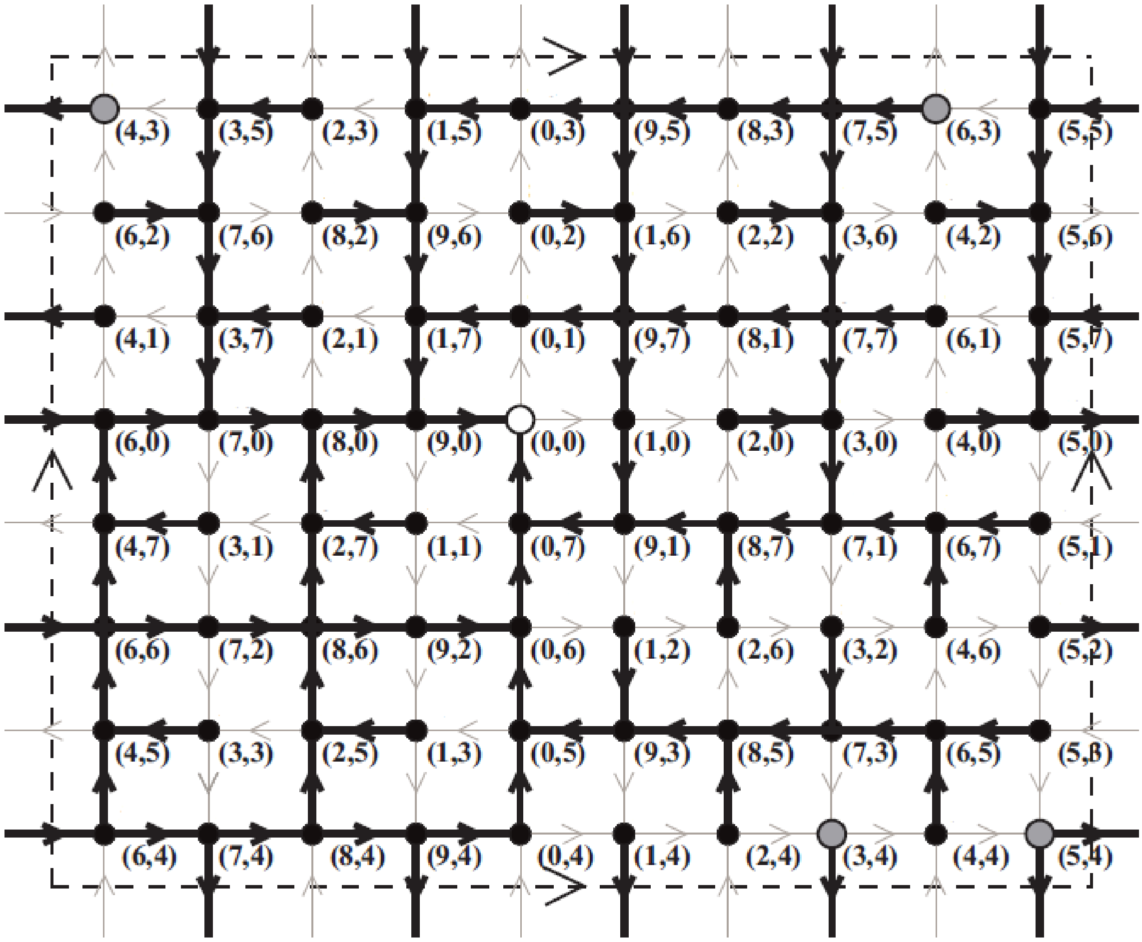


Fig.1.16 The shortest paths to $(0{,}0)$ in $M(10{,}8)$. Just for illustration, the four vertices $(5{,}4)$, $(3{,}4)$, $(6{,}3)$, and $(4{,}3)$ at maximum distance 9 to $(0{,}0)$ are indicated in grey (Comellas, Dalfó, and Fiol [7]).

We show the 8 steps of a Manhattan digraph and, as an example, a Manhattan digraph in **Fig.1.17**.

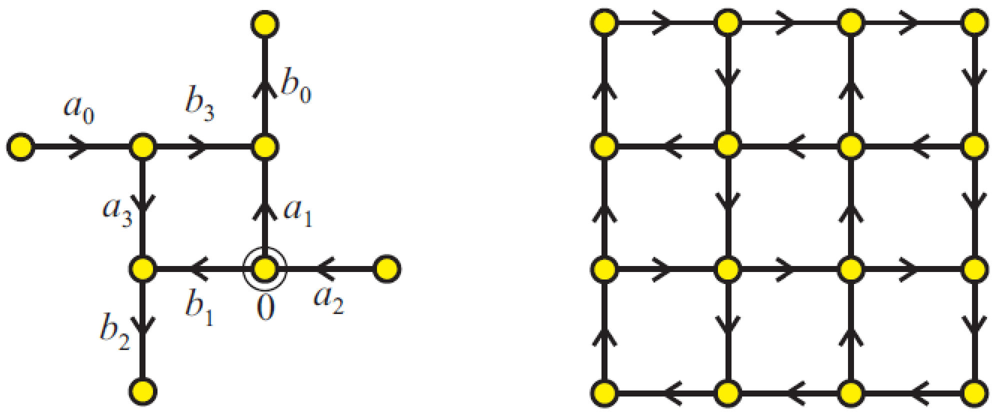


Fig.1.17 Steps of a Manhattan digraph $M(4N; a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3)$ and an example of a Manhattan digraph.

Comellas, Dalfó, and Fiol [7] stated that the Manhattan digraph is the line digraph of the digraph called New Amsterdam. This name comes from the fact that the previous name of the Manhattan island was New Amsterdam. A line digraph $L(G)$ (for example, a Manhattan digraph) is obtained from another digraph (for example, a New Amsterdam digraph) by replacing the arcs of the original digraph with vertices in the second digraph and the arcs adjacencies with vertices adjacencies.

The definition of the New Amsterdam digraph is the following one: A *New Amsterdam* digraph has a set of vertices (with even order) $V = V_0 \cup V_1$ with $V_0 = \{0,2,\dots,N-2\}$ and $V_0 = \{1,3,\dots,N-1\}$. Every vertex $i \in V_0$ is adjacent to vertices (modulo N) $i+\alpha$, $i+\beta \in V_1$ for some different odd integers α, β, and every vertex $j \in V_1$ is adjacent to vertices (modulo N) $j+\gamma$, $j+\delta \in V_0$ for some odd integers γ, δ, so that $\alpha+\beta+\gamma+\delta \equiv 0 \pmod N$. This digraph is illustrated in **Fig.1.18** (left). For further information about these two digraphs, see Morillo, Fiol, and Fàbrega [14] and Comellas, Dalfó, and Fiol [7].

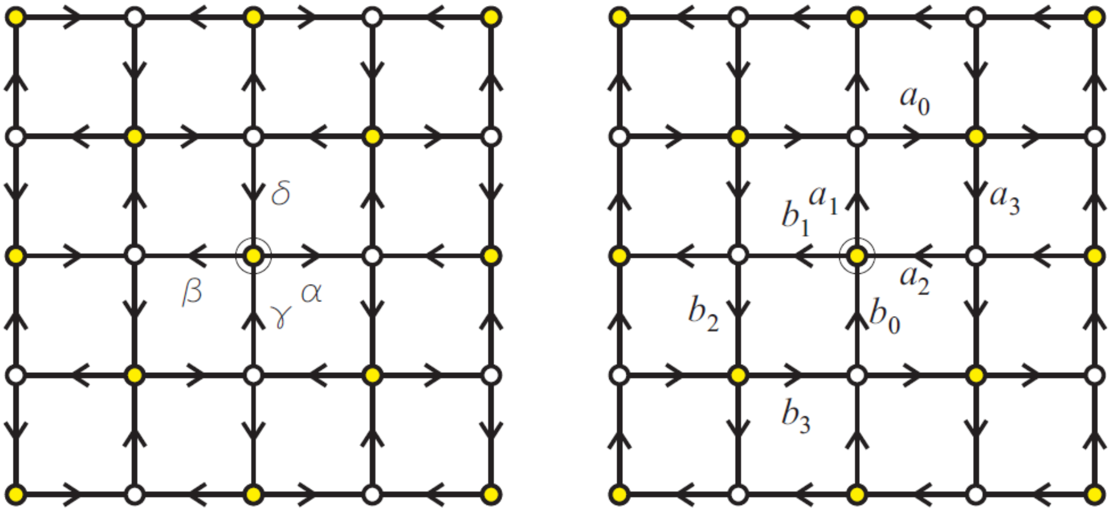


Fig.1.18 New Amsterdam digraph (left) and Manhattan digraph (right) (even vertices are yellow and the odd ones are white).

Morillo, Fiol, and Fàbrega [14] solved the (Δ, D) problem for various digraphs, among them, the Manhattan and the New Amsterdam digraphs. For a Manhattan digraph with diameter $k > 4$, they proved that the maximum number of vertices $N_M(k)$ is:

$$N_M(k) = \begin{cases} 2(k-1) & \text{if } k \text{ is odd,} \\ 2[(k-1)^2 - 1] & \text{if } k \text{ is even.} \end{cases}$$

CHAPTER 2. PROGRAM

Now, we are going to make a brief description of the program we used to solve the problem considered. With this program we are able to find the diameter of any New Amsterdam digraph.

According to the structure of a New Amsterdam digraph, we know that steps α, β, γ and δ always have the following directions: α and β are always horizontal and have opposite directions, whereas γ and δ are always vertical with opposite directions (see **Fig.1.18** (left)).

The main point of our program is to differentiate the vertical steps from the horizontal ones.

At first, we can think that repeating these steps we are able to reach the maximum distance or the diameter, but this is not really true. It is also necessary to carry out the same process again but, instead of starting from vertex 0, which is an even vertex, we must start from an odd vertex such as vertex 1. This is due to the fact that, as we said before, the adjacencies between even and odd vertices are different and, consequently, the maximum distance from vertex 0 or 1 could change.

In this way, if we start to cover all the vertices of the New Amsterdam digraph from an even vertex (vertex 0, for example), we can reach the other even vertices (distance k is even) with the same number of steps of type α and/or β and steps of type γ and/or δ. However, we reach the odd vertices with an additional step of type α and/or β.

On the other hand, if we start from an odd vertex (vertex 1, for example), we reach the other odd vertices (distance k is even) with the same number of steps of type α and/or β and steps of type γ and/or δ. However, we reach the even vertices with an additional step of type γ and/or δ.

In other words, we must make this process twice. Firstly, initializing the program with an even vertex and, lastly, initializing it with an odd vertex. Once we carry out the two procedures, we must choose the greatest of the maximum distances, so we find the diameter of the New Amsterdam digraph for a given number of vertices.

We show a scheme in **Fig.2.1** that illustrates the operation of the program in a more visual way.

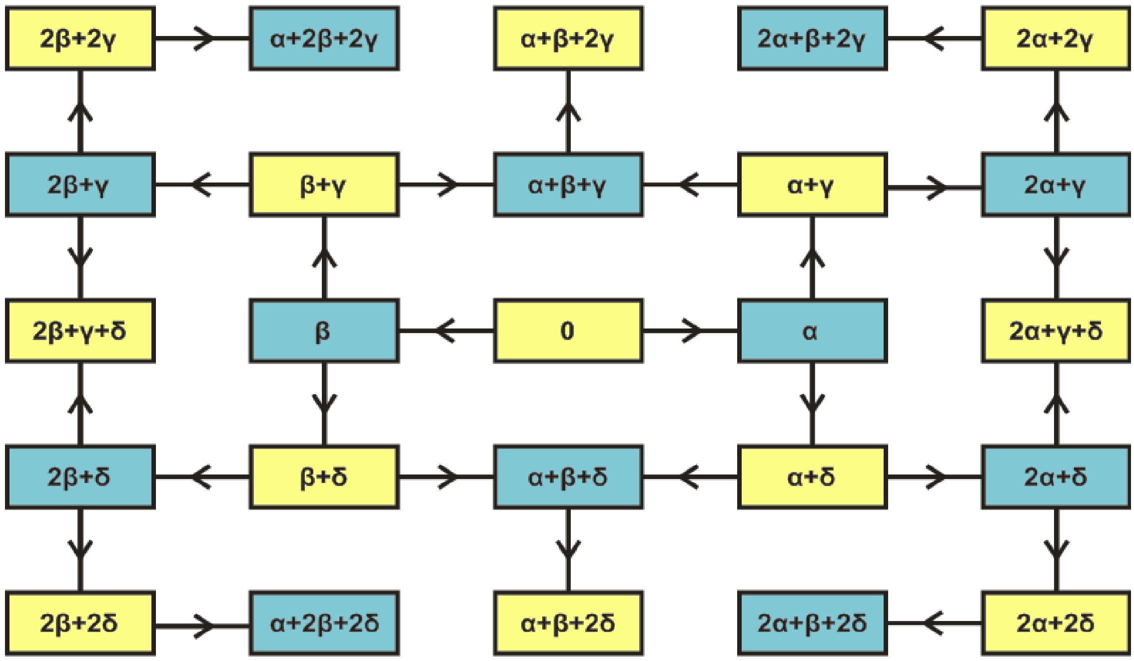


Fig.2.1 Even vertices are yellow and the odd ones are blue. In this figure, the arcs that are not used are not showed.

Once we find the diameter of the New Amsterdam digraph, we are able to find the diameter of the Manhattan digraph. We know that the diameter of a line digraph is the diameter of the original digraph plus one, except for a directed cycle, where both diameters are equal (see Fiol, Yebra, Alegre and Valero [11]). As the Manhattan digraph is the line digraph of the New Amsterdam digraph (see section **1.11**), then

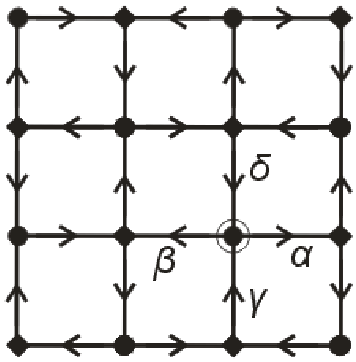
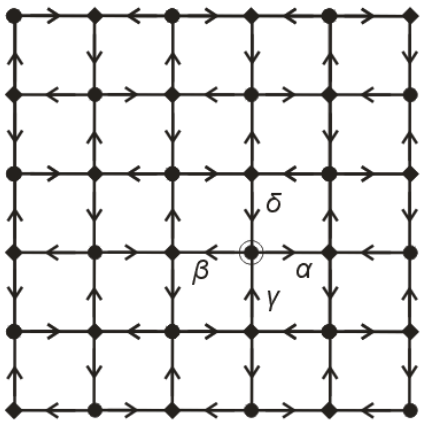
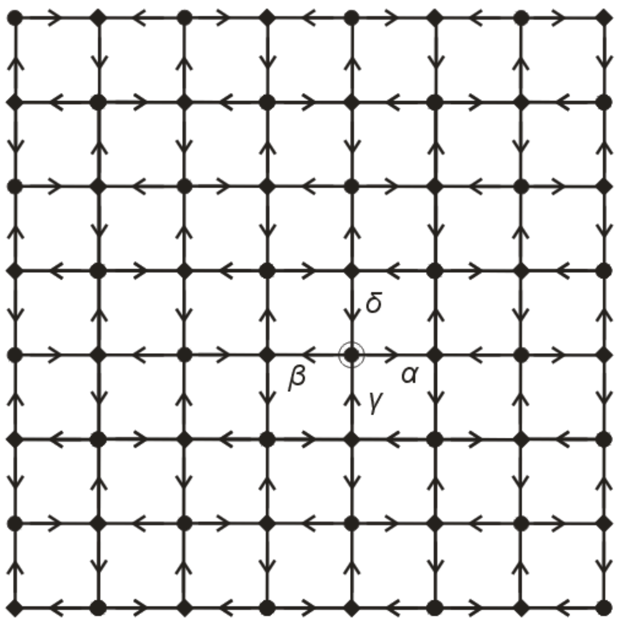
$$D_M = D_{NA} + 1,$$

where D_M is the diameter of the Manhattan digraph and D_{NA} is the diameter of the New Amsterdam digraph.

2.1. Examples

In **Table 2.1.** we show three examples of New Amsterdam digraphs with their corresponding parameters.

Table 2.1. Examples of New Amsterdam digraphs

<table>
<tr><th>New Amsterdam digraph</th><th>New Amsterdam parameters</th></tr>
<tr><td>
</td><td>$N_{NA} = 16$
$D_{NA} = 5$
$\alpha = -1$
$\beta = 1$
$\gamma = 3$
$\delta = -3$</td></tr>
<tr><td>
</td><td>$N_{NA} = 36$
$D_{NA} = 7$
$\alpha = -1$
$\beta = 1$
$\gamma = 5$
$\delta = -5$</td></tr>
<tr><td>
</td><td>$N_{NA} = 64$
$D_{NA} = 9$
$\alpha = -1$
$\beta = 1$
$\gamma = 7$
$\delta = -7$</td></tr>
</table>

2.2. Program code in Free Basic

```
Dim N As Integer                'New Amsterdam vertices
Dim DN As Integer               'New Amsterdam diameter
Dim alfa As Integer             'New Amsterdam step α
Dim beta As Integer             'New Amsterdam step β
Dim gamma As Integer            'New Amsterdam step γ
Dim delta As Integer            'New Amsterdam step δ
Dim aa As Integer               'Minimum diameter step α
Dim bb As Integer               'Minimum diameter step β
Dim cc As Integer               'Minimum diameter step γ
Dim dd As Integer               'Minimum diameter step δ
Dim e As Integer                'Step set counter
Dim k As Integer                'Current diameter value
Dim yy As Integer               'Vertex position
Dim z As Integer                'Step counter
Dim SX As Integer               'Addition of horizontal vertices
Dim SY As Integer               'Addition of vertical vertices
Dim r As Integer                'Vector vertex position
Dim u As Integer                'Horizontal step
Dim v As Integer                'Vertical step

Open "Data.dat" For Output As #1

print "(Delta,N) problem for New Amsterdam digraphs"
print " "

Print #1, "(Delta,N) problem for New Amsterdam digraphs"
Print #1,

For N=8 to 100 step 2
    DN=N\2+1
    For alfa=1 to N-1 step 2
        For beta=alfa to N-1 step 2
            For gamma=1 to N-1 step 2
                delta=-alfa-beta-gamma
```

'To calculate the addition of all the vertices

```
        SX=N*(N-1)/2
        SY=N*(N-1)/2-1
```

'To create a vector of N elements $X(r)$, $r = 0,1,\ldots,N-1$, and to label the vertices as 'not reached from 0' $X(r) = r$

```
        Dim X(0 to N-1) As Integer
        For r=0 to N-1
```

```
  X(r)=r
Next r
```

‘To create a vector of N elements $Y(r)$, $r = 0,1,...,N-1$, and to label the vertices as ‘not reached from 1’ $Y(r) = r$

```
Dim Y(0 to N-1) As Integer
For r=0 to N-1
  Y(r)=r
Next r
Y(1)=0
```

‘From distance 1 to the diameter

```
k=1
Do Until (SX=0 and SY=0 or k=N\2+1)

  If(k\2=k/2) then
    e=k/2
```

‘To calculate the vertices reached from 0 in an even number $k = 2e$ of steps, which are of the form $e\{\alpha,\beta\} + e\{\gamma,\delta\} \pmod N$. e is a counter of the sets $\{\alpha,\beta\}$ and $\{\gamma,\delta\}$ (Example: if we have $e = 2$, there are 2 steps of type α and/or β and 2 steps of type γ and/or δ)

```
    For u=0 to e
      For v=0 to e
        z=u*alfa+(e-u)*beta+v*gamma+(e-v)*delta
        yy=z-(z\N)*N

        if yy<0 then
          yy=N+yy
        endif
```

‘If vertex y has not been reached yet, y is subtracted from SX and it is defined $X(y) = 0$

```
          For r=1 to N-1
            if yy=X(r) then SX=SX-yy: X(r)=0
          Next r
        Next v
      Next u
```

‘To calculate the vertices reached from 1 in an even number $k = 2e$ of steps, which are of the form $1 + e\{\alpha,\beta\} + e\{\gamma,\delta\} \pmod N$)

```
    For u=0 to e
      For v=0 to e
        z=1+u*alfa+(e-u)*beta+v*gamma+(e-v)*delta
        yy=z-(z\N)*N
```

```
        if yy<0 then
          yy=N+yy
        endif
```

‘If vertex y has not been reached yet, y is subtracted from SY and it is defined $Y(y) = 0$

```
          For r=0 to N-1
            if yy=Y(r) then SY=SY-yy: Y(r)=0
          Next r
        Next v
      Next u

    Elseif (k\2<k/2) then
      e=k\2
```

‘To calculate the vertices reached from 0 in an odd number $k = 2e + 1$ of steps, which are of the form $(e + 1)\{\alpha, \beta\} + e\{\gamma, \delta\}\ (\mathrm{mod}\ N)$

```
    For u=0 to e+1
      For v=0 to e
        z=u*alfa+(e+1-u)*beta+v*gamma+(e-v)*delta
        yy=z-(z\N)*N

        if yy<0 then
          yy=N+yy
        endif
```

‘If vertex y has not been reached yet, y is subtracted from SX and it is defined $X(y) = 0$

```
          For r=1 to N-1
            if yy=X(r) then SX=SX-yy: X(r)=0
          Next r
        Next v
      Next u
```

‘To calculate the vertices reached from 1 in an odd number $k = 2e + 1$ of steps, which are of the form $1 + (e + 1)\{\alpha, \beta\} + e\{\gamma, \delta\}\ (\mathrm{mod}\ N)$

```
    For u=0 to e+1
      For v=0 to e
        z=1+u*gamma+(e+1-u)*delta+v*alfa+(e-v)*beta
        yy=z-(z\N)*N

        if yy<0 then
          yy=N+yy
        endif
```

‘If vertex y has not been reached yet, y is subtracted from SY and it is defined $Y(y) = 0$

```
                For r=0 to N-1
                  if yy=Y(r) then SY=SY-yy: Y(r)=0
                Next r
              Next v
            Next u

          end if
```

‘To add one to k

```
          k +=1

        Loop

        if k-1<DN then aa=alfa: bb=beta: cc=gamma:
dd=delta: DN=k-1

      Next gamma
    Next beta
  Next alfa

  print N;DN;aa;bb;cc;dd,

  Print #1, N;DN;aa;bb;cc;dd
  Print #1,

Next N
Close #1

Sleep
```

2.3. Program results

In **Table 2.2.** we show the results obtained when we execute the program with the number of vertices from 10 to 100. The number of vertices N_{NA}, the diameter D_{NA} and the steps α, β, γ, δ of the New Amsterdam digraph, and the steps a and b of the double-step graph are listed.

It is convenient to say that we initialize the program with 8 vertices, because we cannot strictly talk about New Amsterdam digraphs if we have between 1 and 7 vertices.

Table 2.2. Program results.

N_{NA}	D_{NA}	α	β	γ	δ
8	3	-1	1	3	-3
10	3	-1	1	3	-3
12	4	-1	1	3	-3
14	4	-1	3	3	-5
16	5	-1	1	3	-3
18	5	-1	1	3	-3
20	5	-1	1	5	-5
22	5	-1	1	5	-5
24	5	-1	1	5	-5
26	5	-1	1	5	-5
28	6	-1	1	5	-5
30	6	11	21	1	-33
32	7	-1	1	5	-5
34	7	-1	1	5	-5
36	7	-1	1	5	-5
38	7	-1	1	5	-5
40	7	-1	1	7	-7
42	7	-1	1	7	-7
44	7	-1	1	7	-7
46	7	-1	1	7	-7
48	7	-1	1	7	-7
50	7	-1	1	7	-7
52	8	-1	1	7	-7
54	8	-1	3	7	-9
56	9	-1	1	7	-7
58	9	-1	1	7	-7
60	9	-1	1	7	-7
62	9	-1	1	7	-7
64	9	-1	1	7	-7
66	9	-1	1	7	-7
68	9	-1	1	9	-9
70	9	-1	1	9	-9
72	9	-1	1	9	-9
74	9	-1	1	9	-9
76	9	-1	1	9	-9
78	9	-1	1	9	-9
80	9	-1	1	9	-9

N_{NA}	D_{NA}	α	β	γ	δ
82	9	-1	1	9	-9
84	10	-1	1	9	-9
86	10	-1	3	37	-39
88	11	-1	1	9	-9
90	11	-1	1	9	-9
92	11	-1	1	9	-9
94	11	-1	1	9	-9
96	11	-1	1	9	-9
98	11	-1	1	9	-9
100	11	-1	1	9	-9

CHAPTER 3. THE (Δ, N) PROBLEM IN MANHATTAN DIGRAPHS

As said before, the (Δ, N) problem in a Manhattan digraph consists in finding its minimum diameter given the number of vertices and the degree $\Delta = 2$. The (Δ, D) problem consists in finding the maximum number of vertices given the diameter and the degree $\Delta = 2$. This problem was solved for the case of double-step graphs $G(N; \pm a, \pm b)$ by Yebra, Fiol, Morillo, and Alegre [16], who proved that, in this case, for a diameter $D = k$, the maximum number of vertices is $N = 2k^2 + 2k + 1$. Two possible pairs of steps are $a = k$ and $b = k + 1$, and $a = 2k + 1$ and $b = 1$. Furthermore, these results enabled them to solve the (Δ, D) problem for Manhattan digraphs. Regarding the (Δ, N) problem, Bermond, Iliades, and Peyrat [5] proved that with the first pair of steps ($a = k$ and $b = k + 1$), the minimum diameter $D = k$ is also obtained for any number of vertices N so that $2(k-1)^2 + 2(k-1) + 1 < N \leq 2k^2 + 2k + 1$.

In order to obtain Manhattan digraphs, we expand the double-step graphs, so that we transform every vertex into a directed cycle of order 4, and every edge into two arcs in opposite directions. From the known results of the double-step graphs, we obtain results for the Manhattan double-step digraphs.

Moreover, the program explained in the previous chapter, calculates the diameter of the New Amsterdam digraph $NA(2N; \alpha, \beta, \gamma, \delta)$.

Table 3.1. Examples of the diameter and the number of vertices of the double-step graph $G(N; \pm a, \pm b)$, the Manhattan digraph $M(4N; a_i, b_i)$, $0 \leq i \leq 3$, and the toroidal Manhattan digraph $M(N_1, N_2)$.

$G(N; \pm a, \pm b)$		$M(4N; a_i, b_i)$		$M(N_1, N_2)$			
D	N	D	N	D	N_1	N_2	N
1	5	4	20	7	2	10	20
2	13	6	52	14	2	26	52
3	25	8	100	26	2	50	100
				10	10	10	

See three examples of double-step graphs and Manhattan digraphs in **Fig.3.1**, **3.2** and **3.3**. In **Table 3.1**, there are the results of these three examples. We can compare the results of the double-step graph with the ones of the Manhattan digraph and with the ones of the toroidal Manhattan digraph of dimensions N_1 and N_2. We observe that the diameter of the Manhattan digraph is lower than the one of the toroidal Manhattan digraph. In the first three cases of the diameter of $M(N_1, N_2)$, as $N_1 = 2 < 4$, we were not been able to use the formula for the diameter of the toroidal Manhattan digraph given by equation $(\mathbf{1.1})$ and we calculated them specifically for each case.

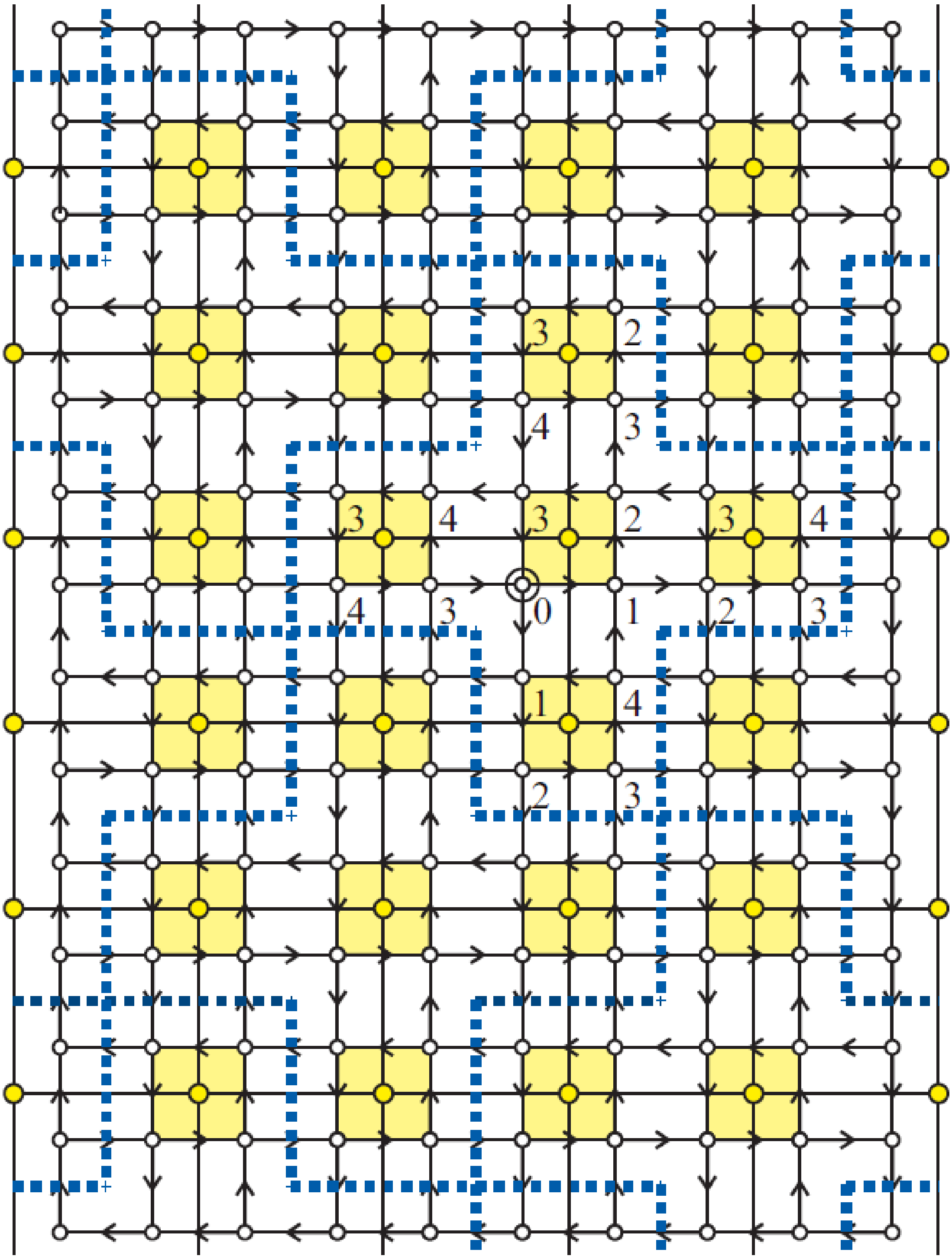


Fig.3.1 Example of the double-step graph with $N = 5$ and the Manhattan digraph with $N = 20$. The vertices of the double-step graph are yellow and the ones of the Manhattan digraph are white. The numbers indicate the distance from vertex 0.

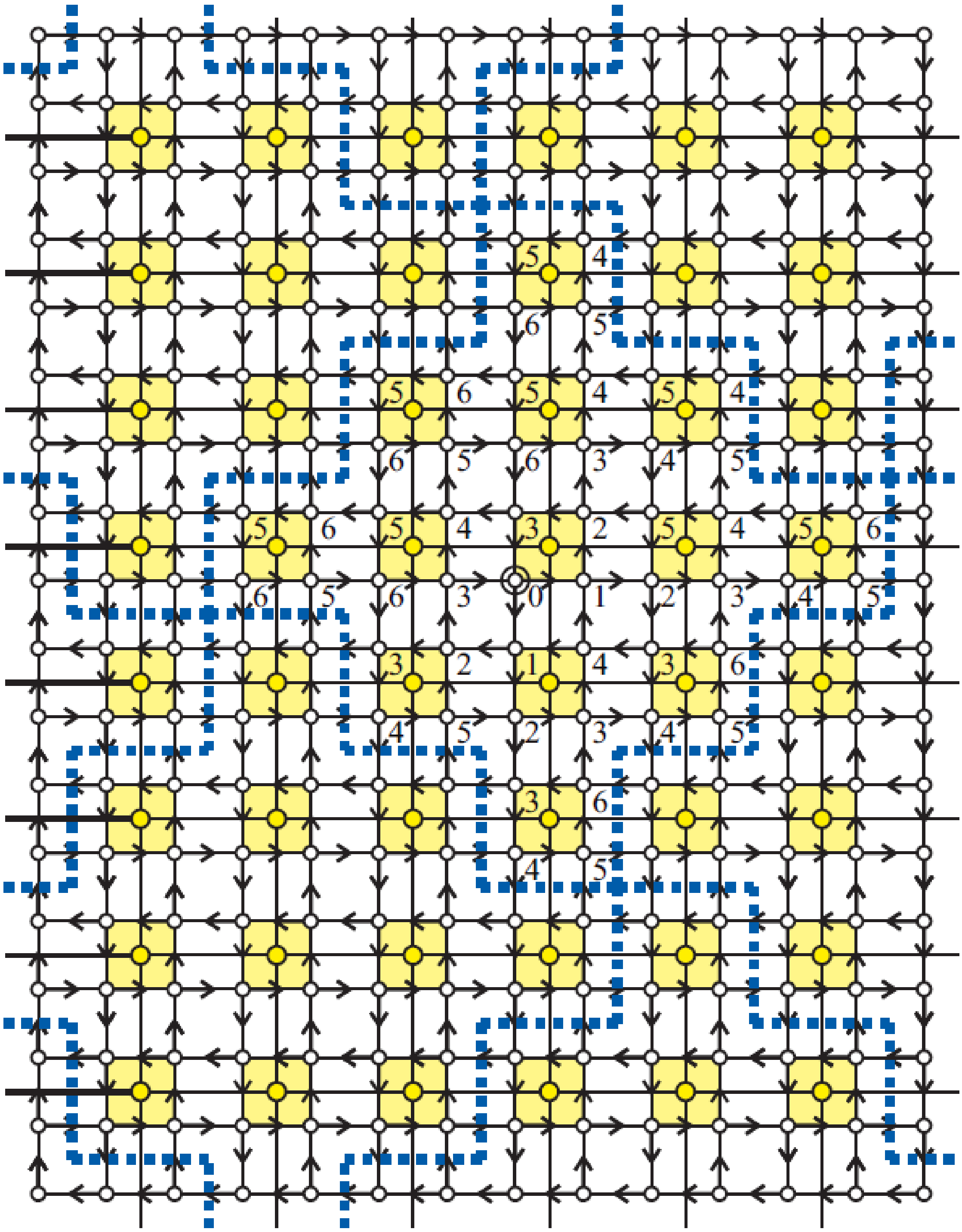


Fig.3.2 Example of the double-step graph with $N = 13$ and the Manhattan digraph with $N = 52$. Vertices of the double-step graph are yellow and the ones of the Manhattan digraph are white. The numbers indicate the distance from vertex 0.

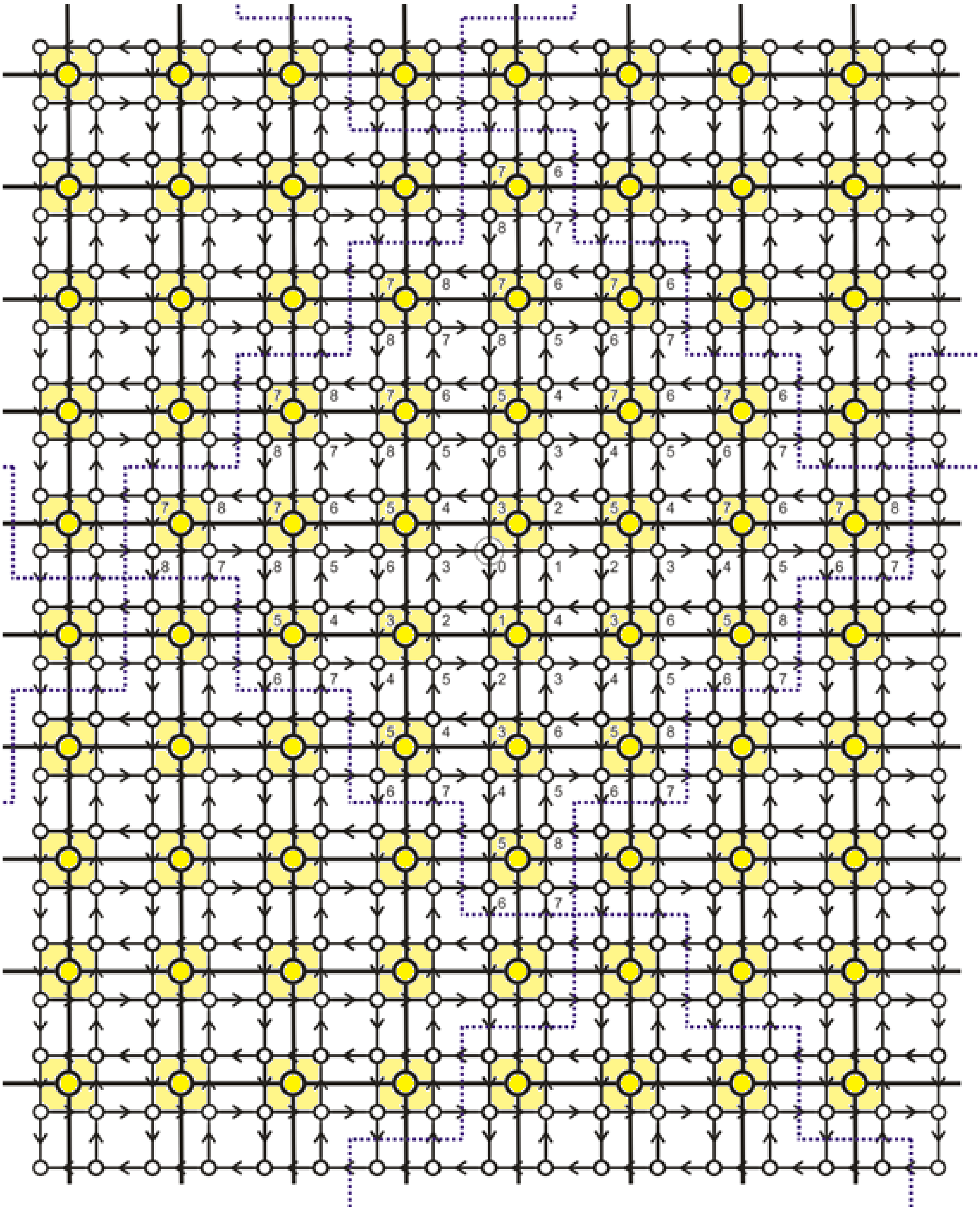


Fig.3.3 Example of the double-step graph with $N = 25$ and the Manhattan digraph with $N = 100$. Vertices of the double-step graph are yellow and the ones of the Manhattan digraph are white. The numbers indicate the distance from vertex 0.

3.1. Relation between a double-step graph and a Manhattan digraph

We show a scheme of a double-step graph and a Manhattan digraph in **Fig.3.4**.

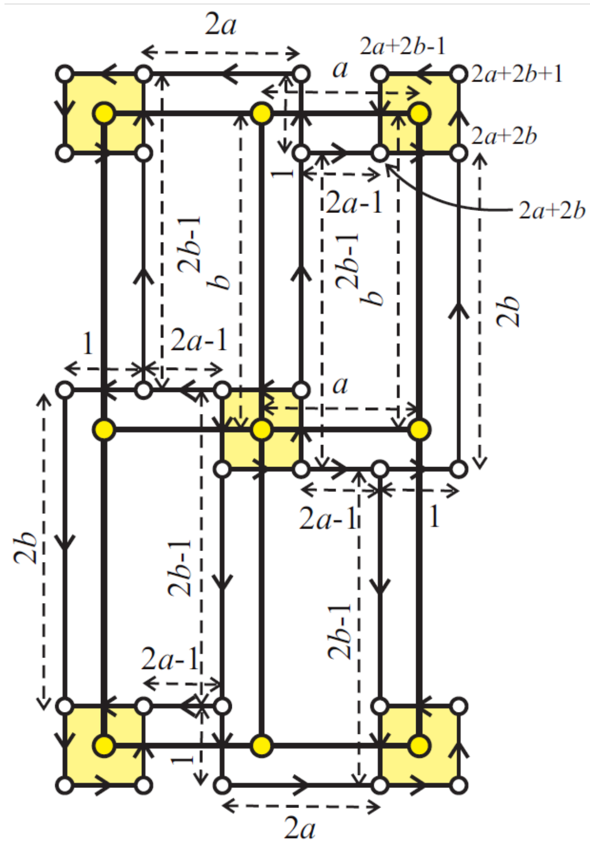


Fig.3.4 Scheme of a double-step graph and a Manhattan digraph.

3.1.2. Result 1:

Let $G(N; \pm a, \pm b)$ be a double-step graph and $M(4N; a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3)$ a Manhattan digraph. Then, a relation between their parameters is the following:

$$a_0 = 1, \qquad a_1 = 1, \qquad a_2 = 1, \qquad a_3 = -3,$$

$$b_0 = 4a - 1, \qquad b_1 = 4b - 1, \qquad b_2 = -4a + 3, \qquad b_3 = -4b - 1.$$

Proof:

We transform a double-step graph $G(N; \pm a, \pm b)$ into a New Amsterdam digraph $NA(2N; \alpha, \beta, \gamma, \delta)$ and, then, we transform the New Amsterdam digraph into a Manhattan digraph $M(4N; a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3)$:

$G(N; \pm a, \pm b)$ → $NA(2N; \alpha, \beta, \gamma, \delta)$ → $M(4N; a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3)$

In order to obtain a New Amsterdam digraph from a double-step graph, we transform every edge into a path of length 2, so that for every edge we create a new vertex and, therefore, the number of vertices is duplicated. With regard to directions, as it can be seen in **Fig.1.18** (left) and **3.5** (left), in a New Amsterdam digraph, even vertices have two horizontal arcs outwards and two vertical arcs inwards, and odd vertices have two horizontal arcs inwards and two vertical arcs outwards.

According to the New Amsterdam digraph definition, steps $\alpha, \beta, \gamma, \delta$ must satisfy the following condition (see Morillo, Fiol, and Fàbrega [14]):

1. $\alpha + \beta + \gamma + \delta \equiv 0 \pmod{2N}$.

Besides, just as we built the New Amsterdam digraph, now it must also satisfy (see **Fig.1.18** (left) and **3.5** (left)):

$\alpha, \beta, \gamma, \delta$ odd numbers,

2. $\alpha + \gamma \equiv 2a \pmod{2N}$,
3. $\beta + \gamma \equiv 2b \pmod{2N}$.

From all the possible solutions, as we have a system with three equations and four unknowns, we give the value $\alpha = 1$. Therefore, the solution is:

$$\alpha = 1, \qquad \beta = 2(b-a)+1, \qquad \gamma = 2a-1, \qquad \delta = -2b-1.$$

Now we transform a New Amsterdam digraph $NA(2N; \alpha, \beta, \gamma, \delta)$ into a Manhattan digraph $M(4N; a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3)$, transforming every arc into a path of length 2 and, therefore, duplicating the number of vertices. In a Manhattan digraph, the directions of the arcs follow the directions of Manhattan streets and avenues. See **Fig.1.18** (right) and **3.5** (right).

According to the Manhattan digraph definition, steps $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ must satisfy the following conditions (see Morillo, Fiol, and Fàbrega [14]):

1. $a_0 + a_1 + a_2 + a_3 \equiv 0 \pmod{4N}$,
2. $b_0 + b_1 + b_2 + b_3 \equiv 0 \pmod{4N}$,
3. $a_0 + a_2 \equiv b_0 + b_2 \pmod{4N}$.

Besides, just as we built the Manhattan digraph, the following conditions must be also satisfied (see again **Fig.1.18** (right) and **3.5** (right)):

$a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$ odd numbers,

4. $a_0 + a_1 = 2\alpha,$
5. $b_1 + b_2 = 2\beta,$
6. $b_3 - a_1 = 2\delta,$
7. $b_0 - a_0 = 2\gamma.$

As we now have a system of seven equations (six of which are independent) and eight unknowns, we give the values $a_0 = 1$ and $a_2 = 1$. Then, we obtain a solution for the steps of a Manhattan digraph in function of the ones of the New Amsterdam digraph:

$$a_0 = 1, \quad a_1 = 2\alpha - 1, \quad a_2 = 1, \quad a_3 = -2\alpha - 1,$$

$$b_0 = 2\gamma + 1, \; b_1 = -2\beta + 2\gamma - 1, \; b_2 = 4\beta - 2\gamma + 1, \; b_3 = 2\alpha + 2\gamma + 1.$$

Taking the previous solution for $\alpha, \beta, \gamma, \delta$ into account, the steps of a Manhattan digraph in function of the ones of a double-step graph are:

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = -3,$$

$$b_0 = 4a - 1, \quad b_1 = 4b - 1, \quad b_2 = -4a + 3, \quad b_3 = -4b - 1.$$

See an example in **Fig.3.6**.

3.1.2. Result 2:

If a double-step graph $G(N; \pm a, \pm b)$ has diameter D_G, then the diameter D_M of a Manhattan digraph $M(4N; a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3)$ satisfies

$$D_M \leq 2D_G + 4.$$

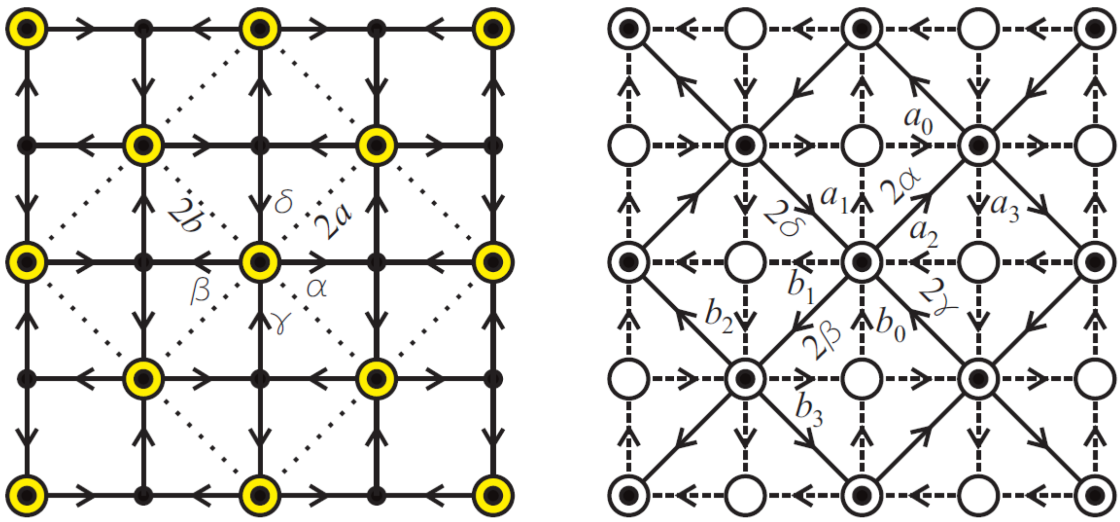


Fig.3.5 Left: A New Amsterdam digraph $NA(2N;\alpha,\beta,\gamma,\delta)$ superimposed on a double-step graph $G(N;\pm a,\pm b)$. Right: A Manhattan digraph $M(4N;a_i,b_i)$, $0\le i\le 3$, superimposed on a New Amsterdam digraph $NA(2N;\alpha,\beta,\gamma,\delta)$. The double-step graph has yellow vertices and the edges are dotted lines, the New Amsterdam digraph has black vertices and continuous arcs, and the Manhattan digraph has white vertices and discontinuous arcs.

Proof:

Let the vertices $i,j\in V(G)$, so that $\mathrm{dist}(i,j)=a+b=D_G$. Then, the maximum distances between the four vertices i_0,i_1,i_2,i_3 of digraph M associated to vertex i of graph G and the ones of the four vertices j_0,j_1,j_2,j_3 of digraph M associated to vertex j of graph G are:

$$\mathrm{dist}(i_0,j_0)=2a+2b,\qquad \mathrm{dist}(i_0,j_1)=2a+2b+1,$$

$$\mathrm{dist}(i_0,j_2)=2a+2b+2,\qquad \mathrm{dist}(i_0,j_3)=2a+2b-1;$$

$$\mathrm{dist}(i_1,j_0)=2a+2b-1,\qquad \mathrm{dist}(i_1,j_1)=2a+2b,$$

$$\mathrm{dist}(i_1,j_2)=2a+2b+1,\qquad \mathrm{dist}(i_1,j_3)=2a+2b-2;$$

$$\mathrm{dist}(i_2,j_0)=2a+2b+2,\qquad \mathrm{dist}(i_2,j_1)=2a+2b+3,$$

$$\mathrm{dist}(i_2,j_2)=2a+2b+4,\qquad \mathrm{dist}(i_2,j_3)=2a+2b+1;$$

$$\mathrm{dist}(i_3,j_0)=2a+2b+1,\qquad \mathrm{dist}(i_3,j_1)=2a+2b+2,$$

$$\mathrm{dist}(i_3,j_2)=2a+2b+3,\qquad \mathrm{dist}(i_3,j_3)=2a+2b.$$

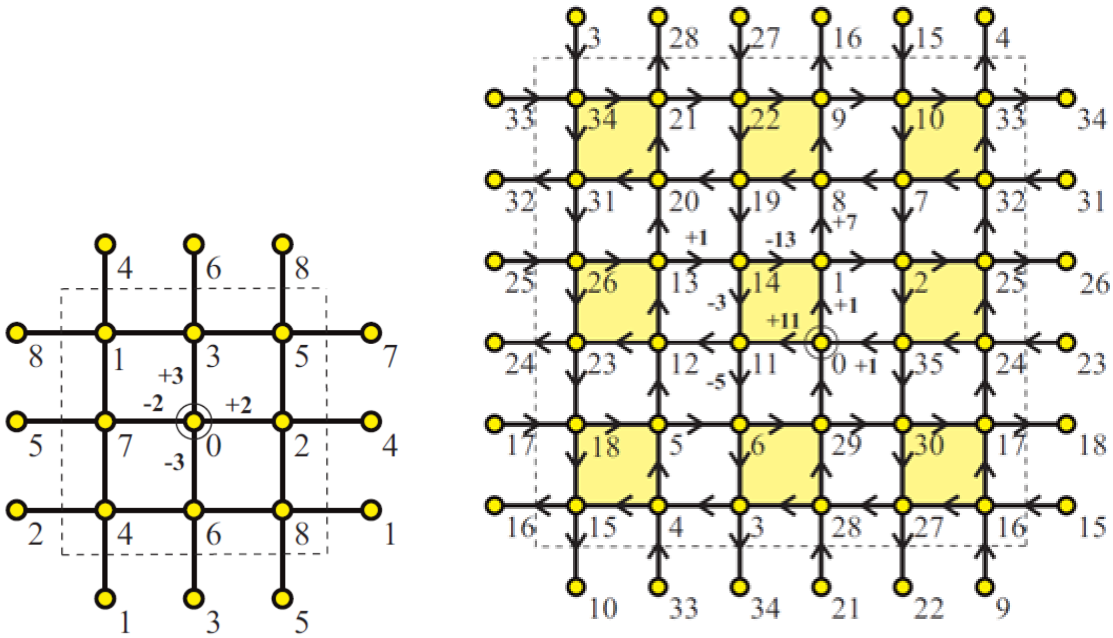


Fig.3.6 Examples of the steps of double-step graph $G(9; \pm 2, \pm 3)$ and the ones of Manhattan digraph $M(36; 1,1,1,-3,7,11,-5,-13)$.

Therefore, $D_M \leq 2a + 2b + 4 = 2D_G + 4$, as claimed. See a scheme in **Fig.3.7**.

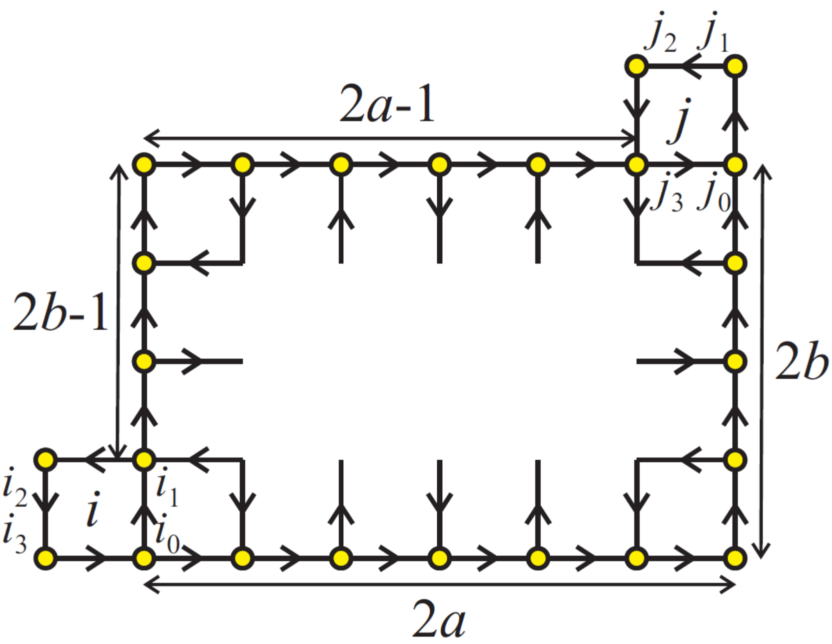


Fig.3.7 Scheme of the proof of Result 2.

CHAPTER 4. CONCLUSIONS

In this section we present the different conclusions obtained from our results.

Given the number of vertices and finding the diameter for each case by means of the program carried out with the free compiler Free Basic, we were able to analyze the evolution of the diameter in function of the number of vertices of the New Amsterdam digraph.

With the program results, we obtain **Fig.4.1** that represents the diameter D_{NA} of New Amsterdam digraphs in function of the number of vertices N_{NA}.

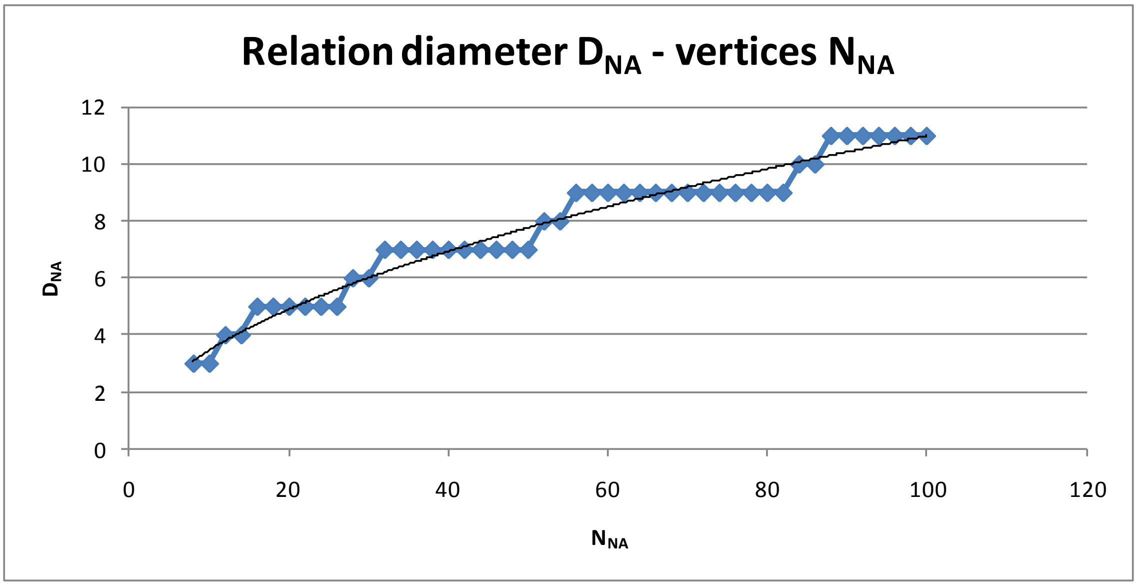


Fig.4.1 The diameter in function of the number of vertices in New Amsterdam digraphs.

As said in Chapter 2, we know that the diameter of a line digraph is the diameter of the original digraph plus one (see Fiol, Yebra, Alegre and Valero [11]). As the Manhattan digraph is the line digraph of the New Amsterdam digraph, then

$$D_M = D_{NA} + 1.$$

Following this equation, we can obtain the diameter of the Manhattan digraph, as we have the diameter of the New Amsterdam that we calculated in our program.

We present **Table 4.1** with the known results, those that we found and the problems that remain open.

Table 4.1. Summary of the results

	(Δ, D)	(Δ, N)	(Δ, D^*)	$(\Delta, N)^*$
Double-step graph	Yebra, Fiol, Morillo, Alegre [16]	Bermond, Iliades, Peyrat [5], Beivide, Herrada, Balcázar, Arruabarrena [4]	-	-
Double-step digraph	Fiol, Yebra, Alegre, Valero [11]	Fiol, Yebra, Alegre, Valero [11], Esqué, Aguiló, Fiol [9], Aguiló, Fiol [1]	TFC Gomis [12]	TFC Gomis [12]
New Amsterdam digraph	Morillo, Fiol, Fàbrega [14]	This work	Open problem	Open problem
Manhattan digraph	Morillo, Fiol, Fàbrega [14]	This work	Open problem	Open problem

The (Δ, D^*) and $(\Delta, N)^*$ problems are similar to the (Δ, D) and (Δ, N) problems, but considering the unilateral diameter D^* instead of the diameter D. The *unilateral diameter* is the minimum one between the diameter of the digraph and the diameter of its *converse* digraph, obtained changing the directions of all the arcs. We observe that in the case of graphs, the diameter and the unilateral diameter are equal and the (Δ, D^*) and $(\Delta, N)^*$ problems coincide, respectively, with the (Δ, D) and (Δ, N) problems.

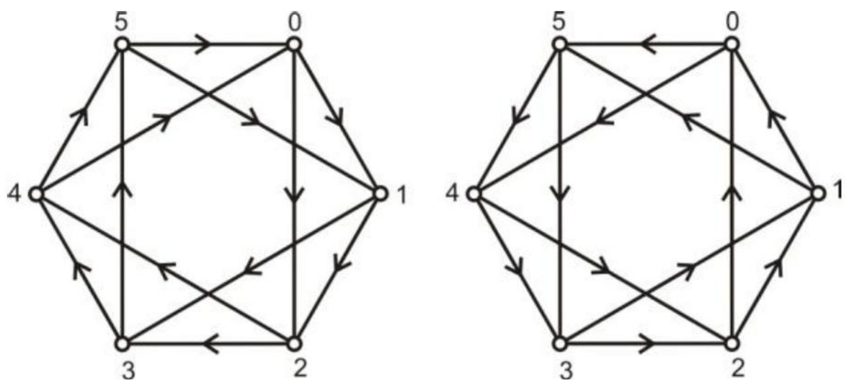


Fig.4.2 Example of a double-step digraph and its converse (Gomis [12]).

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