MASTER'S DEGREE THESIS

# Master of Science in Advanced Mathematics and Mathematical Engineering

**Title:** Some Applications of Linear Algebra in Spectral Graph Theory

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Master Thesis

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# Preface

The application of the theory of matrices and eigenvalues to combinatorics is certainly not new. In the present work the starting point is a theorem that concerns the eigenvalues of partitioned matrices. Interlacing yields information on subgraphs of a graph, and the way such subgraphs are embedded. In particular, one gets bounds on extremal substructures. Applications of this theorem and of some known matrix theorems to matrices associated to graphs lead to new results. For instance, some characterizations of regular partitions, and bounds for some parameters, such as the independence and chromatic numbers, the diameter, the bandwidth, etc. This master thesis is a contribution to the area of algebraic graph theory and the study of some generalizations of regularity in bipartite graphs.

In Chapter 1 we recall some basic concepts and results from graph theory and linear algebra.

Chapter 2 presents some simple but relevant results on graph spectra concerning eigenvalue interlacing. Most of the previous results that we use were obtained by Haemers in [**33**]. In that work, the author gives bounds for the size of a maximal (co)clique, the chromatic number, the diameter and the bandwidth in terms of the eigenvalues of the standard adjacency matrix or the Laplacian matrix. He also finds some inequalities and regularity results concerning the structure of graphs.

The work initiated by Fiol [**26**] in this area leads us to Chapter 3. The discussion goes along the same spirit, but in this case eigenvalue interlacing is used for proving results about some weight parameters and weight-regular partitions of a graph. In this master thesis a new observation leads to a greatly simplified notation of the results related with weight-partitions. We find an upper bound for the weight independence number in terms of the minimum degree.

Special attention is given to regular bipartite graphs, in fact, in Chapter 4 we contribute with an algebraic characterization of regularity properties in bipartite graphs. Our first approach to regularity in bipartite graphs comes from the study of its spectrum. We characterize these graphs using eigenvalue interlacing and we provide an improved bound for biregular graphs inspired in Guo's inequality. We prove a condition for existence of a $k$-dominating set in terms of its Laplacian eigenvalues. In particular, we give an upper bound on the sum of the first Laplacian eigenvalues of a $k$-dominating set and generalize a Guo's result for these structures. In terms of predistance polynomials, we give a result that can be seen as the biregular counterpart of Hoffman's Theorem. Finally, we also provide new characterizations of bipartite graphs inspired in the notion of distance-regularity.

In Chapter 5 we describe some ideas to work with a result from linear algebra known as the Rayleigh's principle. We observe that the clue is to make the "right choice" of the eigenvector that is used in Rayleigh's principle. We can use this method to give a spectral characterization of regular and biregular partitions. Applying this technique, we also derive an alternative proof for the upper bound of the independence number obtained by Hoffman (Chapter 2, Theorem 1.2).

Finally, in Chapter 6 other related new results and some open problems are presented.

# Abstract

**Keywords:** Graph, adjacency matrix, Laplacian matrix, spectrum, bipartite graph, distance-regular graph, eigenvalue interlacing.

This master thesis is a contribution to the study of regularity properties in bipartite graphs. The main results are the characterization of biregular graphs in terms of eigenvalues, $k$-dominating sets, distance-regular graphs and polynomials.

Regarding the study of the graph partitioning problem, we focus on three particular families of structures: regular and biregular partitions, partitions induced by a largest size of the coclique (the independence number) and graph partitions into three sets with an induced bipartite subgraph.

# Contents

# Chapter 1
# Introduction

In this introductory chapter we first recall some basic concepts concerning graphs and algebraic graph theory. We also introduce the notion of graph partition, eigenvalue interlacing and distance-regular graph.

## 1. Graphs

A *graph* $\Gamma$ consists of a *vertex set* $V = V(\Gamma)$ and an *edge set* $E = E(\Gamma)$, where an edge is an unordered pair of distinct vertices of $\Gamma$. The vertex set of a graph $\Gamma$ is referred to as $V = \{u, v, w, \ldots\}$ and its edge set as $E = \{uv, wz, \ldots\}$. Adjacency between vertices $u$ and $v$ is denoted by $u \sim v$.

The *order* of a graph $\Gamma$ is its number of vertices, $|V| = n$. The number of edges of a graph $\Gamma$ is its *size*, denoted by $|E| = m$.

A vertex $u$ is *incident* with an edge $e$ if $u \in e$; then $e$ is an edge at $u$. Two vertices $u, v$ of $\Gamma$ are *adjacent*, or *neighbours*, if $e = uv = \{u, v\}$ is an edge of $\Gamma$. Two edges $e \neq f$ are *incident* if they have one end in common. If all the vertices of $\Gamma$ are pairwise adjacent, then $\Gamma$ is *complete*. A complete graph on $n$ vertices is denoted by $K_n$.

Let $\Gamma = (V, E)$ and $\Gamma' = (V', E')$ be two graphs. We say that $\Gamma$ and $\Gamma'$ are isomorphic, and we write $\Gamma \simeq \Gamma'$, if there exists a bijection $\varphi : V \to V'$ such that $uv \in E \Leftrightarrow \varphi(u)\varphi(v) \in E'$ for all $u, v \in V$. Such a map $\varphi$ is called an *isomorphism*.

If $V' \subseteq V$ and $E' \subseteq E$, then $\Gamma' = (V', E')$ is a *subgraph* of $\Gamma$, written as $\Gamma' \subseteq \Gamma$. If $\Gamma' \subseteq \Gamma$ and $\Gamma'$ contains all edges $uv \in E$ with $u, v \in V'$, then $\Gamma'$ is an *induced subgraph* of $\Gamma$.

Note that for the set $B\backslash\{u\}$ we write $B - u$. Likewise, $\Gamma - u$ denotes the subgraph of $\Gamma = (V, E)$ induced by the vertices $V - u$ and $\Gamma - e$ is obtained from $\Gamma$ by removing the edge $e \in E$.

The *line graph* $L(\Gamma)$ of $\Gamma$ is the graph on $E$ in which $e, f \in E$ are adjacent as vertices if and only if they are adjacent as edges in $\Gamma$.

The set of neighbours of a vertex $u$ in $\Gamma$ is denoted by $\Gamma(u)$, and it is the set of all vertices adjacent to $u$.

The *degree* (or *valency*) $\delta_u$ of a vertex $u$ is the number of edges adjacent to $u$, i.e. the number of neighbours of $u$. If all the vertices of $\Gamma$ have the same degree $k$, then $\Gamma$ is $k$*-regular*. The number

$$\overline{\delta} := \frac{1}{|V|} \sum_{u \in V} \delta_u$$

is the *average degree* of $\Gamma$.

If we sum up all the vertex degrees in $\Gamma$, we count every edge exactly twice: once from each of its ends. Thus,

$$|E| = \frac{1}{2} \sum_{u \in V} \delta_u = \frac{1}{2} \overline{\delta} |V|.$$

LEMMA 1.1. *The number of vertices of odd degree in a graph is always even.*

A *strongly regular graph* is a regular graph where every pair of adjacent vertices has the same number of neighbors in common and the same holds for every pair of non-adjacent vertices.

An $u$-$v$ *walk* (of length $k$) in a graph $\Gamma$ is a sequence $u_0 u_1 \ldots u_{k-1} u_k$ of vertices such that $u_0 = u$, $u_k = v$ and $u_{i-1} u_i \in E$ for all $i = 1, \ldots, k$, i.e. each pair of consecutive vertices are adjacent. If $u = v$, the walk is *closed*. If the vertices in a walk are all distinct, it defines a *path* in $\Gamma$.

A non-empty graph $\Gamma$ is called *connected* if any two of its vertices are linked by a path in $\Gamma$.

The *distance* $\partial(u, v)$ in $\Gamma$ of two vertices $u, v$ is the length of a shortest $u$-$v$ path in $\Gamma$. The maximum distance between any two vertices in $\Gamma$ is the *diameter* of $\Gamma$, denoted by $D = D(\Gamma)$. The *eccentricity* of a vertex $u$ is defined as $\mathcal{E} = \mathrm{ecc}(u) = \max_{v \in V} \partial(u, v)$. Let $\Gamma_k(u)$ be the set of vertices at distance $k$ from $u$, for $0 \le k \le \mathrm{ecc}(u)$, and let $\Gamma_k$ be the *distance-$k$ graph* with the same vertex set as $\Gamma$ and where two vertices are adjacent whenever they are at distance $k$ in $\Gamma$.

A graph $\Gamma$ with diameter $D$ is *distance-regular* whenever, for any two vertices $u, v \in V$ at distance $\partial(u, v) = k$, $0 \le k \le D$, the *intersection numbers* $c_k := |\Gamma(v) \cap \Gamma_{k-1}(u)|$, $a_k := |\Gamma(v) \cap \Gamma_k(u)|$ and $b_k := |\Gamma(v) \cap \Gamma_{k+1}(u)|$ do not depend on the chosen vertices $u$ and $v$ but only on their distance $k$.

Let $r \geq 2$ be an integer. A graph $\Gamma = (V, E)$ is called *r-partite* if $V$ admits a partition into $r$ classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. In particular, a graph $\Gamma$ is called *bipartite* when its vertex set can be partitioned into two disjoint parts $V_1$, $V_2$ such that all edges of $\Gamma$ meet both $V_1$ and $V_2$.

The greatest integer $r$ such that $K_r \subseteq \Gamma$ is the *clique number* $\omega(\Gamma)$ of $\Gamma$, and the *largest coclique* is the size of the independent set of vertices of $\Gamma$, which is denoted by $\alpha(\Gamma)$. Clearly, $\alpha(\Gamma) = \omega(\overline{\Gamma})$.

A colouring of a graph $\Gamma$ is a partition of its vertices into cocliques (colour classes). The smallest number of colors needed to color a graph $\Gamma$ is called its *chromatic number*, denoted by $\chi(\Gamma)$.

Denote by $\Gamma^l$ the product of $l$ copies of $\Gamma$ (that means, a graph with vertex set $\{1, \ldots, n\}^l$), where two vertices are adjacent if all of the coordinates places corresponds to adjacent or coinciding vertices of $\Gamma$. The number

$$\theta(\Gamma) = \sup_l \sqrt[l]{\alpha(\Gamma^l)} = \lim_{l \to \infty} \sqrt[l]{\alpha(\Gamma^l)}$$

is called the *Shannon capacity* of $\Gamma$.

Here $\alpha(\Gamma^l)$ denotes the independence number of $\Gamma^l$. Note that, since $\alpha(\Gamma^l) \geq \alpha^l$, the Shannon capacity always satisfies the bound $\theta \geq \alpha$.

A *cut* is a partition of the vertices of a graph into two disjoint subsets. The *cut-set* of the cut is the set of edges whose ends are in different subsets of the partition.

## 2. Algebraic graph theory

We will consider finite, simple, loopless graphs. Unless stated otherwise all graphs are undirected.

### 2.1. Preliminaries.

We let $\boldsymbol{I}$ denote the identity matrix, $\boldsymbol{J}$ the all-one matrix, $\boldsymbol{O}$ the all-zero matrix, $\boldsymbol{j}$ the all-one vector and $\mathbf{0}$ the all-zero vector.

One basic result from linear algebra is Rayleigh's principle, which can be stated as follows.

THEOREM 2.1 (Rayleigh's principle). *Let $\boldsymbol{A}$ be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and with the orthonormal set of eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$, such that $\boldsymbol{u}_i$ is a $\lambda_i$-eigenvector. Then*

(i) $\dfrac{\boldsymbol{u}^\top \boldsymbol{A} \boldsymbol{u}}{\boldsymbol{u}^\top \boldsymbol{u}} \geq \lambda_i$ *if* $\boldsymbol{u} \in \langle \boldsymbol{u}_1, \ldots, \boldsymbol{u}_i \rangle$.

(*ii*) $\frac{\boldsymbol{u}^\top \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^\top \boldsymbol{u}} \leq \lambda_i$ *if* $\boldsymbol{u} \in \langle \boldsymbol{u}_i, \dots, \boldsymbol{u}_n \rangle$.

*In both cases equality implies that* $\boldsymbol{u}$ *is a* $\lambda_i$*-eigenvector of* $\boldsymbol{A}$*.*

***Proof.*** To prove (*i*), write $\boldsymbol{u}$ as $\boldsymbol{u} = \sum_{k=1}^{i} \alpha_k \boldsymbol{u}_k$. Then, using that $||\boldsymbol{u}_k|| = 1$,

$$
\begin{aligned}
\boldsymbol{u}^\top \boldsymbol{A}\boldsymbol{u} &= \langle \boldsymbol{u}, \boldsymbol{A}\boldsymbol{u} \rangle \\
&= \left\langle \sum_{k=1}^{i} \alpha_k \boldsymbol{u}_k, \sum_{k=1}^{i} \alpha_k \lambda_k \boldsymbol{u}_k \right\rangle \\
&= \sum_{k=1}^{i} |\alpha_k|^2 \lambda_k \\
&\geq \lambda_i \sum_{k=1}^{i} |\alpha_k|^2 \\
&= \lambda_i \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \lambda_i \boldsymbol{u}^\top \boldsymbol{u},
\end{aligned}
$$

which gives $\lambda_i \leq \frac{\boldsymbol{u}^\top \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^\top \boldsymbol{u}}$.

To prove (*ii*), we write $\boldsymbol{u}$ as $\boldsymbol{u} = \sum_{k=i}^{n} \alpha_k \boldsymbol{u}_k$. Then, using that $||\boldsymbol{u}_k|| = 1$,

$$
\begin{aligned}
\boldsymbol{u}^\top \boldsymbol{A}\boldsymbol{u} &= \langle \boldsymbol{u}, \boldsymbol{A}\boldsymbol{u} \rangle \\
&= \left\langle \sum_{k=i}^{n} \alpha_k \boldsymbol{u}_k, \sum_{k=i}^{n} \alpha_k \lambda_k \boldsymbol{u}_k \right\rangle \\
&= \sum_{k=i}^{n} |\alpha_k|^2 \lambda_k \\
&\leq \lambda_i \sum_{k=i}^{n} |\alpha_k|^2 \\
&= \lambda_i \langle \boldsymbol{u}, \boldsymbol{u} \rangle = \lambda_i \boldsymbol{u}^\top \boldsymbol{u}
\end{aligned}
$$

that gives $\lambda_i \geq \frac{\boldsymbol{u}^\top \boldsymbol{A}\boldsymbol{u}}{\boldsymbol{u}^\top \boldsymbol{u}}$.

Suppose that equality holds in both cases. It implies that $\alpha_k = 0$ for all $k$ such that $\lambda_k \neq \lambda_i$. It follows that $\boldsymbol{u} \in \mathrm{Ker}(\lambda_i \boldsymbol{I} - \boldsymbol{A})$, which proves the required result. □

The basic information about the largest eigenvalue of a (possibly directed) graph is provided by Perron-Frobenius theory.

THEOREM 2.2 (Perron-Frobenius). *Let* $\boldsymbol{A}$ *be a non-negative irreducible symmetric* $n \times n$ *matrix. Then the largest eigenvalue* $\lambda_1$ *has multiplicity* 1 *and has an eigenvector whose entries are all positive. For all other eigenvalues we have* $|\lambda_i| \leq \lambda_1$*.*

***Proof.*** Suppose $\boldsymbol{x}$ is an eigenvector of $\boldsymbol{A}$ for the eigenvalue $\lambda_1$. Let $\boldsymbol{A} = (a_{uv})$ and $\boldsymbol{y} = |\boldsymbol{x}|$ (entry-wise). Then

$$\frac{\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}}{\boldsymbol{y}^\top \boldsymbol{y}} = \frac{\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}}{\boldsymbol{x}^\top \boldsymbol{x}} \geq \frac{\boldsymbol{x}^\top \boldsymbol{A}\boldsymbol{x}}{\boldsymbol{x}^\top \boldsymbol{x}} = \lambda_1.$$

Rayleigh's principle gives $\frac{\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}}{\boldsymbol{y}^\top \boldsymbol{y}} \leq \lambda_1$, so $\frac{\boldsymbol{y}^\top \boldsymbol{A}\boldsymbol{y}}{\boldsymbol{y}^\top \boldsymbol{y}} = \lambda_1$. Hence $\boldsymbol{y}$ must be a non-negative eigenvector for the eigenvalue $\lambda_1$. Suppose $y_u = 0$ for some $u$. Then,

$$0 = \lambda_1 y_u = (\boldsymbol{A}\boldsymbol{y})_u = \sum_{v \in V} a_{uv} y_v = \sum_{v \in \Gamma(u)} a_{uv} y_v.$$

It follows that $y_v = 0$ for all $v$ such that $v \sim u$. Repeating this step over and over for such $y_v$'s and using irreducibility, we get that $\boldsymbol{y} = \boldsymbol{0}$, which is a contradiction. Thus, all entries of $\boldsymbol{y}$ are strictly positive, which also implies that any eigenvector $\boldsymbol{x}$ for the eigenvalue $\lambda_1$ cannot have zero entries.

Suppose there are two linearly independent eigenvectors for the eigenvalue $\lambda_1$. Then there is a linear combination $\boldsymbol{z}$ of these eigenvectors, that is also an eigenvector, such that $z_u = 0$ for some $u$, which gives a contradiction. So, $\lambda_1$ must have multiplicity 1. □

### 2.2. Matrices associated with graphs.

Let $\Gamma$ be a (finite, undirected, simple) graph with vertex set $V(\Gamma) = V = \{u, v, \ldots\}$. The *adjacency matrix* of $\Gamma$ is defined as the $n \times n$ matrix $\boldsymbol{A} := \boldsymbol{A}(\Gamma) = (a_{uv})$ in which

$$a_{uv} = \begin{cases} 1 & \text{if } u \sim v, \\ 0 & \text{otherwise.} \end{cases}$$

It follows directly from the definition that $\boldsymbol{A}$ is real a symmetric matrix, so all eigenvalues are real, and that the trace of $\boldsymbol{A}$ is zero,

$$\operatorname{tr} \boldsymbol{A} = \sum_{u \in V} a_{uu} = \sum_{i=1}^{n} \lambda_i = 0.$$

Since the rows and columns of $\boldsymbol{A}$ correspond to an arbitrary labeling of the vertices of $\Gamma$, we are mainly interested in those properties of the adjacency matrix which are invariant under permutations of the rows and columns.

We can extend the definition of the adjacency matrix to the case when $\Gamma$ has multiple edges: we just let $a_{uv}$ be the number of edges connecting $u$ and $v$. We can also have weights on the edges, in which case we let $a_{uv}$ be the weight of the edge $uv$.

The adjacency matrix of a graph gives information about walks of length 1. The powers of the adjacency matrix have a similar property:

LEMMA 2.3. *Let $\boldsymbol{A}$ be the adjacency matrix of the graph $\Gamma$. Then the $(u, v)$-entry of $\boldsymbol{A}^r$ equals the number of walks of length $r$ from vertex $u$ to vertex $v$.*

The *Laplacian* of the graph is defined as the $n \times n$ matrix $\boldsymbol{L}(\Gamma) = (l_{uv})$ in which

$$l_{uv} = \begin{cases} \delta_u & \text{if } u = v, \\ -a_{uv} & \text{if } u \neq v. \end{cases}$$

where $\delta_u$ denotes the degree of the vertex $u$.

The Laplacian matrix of $\Gamma$ is $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$, where $\boldsymbol{D}$ is the diagonal matrix of the degrees of $\Gamma$, so that $\boldsymbol{L}$ has zero row and column sums.

Since $\boldsymbol{A}$ and $\boldsymbol{L}$ are symmetric, their eigenvalues are real. Since $\boldsymbol{L}$ is positive semi-definite, it follows that the Laplacian eigenvalues are nonnegative. Besides, as $\boldsymbol{L}$ has zero row sums, 0 is a Laplacian eigenvalue. In fact the multiplicity of 0 as eigenvalue of $\boldsymbol{L}$ equals the number of connected components of $\Gamma$.

An useful matrix for studying non-regular graphs is the *normalized Laplacian* $\boldsymbol{\mathcal{L}}$, since it uses the degree of each node. The normalized Laplacian of $\Gamma$ is $\boldsymbol{\mathcal{L}}(\Gamma) = (\ell_{uv})$ with entries

$$\ell_{uv} = \begin{cases} 1 & \text{if } u = v \text{ and } \delta_u \neq 0, \\ -\frac{1}{\sqrt{\delta_u \delta_v}} & \text{if } uv \in E(\Gamma), \\ 0 & \text{otherwise}. \end{cases}$$

We can write $\boldsymbol{\mathcal{L}}(\Gamma) = \boldsymbol{T}(\Gamma)\boldsymbol{L}(\Gamma)\boldsymbol{T}(\Gamma)$, where $\boldsymbol{T}(\Gamma) = \operatorname{diag}(t_u, t_v, \ldots)$ such that $t_u = \frac{1}{\sqrt{\delta_u}}$ if $\delta_u \neq 0$ (and $t_v$ can be arbitrary if $\delta_v = 0$).

The three matrices $\boldsymbol{A}(\Gamma)$, $\boldsymbol{L}(\Gamma)$ and $\boldsymbol{\mathcal{L}}(\Gamma)$ are all real and symmetric.

The following theorem from matrix theory plays a key role in some proofs. We denote the eigenvalues of a symmetric matrix $\boldsymbol{M}$ by $\lambda_1(\boldsymbol{M}) \geq \lambda_1(\boldsymbol{M}) \geq \cdots \geq \lambda_n(\boldsymbol{M})$.

THEOREM 2.4 ([**35**]). *Let $\boldsymbol{A}$ and $\boldsymbol{B}$ be two real symmetric matrices of size $n$. Then for any $1 \leq k \leq n$,*

$$\sum_{i=1}^{k} \lambda_i(\boldsymbol{A} + \boldsymbol{B}) \leq \sum_{i=1}^{k} \lambda_i(\boldsymbol{A}) + \sum_{i=1}^{k} \lambda_i(\boldsymbol{B}).$$

**2.3. The spectrum of a graph.**

The eigenvalues of a graph $\Gamma$ are the roots of the *characteristic polynomial* of its adjacency matrix:

$$\phi_\Gamma(x) = \phi(\Gamma, x) = \det(\boldsymbol{A}(\Gamma) - x\boldsymbol{I}).$$

The *spectrum* of $\Gamma$ is the set of eigenvalues of $\boldsymbol{A}$ together with their multiplicities, and we write

$$\operatorname{sp}\Gamma = \operatorname{sp}\boldsymbol{A} = \{\theta_0^{m_0}, \theta_1^{m_1}, \ldots, \theta_d^{m_d}\} \tag{1}$$

where the eigenvalues are in decreasing order, $\theta_0 > \theta_1 > \cdots > \theta_d$, and the superscripts stand for their multiplicities $m_i = m(\theta_i)$. In particular, when $\Gamma$ is $\delta$-regular, the largest eigenvalue is $\theta_0 = \delta$ and has multiplicity $m_0 = 1$ (as $\Gamma$ is connected). Note that $\operatorname{tr}\boldsymbol{A}^k = \sum_{i=0}^{d} m_i\theta_i^k$, and, in particular, for $k = 0$ we have $\operatorname{tr}\boldsymbol{I} = n = \sum_{i=0}^{d} m_i$.

The eigenvalues of $\Gamma$ can also be denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ (including multiplicities). Note that $\theta_0 = \lambda_1$ and $\theta_d = \lambda_n$. Since $\Gamma$ is connected (it means that $\boldsymbol{A}$ is irreducible), by Theorem 2.2, we can assure that $\lambda_1$ is simple, positive and with positive eigenvector. If $\Gamma$ is non-connected, the existence of such an eigenvector is not guaranteed, unless all its connected components have the same maximum eigenvalue. Throughout this work, the positive eigenvector associated with the largest (positive and with multiplicity one) eigenvalue $\lambda_1$ is denoted by $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n)^\top$ (that is, $\boldsymbol{\nu}$ has elements indexed by the vertices of $\Gamma$). This eigenvector is normalized in such a way that its minimum entry (in each connected component of $\Gamma$) is 1. For instance, if $\Gamma$ is regular, we have $\boldsymbol{\nu} = \boldsymbol{j}$, the all-1 vector. The maximum modulus of all eigenvalues of $\Gamma$ is called the *spectral radius*, $\rho(\Gamma) = \lambda_1$.

Let $\boldsymbol{E}_i$ be the idempotent matrix representing the orthogonal projections onto the eigenspace $\mathcal{E}_i$ corresponding to $\theta_i$, $i = 0, \ldots, d$. Since $\boldsymbol{A}(\Gamma)$ is symmetric and, therefore, diagonalizable, we see that the multiplicity of a root of $\phi_\Gamma(x)$ equals the dimension of the corresponding eigenspace. For any graph with eigenvalue $\theta_i$ having multiplicity $m_i$, its corresponding idempotent can be computed as $\boldsymbol{E}_i = \boldsymbol{U}_i\boldsymbol{U}_i^\top$, where $\boldsymbol{U}_i$ is the $m_i \times n$ matrix whose columns form an orthonormal basis of $\mathcal{E}_i$. For instance, when $\Gamma$ is $\delta$-regular and has $n$ vertices, its largest eigenvalue $\theta_0 = \delta$ has eigenvector $\boldsymbol{j}$, the all-1 (column) vector, and corresponding idempotent $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{j}\boldsymbol{j}^\top = \frac{1}{n}\boldsymbol{J}$, where $\boldsymbol{J}$ is the all-1 matrix. Alternatively, we can also compute the idempotents as $\boldsymbol{E}_i = L_i(\boldsymbol{A})$ where $L_i$ is the Lagrange interpolating polynomial of degree $d$ satisfying $L_i(\theta_i) = 1$ and $L_i(\theta_j) = 0$ for $j \neq i$. That is,

$$L_i(x) = \frac{1}{\phi_i}\prod_{\substack{j=0\\ j\neq i}}^{d}(x - \theta_j) = \frac{(-1)^i}{\pi_i}\prod_{\substack{j=0\\ j\neq i}}^{d}(x - \theta_j)$$

where $\phi_i = \prod_{j=0,j\neq i}^{d}(\theta_i - \theta_j)$ and $\pi_i = |\phi_i|$. Then, the idempotents of $\boldsymbol{A}$ satisfy the known properties: $\boldsymbol{E}_i^2 = \boldsymbol{E}_i$, $\boldsymbol{E}_i\boldsymbol{E}_j = \boldsymbol{O}$ for $j \neq i$; $\boldsymbol{A}\boldsymbol{E}_i = \theta_i\boldsymbol{E}_i$; and $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\theta_i)\boldsymbol{E}_i$, for any polynomial $p \in \mathbb{R}[x]$ (see, for example, Godsil [**30**, p. 28]). In particular, taking $p = 1$ we obtain $\sum_{i=0}^{d} \boldsymbol{E}_i = \boldsymbol{I}$ (as expected), and for $p = x$ we have the *spectral decomposition theorem*

$$\boldsymbol{A} = \sum_{i=0}^{d} \theta_i \boldsymbol{E}_i.$$

The entries of the idempotents $m_{uv}(\theta_i) = (\boldsymbol{E}_i)_{uv}$ are called *crossed uv-local multiplicities* and, by taking $p = x^\ell$, $\ell \geq 0$, they allow us to compute the number of $\ell$-walks between any two vertices ([**18**], [**27**]):

$$a_{uv}^{(\ell)} = (\boldsymbol{A}^\ell)_{uv} = \sum_{i=0}^{d} m_{uv}(\theta_i)\theta_i^\ell, \tag{2}$$

In particular, when $u = v$, $m_u(\theta_i) = m_{uu}(\theta_i)$ are the so-called *local multiplicities* of vertex $u$, satisfying $\sum_{i=0}^{d} m_u(\theta_i) = 1$, $u \in V$, and $\sum_{u\in V} m_u(\theta_i) = m_i$, $i = 0, 1, \ldots, d$ [**24**].

## 3. Eigenvalue interlacing

Our starting point is the following theorem, proved by Haemers in [**32, 33**]. This author alludes to the first part of the theorem as a classical result, referring the reader to the book by Courant and Hilbert [**15**].

THEOREM 3.1 (Interlacing). *Let $\boldsymbol{A}$ be a symmetric $n \times n$ matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and respective eigenvectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$. For some integer $m < n$, let $\boldsymbol{S}$ be a real $n \times m$ matrix such that $\boldsymbol{S}^\top\boldsymbol{S} = \boldsymbol{I}$ (its columns are orthonormal), and consider the $m \times m$ matrix $\boldsymbol{B} = \boldsymbol{S}^\top\boldsymbol{A}\boldsymbol{S}$, with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ and respective eigenvectors $\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots, \boldsymbol{v}_m$. Then the following statements hold.*

(*i*) *The eigenvalues of $\boldsymbol{B}$ interlace the eigenvalues of $\boldsymbol{A}$, that is:*

$\lambda_i \geq \mu_i \geq \lambda_{n-m+i}, \quad 1 \leq i \leq m.$

(*ii*) *If $\mu_i = \lambda_i$ or $\mu_i = \lambda_{n-m+i}$ for some $i \in [1, m]$, then $\boldsymbol{B}$ has a $\mu_i$-eigenvector $\boldsymbol{v}$ such that $\boldsymbol{S}\boldsymbol{v}$ is a $\mu_i$-eigenvector of $\boldsymbol{A}$.*

(*iii*) *If for some integer $l$, $\mu_i = \lambda_i$ for $i = 1, \ldots, l$ (or $\mu_i = \lambda_{n-m+i}$ for $i = l, \ldots, m$), then $\boldsymbol{S}\boldsymbol{v_i}$ is a $\mu_i$-eigenvector of $\boldsymbol{A}$ for $i = 1, \ldots, l$ (respectively $i = l, \ldots, m$).*

(*iv*) *If the interlacing is tight, that is, for some $0 \leq k \leq m$, $\lambda_i = \mu_i$ $(1 \leq i \leq k)$ and $\mu_i = \lambda_{n-m+i}$ $(k + 1 \leq i \leq m)$, then $\boldsymbol{S}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{S}$.*

***Proof.*** Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_n$ be an orthonormal set of eigenvectors of $\boldsymbol{A}$. For any $i$, $1 \leq i \leq m$, take a non-zero vector $\boldsymbol{s}_i$ such that

$$\boldsymbol{s}_i \in \langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_i \rangle$$
$$\boldsymbol{S}\boldsymbol{s}_i \in \langle \boldsymbol{u}_i, \dots, \boldsymbol{u}_n \rangle.$$

Then we have to prove the existence of such a $\boldsymbol{s}_i$:

$$\begin{aligned}
&\boldsymbol{S}\boldsymbol{s}_i \in \langle \boldsymbol{u}_1, \dots, \boldsymbol{u}_{i-1} \rangle^{\perp};\\
&\langle \boldsymbol{S}\boldsymbol{s}_i, \boldsymbol{u}_k \rangle = 0 \quad \text{for} \quad k = 1, \dots, i-1;\\
&(\boldsymbol{S}\boldsymbol{s}_i)^{\top}\boldsymbol{u}_k = \boldsymbol{s}_i^{\top}(\boldsymbol{S}^{\top}\boldsymbol{u}_k) = 0 \quad \text{for} \quad k = 1, \dots, i-1;\\
&\langle \boldsymbol{s}_i, \boldsymbol{S}^{\top}\boldsymbol{u}_k \rangle = 0 \quad \text{for} \quad k = 1, \dots, i-1;\\
&\boldsymbol{s}_i \in \langle \boldsymbol{S}^{\top}\boldsymbol{u}_1, \dots, \boldsymbol{S}^{\top}\boldsymbol{u}_{i-1} \rangle^{\perp}.
\end{aligned}$$

Finally, we have to prove that

$$\boldsymbol{s}_i \in \langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_i \rangle \cap \langle \boldsymbol{S}^{\top}\boldsymbol{u}_1, \dots, \boldsymbol{S}^{\top}\boldsymbol{u}_{i-1} \rangle^{\perp}, \tag{3}$$

and since $\langle \boldsymbol{v}_1, \dots, \boldsymbol{v}_i \rangle$ has dimension $i$ and $\langle \boldsymbol{S}^{\top}\boldsymbol{u}_1, \dots, \boldsymbol{S}^{\top}\boldsymbol{u}_{i-1} \rangle^{\perp}$ has dimension greater or equal than $m-i$, there is at least one non-zero vector in the intersection.

Since we have shown that the vector $\boldsymbol{s}_i$ exists, Rayleigh's principle yields

$$\lambda_i \geq \frac{(\boldsymbol{S}\boldsymbol{s}_i)^{\top}\boldsymbol{A}(\boldsymbol{S}\boldsymbol{s}_i)}{(\boldsymbol{S}\boldsymbol{s}_i)^{\top}(\boldsymbol{S}\boldsymbol{s}_i)} = \frac{\boldsymbol{s}_i^{\top}\boldsymbol{B}\boldsymbol{s}_i}{\boldsymbol{s}_i^{\top}\boldsymbol{s}_i} \geq \mu_i.$$

Similarly, proving the existence of a non-zero vector $\boldsymbol{s}_i$, such that

$$\boldsymbol{s}_i \in \langle \boldsymbol{v}_i, \dots, \boldsymbol{v}_m \rangle$$
$$\boldsymbol{S}\boldsymbol{s}_i \in \langle \boldsymbol{u}_1, \dots, \boldsymbol{u}_{n-m+i} \rangle$$

we get

$$\lambda_{n-m+i} \leq \frac{(\boldsymbol{S}\boldsymbol{s}_i)^{\top}\boldsymbol{A}(\boldsymbol{S}\boldsymbol{s}_i)}{(\boldsymbol{S}\boldsymbol{s}_i)^{\top}(\boldsymbol{S}\boldsymbol{s}_i)} = \frac{\boldsymbol{s}_i^{\top}\boldsymbol{B}\boldsymbol{s}_i}{\boldsymbol{s}_i^{\top}\boldsymbol{s}_i} \leq \mu_i,$$

which completes the proof of $(i)$.

If $\lambda_i = \mu_i$, then $\boldsymbol{s}_i$ and $\boldsymbol{S}\boldsymbol{s}_i$ are $\lambda_i$-eigenvectors of $\boldsymbol{B}$ and $\boldsymbol{A}$ respectively, proving $(ii)$.

We prove $(iii)$ by induction on $(l)$. Assume $\boldsymbol{S}\boldsymbol{v}_i = \boldsymbol{u}_i$ for $i = 1, \dots, l-1$, then we may take $\boldsymbol{s}_l = \boldsymbol{v}_l$ in (3), but in proving $(ii)$ we saw that $\boldsymbol{S}\boldsymbol{s}_l$ is a $\lambda_l$-eigenvector of $\boldsymbol{A}$. (The statement between parenthesis follows by considering $-\boldsymbol{A}$ and $-\boldsymbol{B}$.) Thus we have $(iii)$.

Let the interlacing be tight, then by (*iii*) $\boldsymbol{S}\boldsymbol{v}_1, \ldots, \boldsymbol{S}\boldsymbol{v}_m$ is an orthonormal set of eigenvectors of $\boldsymbol{A}$ for the eigenvalues $\mu_1, \ldots, \mu_m$. So we have $\boldsymbol{S}\boldsymbol{B}\boldsymbol{v}_i = \mu\boldsymbol{S}\boldsymbol{v}_i = \boldsymbol{A}\boldsymbol{S}\boldsymbol{v}_i$ for $i = 1, \ldots, m$. Since the vectors $\boldsymbol{v}_i$ form a basis, it follows that $\boldsymbol{S}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{S}$. □

If in Interlacing Theorem we take $\boldsymbol{S} = [\boldsymbol{I}\ \boldsymbol{O}]$, then $\boldsymbol{B}$ is just a principal submatrix of $\boldsymbol{A}$ and we have:

COROLLARY 3.2. *If $\boldsymbol{B}$ is a principal submatrix of a symmetric matrix $\boldsymbol{A}$, then the eigenvalues of $\boldsymbol{B}$ interlace the eigenvalues of $\boldsymbol{A}$.*

Let $\mathcal{P} = \{V_1, \ldots, V_m\}$ be a partition of the vertex set $V$, with each $V_i \neq 0$. Let $\boldsymbol{A}$ be partitioned according to $\mathcal{P}$:

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A_{1,1}} & \cdots & \boldsymbol{A_{1,m}} \\ \vdots & & \vdots \\ \boldsymbol{A_{m,1}} & \cdots & \boldsymbol{A_{m,m}} \end{bmatrix},$$

where $\boldsymbol{A}_{i,j}$ denotes the submatrix (block) of $\boldsymbol{A}$ formed by rows in $V_i$ and columns in $V_j$. The *characteristic matrix* $\widetilde{\boldsymbol{S}} = (\widetilde{s}_{uj})$ is the $n \times m$ matrix whose $j$-th column is the characteristic vector of $V_j$, for $j = 1, \ldots, m$, that is

$$\widetilde{s}_{uj} = \begin{cases} 1 & \text{if } u \in V_j, \\ 0 & \text{otherwise.} \end{cases}$$

The *quotient matrix* is the $m \times m$ matrix $\widetilde{\boldsymbol{B}} = (\widetilde{b}_{ij})$ whose entries are the average row sums of the blocks of $\boldsymbol{A}$, more precisely:

$$\widetilde{b}_{ij} = \frac{1}{|V_i|}\boldsymbol{j}^\top \boldsymbol{A}_{i,j}\boldsymbol{j} = \frac{1}{|V_i|}(\widetilde{\boldsymbol{S}}^\top \boldsymbol{A}\widetilde{\boldsymbol{S}})_{i,j}.$$

The partition is called *regular* (or *equitable*) if each block $\boldsymbol{A}_{i,j}$ of $\boldsymbol{A}$ has constant row (and column) sum, that is, $\boldsymbol{A}\widetilde{\boldsymbol{S}} = \widetilde{\boldsymbol{S}}\widetilde{\boldsymbol{B}}$.

EXAMPLE 3.3. *The adjacency matrix of the complete bipartite graph $K_{p,q}$ has an equitable partition with $m = 2$. The quotient matrix $\widetilde{\boldsymbol{B}}$ is*

$$\widetilde{\boldsymbol{B}} = \begin{bmatrix} 0 & q \\ p & 0 \end{bmatrix},$$

*and has eigenvalues $\pm\sqrt{pq}$, which are the nonzero eigenvalues of $K_{p,q}$.*

COROLLARY 3.4. *Suppose $\widetilde{\boldsymbol{B}}$ is the quotient matrix of a symmetric partitioned matrix $\boldsymbol{A}$.*

(*i*) *The eigenvalues of $\widetilde{\boldsymbol{B}}$ interlace the eigenvalues of $\boldsymbol{A}$.*

(*ii*) *If the interlacing is tight then the partition is regular.*

***Proof.*** Let $\boldsymbol{D} = \operatorname{diag}(|V_1|, \ldots, |V_m|)$ and $\boldsymbol{S} = \widetilde{\boldsymbol{S}}\boldsymbol{D}^{-\frac{1}{2}}$. Then the eigenvalues of $\boldsymbol{B} = \boldsymbol{S}^\top \boldsymbol{A}\boldsymbol{S}$ interlace those of $\boldsymbol{A}$. This proves $(i)$, because $\boldsymbol{B}$ and $\widetilde{\boldsymbol{B}} = \boldsymbol{D}^{-\frac{1}{2}}\boldsymbol{B}\boldsymbol{D}^{\frac{1}{2}}$ have the same spectrum. If the interlacing is tight then $\boldsymbol{S}\boldsymbol{B} = \boldsymbol{A}\boldsymbol{S}$, hence $\boldsymbol{A}\widetilde{\boldsymbol{S}} = \widetilde{\boldsymbol{S}}\widetilde{\boldsymbol{B}}$. □

Note that the converse of Corollary 3.4.$(ii)$ is not true, a regular partition does not imply tight interlacing. Take for example the hypercube graph $Q_3$, with spectrum of the adjacency matrix $\operatorname{sp} Q_3 = \{3, 1^3, -1^3, -3\}$. If we consider the partition of the hypercube into antipodal pairs of vertices we get a $4 \times 4$ quotient matrix $\widetilde{\boldsymbol{B}}$ with spectrum $\{3, -1^3\}$. Thus, the last eigenvalue of $\widetilde{\boldsymbol{B}}$ is not equal to the last eigenvalue of $\boldsymbol{A}$, so there is not tight interlacing.

LEMMA 3.5. *If, for a regular partition, $\boldsymbol{\nu}$ is an eigenvector of $\widetilde{\boldsymbol{B}}$ for an eigenvalue $\lambda$, then $\widetilde{\boldsymbol{S}}\boldsymbol{\nu}$ is an eigenvector of $\boldsymbol{A}$ for the same eigenvalue $\lambda$.*

***Proof.*** $\widetilde{\boldsymbol{B}}\boldsymbol{\nu} = \lambda\boldsymbol{\nu}$ implies $\boldsymbol{A}\widetilde{\boldsymbol{S}}\boldsymbol{\nu} = \widetilde{\boldsymbol{S}}\widetilde{\boldsymbol{B}}\boldsymbol{\nu} = \lambda\widetilde{\boldsymbol{S}}\boldsymbol{\nu}$. □

EXAMPLE 3.6. *Here there is an example of interlacing in a graph $\Gamma$ that contains one vertex of degree 1, called $u$, which is joined to the rest of the vertices of $\Gamma$ through the vertex $v$.*

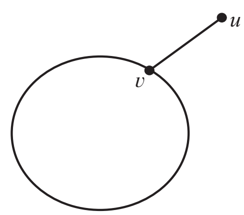



FIG. 1. A graph with a vertex of degree 1

*We can compute the characteristic polynomial as follows:*

$$\phi(\Gamma, x) = \det \begin{pmatrix} x & -1 & 0 & \cdots & 0 \\ \hline -1 & x & & & \\ 0 & & \ddots & & \\ \vdots & & & & \\ 0 & & & & \end{pmatrix} = x\phi(\Gamma - u, x) - \phi(\Gamma - \{u, v\}, x).$$

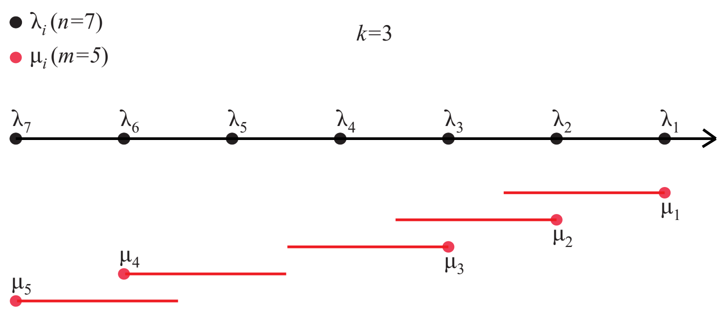
*Let $P_n$ be an $n$-path. Then,*

$$\Gamma = P_n : \phi(P_n, x) = x\phi(P_{n-1}, x) - \phi(P_{n-2}, x)$$

*where $\phi_0 := 1$, $\phi_1 = x$, $\phi_2 = x^2 - 1$, $\phi_3 = x^3 - 2x$, $\phi_4 = x^4 - 3x^2 + 1$, $\ldots$ are orthogonal polynomials (see* [**13**] *for more details).*


FIG. 2. A path

EXAMPLE 3.7. *The following figure shows an example of tight interlacing.*




# 4. Distance-regular graphs

Let us first give a combinatorial interpretation of distance-regularity. Distance-regular graphs were introduced by Biggs [**5**] by changing a symmetry-type requirement, that of distance-transitivity, to a regularity-type condition concerning the cardinality of some vertex subsets. A graph $\Gamma$ with diameter $D$ is *distance-transitive* when any two pairs of vertices $(u, v)$ and $(x, y)$ at the same distance $\partial(u, v) = \partial(x, y) \leq D$ are indistinguishable from each other; that is, there is an automorphism of the graph that takes $u$ to $x$ and $v$ to $y$. Thus, a distance-transitive graph "looks the same" when viewed from each one of such pairs. In particular, for any vertex pair $(u, v)$ and integers $0 \leq i, j \leq D$, the number $p_{ij}(u, v)$ of vertices at distance $i$ from $u$ and at distance $j$ from $v$ only depends on $k := \partial(u, v)$, and we write $p_{ij}(u, v) = p_{ij}^k$ for the *intersection numbers*. Such a condition is precisely the combinatorial property that defines a *distance-regular* graph.

In order to give some algebraic characterizations of distance-regularity, we now consider the following algebras. Let $\Gamma$ be a graph with diameter $D$, adjacency matrix $\boldsymbol{A}$ and $d+1$ distinct eigenvalues. Let $\boldsymbol{A}_i$, $i = 0, 1, \ldots, D$, be the distance-$i$ matrix of $\Gamma$, with entries $(\boldsymbol{A}_i)_{uv} = 1$ if $\partial(u, v) = i$ and $(\boldsymbol{A}_i)_{uv} = 0$ otherwise. Then,

$$\mathcal{A} = \mathbb{R}_d[\boldsymbol{A}] = \langle \boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d \rangle$$

is an algebra, with the ordinary product of matrices and orthogonal basis

$$\{\boldsymbol{E}_0, \boldsymbol{E}_1, \ldots, \boldsymbol{E}_d\}$$

and

$$\{p_0(\boldsymbol{A}), p_1(\boldsymbol{A}), \ldots, p_d(\boldsymbol{A})\},$$

called the *adjacency algebra*, whereas

$$\mathcal{D} = \langle \boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}_2, \ldots, \boldsymbol{A}_D \rangle$$

forms an algebra with the entrywise or Hadamard product "$\circ$" of matrices, defined by $(\boldsymbol{X} \circ \boldsymbol{Y})_{uv} = (\boldsymbol{X})_{uv}(\boldsymbol{Y})_{uv}$. We call $\mathcal{D}$ the *distance* $\circ$*-algebra*. Note that, when $\Gamma$ is regular, $\boldsymbol{I}, \boldsymbol{A}, \boldsymbol{J} \in \mathcal{A} \cap \mathcal{D}$ since $\boldsymbol{J} = H(\boldsymbol{A}) = \sum_{i=0}^{D} \boldsymbol{A}_i$. Thus, $\dim(\mathcal{A} \cap \mathcal{D}) \geq 3$, if $\Gamma$ is not a complete graph (in this exceptional case, $\boldsymbol{J} = \boldsymbol{I} + \boldsymbol{A}$). In this algebraic context, an important result is that $\Gamma$ is distance-regular if and only if $\mathcal{A} = \mathcal{D}$, which is therefore equivalent to $\dim(\mathcal{A} \cap \mathcal{D}) = d + 1$ (and hence $d = D$); see, for instance, Biggs [**4**] or Brouwer, Cohen and Neumaier [**8**]. This leads to the following definitions of distance-regularity where, for types $(a)$ and $(b)$, $p_{ji}$ and $q_{ij}$ are constants, $p_i$ are the predistance polynomials, and $q_j$ are the polynomials defined by

$$q_j(\theta_i) = m_j \frac{p_i(\theta_j)}{p_i(\theta_0)}.$$

$$\begin{aligned}
(a)\ \Gamma \text{ distance-regular} \quad &\iff \quad \boldsymbol{A}_i \boldsymbol{E}_j = p_{ji} \boldsymbol{E}_j, \qquad i, j = 0, 1, \ldots, d(= D),\\
&\iff \quad \boldsymbol{A}_i = \sum_{j=0}^{d} p_{ji} \boldsymbol{E}_j, \qquad i = 0, 1, \ldots, d(= D),\\
&\iff \quad \boldsymbol{A}_i = \sum_{j=0}^{d} p_i(\theta_j) \boldsymbol{E}_j, \qquad i = 0, 1, \ldots, d(= D),\\
&\iff \quad \boldsymbol{A}_i \in \mathcal{A}, \qquad i = 0, 1, \ldots, d(= D).
\end{aligned}$$

$$\begin{aligned}
(b)\ \Gamma \text{ distance-regular} \quad &\iff \quad \boldsymbol{E}_j \circ \boldsymbol{A}_i = q_{ij} \boldsymbol{A}_i, \qquad i, j = 0, 1, \ldots, d,\\
&\iff \quad \boldsymbol{E}_j = \sum_{i=0}^{d} q_{ij} \boldsymbol{A}_i, \qquad j = 0, 1, \ldots, d,\\
&\iff \quad \boldsymbol{E}_j = \frac{1}{n} \sum_{i=0}^{d} q_j(\theta_i) \boldsymbol{A}_i, \qquad j = 0, 1, \ldots, d,\\
&\iff \quad \boldsymbol{E}_j \in \mathcal{D}, \qquad j = 0, 1, \ldots, d.
\end{aligned}$$

In fact, for general graphs with $D \leq d$, the conditions of type $(a)$ are a characterization of the so-called *distance-polynomial graphs*, introduced by Weichsel [**45**] (see also Dalfó, van Dam, Fiol, Garriga and Gorissen [**19**]). This is equivalent to $\mathcal{D} \subset \mathcal{A}$ (but not necessarily $\mathcal{D} = \mathcal{A}$), that is, every distance matrix $\boldsymbol{A}_i$ is a polynomial in

$\boldsymbol{A}$. In contrast with this, the conditions of type $(b)$ are equivalent to $\mathcal{A} \subset \mathcal{D}$ and, hence, to $\mathcal{A} = \mathcal{D}$ (which implies $d = D$) as $\dim \mathcal{A} \geq \dim \mathcal{D}$.

Note also that the second implication in $(a)$ is obtained from the first one by using that $\sum_{j=0}^{d} \boldsymbol{E}_j = \boldsymbol{I}$, whereas the second implication in $(b)$ comes from $\sum_{i=0}^{d} \boldsymbol{A}_i = \boldsymbol{J}$.

Moreover, with the $a_i^{(\ell)}$, $i, \ell = 0, 1, \ldots, d$, being constants, we also have:

$$
\begin{aligned}
(c)\ \Gamma \text{ distance-regular} \quad &\Longleftrightarrow \quad \boldsymbol{A}^{\ell} \circ \boldsymbol{A}_i = a_i^{(\ell)} \boldsymbol{A}_i, \qquad i, \ell = 0, 1, \ldots, d, \\
&\Longleftrightarrow \quad \boldsymbol{A}^{\ell} = \sum_{i=0}^{d} a_i^{(\ell)} \boldsymbol{A}_i, \qquad \ell = 0, 1, \ldots, d, \\
&\Longleftrightarrow \quad \boldsymbol{A}^{\ell} = \frac{1}{n} \sum_{i=0}^{d} \sum_{j=0}^{d} q_{ij} \theta_j^{\ell} \boldsymbol{A}_i, \qquad \ell = 0, 1, \ldots, d, \\
&\Longleftrightarrow \quad \boldsymbol{A}^{\ell} \in \mathcal{D}, \qquad \ell = 0, 1, \ldots, d,
\end{aligned}
$$

where we used (2) with $a_{uv}^{(\ell)} = a_i^{(\ell)}$ and $m_{uv}(\theta_j) = q_{ij}$ for vertices $u, v$ at distance $\partial(u, v) = i$.

# Chapter 2
# Previous Results

There are many useful connections between the eigenvalues of a graph and its combinatorial properties. One of these follows from interlacing. In this chapter we see several applications of eigenvalue interlacing to matrices associated to graphs. Bounds are obtained for some parameters of graphs, such as the size of a maximal (co)clique, the chromatic number, the diameter and the bandwidth in terms of the eigenvalues of the standard adjacency matrix or the Laplacian matrix. We also study inequalities and regularity results concerning the structure of graphs.

## 1. Eigenvalue interlacing in graph parameters

### 1.1. Independence number: the largest coclique.

The following theorems show some upper bounds for the independence number $\alpha(\Gamma)$ (see [**33**]).

The following bound is due to Cvetković:

Theorem 1.1. $\alpha(\Gamma) \leq \min\{|\{i|\lambda_i \geq 0\}|, |\{i|\lambda_i \leq 0\}|\}$.

***Proof.*** Since $\boldsymbol{A}$ has a principal submatrix $\boldsymbol{B} = \boldsymbol{O}$ of size $\alpha(\Gamma)$, Corollary 3.2 gives $\lambda_\alpha \geq \mu_\alpha = 0$ and $\lambda_{n-\alpha+1} \leq \mu_1 = 0$. □

The following bound is an unpublished result of Hoffman.

Theorem 1.2 (Hoffman's upper bound for regular graphs). *If $\Gamma$ is $\delta$-regular, then*

$$\alpha(\Gamma) \leq n\frac{-\lambda_n}{\lambda_1 - \lambda_n}$$

*and if a coclique $C$ meets this bound then every vertex not in $C$ is adjacent to precisely $-\lambda_n$ vertices of $C$.*

***Proof.*** We apply Corollary 3.4. Let $\delta = \lambda_1$ be the degree of $\Gamma$ and put $\alpha = \alpha(\Gamma)$. The coclique gives rise to a partition of $\boldsymbol{A}$ with quotient matrix

$$\boldsymbol{B} = \begin{bmatrix} 0 & \delta \\ \frac{\delta\alpha}{n-\alpha} & \delta - \frac{\delta\alpha}{n-\alpha} \end{bmatrix}.$$

$\boldsymbol{B}$ has eigenvalues $\mu_1 = \delta$ (row sum) and $\mu_2 = -\frac{\delta\alpha}{n-\alpha}$ $(\mathrm{tr}\,\boldsymbol{B} - \delta)$ and so $\lambda_n \leq \mu_2$ gives the required inequality. If equality holds then $\mu_2 = \lambda_n$ and since $\mu_1 = \lambda_1$ the interlacing is tight and hence the partition is regular. □

There are many examples where equality holds. For instance, a 4-coclique in the Petersen graph is tight for both bounds. The bound of Theorem 1.2 can be generalized to arbitrary graphs in the following way:

THEOREM 1.3. *If $\Gamma$ has smallest degree $\delta_{min}$ then*

$$\alpha(\Gamma) \leq n \frac{-\lambda_1\lambda_n}{\delta_{min}^2 - \lambda_1\lambda_n}.$$

If $\Gamma$ is regular of degree $\delta$ then $\delta = \lambda_1$ and the above theorem reduces to Hoffman's bound.

More generally, one can obtain results on the size of induced subgraphs, analogues to Hoffman's bound.

### 1.2. Induced graphs.

THEOREM 1.4. *Let $\Gamma$ be a $\delta$-regular on $n$ vertices and suppose that it has an induced subgraph $\Gamma'$ with $n'$ vertices and $m'$ edges. Then*

$$\lambda_2 \geq \frac{2m'\frac{n}{n'} - n'\delta}{n - n'} \geq \lambda_n.$$

*If equality holds on either side then $\Gamma'$ is regular and so is the subgraph induced by the vertices not in $\Gamma'$.*

***Proof.*** We now have the quotient matrix

$$\boldsymbol{B} = \begin{bmatrix} \frac{2m'}{n'} & \delta - \frac{2m'}{n'} \\ \frac{n'\delta - 2m'}{n-n'} & \delta - \frac{n'\delta-2m'}{n-n'} \end{bmatrix},$$

with eigenvalues $\delta$ and $\frac{2m'}{n'} - \frac{n'\delta-2m'}{n-n'}$ and Corollary 3.4 gives the result. □

If $m' = 0$ we get the Hoffman's bound back. If $m' = \frac{1}{2}n'(n'-1)$ Theorem 1.4 gives that the size of a clique is bounded above by

$$n\frac{1+\lambda_2}{n-\delta+\lambda_2},$$

which is again the Hoffman's bound applied to the complement of $\Gamma$.

### 1.3. Chromatic number.

Notice that upper bounds for $\alpha(\Gamma)$ give lower bounds for $\chi(\Gamma)$. It is known that the ratio between the largest and smallest eigenvalue can be used to estimate the chromatic number (Hoffman [**38**]). The following theorem [**33**] shows a lower bound of $\chi(\Gamma)$ for regular and non-regular graphs.

THEOREM 1.5.

(*i*) *If* $\Gamma$ *is not empty then* $\chi(\Gamma) \geq 1 - \frac{\lambda_1}{\lambda_n}$.

(*ii*) *If* $\lambda_2 > 0$ *then* $\chi(\Gamma) \geq 1 - \frac{\lambda_{n-\chi(\Gamma)+1}}{\lambda_2}$.

The first inequality is due to Hoffman [**38**].

In [**33**] Haemers also finds a lower bound of $\chi(\Gamma)$ for strongly regular graphs:

COROLLARY 1.6. *If* $\Gamma$ *is a strongly regular graph, not the pentagon or a complete multipartite graph, then*

$$\chi(\Gamma) \geq 1 - \frac{\lambda_n}{\lambda_2}.$$

### 1.4. Shannon capacity.

This is a concept from information theory. Lovász [**40**] proved that the Hoffman's bound is also an upper bound for the Shannon capacity of $\Gamma$. For $\Gamma$ regular, it is shown in the next theorem [**33**].

THEOREM 1.7. *Let* $\Gamma$ *be regular of degree* $\delta$*, then*

$$\theta(\Gamma) \leq n\frac{-\lambda_n}{\delta - \lambda_n}.$$

***Proof.*** First note that the above proof of 1.2 remains valid if the ones in $\boldsymbol{A}$ are replaced by arbitrary real numbers, as long as $\boldsymbol{A}$ remains symmetric with constant row sum. So we may apply Hoffman's bound to $\boldsymbol{A}_l = (\boldsymbol{A} - \lambda_n)\boldsymbol{I}^{\otimes l} - (-\lambda_n)^l \boldsymbol{I}$ to get a bound for $\alpha(\Gamma^l)$. It easily follows that $\boldsymbol{A}_l$ has row sum $(\delta - \lambda_n)^l - (-\lambda_n)^l$ and smallest eigenvalue $-(-\lambda_n)^l$. So we find $\alpha(\Gamma^l) \leq \left(n\frac{-\lambda_n}{\delta-\lambda_n}\right)^l$. □

### 1.5. Diameter.

Let $\Gamma$ be an undirected graph with $n$ vertices, adjacency matrix $\boldsymbol{A}$ and diameter $D(\Gamma)$. We allow $\Gamma$ to be a *multigraph*, that is, $\Gamma$ may have multiple edges and loops (a loop counts for one edge in the degree).

The following results deal with the study of the diameter in regular graphs. In this context, a basic result concerning the distance between sets is showed in [**17**].

THEOREM 1.8. *Let $\Gamma$ be connected and regular of degree $\delta$. Let $m$ be a nonnegative integer and let $X$ and $Y$ be sets of sizes $x$ and $y$, respectively, such that the distance between any vertex of $X$ and any vertex of $Y$ is at least $m+1$. If $p$ is a polynomial of degree $m$ such that $p(\delta) = 1$, then*

$$\frac{xy}{(n-x)(n-y)} \leq \max_{i \neq 1} p^2(\lambda_i).$$

A consequence of this result is the following theorem.

THEOREM 1.9. *Let $\Gamma$ be connected and regular of degree $\delta$ (not complete), then*

$$D(\Gamma) < \frac{log_2(n-1)}{log\left[\frac{\sqrt{\delta-\lambda_n}+\sqrt{\delta-\lambda_2}}{\sqrt{\delta-\lambda_n}-\sqrt{\delta-\lambda_2}}\right]} + 1.$$

If $\Gamma$ is not a regular graph, it can be transformed into a regular graph $\Gamma$ by adding a suitable number of loops to every vertex. If $\delta$ is the maximum degree in $\Gamma$, we add $\delta$-degree($i$) loops to every vertex $i$, so that $\Gamma$ is regular of degree $\delta$. Moreover, there is a relation between the eigenvalues of the Laplacian matrix $\boldsymbol{L}$ of $\Gamma$ and the eigenvalues of the adjacency matrix $\boldsymbol{A}$ of $\Gamma$. The Laplacian matrix $\boldsymbol{L}$ of $\Gamma$ is defined by $\boldsymbol{L} = \delta\boldsymbol{I} - \boldsymbol{A}$, so $0 = \theta_1 \leq \theta_2 \leq \cdots \leq \theta_n$ are the Laplacian eigenvalues of $\Gamma$, then $\theta_i = \delta - \lambda_i$, $i = 1, 2, \ldots, n$. We can get bounds in terms of the Laplacian eigenvalues of $\Gamma$. For example, the above theorem ([**33**], [**17**]) now says the following:

THEOREM 1.10. *If $\Gamma$ is a connected graph with diameter $D(\Gamma) > 1$, then*

$$D(\Gamma) < \frac{log_2(n-1)}{log(\sqrt{\theta_n}+\sqrt{\theta_2}) - log(\sqrt{\theta_n}-\sqrt{\theta_2})} + 1.$$

In [**21**] upper bounds for the diameter of regular and bipartite graphs are found using eigenvalue interlacing and Chebyshev polynomials. This method also gives upper bounds for the number of vertices at a given minimum distance from a given vertex set. These results have some applications to the covering radius of error-correcting codes.

### 1.6. Bandwidth.

A symmetric matrix $\boldsymbol{A}$ is said to have *bandwidth* $b\omega$ if $a_{ij} = 0$ for all $i$, $j$ such that $|i-j| \geq b\omega$. The bandwidth $b\omega(\Gamma)$ of a graph $\Gamma$ is the smallest possible bandwidth for its adjacency matrix (or Laplacian matrix). This number (or rather, the vertex order realizing it) is of interest for some combinatorial optimization problems.

LEMMA 1.11. *Let $X$ and $Y$ be disjoint sets of vertices of $\Gamma$, such that there is no edge between $X$ and $Y$, then*

$$\frac{|X||Y|}{(n-|X|)(n-|Y|)} \leq \left(\frac{\theta_n - \theta_2}{\theta_n + \theta_2}\right)^2.$$

In the following theorem [**33**] a lower bound of $b\omega(\Gamma)$ is found:

THEOREM 1.12. *Suppose* $\Gamma$ *is not the empty graph and define* $b = \lceil n\frac{\theta_2}{\theta_n}\rceil$, *then*

$$b\omega(\Gamma) \geq \begin{cases} b & \text{if } n-b \text{ is even,} \\ b-1 & \text{if } n-b \text{ is odd.} \end{cases}$$

***Proof.*** Order the vertices of $\Gamma$ such that $\boldsymbol{L}$ has bandwidth $b\omega = b\omega(\Gamma)$. If $n - b\omega$ is even, let $X$ be the first $\frac{1}{2}(n-b\omega)$ vertices and let $Y$ be the last $\frac{1}{2}(n-b\omega)$ vertices. Then 1.11 applies and thus we find the first inequality. If $n - b\omega$ is odd, take for $X$ and $Y$ the first and last $\frac{1}{2}(n - b\omega - 1)$ vertices and the second inequality follows. If $b$ and $b\omega$ have different parity, then $b\omega - b \geq 1$ and so the improved inequality holds. □

## 2. Regularity

Corollary 3.4.(*ii*) of Chapter 1 gives a sufficient condition for a partition of a matrix $\boldsymbol{A}$ to be regular. This has turned out to be handy for proving various kinds of regularity. Here we give some examples. If we apply Corollary 3.4 to the trivial one-class partition of the adjacency matrix of a graph $\Gamma$ with $n$ vertices and $m$ edges we obtain

$$\frac{2m}{n} \leq \lambda_1,$$

and equality implies that $\Gamma$ is regular. This is a well-known result, see Cvetković, Doob and Sachs [**17**]. In fact, since $2m = \text{tr}(\boldsymbol{A}^2) = \sum_{i=1}^{n} \lambda^2$, it implies that $\Gamma$ is regular if and only if

$$\sum_{i=1}^{n} \lambda_i^2 = n\lambda_1.$$

Next we consider less trivial partitions. For a vertex $v$ of $\Gamma$, we denote by $X_i(v)$ the set of vertices at distance $i$ from $v$. The *neighbour partition* of $\Gamma$ with respect to $v$ is the partition into $X_0(v)$, $X_1(v)$ and the remaining vertices. If $\Gamma$ is connected, the partition into the $X_i(v)$'s is called the *distance partition* with respect to $v$. A graph is *distance-regular around* $v$ if the distance partition with respect to $v$ is regular. If $\Gamma$ is distance-regular around each vertex with the same quotient matrix, then $\Gamma$ is called *distance-regular.* A *strongly regular graph* is a distance-regular graph of diameter 2. A distance-regular graph of diameter $d$ has precisely $d + 1$ distinct eigenvalues, being the eigenvalues of the quotient matrix of the distance partition. See Brouwer, Cohen and Neumaier [**8**] for more about distance-regular graphs.

The following theorem provides a condition to determine if a regular graph is strongly regular using its degree and the number of triangles through a vertex $v$ [**33**].

THEOREM 2.1. *Suppose* $\Gamma$ *is regular of degree* $\delta$ $(0 < \delta < n-1)$ *and let* $t_v$ *be the number of triangles through the vertex* $v$. *Then*

$$n\delta - 2\delta^2 + 2t_v \leq -\lambda_2\lambda_n(n - \delta - 1).$$

*If equality holds for every vertex,* $\Gamma$ *is strongly regular.*

***Proof.*** The neighbour partition has the following quotient matrix

$$\boldsymbol{B} = \begin{bmatrix} 0 & \delta & 0 \\ 1 & \frac{2t_v}{\delta} & \frac{\delta^2-\delta-2t_v}{\delta} \\ 0 & \frac{\delta^2-\delta-2t_v}{n-\delta-1} & \frac{n\delta-2\delta^2+2t_v}{n-\delta-1} \end{bmatrix}.$$

Interlacing gives:

$$\delta\frac{n\delta - 2\delta^2 + 2t_v}{n - \delta - 1} = -\det(\boldsymbol{B}) = -\delta\mu_2\mu_3 \leq -\delta\lambda_2\lambda_n.$$

This proves the inequality. If equality holds then $\lambda_2 = \mu_2$ and $\lambda_n = \mu_3$ so (since $\delta = \lambda_1 = \mu_1$) the interlacing is tight and the neighbour partition is regular with quotient matrix $\boldsymbol{B}$. By definition, equality for all vertices implies that $\Gamma$ is strongly regular. □

The average number of triangles through a vertex is

$$\frac{1}{2n}\operatorname{tr}(\boldsymbol{A}^3) = \frac{1}{2n}\sum_{i=1}^{n}\lambda_i^3.$$

So if we replace $t_v$ by this expression the above inequality remains valid. Equality then means automatically equality for all vertices so strong regularity. In [**34**] Haemers looked for similar results for distance-regular graphs of diameter $d > 2$, in order to find sufficient conditions for distance-regularity in terms of eigenvalues. Therefore, one needs to prove regularity of the distance partition. The problem is, however, that in general all eigenvalues $\neq \lambda_1$ of a distance-regular graphs have a multiplicity greater than 1, whilst the quotient matrix has all multiplicities equal to 1. So for $d \geq 3$ there is not much chance for tight interlacing. But because of the special nature of the partition we still can conclude regularity, as we see in the next result [**33**].

LEMMA 2.2. *Let* $\boldsymbol{A}$ *be a symmetric partitioned matrix such that* $\boldsymbol{A}_{ij} = \boldsymbol{O}$ *if* $|i-j| > 1$ *and let* $\boldsymbol{B}$ *be the quotient matrix. For* $i = 1, \dots, m$*, let* $\boldsymbol{v}_i = (v_{i,1}, \dots, v_{i,m})^\top$ *denote a* $\mu_i$*-eigenvector of* $\boldsymbol{B}$*. If* $\lambda_0 = \mu_0$*,* $\lambda_1 = \mu_1$ *and* $\lambda_n = \mu_m$ *and if any three consecutive rows of* $[\boldsymbol{v}_1 \boldsymbol{v}_2 \boldsymbol{v}_m]$ *are independent, then the partition is regular.*

***Proof.*** By $(ii)$ of Theorem 3.1 $\boldsymbol{A}\widetilde{\boldsymbol{S}}\boldsymbol{v}_i = \mu_i \widetilde{\boldsymbol{S}}\boldsymbol{v}_i$ for $i = 1, 2, m$. By considering the $l^{\text{th}}$ block row of $\boldsymbol{A}$ we get

$$v_{i,l-1}\boldsymbol{A}_{l,l-1}\boldsymbol{j} + v_{i,l}\boldsymbol{A}_{l,l}\boldsymbol{j} + v_{i,l+1}\boldsymbol{A}_{l,l+1}\boldsymbol{j} = \mu_i v_{i,l}\boldsymbol{j} \quad \text{for} \quad i = 1, 2, m,$$

(where the undefined terms have to be taken equal to zero). Since for $i = 1, 2, m$ and $j = l-1, l, l+1$, the matrix $(v_{i,j})$ is non-singular, we find $\boldsymbol{A}_{l,j}\boldsymbol{j} \in \langle \boldsymbol{j} \rangle$ for $j = l-1, l, l+1$ (and hence for $j = 1, \dots, m$). Thus the partition is regular. □

Interlacing Theorem is a tool for proving regularity of a partition using eigenvalues. If we want to prove distance-regularity of a graph $\Gamma$, we want to apply that if we have tight interlacing then the partition is regular to its distance partitions. This, however, will hardly ever work if the diameter is bigger than 2, since if $\Gamma$ is connected, the quotient matrix $\boldsymbol{B}$ has $d+1$ distinct eigenvalues, whilst all but the largest eigenvalue of the adjacency matrix $\boldsymbol{A}$ have in general a multiplicity greater than 1, in which case equality in the case $(i)$ of Interlacing Theorem can only hold for $\mu_0$, $\mu_1$ and $\mu_d$. In the above lemma, Haemers proves a result in terms of these three eigenvalues only. Maybe this result could extended for the case when we have diameter 5.

In [**34**] it was proved that the independence condition in the above lemma is always fulfilled if we consider the distance partition of a graph. So we have

THEOREM 2.3. *Let* $\Gamma$ *be a connected graph and let* $\boldsymbol{B}$ *be a quotient matrix of the distance partition with respect to a vertex* $v$*. If* $\lambda_0 = \mu_0$*,* $\lambda_1 = \mu_1$ *and* $\lambda_n = \mu_m$ *then* $\Gamma$ *is distance-regular over* $v$*.*

Using this result it was proved that $\Gamma$ has the same spectrum and the same number of vertices at maximal distance from each vertex as a distance regular graph $\Gamma'$ of diameter 3, then $\Gamma$ is distance-regular.

# Chapter 3
# Partitions

## 1. Weight-Partitions

Let $\mathcal{P}$ be a partition of the vertex set $V = V_1 \cup \cdots \cup V_m$, $1 \le m \le n$. Consider the map $\boldsymbol{\rho} : V \longrightarrow \mathbb{R}^+$ defined by $\boldsymbol{\rho}U := \sum_{u \in U} \rho_u \boldsymbol{e}_u$. In particular, for weight-partitions we consider the map $\boldsymbol{\rho} : \mathcal{P}(V) \longrightarrow \mathbb{R}^n$ defined by $\boldsymbol{\rho}U := \sum_{u \in U} \nu_u \boldsymbol{e}_u$ for any $U \neq \emptyset$, where $\boldsymbol{e}_u$ represents the $u$-th canonical (column) vector, and $\boldsymbol{\rho}\emptyset = \mathbf{0}$. Note that, with $\boldsymbol{\rho}u := \boldsymbol{\rho}\{u\}$, we have $||\boldsymbol{\rho}u|| = \nu_u$, so that we can see $\boldsymbol{\rho}$ as a function which assigns weights to the vertices of $\Gamma$. In doing so we "regularize" the graph, in the sense that the *weight-degree* of each vertex $u \in V$ becomes a constant:

$$\delta^*_u := \frac{1}{\nu_u} \sum_{v \in \Gamma(u)} \nu_v = \lambda_1 \tag{4}$$

Given $\mathcal{P} = \{V_1, \ldots, V_m\}$, for $u \in V_i$ we define the *weight-intersection numbers* as follows:

$$b^*_{ij}(u) := \frac{1}{\nu_u} \sum_{v \in \Gamma(u) \cap V_j} \nu_v \qquad (1 \le i, j \le m). \tag{5}$$

Observe that the sum of the weight-intersection numbers for all $1 \le j \le m$ gives the weight-degree of each vertex $u \in V_i$:

$$\sum_{j=1}^{m} b^*_{ij}(u) = \frac{1}{\nu_u} \sum_{v \in \Gamma(u)} \nu_v = \delta^*_u = \lambda_1.$$

A matrix characterization of weight-regular partitions, which are defined in the next section, can be done via the following matrix associated with (any) partition

$\mathcal{P}$. The *weight-characteristic matrix* of $\mathcal{P}$ is the $n \times m$ matrix $\widetilde{\boldsymbol{S}}^* = (\widetilde{s}^*_{uj})$ with entries

$$\widetilde{s}^*_{uj} = \begin{cases} \nu_u & \text{if } u \in V_j, \\ 0 & \text{otherwise.} \end{cases}$$

and, hence, satisfying $(\widetilde{\boldsymbol{S}}^*)^\top \widetilde{\boldsymbol{S}}^* = \boldsymbol{D}^2$, where $\boldsymbol{D} = \operatorname{diag}(||\boldsymbol{\rho} V_1||, \ldots, ||\boldsymbol{\rho} V_m||)$.

From such a weight-characteristic matrix we define the *weight-quotient matrix* of $\boldsymbol{A}$, with respect to $\mathcal{P}$, as $\widetilde{\boldsymbol{B}}^* := (\widetilde{\boldsymbol{S}}^*)^\top \boldsymbol{A} \widetilde{\boldsymbol{S}}^* = (\widetilde{b}^*_{ij})$. Notice that this matrix is symmetric with entries

$$\widetilde{b}^*_{ij} = \sum_{u,v \in V} \widetilde{s}^*_{ui} a_{uv} \widetilde{s}^*_{vj} = \sum_{u \in V_i, v \in V_j} a_{uv} \nu_u \nu_v = \sum_{uv \in E(V_i, V_j)} \nu_u \nu_v = \widetilde{b}^*_{ji}$$

where $E(V_i, V_j)$ stands for the set of edges with ends in $V_i$ and $V_j$ (when $V_i = V_j$ each edge counts twice). Also, in terms of the weight-intersection numbers,

$$\begin{aligned} \widetilde{b}^*_{ij} &= \sum_{u \in V_i} \nu_u \sum_{v \in \Gamma(u) \cap V_j} \nu_v = \sum_{u \in V_i} \nu_u^2 b^*_{ij}(u) \\ &= \sum_{v \in V_j} \nu_v \sum_{u \in \Gamma(v) \cap V_i} \nu_u = \sum_{v \in V_j} \nu_v^2 b^*_{ji}(v) = \widetilde{b}^*_{ji}. \end{aligned} \tag{6}$$

Let us consider a new $n \times m$ matrix, $\overline{\boldsymbol{S}}^* = (\overline{s}^*_{uj})$, called *normalized weight-characteristic matrix*, obtained by just normalizing the columns of $\widetilde{\boldsymbol{S}}^*$, that is, $\overline{\boldsymbol{S}}^* = \widetilde{\boldsymbol{S}}^* \boldsymbol{D}^{-1}$. Thus,

$$\overline{s}^*_{uj} = \begin{cases} \frac{\nu_u}{||\boldsymbol{\rho} V_j||} & \text{if } u \in V_j, \\ 0 & \text{otherwise.} \end{cases}$$

that satisfies $(\overline{\boldsymbol{S}}^*)^\top \overline{\boldsymbol{S}}^* = \boldsymbol{I}$.

We define the *normalized weight-quotient matrix* of $\boldsymbol{A}$ with respect to $\mathcal{P}$, $\overline{\boldsymbol{B}}^* = (\overline{b}^*_{ij})$, as

$$\overline{\boldsymbol{B}}^* = (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{A} \overline{\boldsymbol{S}}^* = \boldsymbol{D}^{-1} (\widetilde{\boldsymbol{S}}^*)^\top \boldsymbol{A} \widetilde{\boldsymbol{S}}^* \boldsymbol{D}^{-1} = \boldsymbol{D}^{-1} \widetilde{\boldsymbol{B}}^* \boldsymbol{D}^{-1},$$

and hence

$$\overline{b}^*_{ij} = \frac{\widetilde{b}^*_{ij}}{||\boldsymbol{\rho} V_i|| ||\boldsymbol{\rho} V_j||}.$$

LEMMA 1.1. *In a weight-partition, we can assure that $\overline{\boldsymbol{B}}^*$ has eigenvector $\boldsymbol{\mu} = (||\boldsymbol{\rho} V_1||, \ldots, ||\boldsymbol{\rho} V_m||)^\top$ of eigenvalue $\lambda_1$.*

***Proof.*** To show it, we can check each entry of $\overline{\boldsymbol{B}}^*$,

$$\begin{aligned}
(\overline{\boldsymbol{B}}^* \boldsymbol{\mu})_i &= \sum_{j=1}^{m} \overline{b}^*_{ij} ||\boldsymbol{\rho} V_j|| \\
&= \sum_{j=1}^{m} \frac{\sum_{u \in V_i} \nu_u^2 b^*_{ij}(u)}{||\boldsymbol{\rho} V_i|| ||\boldsymbol{\rho} V_j||} ||\boldsymbol{\rho} V_j|| \\
&= \sum_{u \in V_i} \frac{\nu_u^2}{||\boldsymbol{\rho} V_i||} \sum_{j=1}^{m} b^*_{ij}(u) \\
&= \lambda_1 ||\boldsymbol{\rho} V_i||.
\end{aligned}$$

An alternative way to show the above result, is to do it through matrices. We know that $(\overline{\boldsymbol{S}}^*)^\top \overline{\boldsymbol{S}}^* = \boldsymbol{I}$ and $\boldsymbol{A\nu} = \lambda_1 \boldsymbol{\nu}$, and we consider $\overline{\boldsymbol{B}}^* = (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{A} \overline{\boldsymbol{S}}^*$. Denote $\boldsymbol{\mu} = (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{\nu}$. Observe that $||\boldsymbol{\nu}||^2 \overline{\boldsymbol{S}}^* (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{\nu}$ is equivalent to do the projection of $\boldsymbol{\nu}$ onto the eigenspace $\varepsilon_{\lambda_1}$, $\boldsymbol{E}_{\lambda_1} \boldsymbol{\nu} = ||\boldsymbol{\nu}||^2 \overline{\boldsymbol{S}}^* (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{\nu}$, hence $\overline{\boldsymbol{S}}^* (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{\nu} = \boldsymbol{\nu}$. Then,

$$\begin{aligned}
\overline{\boldsymbol{B}}^* \boldsymbol{\mu} = \overline{\boldsymbol{B}}^* (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{\nu} = (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{A} \overline{\boldsymbol{S}}^* (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{\nu} = (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{A} \frac{\boldsymbol{E}_{\lambda_1} \boldsymbol{\nu}}{||\boldsymbol{\nu}||^2} \\
= (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{A\nu} = \lambda_1 (\overline{\boldsymbol{S}}^*)^\top \boldsymbol{\nu} = \lambda_1 \boldsymbol{\mu},
\end{aligned}$$

which proves the result. □

Note that we defined two forms for the weight-characteristic matrix and the weight-quotient matrix: the non-normalized matrix and the normalized one. We will use either of them.

## 2. Weight-Regular Partitions

Using the weights introduced in the above section (see Eq. (4)), we can also consider the so-called weight-regular partitions of a graph. A partition $\mathcal{P}$ is called *weight-regular* whenever the weight-intersection numbers do not depend on the chosen vertex $u \in V_i$, but only on the subsets $V_i$ and $V_j$. In such a case, we denote them by

$$b_{ij}^*(u) = b_{ij}^* \qquad \forall u \in V_i$$

and we consider the $m \times m$ matrix $\boldsymbol{B}^* = (b_{ij}^*)$, called the *weight-regular-quotient matrix* of $\boldsymbol{A}$ with respect to $\mathcal{P}$.

Weight-regular partitions were introduced by Fiol and Garriga [**25**] with the name of *pseudo-regular partitions*, as a generalization of the standard notion of regular (or equitable) partitions. Regular partitions were studied in some detail in Godsil [**30**]. Roughly speaking, the definition of regular partition is the same as that of weight-regular partition, but now all the vertices have constant weight 1 ($\rho \equiv 1$). More precisely, a partition $V = V_1 \cup \cdots \cup V_m$ of the vertex set of a graph $\Gamma = (V, E)$, is *regular* (or *equitable*) if the numbers in (5), defined by

$$b_{ij}^*(u) := |\Gamma(u) \cap V_j| \qquad (u \in V_i, 1 \leq i, j \leq m)$$

only depend on the values $i$ and $j$. Then we denote $b_{ij}^*(u) := b_{ij}$. Thus, $\boldsymbol{\nu} = \boldsymbol{j}$ when $\Gamma$ is regular.

The following theorem was proved for graphs by Godsil [**30**]. We observe that it holds also for non-negative symmetric matrices.

THEOREM 2.1. *Let $\boldsymbol{A}$ be an irreducible, non-negative symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\mathcal{P}$ be a regular partition for $\boldsymbol{A}$. Let $\boldsymbol{B}$ be the quotient matrix of $\boldsymbol{A}$ with respect to $\mathcal{P}$, with eigenvalues $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$, $m < n$. Then $\lambda_1 = \mu_1$.*

***Proof.*** Let $\mathcal{P}$ be a partition of the vertex set $V = V_1 \cup \cdots \cup V_m$, $1 \leq m \leq n$ and suppose that $\mathcal{P}$ is regular for $\boldsymbol{A}$. Since $\boldsymbol{A}$ is irreducible, $\boldsymbol{B}$ is also irreducible; let $\boldsymbol{y} = (y_1, \ldots, y_m)$ be a positive unit eigenvector to $\mu_1$. Then the vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ defined by

$$x_u = \frac{1}{\sqrt{|V_j|}} y_j \quad \text{for} \quad u \in V_j$$

is a positive unit vector such that $\boldsymbol{A}\boldsymbol{x} = \mu_1 \boldsymbol{x}$, implying that $\mu_1$ is an eigenvalue of $\boldsymbol{A}$ with eigenvector $\boldsymbol{x}$. Perron-Frobenius Theorem implies that $\mu_1 = \lambda_1$, completing the proof. □

LEMMA 2.2. *If, for a regular partition, $\boldsymbol{\nu}$ is an eigenvector of $\boldsymbol{B}$ for an eigenvalue $\lambda$, then $\boldsymbol{S}\boldsymbol{\nu} = (\nu_1 \boldsymbol{j} | \ldots | \nu_m \boldsymbol{j})^\top$, with the $\boldsymbol{j}$'s being all 1-vectors with appropriate lengths, depending on the size of $n_i$, $i = 1, \ldots, m$, is an eigenvector of $\boldsymbol{A}$ for the same eigenvalue $\lambda$.*

***Proof.*** By Theorem 2.1, we can assure that the quotient matrix $\boldsymbol{B}$ has a positive eigenvector, denoted by $\boldsymbol{\nu}$. Then, $\boldsymbol{B}\boldsymbol{\nu} = \lambda \boldsymbol{\nu}$ implies $\boldsymbol{A}\boldsymbol{S}\boldsymbol{\nu} = \boldsymbol{S}\boldsymbol{B}\boldsymbol{\nu} = \lambda \boldsymbol{S}\boldsymbol{\nu}$. □

LEMMA 2.3. *Let $\mathcal{P}$ be a regular partition of a graph $\Gamma$, with intersection numbers $b_{ij}$, $1 \le i, j \le m$. Let $\Gamma$ have positive eigenvector $\boldsymbol{\nu}$ with entries denoted as above. Then $\mathcal{P}$ is also a weight-regular partition of $\Gamma$ with intersection numbers*

$$b^*_{ij} = \frac{\nu_j}{\nu_i} b_{ij} \qquad (1 \le i, j \le m). \tag{7}$$

***Proof.*** Let $u \in V_i$, and recall that the weight-regular quotient matrix is denoted as $\boldsymbol{B}^* = (b^*_{ij})$. Then,

$$b^*_{ij}(u) = \frac{1}{\nu_u} \sum_{v \in \Gamma(u) \cap V_j} \nu_v = \frac{1}{\nu_i} b_{ij} \nu_j, \qquad 1 \le i, j \le m.$$

□

Note that there are some particular cases of trivial partitions that can be immediately proved. This can be summarized by the following table:

| | Regular partition | Weight-regular partition |
|---|---|---|
| $m = 1$ | $\Longleftrightarrow \Gamma$ regular | always |
| $m = 2$ | $\Longleftrightarrow \Gamma$ biregular | $\Longleftrightarrow \Gamma$ bipartite |
| $m = n$ | always | $\Longleftrightarrow \Gamma$ regular |

When the eigenvector $\boldsymbol{\nu}$ of a regular partition $\mathcal{P}$ bears the above mentioned regularity, then $\mathcal{P}$ is also a weight-regular partition, and the relation between the corresponding intersection numbers is given by (7).

EXAMPLE 2.4. *Let $\Gamma$ be a graph partitioned as follows*

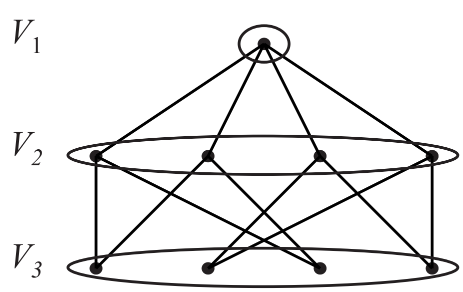



FIG. 1. Partition of a graph $\Gamma$

*and consider $\boldsymbol{\nu} = (\nu_1 \boldsymbol{j} | \nu_2 \boldsymbol{j} | \nu_3 \boldsymbol{j})^\top$ its positive eigenvector with entries $\nu_1 = 2$, $\nu_2 = \sqrt{2}$, $\nu_3 = 1$ with the $\boldsymbol{j}$'s being all 1-vectors with appropriate lengths, depending of $|V_i|$, $1 \le i \le 3$.*

*As it is a regular partition, the intersection numbers are just* $b_{ij} := |\Gamma(u) \cap V_j|$, *where* $u \in V_i$, $1 \le i, j \le 3$. *It follows that* $b_{12} = 4$, $b_{21} = 1$, $b_{23} = 2$ *and* $b_{32} = 2$ *are the non-null intersection numbers.*

*Using the above lemma, we can consider it as a weight-regular partition and then find the corresponding non-null intersection numbers:* $b^*_{12} = \frac{\sqrt{2}\cdot 4}{2}$, $b^*_{21} = \frac{2\cdot 1}{\sqrt{2}}$, $b^*_{23} = \frac{1\cdot 4}{\sqrt{2}}$ *and* $b^*_{32} = \frac{\sqrt{2}\cdot 4}{1}$.

To show, however, that this is not always the case, let us consider the following example of weight-regular partition which is not equitable.

EXAMPLE 2.5. *Take the binary tree* $T$ *of depth two, with vertices* $*$ *(father),* $*0$, $*1$ *(sons), and* $*00$, $*01$, $*10$, $*11$ *(grandsons), radius* $r = 2$, *maximum eigenvalue* $\lambda_0 = 2$, *and positive eigenvector* $\boldsymbol{\nu}$ *with entries* $\nu_* = \nu_{*0} = \nu_{*1} = 1$, $\nu_{*00} = \nu_{*01} = \nu_{*10} = \nu_{*11} = \frac{1}{2}$. *Then, by using known results about the spectrum and eigenvectors of the cartesian product of graphs (see* **[16]***), it is shown that the graph* $\Gamma = T \times \cdots \times T$ *(t factors) has radius* $r' = 2t$, *maximum eigenvalue* $\lambda'_0 = 2t$, *and eigenvector* $\boldsymbol{\nu}'$ *with* $\nu_{u_1}\nu_{u_2}\cdots\nu_{u_t}$ *as the component associated with the vertex* $(u_1, u_2, \ldots, u_t)$, $u_i \in V(T)$. *By using these data, an easy computation shows that the partition induced in* $\Gamma$ *by the central vertex* $(*, *, \ldots, *)$ *is indeed weight-regular (but not regular), and its non-null intersection numbers are* $b^*_{k-1,k} = b^*_{r'-k+1,r'-k} = k$, $1 \le k \le r'$.

Note that, in a weight-regular partition, the following holds from $\widetilde{b}^*_{ij}(u) = b^*_{ij}$ and (6):

$$\widetilde{b}^*_{ij} = b^*_{ij} \sum_{u \in V_i} \nu_u^2 = b^*_{ij}||\boldsymbol{\rho} V_i||^2 = b^*_{ji}||\boldsymbol{\rho} V_j||^2.$$

For the case of a regular partition, this is equivalent to

$$b_{ij}|V_i| = b_{ji}|V_j|$$

which counts in two ways the number $|E(V_i, V_j)|$ of edges between $V_i$ and $V_j$.

A weight-regular partition can be characterized by the following lemma.

LEMMA 2.6. *Let* $\Gamma = (V, E)$ *be a graph with adjacency matrix* $\boldsymbol{A}$ *and positive eigenvector* $\boldsymbol{\nu}$, *and consider a vertex partition* $\mathcal{P}$ *with weight-characteristic matrix* $\widetilde{\boldsymbol{S}}^*$. *Then* $\mathcal{P}$ *is weight-regular partition if and only if there exists an* $(m \times m)$ *matrix* $\boldsymbol{C}^*$ *such that* $\widetilde{\boldsymbol{S}}^*\boldsymbol{C}^* = \boldsymbol{A}\widetilde{\boldsymbol{S}}^*$. *Moreover, in this case* $\boldsymbol{C}^* = \boldsymbol{B}^*$.

***Proof.*** Let $\boldsymbol{C}^* = (c^*_{ij})$ be an $m \times m$ matrix. Let $u \in V_i$ and $1 \le j \le m$. Then, the result follows from the equalities:

$$(\widetilde{\boldsymbol{S}}^* \boldsymbol{C}^*)_{uj} = \sum_{k=1}^{m} \widetilde{s}^*_{uk} c^*_{kj} = \nu_u c^*_{ij}$$

$$(\boldsymbol{A}\widetilde{\boldsymbol{S}}^*)_{uj} = \sum_{v \in V} a_{uv} \widetilde{s}^*_{vj} = \sum_{v \in \Gamma(u) \cap V_j} \nu_v = \nu_u b^*_{ij}(u)$$

where we have used the definition of $b^*_{ij}(u)$. Then the entries of the weight-quotient matrix become constant and equal the entries of the weight-regular-quotient matrix, $b^*_{ij}(u) = c^*_{ij} = b^*_{ij}$. □

The following result is a direct consequence of Interlacing Theorem.

LEMMA 2.7. *Let $\Gamma = (V, E)$ be a graph with adjacency matrix $\boldsymbol{A}$ and positive eigenvector $\boldsymbol{\nu}$, and consider a vertex partition $\mathcal{P}$ of $V$ inducing the normalized weight-quotient matrix $\overline{\boldsymbol{B}}^*$. Then the following holds:*

$(i)$ *The eigenvalues of $\overline{\boldsymbol{B}}^*$ interlace the eigenvalues of $\boldsymbol{A}$.*

$(ii)$ *If the interlacing is tight, then the partition $\mathcal{P}$ is weight-regular.*

***Proof.*** We only need to prove $(ii)$, since $(i)$ is already proved by Interlacing Theorem. If the interlacing is tight we know that $\overline{\boldsymbol{S}}^* \overline{\boldsymbol{B}}^* = \boldsymbol{A}\overline{\boldsymbol{S}}^*$. Moreover,

$\overline{\boldsymbol{S}}^* = \widetilde{\boldsymbol{S}}^* \boldsymbol{D}^{-1}$ with $\boldsymbol{D} = \mathrm{diag}(||\boldsymbol{\rho} V_1||, \ldots, ||\boldsymbol{\rho} V_m||)$. Hence,

$$\widetilde{\boldsymbol{S}}^* \boldsymbol{D}^{-1} \overline{\boldsymbol{B}}^* = \boldsymbol{A}\widetilde{\boldsymbol{S}}^* \boldsymbol{D}^{-1} \Longrightarrow \widetilde{\boldsymbol{S}}^* \boldsymbol{D}^{-1} \overline{\boldsymbol{B}}^* \boldsymbol{D} = \boldsymbol{A}\widetilde{\boldsymbol{S}}^*$$

then $\boldsymbol{C}^* = \boldsymbol{B}^* = \boldsymbol{D}^{-1} \overline{\boldsymbol{B}}^* \boldsymbol{D}$ and the partition is weight-regular. We prove that in the case of a weight-regular partition $\overline{\boldsymbol{B}}^*$ is directly related with $\boldsymbol{B}^*$, and its entries are also constant. □

# 3. Eigenvalue interlacing in weight parameters of graphs

For each parameter of a graph involving the cardinality of some vertex sets, we can define its corresponding weight parameter by giving some weights (that means, the entries of the positive eigenvector) to the vertices and replacing cardinalities by square norms. The main idea is that such weights regularize the graph, and hence allow us to define a kind of regular partition. It has been showed that interlacing can provide results on some weight parameters. Thus, using these weights we can also consider the so-called weight-regular partitions of a graph, which generalize the standard notion of regular partitions. In [**26**] Fiol finds some bounds for graph parameters in the non-regular case.

The eigenvalues of the adjacency matrix $\boldsymbol{A}$ of $\Gamma$ will be denoted by $\lambda_1 \geq \cdots \geq \lambda_n$. If $\Gamma$ is connected, Perron-Frobenius Theorem assures that $\lambda_1$ is simple, positive and with positive eigenvector. If $\Gamma$ is not connected, the existence of such an eigenvector is not guaranteed, unless all its connected components have the same maximum eigenvalue. For these results it is supposed that the eigenvalue $\lambda_1$ has a positive eigenvector, denoted by $\boldsymbol{\nu}$, which is normalized in such a way its minimum entry is 1.

In this context, the notion of a "weight parameter" can be introduced. For each parameter of a graph $\Gamma$, say $\xi$, defined as the maximum cardinality of a set $U \subset V$ satisfying a given property $P$, we define the corresponding *weight parameter*, denoted by $\xi^*$, as the maximum value of $||\boldsymbol{\rho} U||^2$ of a vertex set $U$ satisfying $P$. Note that, when the graph is regular, we have $\boldsymbol{\nu} = \boldsymbol{j}$ and then $\xi^* \equiv \xi$.

Using the results derived above, mainly Lemma 2.7, most of the results obtained for regular graphs can be extended to general graphs (with a positive eigenvector). The only difference is that we must now consider weight parameters and weight-regular partitions. Inspired by Haemers work [**33**], Fiol [**26**] derived an upper bound for the weight independence number and for the weight Shannon capacity of a graph. As a straightforward consequence of the former, we then obtain the well-known Hoffman's upper bound for the chromatic number. We also contribute with an upper bound for the weight independence number in terms of the smallest degree.

### 3.1. The weight independence number: the largest coclique.

Define the *weight independence number of* $\Gamma$ as

$$\alpha^* := \max_{U \subset V} \{||\boldsymbol{\rho} U||^2 : U \text{ is an independent set}\}.$$

Recall also that $\Gamma$ is distance-regular around $C$, with eccentricity $\mathcal{E} = \mathrm{ecc}(C)$, if the distance partition $V = C_0 \cup C_1 \cup \cdots \cup C_{\mathcal{E}}$ is regular, that is the numbers

$$c_k := |\Gamma(v) \cap \Gamma_{k-1}(u)|, \quad a_k := |\Gamma(v) \cap \Gamma_k(u)|, \quad b_k := |\Gamma(v) \cap \Gamma_{k+1}(u)|,$$

do not on the chosen vertices $u, v \in V$, only on their distance $k$. The set $C$ is also referred to as a *completely regular set* or *completely regular code* (see [**30**]). In other words, a vertex subset $C \subset V$ is said to be a completely regular code if the distance partition around $C$, that is, $V = C \cup N_1(C) \cup \cdots \cup N_{\mathrm{ecc}_C}(C)$, is weight-regular.

THEOREM 3.1. *Let $\Gamma$ be a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$ and positive eigenvector $\boldsymbol{\nu}$. Then, its weight independence number satisfies*

$$\alpha^* \leq \frac{\|\boldsymbol{\nu}\|^2}{1 - \frac{\lambda_1}{\lambda_n}} \tag{8}$$

*If the bound is attained for some independent set $C$, then $C$ is a completely weight-regular code with eccentricity $\mathrm{ecc}_C = 2$.*

***Proof.*** Let $C \subset V$ such that $\alpha^* = ||\boldsymbol{\rho}C||^2$, and let $\mathcal{P}$ be the partition $V_1 \cup V_2 = C \cup \overline{C}$, where $\overline{C} := V\backslash C$. Then, the normalized weight-quotient matrix of $\boldsymbol{A} := \boldsymbol{A}(\Gamma)$ with respect to $\mathcal{P}$ turns out to be

$$\overline{\boldsymbol{B}}^* = \lambda_1 \begin{pmatrix} 0 & \frac{||\boldsymbol{\rho}C||^2}{||\boldsymbol{\rho}C||||\boldsymbol{\rho}\overline{C}||} \\ \frac{||\boldsymbol{\rho}C||^2}{||\boldsymbol{\rho}C||||\boldsymbol{\rho}\overline{C}||} & \frac{||\boldsymbol{\rho}\overline{C}||^2-||\boldsymbol{\rho}C||^2}{||\boldsymbol{\rho}\overline{C}||^2} \end{pmatrix}$$

with eigenvalues $\mu_1 = \lambda_1$ and

$$\mu_2 = \operatorname{tr} \overline{\boldsymbol{B}}^* - \lambda_1 = \frac{-\lambda_1||\boldsymbol{\rho}C||^2}{||\boldsymbol{\nu}||^2 - ||\boldsymbol{\rho}C||^2} = \frac{-\lambda_1\alpha^*}{||\boldsymbol{\nu}||^2 - \alpha^*}.$$

Hence, since $\mu_2 \geq \lambda_n$ by Lemma 2.7, the result follows. In addition, if equality holds, then the interlacing is tight (since $\mu_1 = \lambda_1$ and $\mu_2 = \lambda_n$) and therefore the partition is weight-regular. In particular, from the corresponding weight-regular-quotient matrix $\boldsymbol{B}^* = \boldsymbol{D}\overline{\boldsymbol{B}}^*\boldsymbol{D}^{-1}$, we get that, for every vertex $u \in \overline{C}$,

$$b_{21}^* = \frac{1}{\nu_u} \sum_{v\in\Gamma(u)\cap C} \nu_v = \frac{\lambda_1||\boldsymbol{\rho}C||^2}{||\boldsymbol{\rho}\overline{C}||^2} = -\lambda_n \neq 0.$$

Consequently, $\operatorname{ecc}_C = 2$ and $\mathcal{P}$ is the distance partition around $C$. □

Let $\nu_{max} := \max_{u\subset V}\{\nu_u\}$. Then, since $\alpha^* \geq \nu_{max}^2$ the above theorem gives

$$1 - \frac{\lambda_1}{\lambda_n} \leq \frac{||\boldsymbol{\nu}||^2}{\nu_{max}^2} \leq n$$

for any such graph $\Gamma$, with equality holding in both if and only if $\Gamma$ is the complete graph $K_n$.

From the above Theorem 3.1 we can derive an upper bound for the weight independence number in terms of the smallest degree of $\Gamma$.

THEOREM 3.2. *Let $\Gamma$ be a graph with eigenvalues $\lambda_1 \geq \cdots \geq \lambda_n$, positive eigenvector $\boldsymbol{\nu}$ and smallest degree $\delta_{min}$. Then, its weight independence number satisfies*

$$\alpha^* \leq \frac{||\boldsymbol{\nu}||^2}{1 - \frac{\delta_{min}^2}{\lambda_1\lambda_n}}.$$

***Proof.*** Now the quotient matrix $\overline{\boldsymbol{B}}^*$ is the same as above, with $\mu_1 = \lambda_1$ and $\mu_2 = \operatorname{tr}\overline{\boldsymbol{B}}^* - \lambda_1 = \frac{-\lambda_1\alpha^*}{||\boldsymbol{\nu}||^2-\alpha^*}$. Since $\delta_{min} \leq \lambda_1$, using interlacing we get

$$-\lambda_1\lambda_2 \geq -\mu_1\mu_2 = -\det(\boldsymbol{B}) = \frac{\lambda_1^2\alpha^*}{||\boldsymbol{\nu}||^2 - \alpha^*} \geq \frac{\delta_{min}^2\alpha^*}{||\boldsymbol{\nu}||^2 - \alpha^*},$$

which yields the required inequality. □

### 3.2. Chromatic number.

As a corollary of Theorem 3.1 we can get the known result of Hoffman [**38**], which provides a lower bound on the chromatic number $\chi$ of any graph $\Gamma$.

Corollary 3.3. [**38**] *Let* $\Gamma$ *be a graph with eigenvalues* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$*. Then, its chromatic number satisfies*

$$\chi \geq 1 - \frac{\lambda_1}{\lambda_n}. \tag{9}$$

***Proof.*** Suppose first $\Gamma$ is connected, with positive eigenvector $\boldsymbol{\nu}$. Since, for any minimum coloring of $\Gamma$, each color class $U_i$, $1 \leq i \leq \chi$, is an independent set, we have $||\boldsymbol{\rho} U_i||^2 \leq \alpha^*$. Hence, $\chi \geq \frac{||\boldsymbol{\nu}||^2}{\alpha^*}$ and Eq. (8) yields the result. Otherwise, if $\Gamma$ is disconnected, we only need to apply Eq. (9) to any connected component with maximum eigenvalue $\lambda_1$. □

A direct proof of Eq. (9) was given by Haemers [**32**]. His proof also uses eigenvalue interlacing, and so it is different from Hoffman's original one. However, excepting for the regular case, Haemer's proof is not related to any independence-like number. As cited by that author in Ref. [**41**], his proof has become a common example of application of the interlacing technique.

When $\Gamma$ is regular, Theorem 3.1 reduces to the following bound for the (standard) independence number:

$$\alpha \leq \frac{n}{1 - \frac{\lambda_1}{\lambda_n}}. \tag{10}$$

The first published proof is due to Lovász [**40**] who derived the same upper bound for the *Shannon capacity* of $\Gamma$.

### 3.3. Shannon capacity.

The weight version of the Shannon capacity can be defined as

$$\theta^* := \sup_l \sqrt[l]{\alpha^*(\Gamma^l)}$$

and, as expected, it can be shown to be bounded above by the weight analogue of Lovász bound, as the next theorem shows. (To prove it, recall that the *Kronecker product* of two matrices $\boldsymbol{A} \otimes \boldsymbol{B}$ is obtained by replacing each entry $(\boldsymbol{A})_{uv}$ with the matrix $(\boldsymbol{A})_{uv}\boldsymbol{B}$. Then if $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ are eigenvectors of $\boldsymbol{A}$ and $\boldsymbol{B}$, with corresponding eigenvalues $\lambda$ and $\mu$, respectively, then $\boldsymbol{\nu} \otimes \boldsymbol{\eta}$ —viewing $\boldsymbol{\nu}$ and $\boldsymbol{\eta}$ as 1-column matrices— is an eigenvector of $\boldsymbol{A} \otimes \boldsymbol{B}$, with eigenvalue $\lambda\mu$.)

We can find the following result in [**26**].

THEOREM 3.4. *Let* $\Gamma$ *be a graph with eigenvalues* $\lambda_1 \geq \cdots \geq \lambda_n$ *and positive eigenvector* $\boldsymbol{\nu}$*. Then, its weight Shannon capacity satisfies*

$$\theta^* \leq \frac{||\boldsymbol{\nu}||^2}{1 - \frac{\lambda_1}{\lambda_n}}.$$

***Proof.*** The proof goes along the same lines as that given by Haemers [**33**] in the regular case. Note that the above results remain valid for any symmetric matrix $\boldsymbol{A}^*$ with $(\boldsymbol{A}^*)_{uv} = 0$ if $u \not\sim v$, which has maximum eigenvalue with a positive eigenvector. Then the application of Theorem 3.1 to the matrix

$$\boldsymbol{A}^*(\Gamma^l) := (\boldsymbol{A} - \lambda_n \boldsymbol{I}) \otimes \overset{l}{\cdots} \otimes (\boldsymbol{A} - \lambda_n \boldsymbol{I}) - (-\lambda_n)^l$$

with maximum eigenvalue $(\lambda_1 - \lambda_n)^l - (-\lambda_n)^l$, positive eigenvector $\boldsymbol{\nu} \times \overset{l}{\cdots} \times \boldsymbol{\nu}$, and minimum eigenvalue $-(-\lambda_n)^l$ gives

$$\alpha^*(\Gamma^l) \leq \left( \frac{||\boldsymbol{\nu}||^2}{1 - \frac{\lambda_1}{\lambda_n}} \right)^l,$$

whence the result follows. □

Note that, since $\alpha^* \leq \theta^*$ and $\theta \leq \theta^*$, the above result yields also bounds for both $\alpha^*$ (that is Theorem 3.1) and $\theta$, the (standard) Shannon capacity of a (not necessarily regular) graph.

# Chapter 4
# Regularity Properties in Bipartite Graphs

Bipartite graphs are combinatorial objects that show some interesting symmetries. For instance, their spectra are symmetric about zero, as the the corresponding eigenvectors come into pairs. Moreover, vertices in the same (respectively, different) independent set are always at even (respectively, odd) distance. Both properties have well-known consequences in most parameters of such graphs. Roughly speaking, we could say that the conditions for a given property to be satisfied in a general graph can be somehow relaxed to guarantee the same property for a bipartite graph. In this chapter we focus on this phenomenon in the framework of regular and distance-regular graphs, for which several characterizations of combinatorial or algebraic nature are known.

We also see some characterizations of bipartite graphs (and also of bipartite distance-regular graphs) involve such parameters as the numbers of walks between vertices (entries of the powers of the adjacency matrix $\boldsymbol{A}$), the crossed local multiplicities (entries of the idempotents $\boldsymbol{E}_i$ or eigenprojectors) and so on.

## 1. Preliminaries

Recall that $\Gamma$ is bipartite if and only if it does not contain odd cycles. Then, its adjacency matrix is of the form

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix}.$$

Here and hereafter, it is assumed that the block matrices have the appropriate dimensions. Moreover, for any polynomial $p \in \mathbb{R}_d[x]$ with even and odd parts $p_0$ and $p_1$, we have

$$p(\boldsymbol{A}) = p_0(\boldsymbol{A}) + p_1(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{C} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{D} \end{pmatrix} + \begin{pmatrix} \boldsymbol{O} & \boldsymbol{M} \\ \boldsymbol{M}^\top & \boldsymbol{O} \end{pmatrix}. \tag{11}$$

Also, the spectrum of $\Gamma$ is symmetric about zero: $\theta_i = -\theta_{d-i}$ and $m_i = m_{d-i}$, $i = 0, 1, \ldots, d$. (In fact, a well-known result states that a connected graph $\Gamma$ is bipartite if and only if $\theta_0 = -\theta_d$; see, for instance, Cvetković, Doob and Sachs Cvetković et al. [**17**].) This is due to the fact that, if $(\boldsymbol{u}|\boldsymbol{v})^\top$ is a (right) eigenvector with eigenvalue $\theta_i$, then $(\boldsymbol{u}| - \boldsymbol{v})^\top$ is an eigenvector for the eigenvalue $-\theta_i$.

From any of the expressions of $\boldsymbol{E}_i$, we deduce that, when $\Gamma$ is bipartite, such parameters satisfy:

- $m_{uv}(\theta_i) = m_{uv}(\theta_{d-i})$, $\quad i = 0, 1, \ldots, d$, if $\partial(u, v)$ is even.
- $m_{uv}(\theta_i) = -m_{uv}(\theta_{d-i})$, $\quad i = 0, 1, \ldots, d$, if $\partial(u, v)$ is odd.

In particular, the local multiplicities bear the same symmetry as the standard multiplicities: $m_u(\theta_i) = m_u(\theta_{d-i})$ for any vertex $u \in V$ and eigenvalue $\theta_i$, $i = 0, 1, \ldots, d$.

From the above, notice that, when $\Gamma$ is regular and bipartite, we have $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J}$ and

$$\boldsymbol{E}_d = \frac{1}{n}\begin{pmatrix} \boldsymbol{J} & -\boldsymbol{J} \\ -\boldsymbol{J} & \boldsymbol{J} \end{pmatrix}. \tag{12}$$

## 2. Spectrum and regularity

A direct consequence of Interlacing Theorem (Chapter 1, Theorem 3.1) is the following result.

Corollary 2.1. *Let $\boldsymbol{A}$ be a symmetric matrix with eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and partitioned as follows*

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_{1,1} & \cdots & \boldsymbol{A}_{1,m} \\ \vdots & & \vdots \\ \boldsymbol{A}_{m,1} & \cdots & \boldsymbol{A}_{m,m} \end{bmatrix},$$

*such that $\boldsymbol{A}_{i,i}$ is square for $i = 1, \ldots, m$. For some integer $m < n$, define the $m \times m$ matrix $\boldsymbol{B} = \widetilde{\boldsymbol{B}} = (b_{ij})$ such that the entries $b_{ij}$ are the average row sum of $\boldsymbol{A}_{i,j}$, for $i, j = 1, \ldots, m$. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be the eigenvalues of $\boldsymbol{B}$. Then,*

*(i) The eigenvalues of $\boldsymbol{B}$ interlace the eigenvalues of $\boldsymbol{A}$.*

*(ii) If the interlacing is tight, then $\boldsymbol{A}_{i,j}$ has constant row and column sums for $i, j = 1, \ldots, m$.*

*(iii) If, for $i, j = 1, \ldots, m$, $\boldsymbol{A}_{i,j}$ has constant row and column sums, then any eigenvalue of $\boldsymbol{B}$ is also an eigenvalue of $\boldsymbol{A}$.*

***Proof.*** Let $n_i$ be the size of $\boldsymbol{A}_{i,i}$. Define

$$\widetilde{\boldsymbol{S}}^{\top} = \left(\begin{array}{ccc|ccc|c|ccc} 1 & \cdots & 1 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & \cdots & 1 & \cdots & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & \vdots & & \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 1 & \cdots & 1 \end{array}\right),$$

where the block $i$ has size $n_i$, $i = 1, \ldots, n$. Consider $\boldsymbol{D} = \operatorname{diag}(\sqrt{n_1}, \ldots, \sqrt{n_m})$, and $\boldsymbol{S} = \widetilde{\boldsymbol{S}}^{\top}\boldsymbol{D}^{-1}$. Then $\widetilde{\boldsymbol{S}}^{\top}\widetilde{\boldsymbol{S}} = \boldsymbol{D}^2$. We easily see that $(\widetilde{\boldsymbol{S}}^{\top}\boldsymbol{A}\widetilde{\boldsymbol{S}})_{ij}$ equals the sum of the entries of $\boldsymbol{A}_{i,j}$. Hence

$$\boldsymbol{B} = \widetilde{\boldsymbol{S}}^{\top}\boldsymbol{A}\widetilde{\boldsymbol{S}}\boldsymbol{D}^{-2}.$$

By 3.1.$(i)$ we know that the eigenvalues of $\boldsymbol{S}^{\top}\boldsymbol{A}\boldsymbol{S}$ interlace the eigenvalues of $\boldsymbol{A}$. But $\boldsymbol{B}$ has the same eigenvalues as $\boldsymbol{S}^{\top}\boldsymbol{A}\boldsymbol{S}$, since

$$\boldsymbol{S}^{\top}\boldsymbol{A}\boldsymbol{S} = \boldsymbol{D}^{-1}\widetilde{\boldsymbol{S}}^{\top}\boldsymbol{A}\widetilde{\boldsymbol{S}}\boldsymbol{D}^{-1} = \boldsymbol{D}^{-1}\boldsymbol{B}\boldsymbol{D}.$$

This proves $(i)$.

It is easily checked that $\boldsymbol{A}\boldsymbol{S} = \boldsymbol{S}(\boldsymbol{D}^{-1}\boldsymbol{B}\boldsymbol{D})$ reflects that $\boldsymbol{A}_{i,j}$ has constant row sum for all $i, j = 1, \ldots, m$. Hence 3.1.$(iii)$ implies $(ii)$.

On the hand, if $\boldsymbol{A}\boldsymbol{S} = \boldsymbol{S}\boldsymbol{D}^{-1}\boldsymbol{B}\boldsymbol{D}$ and $\boldsymbol{B}\boldsymbol{U} = \mu_i\boldsymbol{U}$ for some matrix $\boldsymbol{U}$ and integer $i$, then $\boldsymbol{A}(\boldsymbol{S}\boldsymbol{D}^{-1}\boldsymbol{U}) = \mu_i\boldsymbol{S}\boldsymbol{D}^{-1}\boldsymbol{U}$, and $\operatorname{rank}\boldsymbol{U} = \operatorname{rank}\boldsymbol{S}\boldsymbol{D}^{-1}\boldsymbol{U}$. This proves $(iii)$. □

This result will be used in the next proposition.

Recall that the average degree satisfies

$$\overline{\delta} = \frac{1}{n}\sum_{u \in V}\delta_u = \frac{1}{n}\operatorname{tr}\boldsymbol{A}^2 = \frac{1}{n}\sum_{i=0}^{d} m_i\theta_i^2,$$

and it holds that $\overline{\delta} \leq \lambda_1$ by Interlacing Theorem. In this case, the matrix quotient is $\boldsymbol{B} = (\overline{\delta})$, with the the eigenvalue $\mu_1$. In particular, $\overline{\delta} = \lambda_1$ if and only if $\Gamma$ is $\overline{\delta}$-regular.

We show that there is an analog result for bipartite graphs. A bipartite graph $\Gamma = (V_1 \cup V_2, E)$ is called $(\delta_1, \delta_2)$-*biregular* when all $n_1$ vertices of $V_1$ has degree $\delta_1$, and the $n_2$ vertices of $V_2$ has degree $\delta_2$. So, $n_1\delta_1 = n_2\delta_2$. For a bipartite graph, define $\overline{\delta}_1$ and $\overline{\delta}_2$ as the average degree of the vertices of $V_1$ and $V_2$, respectively.

PROPOSITION 2.2. *Let $\Gamma = (V_1 \cup V_2, E)$ be a bipartite graph with $n = n_1 + n_2$ vertices, average degrees $\overline{\delta}_1$ and $\overline{\delta}_2$ and maximum eigenvalue $\lambda_1$. Then,*

$$\overline{\delta}_1\overline{\delta}_2 \le \lambda_1^2 \tag{13}$$

*and equality holds if and only if* $\Gamma$ *is* $(\overline{\delta}_1, \overline{\delta}_2)$*-biregular.*

***Proof.*** As $\Gamma$ is a bipartite graph it follows that $\boldsymbol{A}_{1,1} = \boldsymbol{A}_{2,2} = \boldsymbol{O}$. Let

$$\boldsymbol{A} = \begin{bmatrix} \mathbf{0} & \boldsymbol{A}_{1,2} \\ \boldsymbol{A}_{2,1} & \mathbf{0} \end{bmatrix}$$

be the adjacency matrix of $\Gamma$. Consider the following quotient matrix

$$\boldsymbol{B} = \begin{pmatrix} 0 & \overline{\delta}_1 \\ \overline{\delta}_2 & 0 \end{pmatrix},$$

whose entries are the average row sums of the block matrices of $\boldsymbol{A}$. Since the eigenvalues of $\boldsymbol{B}$ are $\pm\sqrt{\overline{\delta}_1\overline{\delta}_2}$, Theorem 2.1.($i$) gives

$$\mu_1 = \sqrt{\overline{\delta}_1\overline{\delta}_2} \le \lambda_1.$$

Moreover, in case of equality, $\mu_2 = -\mu_1 = -\lambda_1 = \lambda_n$ so that the interlacing is tight and Theorem 2.1.($ii$) implies the biregularity of $\Gamma$. $\square$

Regarding the above proposition, note that if more is known about the structure of $\Gamma$ or of some of its induced subgraphs, it is often possible to get better results by a more detailed application of Interlacing Theorem. In relation to the size of $\Gamma'$, better bounds can be obtained if more is known about the structure of $\Gamma'$ by considering a refinement of the partition, for instance, the case if $\Gamma'$ is bipartite. The following result, which is a generalization of a result due to Haemers [**32**] for the case of a bipartite induced subgraph, illustrates it.

PROPOSITION 2.3. *Let* $\Gamma$ *be a* $\delta$*-regular graph on* $n$ *vertices, with eigenvalues* $\lambda_1 \ge \cdots \ge \lambda_n$. *Let* $\Gamma'$ *be a bipartite induced subgraph of* $\Gamma$ *with* $n_1 + n_2$ *vertices and average degrees* $\overline{\delta}_1$, $\overline{\delta}_2$. *Let* $x_1$ *and* $x_2$, $x_1 \ge x_2$ *be the zeros of*

$$Ax^2 + Bx + C,$$

*where*

$$A = n_3,$$
$$B = n_1\delta + n_2\delta - n_1\overline{\delta}_1 - n_2\overline{\delta}_2,$$
$$C = \delta(\overline{\delta}_2 n_2 + \overline{\delta}_1 n_1) - (\overline{\delta}_2^2 n_2 + \overline{\delta}_1^2 n_1 + \overline{\delta}_1\overline{\delta}_2 n_3).$$

*Then*

$$\lambda_2 \ge x_1 \quad \textit{and} \quad \lambda_n \le x_2.$$

***Proof.*** Note that $n = n_1 + n_2 + n_3$. Without loss of generality, let $\Gamma$ have adjacency matrix

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{O} & \boldsymbol{A}_{1,2} & \boldsymbol{A}_{1,3} \\ \boldsymbol{A}_{2,1} & \boldsymbol{O} & \boldsymbol{A}_{2,3} \\ \boldsymbol{A}_{3,1} & \boldsymbol{A}_{3,2} & \boldsymbol{A}_{3,3} \end{bmatrix},$$

where the diagonal block matrices are square of sizes $n_1$, $n_2$ and $n_3$, respectively. Consider the quotient matrix

$$\boldsymbol{B} = \begin{bmatrix} 0 & \overline{\delta}_1 & \delta - \overline{\delta}_1 \\ \overline{\delta}_2 & 0 & \delta - \overline{\delta}_2 \\ \frac{n_1}{n_3}(\delta - \overline{\delta}_1) & \frac{n_2}{n_3}(\delta - \overline{\delta}_2) & \delta - \frac{n_1\delta + n_2\delta - n_1\overline{\delta}_1 - n_2\overline{\delta}_2}{n_3} \end{bmatrix},$$

with eigenvalues $\mu_1 \geq \mu_2 \geq \mu_3$. Using Interlacing Theorem, we know that the eigenvalues of $\boldsymbol{B}$ interlace the eigenvalues of $\boldsymbol{A}$. Note that $\mu_1 = \lambda_1 = \delta$ and hence

$$\mu_2\mu_3 = \frac{\det(\boldsymbol{B})}{\delta} = \frac{\delta(n_1\overline{\delta}_1 + n_2\overline{\delta}_2) - (\overline{\delta}_2^2 n_2 + \overline{\delta}_1^2 n_1 + \overline{\delta}_1\overline{\delta}_2 n_3)}{n_3},$$

$$\mu_2 + \mu_3 = \operatorname{tr}(\boldsymbol{B}) - \delta = \frac{n_1\overline{\delta}_1 + n_2\overline{\delta}_2 - n_1\delta - n_2\delta}{n_3}.$$

This yields $x_1 = \mu_2$ and $x_2 = \mu_3$, and the interlacing gives the required result. □

EXAMPLE 2.4. *Let $\Gamma = P$ be the Petersen graph, 3-regular, $n = 10$. The Petersen graph has spectrum $\{3, 1^5, -2^4\}$.*

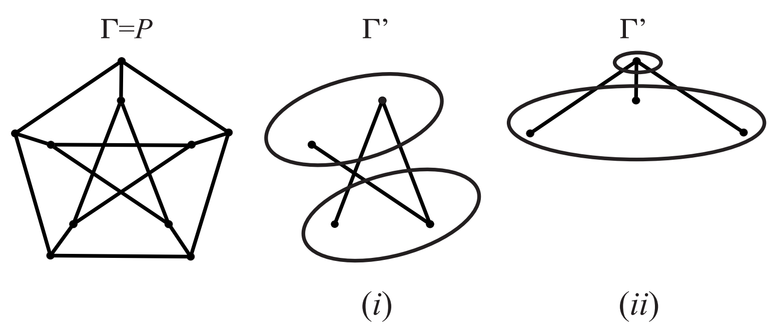



FIG. 1. Petersen graph and two possible bipartite induced subgraphs

*We wish to find a bipartite induced subgraph $\Gamma'$, for example the one shown in Figure 1.(i). Consider the quotient matrix*

$$\boldsymbol{B} = \begin{pmatrix} 0 & \frac{3}{2} & \frac{3}{2} \\ \frac{3}{2} & 0 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & 2 \end{pmatrix},$$

*which has eigenvalues* $\mu_1 = 3$*,* $\mu_2 = \frac{1}{2}$ *and* $\mu_3 = -\frac{3}{2}$ *(note that the two smallest eigenvalues are the zeros of the polynomial* $6x^2 - 6x - \frac{9}{2}$*). Then, by Proposition 2.3,*

$$\lambda_2 \geq \frac{1}{2} \quad \textit{and} \quad \lambda_{10} \leq -\frac{3}{2}.$$

*Take now another bipartite induced subgraph* $\Gamma'$ *of* $\Gamma$*, for example the induced subgraph drawn in Figure 1.(ii). Now the quotient matrix is*

$$\boldsymbol{B} = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 2 \\ 0 & 1 & 2 \end{pmatrix},$$

*with eigenvalues* $\mu_1 = 3$*,* $\mu_2 = 1$ *and* $\mu_3 = -2$*. Then, by Proposition 2.3,*

$$\lambda_2 \geq 1 \quad \textit{and} \quad \lambda_{10} \leq -1.$$

The following corollary follows from Proposition 2.3.

COROLLARY 2.5. *Let* $\Gamma$ *be a* $\delta$*-regular graph on* $n$ *vertices, with eigenvalues* $\lambda_1 \geq \cdots \geq \lambda_n$*. Then, the best upper bounds for* $\lambda_2$ *and* $\lambda_n$ *in Proposition 2.3 are reached taking the maximum induced complete bipartite subgraph* $\Gamma'$ *of* $\Gamma$*.*

# 3. Eigenvalues and the Laplacian of a graph

### 3.1. Some inequalities for Laplacian eigenvalues.

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ be the Laplacian eigenvalues. Let $d_1 \geq d_2 \geq \cdots \geq d_n$ be the degrees, ordered non-increasingly.

There are some inequalities for the eigenvalues of the Laplacian matrix. The first one is

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} d_i. \tag{14}$$

Note that if $m = n$ we have equality in (14), because then it says that the trace is the sum of the eigenvalues. To get the $n \times n$ Laplacian matrix $\boldsymbol{L}$, we order the vertices according to their degrees. Let $\boldsymbol{B}$ be the $m \times m$ submatrix of $\boldsymbol{L}$ indexed by the subindexes corresponding to the $m$ largest degrees:

$$\boldsymbol{L} = \left( \begin{array}{c|c} \boldsymbol{B} & \\ \hline & \end{array} \right),$$

Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ be the eigenvalues of $\boldsymbol{B}$. Then it holds that

$$\operatorname{tr} \boldsymbol{B} = \sum_{i=1}^{m} d_i = \sum_{i=1}^{m} \mu_i,$$

and since $\boldsymbol{B}$ is the principal submatrix of $\boldsymbol{L}$, the eigenvalues of $\boldsymbol{B}$ interlace the eigenvalues of $\boldsymbol{L}$, so it gives (14).

The next result is due to Guo, who proved that if the graph is connected and $m \neq n$ then

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i=1}^{m} d_i + 1. \tag{15}$$

Note that if we take $m = 1$ in (15), we get that $\lambda_1 \geq d_1 + 1$. Guo conjectured another generalization looking at individual eigenvalues, which was proved by Brouwer and Haemers [**12**]. They showed that if $\lambda_j$ is the $j$-th largest Laplacian eigenvalue, and $d_j$ is the $j$-th largest degree ($1 \leq j \leq n$) of a connected graph $\Gamma$ on $n$ vertices, then $\lambda_j \geq d_j - j + 2$ ($1 \leq j \leq n-1$).

### 3.2. Dominating sets.

A *dominating set* in a graph $\Gamma$ is a vertex subset $D \subseteq V$ such that every vertex in $V \backslash D$ is adjacent to some vertex in $D$. The *domination number* of $\Gamma$, written as $\gamma(\Gamma)$, is the minimum size of a dominating set in $\Gamma$.

A *$k$-dominating set* in a graph $\Gamma$ is a vertex subset $D \subseteq V$ such that every vertex not in $D$ has at least $k$ neighbours in $D$, that is, $D \subseteq V$ is a $k$-dominating set if for every $v \in V \backslash D$ there exist $u_1, \ldots, u_k \in D$ such that $u_i \sim v$ for all $i = 1, \ldots, k$ (see Figure 2).

The next proposition can be seen as a generalization of the Guo's result for the case of $k$-dominating sets. This results gives a condition on the existence of $k$-dominating sets.

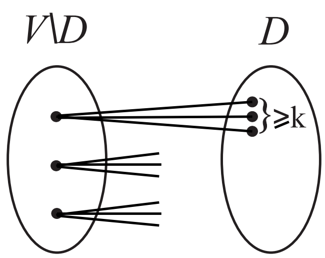



FIG. 2. $D$ is a $k$-dominating set

PROPOSITION 3.1. *Let $\Gamma$ be a finite simple graph on $n$ vertices, with vertex degrees $d_1 \geq d_2 \geq \cdots \geq d_n$, and Laplacian eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$. Let $D$ be a $k$-dominating set in $\Gamma$ with $m = |D|$. Then,*

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i \in D} d_i + k. \tag{16}$$

***Proof.*** Consider the principal submatrix $\boldsymbol{L}_D$ of $\boldsymbol{L}$ with rows and columns indexed by $D$. Consider the quotient matrix $\boldsymbol{B} = \widetilde{\boldsymbol{B}} = (b_{ij})$ of $\boldsymbol{L}$ for the partition of the vertex set $V$ into $m+1$ parts: $\{i\}$ for $i \in D$ and $V \backslash D$. Let $\boldsymbol{S}$ be the $n \times (m+1)$ characteristic matrix. Let $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_{m+1}$ be the eigenvalues of $\boldsymbol{B}$. We have

$$\boldsymbol{B} = \left( \begin{array}{ccc|c} & & & b_{1(m+1)} \\ & \boldsymbol{L}_D & & \vdots \\ & & & b_{m(m+1)} \\ \hline b_{(m+1)1} & \cdots & b_{(m+1)m} & x \end{array} \right),$$

where $b_{ij} = \frac{1}{|V_i|}(\boldsymbol{S}^\top \boldsymbol{L} \boldsymbol{S})_{i,j}$ and $x = b_{(m+1)(m+1)}$.

We know that

$$\operatorname{tr} \boldsymbol{B} = \sum_{i=1}^{m+1} \mu_i = \sum_{i \in D} d_i + x.$$

From the definition of a $k$-dominating set, it follows that $x \geq k$. Then,

$$\operatorname{tr}\boldsymbol{B} = \sum_{i\in D} d_i + x \geq \sum_{i\in D} d_i + k,$$

and since the quotient matrix $\boldsymbol{B}$ has row sum equal to 0, it implies that $\boldsymbol{B}$ has an eigenvalue equal to 0. Then, by interlacing,

$$\sum_{i=1}^{m} \lambda_i \geq \sum_{i\in D} d_i + k,$$

which finishes the proof. □

For $\delta$-regular graphs, the above result leads to

$$\sum_{i=1}^{m} \lambda_i \geq m\delta + k,$$

which improves the inequality (15) due to Guo,

$$\sum_{i=1}^{m} \lambda_i \geq m\delta + 1,$$

in the case when there exists a $k$-dominating set.

The following example illustrates it.

EXAMPLE 3.2. *Let $Q_3$ be the hypercube graph with $2^3$ vertices. The eigenvalues of its Laplacian matrix are $\{6^3, 2^3, 0\}$, and its degree sequence has a constant value of $\delta = 3$. Let $D$ be a 3-dominating set with $m = |D| = 4$.*

*Bound (15) gives that $\sum_{i=1}^{4} \lambda_i \geq 3 \cdot 4 + 1 = 13$, whilst our bound (16) gives $\sum_{i=1}^{4} \lambda_i \geq 3 \cdot 4 + 3 = 15$.*

One easily check that for the regular case and with $k = 1$, Proposition 3.1 leads to Guo's inequality. It should not be surprising that for this case we obtain the same result as Guo, because he also uses eigenvalue interlacing. Let us see it with an example.

EXAMPLE 3.3. *Let $\Gamma = P$ be the Petersen graph, a known regular graph with $\delta = 3$. The Laplacian eigenvalues of $P$ are $\{5^4, 2^5, 0\}$. Let $D$ be the 1-dominating set in $P$, with $m = |D| = 3$. Then, Proposition 3.1 reduces to Guo's inequality (15),*

$$\sum_{i=1}^{3} \lambda_i \geq 3 \cdot 3 + 1 = 10.$$

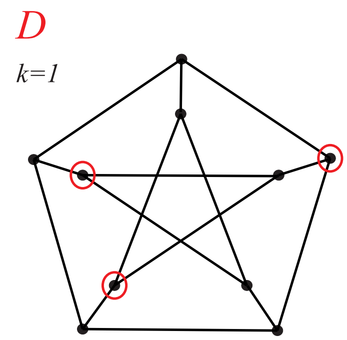



FIG. 3. 1-dominating set in the Petersen graph

The next result is a consequence of Proposition 3.1 for the particular case of biregular graphs.

COROLLARY 3.4. *Let $\Gamma = (V_1 \cup V_2, E)$ be a $(\delta_1, \delta_2)$-biregular graph with $|V_1| = n_1$ and $|V_2| = n_2$. Let $\lambda_1 \geq \cdots \geq \lambda_n$ be the Laplacian eigenvalues of $\Gamma$. Since the set $V_2$ is $\delta_1$-dominant, then*

$$\sum_{i=1}^{n_2} \lambda_i \geq n_2\delta_2 + \delta_1.$$

# 4. Bipartite distance-regular graphs

A general phenomenon is that the above conditions for being distance-regular can be relaxed giving more 'economic' characterizations (see [**29**]). Thus, the purpose of the following three theorems is twofold: First to show how, for general graphs, such conditions can be relaxed if we assume some extra natural hypothesis (such as regularity) and, second, to study what happens in the case of bipartite graphs. Notice that, in all the characterizations, the first results, $(a1)$, $(b1)$, $(c1)$, imply the necessity of the other conditions. Most of the results for general graphs were known, and the results for bipartite graphs are obtained as consequences.

First we give a characterization of distance-regular graphs in terms of predistance polynomials (see type $(a)$ in Chapter 1, Section 4).

THEOREM 4.1. ***(i)*** *A graph $\Gamma$ with predistance polynomials $p_0, p_1, \ldots, p_d$ is distance-regular if and only if any of the following conditions holds:*

$(a1)$ $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = 2, 3, \ldots, d$.

*(a2)* $\Gamma$ *is regular and* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = 2, 3, \ldots, d-1$.
*(a3)* $\Gamma$ *is regular and* $\boldsymbol{A}_d = p_d(\boldsymbol{A})$.
*(a4)* $\Gamma$ *is regular and* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = d-2, d-1$.

***(ii)*** *A bipartite graph* $\Gamma$ *with predistance polynomials* $p_0, p_1, \ldots, p_d$ *is distance-regular if and only if*

*(a5)* $\Gamma$ *is regular and* $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ *for* $i = 3, 4, \ldots, d-2$.

***Proof.*** Statement $(a1)$ with $i = 0, 1, \ldots, d$ is a well-known result; see, for example, Bannai and Ito [**2**]. For our case, just notice that always $p_0(\boldsymbol{A}) = \boldsymbol{A}_0 = \boldsymbol{I}$ and, as $\boldsymbol{I} + \boldsymbol{A} + \sum_{i=2}^{d} p_i(\boldsymbol{A}) = \boldsymbol{J}$, $\Gamma$ is regular and hence $p_1(\boldsymbol{A}) = \boldsymbol{A}_1 = \boldsymbol{A}$; Condition $(a2)$ is a consequence of $(a1)$ taking into account that, under the hypotheses, $\boldsymbol{A}_d = \boldsymbol{J} - \sum_{i=0}^{d-1} \boldsymbol{A}_i = H(\boldsymbol{A}) - \sum_{i=0}^{d-1} p_i(\boldsymbol{A}) = p_d(\boldsymbol{A})$ (see Dalfó et al. [**19**]); $(a3)$ was first proved by Fiol et al. in [**23**]; and $(a4)$ is a consequence of a more general result in [**19**] characterizing $m$-partially distance-regularity ($\Gamma$ is called $m$-partially distance-regular if $\boldsymbol{A}_i = p_i(\boldsymbol{A})$ for any $i = 0, 1, \ldots, m$). Thus, we only need to prove $(a5)$. This is a consequence of $(a2)$ since, if $\Gamma$ is $\delta$-regular, $\boldsymbol{A}_2 = p_2(\boldsymbol{A}) = \boldsymbol{A}^2 - \delta\boldsymbol{I}$. Moreover, from (17) and assuming first that $d$ is even,

$$\boldsymbol{A}_{d-1} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix} - \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-3} \boldsymbol{A}_i = H_1(\boldsymbol{A}) - \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-3} p_i(\boldsymbol{A}) = p_{d-1}(\boldsymbol{A})$$

whereas, if $d$ is odd,

$$\boldsymbol{A}_{d-1} = \begin{pmatrix} \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J} \end{pmatrix} - \sum_{\substack{i=0 \\ i \text{ even}}}^{d-3} \boldsymbol{A}_i = H_0(\boldsymbol{A}) - \sum_{\substack{i=0 \\ i \text{ even}}}^{d-3} p_i(\boldsymbol{A}) = p_{d-1}(\boldsymbol{A}),$$

and the proof is complete. □

With respect to the characterizations of type $(b)$ in Chapter 1, Section 4), we can state the following result:

THEOREM 4.2. ***(i)*** *A graph* $\Gamma$ *with idempotents* $\boldsymbol{E}_0, \boldsymbol{E}_1, \ldots, \boldsymbol{E}_d$ *is distance-regular if and only if any of the following conditions holds:*

*(b1)* $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 0, 1, \ldots, d$.
*(b2)* $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 0, 1, \ldots, d-1$.
*(b3)* $\Gamma$ *is regular and* $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 1, 2, \ldots, d-1$.
*(b4)* $\Gamma$ *is regular and* $\boldsymbol{E}_j \in \mathcal{D}$ *for* $j = 1, d$.

***(ii)*** *A bipartite graph* $\Gamma$ *with idempotent* $\boldsymbol{E}_1$ *is distance-regular if and only if*

*(b5)* $\Gamma$ *is regular and* $\boldsymbol{E}_1 \in \mathcal{D}$.

***Proof.*** Statement $(b1)$ is also well-known and comes from the fact that $\Gamma$ is distance-regular if and only if $\mathcal{A} = \mathcal{D}$; Condition $(b2)$ is a consequence of $(b1)$ since, under the hypotheses, $\boldsymbol{E}_d = \boldsymbol{I} - \sum_{j=0}^{d-1} \boldsymbol{E}_j \in \mathcal{D}$; $(b3)$ comes from $(b2)$ since, if $\Gamma$ is regular, then $\boldsymbol{E}_0 = \frac{1}{n}\boldsymbol{J} = \frac{1}{n}H(\boldsymbol{A}) \in \mathcal{D}$; $(b4)$ was proved by the Fiol in [**28**]. Finally, $(a5)$ can be seen as a consequence of $(b4)$ since, under the hypotheses, (12) yields

$$\boldsymbol{E}_d = \sum_{\substack{i=0 \\ i \text{ even}}}^{d} \boldsymbol{A}_i - \sum_{\substack{i=0 \\ i \text{ odd}}}^{d} \boldsymbol{A}_i \in \mathcal{D}$$

and the proof is complete. □

Now let us go to the characterizations which are given in terms of the numbers $a_{uv}^{(j)} = (\boldsymbol{A}^j)_{uv}$ of walks of length $j \geq 0$ between vertices $u, v$ at distance $\partial(u,v) = i$, $i = 0, 1, \ldots, D$ (see type $(c)$ in Chapter 1, Section 4). When such numbers do not depend on $u, v$ but only on $i$ and $j$, we write $a_{uv}^{(j)} = a_i^{(j)}$. In particular, notice that always $a_0^{(0)} = a_1^{(1)} = 1$ and $\Gamma$ is $\delta$-regular if and only if $a_2^{(2)} = \delta$.

Theorem 4.3. ***(i)*** *A graph $\Gamma$, with diameter $D$ and $d+1$ distinct eigenvalues, is distance-regular if and only if, for any two vertices $u, v$ at distance $\partial(i,j) = i$, any of the following conditions holds:*

$(c1)$ $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = 0, 1, \ldots, D$ *and* $j \geq i$.
$(c2)$ $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = 0, 1, \ldots, D$ *and* $j = i, i+1, \ldots, d$.
$(c3)$ $D = d$, *and* $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = 0, 1, \ldots, D$ *and* $j = i, i+1, \ldots, d-1$.
$(c4)$ $\Gamma$ *is regular,* $D = d$, *and* $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = 0, 1, \ldots, D-1$ *and* $j = i, i+1$.

***(ii)*** *A bipartite graph $\Gamma$ is distance-regular if and only if*

$(c5)$ $\Gamma$ *is regular,* $D = d$, *and* $a_{uv}^{(j)} = a_i^{(j)}$ *for* $i = j = 2, 3, \ldots, D-2$.

***Proof.*** Characterization $(c1)$ was first proved by Rowlinson [**44**]; Statement $(c2)$ is a straightforward consequence of $(b1)$ since $\mathcal{A} = \langle \boldsymbol{I}, \boldsymbol{A}, \boldsymbol{A}^2, \ldots, \boldsymbol{A}^d \rangle$; $(c3)$ comes from $(c2)$ since, if $\Gamma$ is regular and $D = d$, the number of $d$-walks between any two vertices $u, v$ at distance $d$, is a constant:

$$a_{uv}^{(d)} = (\boldsymbol{A}^d)_{uv} = \frac{\pi_0}{n}[H(\boldsymbol{A})]_{uv} = \frac{\pi_0}{n}(\boldsymbol{J})_{uv} = \frac{\pi_0}{n} = a_d^{(d)};$$

$(c4)$ derives from a similar result in [**28**] (not requiring $D = d$) and the above reasoning on $a_{uv}^{(d)}$. Finally, $(c5)$ is a consequence of $(c4)$ since, when $\Gamma$ is bipartite, there are no walks of length $j = i+1$ between vertices at distance $i$ and, thus, $a_i^{(i+1)} = 0$. Moreover, if $\Gamma$ is $\delta$-regular and $D = d$, $a_{d-1}^{(d-1)} = \frac{1}{\delta} a_d^{(d)} = \frac{\pi_0}{n\delta}$. □

Problem 4.4. *Give similar characterizations of types* $(a)$*,* $(b)$ *and* $(c)$ *for distance biregular graphs.*

# 5. Polynomials and regularity

The *predistance polynomials* $p_0, p_1, \ldots, p_d$, $\deg p_i = i$, associated with a given graph $\Gamma$ with spectrum $\operatorname{sp} G$ as in (1), are a sequence of orthogonal polynomials with respect to the scalar product

$$\langle f,g\rangle = \frac{1}{n}\operatorname{tr}[f(\boldsymbol{A})g(\boldsymbol{A})] = \frac{1}{n}\sum_{i=0}^{d} m_i f(\theta_i)g(\theta_i),$$

normalized in such a way that $\|p_i\|^2 = p_i(\theta_0)$ (this makes sense as it is known that always $p_i(\theta_0) > 0$). Notice that, in particular, $p_0 = 1$ and, if $\Gamma$ is $\delta$-regular, $p_1 = x$. Indeed,

- $\langle 1, x\rangle = \frac{1}{n}\sum_{i=0}^{d} m_i\theta_i = 0$.
- $\|1\|^2 = \frac{1}{n}\sum_{i=0}^{d} m_i = 1$.
- $\|x\|^2 = \frac{1}{n}\sum_{i=0}^{d} m_i\theta_i^2 = \delta = \theta_0$.

Moreover, if $\Gamma$ is bipartite, the symmetry of such a scalar product yields that $p_i$ is even (respectively, odd) for even (respectively, odd) degree $i$.

In terms of the predistance polynomials, the *preHoffman polynomial* is $H = p_0 + p_1 + \cdots + p_d$, and satisfies $H(\theta_0) = n$ (the order of the graph) and $H(\theta_i) = 0$ for $i = 1, 2, \ldots, d$ (see [**13**]).

In [**37**], Hoffman proved that a (connected) graph $\Gamma$ is regular if and only if $H(\boldsymbol{A}) = \boldsymbol{J}$, in which case $H$ becomes the *Hoffman polynomial.* (In fact, $H$ is the unique polynomial of degree at most $d$ satisfying this property.) Furthermore, when $\Gamma$ is regular and bipartite, the even and odd parts of $H$, $H_0$ and $H_1$, satisfy, by (11):

$$H_0(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J} \end{pmatrix} \quad \text{and} \quad H_1(\boldsymbol{A}) = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix}. \tag{17}$$

The following proposition can be seen as the biregular counterpart of Hoffman's result. Recall that a bipartite graph $\Gamma = (V_1 \cup V_2, E)$ is called $(\delta_1, \delta_2)$-*biregular* when all the $n_1$ vertices of $V_1$ has degree $\delta_1$, and the $n_2$ vertices of $V_2$ has degree $\delta_2$. So, counting in two ways the number of edges $m = |E|$ we have that $n_1\delta_1 = n_2\delta_2$. For a bipartite graph, define $\overline{\delta}_1$ and $\overline{\delta}_2$ as the average degree of the vertices of $V_1$ and $V_2$, respectively.

Theorem 5.1. *Let $\Gamma$ be a bipartite graph with $n = n_1 + n_2$ vertices, predistance polynomials $p_0, p_1, \ldots, p_d$, and consider the odd part of its preHoffman polynomial; that is, $H_1 = \sum_{i \text{ odd}} p_i$. Then, $\Gamma$ is biregular if and only if*

$$H_1(\boldsymbol{A}) = \alpha \begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix} \tag{18}$$

*with $\alpha = \frac{n_1+n_2}{2\sqrt{n_1 n_2}}$.*

***Proof.*** Assume first that $\Gamma$ is biregular with degrees, say, $\delta_1$ and $\delta_2$. Then, $\theta_0 = -\theta_d = \sqrt{\delta_1\delta_2}$ with respective (column) eigenvectors $\boldsymbol{u} = (\sqrt{\delta_1}\boldsymbol{j}|\sqrt{\delta_2}\boldsymbol{j})^\top$ and

$\boldsymbol{v} = (\sqrt{\delta_1}\boldsymbol{j}|-\sqrt{\delta_2}\boldsymbol{j})$, with the $\boldsymbol{j}$'s being all-1 (row) vectors with appropriate lengths. Therefore, the respective idempotents are

$$\begin{aligned}
\boldsymbol{E}_0 &= \frac{1}{\|\boldsymbol{u}\|^2}\boldsymbol{u}\boldsymbol{u}^\top = \frac{1}{n_1\delta_1 + n_2\delta_2}\begin{pmatrix} \delta_1\boldsymbol{J} & \sqrt{\delta_1\delta_2}\boldsymbol{J} \\ \sqrt{\delta_1\delta_2}\boldsymbol{J} & \delta_2\boldsymbol{J} \end{pmatrix}, \\
\boldsymbol{E}_d &= \frac{1}{\|\boldsymbol{v}\|^2}\boldsymbol{v}\boldsymbol{v}^\top = \frac{1}{n_1\delta_1 + n_2\delta_2}\begin{pmatrix} \delta_1\boldsymbol{J} & -\sqrt{\delta_1\delta_2}\boldsymbol{J} \\ -\sqrt{\delta_1\delta_2}\boldsymbol{J} & \delta_2\boldsymbol{J} \end{pmatrix}.
\end{aligned}$$

As $H_1(x) = \frac{1}{2}[H(x) - H(-x)]$ with $H(\theta_0) = n$ and $H(\theta_i) = 0$ for any $i \neq 0$, we have that $H_1(\theta_0) = n/2$, $H_1(\theta_i) = 0$ for $i \neq 0, d$, and $H_1(\theta_d) = -n/2$. Hence, using the properties and the above expressions of the idempotents,

$$\begin{aligned}
H_1(\boldsymbol{A}) &= \sum_{i=0}^{d} H_1(\theta_i)\boldsymbol{E}_i = H_1(\theta_0)\boldsymbol{E}_0 + H_1(\theta_d)\boldsymbol{E}_d \\
&= \frac{n}{2}(\boldsymbol{E}_0 - \boldsymbol{E}_d) = \frac{n\sqrt{\delta_1\delta_2}}{n_1\delta_1 + n_2\delta_2}\begin{pmatrix} \boldsymbol{O} & \boldsymbol{J} \\ \boldsymbol{J} & \boldsymbol{O} \end{pmatrix}.
\end{aligned}$$

Thus, the result follows since $n_1\delta_1 = n_2\delta_2$. Conversely, if (18) holds, and $\boldsymbol{A} = \begin{pmatrix} \boldsymbol{O} & \boldsymbol{B} \\ \boldsymbol{B}^\top & \boldsymbol{O} \end{pmatrix}$, the equality $\boldsymbol{A}H_1(\boldsymbol{A}) = H_1(\boldsymbol{A})\boldsymbol{A}$ yields

$$\begin{pmatrix} \boldsymbol{B}\boldsymbol{J} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{B}^\top\boldsymbol{J} \end{pmatrix} = \begin{pmatrix} \boldsymbol{J}\boldsymbol{B}^\top & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{J}\boldsymbol{B} \end{pmatrix}.$$

Thus, $(\boldsymbol{B}\boldsymbol{J})_{uv} = (\boldsymbol{J}\boldsymbol{B}^\top)_{uv}$ implies that $\delta(u) = \delta(v)$ for any two vertices $u, v \in V_1$, whereas $(\boldsymbol{B}^\top\boldsymbol{J})_{wz} = (\boldsymbol{J}\boldsymbol{B})_{wz}$ means that $\delta(w) = \delta(z)$ for any two vertices $w, z \in V_2$. Thus, $\Gamma$ is biregular and the proof is complete. □

Notice that the constant $\alpha$ is the ratio between the arithmetic and geometric means of the numbers $n_1, n_2$. Hence, (18) holds with $\alpha = 1$ if and only if $n_1 = n_2$ or, equivalently, $\Gamma$ is regular.

In fact, the above result could be reformulated (and proved) by saying that a (general) bipartite graph is connected and biregular if and only if there exists a polynomial satisfying (18).

# Chapter 5
# Eigenvector Function in Rayleigh's Principle

We know that the average degree of $\Gamma$, namely $\overline{\delta} = \frac{2m}{n}$, always satisfies the bound

$$\overline{\delta} \leq \lambda_1, \tag{19}$$

and equality holds if and only if $\Gamma$ is $\lambda_1$-regular.

From this result, the following questions arises.

Question 0.2. *Can we have $\overline{\delta} = \lambda_1$ for some $i \neq 1$? For paths, the answer is not, but Haemers provided an example of an unicyclic graph, namely $C_4 e S_5$ (e is an edge between a vertex of the square $C_4$ and an end of the star $S_5$) which has average degree and second eigenvalue 2, but is not regular.*

Question 0.3. *Is there some interval $I = [\alpha, \lambda_1]$, where $\alpha$ depends on the spectrum, such that $\overline{\delta} \in I$ for any $\Gamma$?*

There are probably several papers on bounds like this; some involve the spectral radius only. For example Hong [**39**] showed that $\lambda_1 \leq \sqrt{2m - n + 1}$, where $m$ is the number of edges and $n$ the number of vertices. So this gives an example of a lower bound for the average degree that we look for. It doesn't involve $\lambda_2$ though.

We can prove inequality (19), Proposition 2.2 and other similar results by using the well-known result from linear algebra known as the Rayleigh's principle. Such a result states that, if $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_n$ are the eigenvectors corresponding to $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, respectively, and for some $1 \leq i \leq j \leq n$ we have $\boldsymbol{u} \in \langle \boldsymbol{u}_i, \boldsymbol{u}_{i+1}, \ldots, \boldsymbol{u}_j \rangle$, then

$$\lambda_j \leq \frac{\langle \boldsymbol{u}, \boldsymbol{A}\boldsymbol{u} \rangle}{||\boldsymbol{u}||^2} \leq \lambda_i. \tag{20}$$

Moreover, equality on the left (respectively, right) implies that $\boldsymbol{u}$ is a $\lambda_j$-eigenvector (respectively, $\lambda_i$-eigenvector) of $\boldsymbol{A}$.

Thus the clue is to make the "right choice" of $\boldsymbol{u}$.

We can answer Question 0.3 in some particular cases, generalizing a classical result on graph spectra.

## 1. Graphs

If $\boldsymbol{u} = \boldsymbol{j}$, we have $\langle \boldsymbol{j}, \boldsymbol{A}\boldsymbol{j} \rangle = \sum_{u\in V} \delta_u$, $||\boldsymbol{j}||^2 = n$, and we get (19).

## 2. Bipartite graphs

Assume that $\Gamma$ is bipartite with stable sets $V_1$, $V_2$, number of vertices $n_1 = |V_1|$, $n_2 = |V_2|$, and average degrees $\overline{\delta}_1 = \frac{1}{n_1}\sum_{u\in V_1} \delta_u$, $\overline{\delta}_2 = \frac{1}{n_2}\sum_{u\in V_2} \delta_u$.

Notice that $n_1\overline{\delta}_1 + n_2\overline{\delta}_2 = 2m$. Then, if $\boldsymbol{u} = (\sqrt{\overline{\delta}_1}\boldsymbol{j}|\sqrt{\overline{\delta}_2}\boldsymbol{j})^\top$ with $||\boldsymbol{u}||^2 = 2m$ we get

$$\frac{\langle \boldsymbol{u}, \boldsymbol{A}\boldsymbol{u} \rangle}{||\boldsymbol{u}||^2} = \sqrt{\overline{\delta}_1\overline{\delta}_2} \leq \lambda_1.$$

By Proposition 2.2, equality is attained when $\Gamma$ is biregular.

## 3. Independence number: the largest coclique

Let $\Gamma$ be a $\delta$-regular graph with independent number $\alpha$. Suppose that a maximum independent set is $U = \{1, 2, \ldots, \alpha\}$ and take $\boldsymbol{u} = (x\boldsymbol{j}|\boldsymbol{j})^\top$ where $x$ is a variable.

In order to make a good choice for $\boldsymbol{u}$, we consider the function

$$\phi(x) = \frac{\langle \boldsymbol{u}, \boldsymbol{A}\boldsymbol{u} \rangle}{||\boldsymbol{u}||^2} = \frac{2\alpha\delta x + (n-2\alpha)\delta}{\alpha x^2 + n - \alpha}$$

which attain a maximum at $x_1 = 1$ and a minimum at $x_2 = 1 - \frac{n}{\alpha}$. The former, $\phi(x_1) = \delta$, is of no use, but the later gives

$$\lambda_n \leq \phi(x_2) = \frac{\delta}{1 - \frac{n}{\alpha}} \tag{21}$$

whence

$$\alpha \leq \frac{n}{1 - \frac{\delta}{\lambda_n}}$$

as we already knew. It is interesting to note that, in this case, the entries of $\boldsymbol{u}$ add up to zero, $\alpha(1 - \frac{n}{\alpha}) + (n - \alpha) = 0$ and hence the "right choice" is when

$$\boldsymbol{u} \in \boldsymbol{j}^{\perp} = \langle \boldsymbol{u}_2, \boldsymbol{u}_3, \ldots, \boldsymbol{u}_n \rangle.$$

Note that equality on (21) implies that $\boldsymbol{u} = (x_2, \overset{(\alpha)}{\ldots}, x_2, 1, \overset{(n-\alpha)}{\ldots}, 1)^{\top}$ is a $\lambda_n$-eigenvector of $\boldsymbol{A}$.

Recall from the definition of an eigenvector that $\boldsymbol{A}\boldsymbol{u} = \lambda \boldsymbol{u}$, which implies that $\sum_{j \sim i} u_j = \lambda u_i$. Thus, if equality holds (see Figure 1),

$$r(1 - \frac{n}{\alpha})\alpha = \lambda_n(n - \alpha),$$

and we get

$$r = -\lambda_n.$$

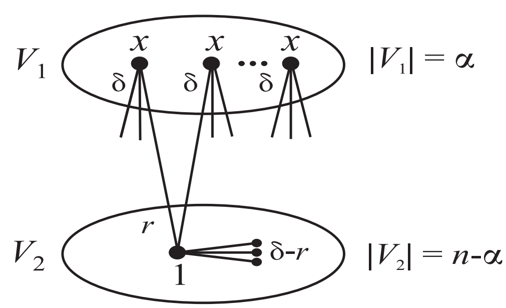



FIG. 1

EXAMPLE 3.1. *Let $\Gamma = P$ be the Petersen graph, with eigenvalues* $3, 1, -2$. *A maximum independent set in $P$ has cardinal $\alpha = 4$.*

*Then, the above bound holds $\alpha \leq \frac{10}{1+\frac{3}{2}} = 4$.*

# Chapter 6
# Other Results and Open Problems

## 1. Polynomials and regularity

With the following proposition we prove one implication of the generalization of the Hoffman's result for a weight-regular partition of $\Gamma$.

Proposition 1.1. *Let $\Gamma$ be a graph with a partition of its vertices into $m$ sets, $\{V_1, \dots, V_m\}$, such that $n = n_1 + \cdots + n_m$. If $\Gamma$ has a weight-regular partition into $m$ sets, then there exists a polynomial $H \in \mathbb{R}_d[x]$ such that*

$$H(\boldsymbol{A}) = \begin{pmatrix} b^*_{11}\boldsymbol{J} & b^*_{12}\boldsymbol{J} & \cdots & b^*_{1m}\boldsymbol{J} \\ b^*_{21}\boldsymbol{J} & b^*_{22}\boldsymbol{J} & \cdots & b^*_{2m}\boldsymbol{J} \\ \vdots & & \ddots & \\ b^*_{m1}\boldsymbol{J} & b^*_{m2}\boldsymbol{J} & \cdots & b^*_{mm}\boldsymbol{J} \end{pmatrix}. \tag{22}$$

***Proof.*** Assume that $\Gamma$ has a weight-regular partition of its vertices. Let $\boldsymbol{A}$ be the adjacency matrix of $\Gamma$ and $\boldsymbol{B}^*$ its weight-regular quotient matrix. By Perron-Frobenius Theorem we know that the maximum eigenvalue $\theta_0$ of $\boldsymbol{A}$ has algebraic and geometric multiplicity one, and also that there is an eigenvector $\boldsymbol{\nu}$ belonging to $\theta_0$ with all coordinates positive. Note that $\operatorname{ev} \boldsymbol{B}^* \subseteq \operatorname{ev} \boldsymbol{A}$. In a weight-regular partition, this eigenvector is $\boldsymbol{\nu} = (\nu_1 \boldsymbol{j} | \dots | \nu_m \boldsymbol{j})^\top$, with the $\boldsymbol{j}$'s being all 1-vectors with appropriate lengths, depending on the size of $n_i$, $i = 1, \dots, m$. This leads to a partition of $\boldsymbol{A}$ with quotient matrix

$$\boldsymbol{B}^* = \begin{pmatrix} b^*_{11} & b^*_{12} & \cdots & b^*_{1m} \\ b^*_{21} & b^*_{22} & \cdots & b^*_{2m} \\ \vdots & & \ddots & \\ b^*_{m1} & b^*_{m2} & \cdots & b^*_{mm} \end{pmatrix}.$$

By the spectral decomposition theorem we can write $\boldsymbol{A} = \sum_{i=0}^{d} \theta_i \boldsymbol{E}_i = \theta_0 \boldsymbol{E}_0 + \cdots + \theta_d \boldsymbol{E}_d$. We have that the weight-Hoffman polynomial can be computed as

$H = \alpha \prod_{l=1}^{d}(x-\theta_i)$ for some non-zero constant $\alpha$. Using the fact that $p(\boldsymbol{A}) = \sum_{i=0}^{d} p(\theta_i)\boldsymbol{E}_i$ for any polynomial $p \in \mathbb{R}_d[x]$, then

$$H(\boldsymbol{A}) = H(\theta_0)\boldsymbol{E}_0 + H(\theta_1)\boldsymbol{E}_1 + \cdots + H(\theta_d)\boldsymbol{E}_d = H(\theta_0)\boldsymbol{E}_0,$$

where $H(\theta_0) = \alpha \prod_{l=1}^{d}(\theta_0 - \theta_i) = \alpha\pi_0$.

Then, the problem reduces to find the idempotent $\boldsymbol{E}_0$. It can be computed as

$$\begin{aligned}\boldsymbol{E}_0 &= \frac{1}{||\boldsymbol{\nu}||^2}\boldsymbol{\nu}\boldsymbol{\nu}^T = (\nu_1\boldsymbol{j}|\cdots|\nu_m\boldsymbol{j})(\nu_1\boldsymbol{j}|\cdots|\nu_m\boldsymbol{j})^T \\ &= \frac{1}{||\boldsymbol{\nu}||^2}\begin{pmatrix} \nu_1\nu_1\boldsymbol{J} & \cdots & \nu_1\nu_m\boldsymbol{J} \\ \vdots & \ddots & \vdots \\ \nu_m\nu_1\boldsymbol{J} & \cdots & \nu_m\nu_m\boldsymbol{J} \end{pmatrix}\end{aligned}$$

where $\boldsymbol{J}$'s are the all 1-matrix with appropriate sizes. If we denote $b^*_{ij} = \nu_i\nu_j$ for $i,j = 1,\ldots,m$, and we consider that $\alpha = \frac{||\boldsymbol{\nu}||^2}{\pi_0}$, it follows that

$$H(\boldsymbol{A}) = \begin{pmatrix} b^*_{11}\boldsymbol{J} & b^*_{12}\boldsymbol{J} & \cdots & b^*_{1m}\boldsymbol{J} \\ b^*_{21}\boldsymbol{J} & b^*_{22}\boldsymbol{J} & \cdots & b^*_{2m}\boldsymbol{J} \\ \vdots & & \ddots & \\ b^*_{m1}\boldsymbol{J} & b^*_{m2}\boldsymbol{J} & \cdots & b^*_{mm}\boldsymbol{J} \end{pmatrix},$$

□

# 2. Eigenvalue interlacing in graph parameters

### 2.1. $k$-independence number.

It is known that the size of the largest coclique (independent set of vertices) satisfies the bound

$$\alpha(G) \le \min\{|i : \lambda_i \ge 0|, |i : \lambda_i \le 0|\}. \tag{23}$$

We can find a similar bound for the $k$-independence number $\alpha_k$, $k \ge 1$; that is, the maximum number of vertices which are mutually at distance greater than $k$ (so, $\alpha_1 = \alpha$).

If we know the spectrum of the distance-$k$ graph $\Gamma_k$, there is nothing to say, Just apply (23). This is the case, for instance, when $\Gamma$ is punctually distance-regular since then $\boldsymbol{A}_k = p_k(\boldsymbol{A})$ (or, more generally, if $\Gamma$ is $k$-punctually distance-polynomial).

In a more general setting, at a first step we can work with the powers of $\boldsymbol{A}$. The following proposition gives an upper bound for the 2-independence number.

PROPOSITION 2.1. *Let $\Gamma$ be a graph with minimum degree $\delta$ and maximum degree $\Delta$. Then*

$$\alpha_2(G) \leq \min\{|i : \lambda_i \geq \Delta|, |i : \lambda_i \leq \delta|\}.$$

***Proof.*** Suppose that the graph has a maximum independent set $U = \{1, 2, \ldots, \alpha_2\}$ with the vertices which are mutually at distance greater than 2. Then the matrix $\boldsymbol{A}^2$ has a principal submatrix of the form $\mathrm{diag}(\delta_1, \ldots, \delta_{\alpha_2})$. Hence, interlacing leads to

$$\alpha_2(G) \leq \min\{|i : \lambda_i \geq \Delta|, |i : \lambda_i \leq \delta|\}.$$

□

# 3. Open problems

PROBLEM 3.1. *Prove or disprove that, given any graph $\Gamma = (V, E)$, we can find a matrix $\boldsymbol{M}$ with entries $m_{uv} = 0$ when $uv \notin E$ such that the upper bound (23) is sharp.*

Let $\boldsymbol{B} = \boldsymbol{S}^\top \boldsymbol{A}\boldsymbol{S}$ be the quotient matrix of $\boldsymbol{A}$ with respect to a partition $\mathcal{P}$ Then we have the following known facts:

(1) The eigenvalues of $\boldsymbol{B}$, $\mathrm{ev}\,\boldsymbol{B} = \{\mu_1, \mu_2, \ldots, \mu_m\}$, interlace the eigenvalues of $\boldsymbol{A}$, $\mathrm{ev}\,\boldsymbol{A} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$.
(2) If the interlacing is tight, then $\mathcal{P}$ is equitable.
(3) If $\mathcal{P}$ is a distance partition, then $\mathcal{P}$ is equitable if and only if the interlacing is $(2, 1)$-exact in the sense of [**6**], that is $\mu_1 = \lambda_1$, $\mu_2 = \lambda_2$ and $\mu_m = \lambda_n$.

PROBLEM 3.2. *Find necessary and sufficient conditions for a partition $\mathcal{P}$ being equitable in terms of the bandwidth $b$ of its quotient matrix $\boldsymbol{B}$. Note that Fact* 3 *above would correspond to the case $b = 3$.*

In Chapter 4, the result shown in Theorem 4.1 suggests the following question:

PROBLEM 3.3. *Prove or disprove: A regular bipartite graph $\Gamma$ with predistance polynomial $p_{d-1}$ is distance-regular if and only if $\boldsymbol{A}_{d-1} = p_{d-1}(\boldsymbol{A})$.*

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# Notation

| | |
|---|---|
| $\boldsymbol{A}$ | Adjacency matrix of graph $\Gamma$ |
| $a_{uv}$ | $(u,v)$-entry of matrix $\boldsymbol{A}$ |
| $\boldsymbol{L}$ | Laplacian matrix of graph $\Gamma$ |
| $l_{uv}$ | $(u,v)$-entry of matrix $\boldsymbol{L}$ |
| $d+1$ | Number of different eigenvalues of adjacency matrix $\boldsymbol{A}$ |
| $D = D(\Gamma)$ | Diameter of a graph $\Gamma$ |
| $\partial(u,v)$ | Distance between vertices $u$ and $v$ |
| $\delta$ | Degree of (regular) graph $\Gamma$ |
| $\delta_u$ | Degree of vertex $u$ |
| $\overline{\delta}$ | Average degree of graph $\Gamma$ |
| $E = E(\Gamma)$ | Edge set of a graph $\Gamma$ |
| $\mathcal{E}_i$ | Eigenspace of eigenvalue $\theta_i$ |
| $\mathrm{ecc}(u)$ | Eccentricity of vertex $u$ |
| $\mathrm{ev}\,\Gamma = \mathrm{ev}\,\boldsymbol{A}$ | Set of different eigenvalues of graph $\Gamma$ |
| $\Gamma$ | Graph |
| $\Gamma_k$ | Distance-$k$ graph of $\Gamma$ |
| $\Gamma_k(u)$ | Set of vertices at distance $k$ from vertex $u$ |
| $u, v$ | Vertices of $\Gamma$ |
| $H$ | Hoffman polynomial |
| $\boldsymbol{I}$ | Identity matrix |
| $\boldsymbol{j}$ | All-1 vector |
| $\boldsymbol{J}$ | All-1 matrix |
| $\theta_i^{m_i}$ | Eigenvalue of adjacency matrix $\boldsymbol{A}$ with multiplicity $m_i = m(\theta_i)$ |
| $m_u(\theta_i)$ | $u$-local multiplicity of $\theta_i$ |
| $n$ | Number of vertices in $\Gamma$ |
| $N_k(u)$ | Set of vertices at distance at most $k$ from $u$ |
| $\boldsymbol{O}$ | 0-matrix |
| $\mathbf{0}$ | 0-vector |
| $\phi_\Gamma$ | Characteristic polynomial of $\Gamma$ |
| $\mathrm{sp}\,\Gamma = \mathrm{sp}\,\boldsymbol{A}$ | Spectrum of the adjacency matrix of graph $\Gamma$ |
| $\mathrm{tr}\,\boldsymbol{A}$ | Trace of matrix $\boldsymbol{A}$ |
| $V = V(\Gamma)$ | Vertex set of a graph $\Gamma$ |
| $\sim$ | Adjacency between vertices |