MASTER'S DEGREE THESIS

MSc in Applied Mathematics

Title: On the length spectrum of analytic convex billiard tables

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Master Thesis

On the length spectrum of analytic convex billiard tables

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Al Pau i al Rafa, per ser positius i sempre estar predisposats a ajudar

Abstract

Key words: Area-preserving twist maps, billiards, length spectrum, Melnikov, exponential smallness, periodic orbits

MSC2000: 37E40, 37J10, 37J40

Billiard maps are a type of area-preserving twist maps and, thus, they inherit a vast number of properties from them, such as the Lagrangian formulation, the study of rotational invariant curves, the types of periodic orbits, etc. For strictly convex billiards, there exist at least two (p, q)-periodic orbits. We study the billiard properties and the results found up to now on measuring the lengths of all the (p, q)-trajectories on a billiard. By using a standard Melnikov method, we find that the first order term of the difference on the lengths among all the (p, q)-trajectories orbits is exponentially small in certain perturbative settings. Finally, we conjecture that the difference itself has to be exponentially small and also that these exponentially small phenomena must be present in many more cases of perturbed billiards than those we have presented on this work.

Contents

Introduction

Birkhoff [**6**] introduced the problem of convex billiard tables to describe a motion of a free particle inside a closed convex curve. The motion follows the law "the angle of incidence equals to the angle of reflection". Billiards are a simple concept which already contains some of the most important questions on dynamics. Birkhoff already reflected this idea when saying "in the billiard problem the formal side, usually so formidable in dynamics, almost completely disappears and only the interesting questions need to be considered"[**6**, p.170].

The billiard motion can be modeled by an area-preserving twist map on an open cylinder. This characterization is useful in different settings.

On the one hand, the twist condition characterizes the rotational invariant circles, RICs, as graphs of a function defined on the angular coordinate (Birkhoff Theorem [**20**, §IV]). As invariant structures, it is interesting to study the restricted dynamics. Since they are conjugate to a circle diffeomorphism, a rotation number can be associated to each RIC. For an integrable area-preserving twist map, all the RICs with a Diophantine rotation number are preserved under small perturbation. This perturbative result is due to Moser [**25**, §32,33] and it is part of the KAM theory.

On the other hand, area-preserving twist maps admit a variational principle which is analogous to the Lagrangian-action formulation of analytical mechanics. Orbits are stationary points of the action functional and the minima and minimax points lead to a class of orbits that are of great importance. In particular, for (p, q)-periodic orbits, the minima and minimax points, different in nature, imply the existence of at least two (p, q)-periodic orbits (Poincaré Birkhoff Theorem [**20**, §VI]).

In the billiard setting, the action coincides with the sum of the length of the chords between two consecutive impact points. Also, in the billiard table, (p, q)-periodic orbits correspond to polygons of q sides making p turns inside the boundary of the table and its rotation number is p/q.

The existence of RICs on the billiard map is closely related to a curve called caustic inside the table. A caustic has the property that once a trajectory is tangent to it, it stays tangent to the caustic after every reflection. We can associate two rotational invariant curves on the phase space to each convex caustic and also a rotation number. The existence of convex caustics with Diophantine rotation numbers is guaranteed close to the billiard table boundary [**17**]. Also, their existence is guaranteed for any trajectory in the circular and elliptic billiards as these maps are Liouville integrable and convex caustics are related to RICs.

In the context of circular and elliptic maps, since the (p, q)-periodic orbits are a continuous family on a RIC, all the trajectories have the same length. Contrary to Diophantine RICs, these resonant RICs generically break up under arbitrarily small perturbations. Thus, the length of the different (p, q)-periodic orbits is not same and one can try to measure the maximum difference. The attempts on this measure rely on a Melnikov method [**24**][**22**]. The Melnikov technique is based on the study of the lower order terms on a Taylor expansion according to the perturbative parameter. When not only the perturbative parameter tends to zero but also the period tends to infinity, the lower order terms might be not the important ones and all the terms have to be taken into account. In these situations, the literature (see [**8**] for instance) has always turned to the study of the Birkhoff normal form [**25**, §23].

There exist results on the maximum difference among all the (p, q)-periodic orbits not only for Liouville integrable maps but for any strictly convex smooth billiard. The difference is beyond any order with respect to q when the (p, q)-periodic orbits are approaching to the boundary [**18**], or close to an elliptic $(1, 2)$-periodic orbit [**8**]. Taking these results to an analytic context leads to think that this maximum difference will be exponentially small, as it happens in other problems [**11**].

This memoir is a first step on the study of the length of the periodic orbits existing in any convex billiard which has to be continued in the next years. In Part 1, we give a complete review on the necessary concepts surrounding our matter of subject and we also highlight the existing results on the length spectrum of billiards. At Part 2 we develop some of the tools exhibited in order to give some first results in very concrete settings.

Part 1, State of the art, is divided into four different chapters. Chapter 1 compiles the most important notions on area-preserving maps while in Chapter 2 we focus on the billiard map and its the geometric properties. We will pay special attention to the circular and elliptic billiards. Chapters 1 and 2 mainly follow [**20**] and [**14**]. Further information on billiards can be found in [**15**], [**26**], [**16**] and [**6**].

In Chapter 3, we review some tools used on dynamical systems when studying the effect that small perturbations cause on the unperturbed invariant sets. In particular we review the Moser's Twist Theorem [**25**, §32,33] and the Melnikov subharmonic potential [**22**]. We also review the Birkhoff normal form [**25**, §23].

Finally, Chapter 4 gives the State of the art results we are based on to start our study. Results on exponentially small phenomena [**11**], the existence of caustics [**17**], the break up of resonant tori [**24**][**22**] and the length spectrum [**18**][**8**] are given.

Part 2 focusses on delimiting the subject studied and showing some first results. In Chapter 5 we discuss the problem we want to study and the approach we will be following. The results obtained when applying the proposed method to generic perturbations of the billiard on the circle and some particular ones on the ellipse are shown on the last chapter.

Part 1

State of the art

Chapter 1
Area-preserving twist maps

1.1. Basic definitions

We will consider diffeomorphisms defined on an open cylinder of the form $C = \mathbb{T}\times Y$, $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ and $Y = (y_-, y_+) \subset \mathbb{R}$. Its universal cover, $\widetilde{C} = \mathbb{R} \times Y$, is a strip of the plane. We will use coordinates (s, y) for C and same coordinates with a tilde, $(\widetilde{s}, y)$ for $\widetilde{C}$. In fact, the tilde will always denote the lift of a point, a function or a set to the universal cover. Horitzontal and vertical directions will be called the angular and radial directions respectively. The image of a point (s, y) will be denoted by (s_1, y_1).

Definition 1.1.1. A *lift* F of a map $T : C \to C$ is a map $F : \widetilde{C} \to \widetilde{C}$ such that if $\pi : \widetilde{C} \to C$ is the projection to the quotient space C, then the following diagram commutes,

$$\begin{array}{ccc} \widetilde{C} & \xrightarrow{F} & \widetilde{C} \\ \downarrow\pi & & \downarrow\pi \\ C & \xrightarrow{T} & C \end{array}$$

Immediate consequences of this definition are the following. Let $T = (T_1, T_2)$ and $F = (F_1, F_2)$. Then, F_1 commutes with integer shifts in the angular direction while F_2 is 1-periodic in the first variable.

Observe that for F and G lifts of T, $F - G \equiv (k, 0)$, $k \in \mathbb{Z}$. In general, we will fix the lift, $\widetilde{T}$, in such a way that $\widetilde{T}(0, 0) \in [0, 1) \times Y$.

Definition 1.1.2. A diffeomorphism $T : C \to C$ of the open cylinder to itself is called an *area-preserving twist map* if

(1) T preserves area,
(2) T preserves orientation,
(3) T preserves boundary components, and
(4) its lift $\widetilde{T} : \widetilde{C} \to \widetilde{C}$ has the *twist property*: $\partial_2\widetilde{T}_1(\widetilde{s}, y) \neq 0$, where ∂_2 denotes the derivation with respect the second variable.

Remark 1.1.1. If the map could be continuously extended to the closed cylinder $T : \mathbb{T} \times \bar{Y} \to \mathbb{T} \times \bar{Y}$, the preservation of the boundary components will be nothing but the condition $T(\mathbb{T}\times\{y_-\}) = \mathbb{T}\times\{y_-\}$ and $T(\mathbb{T}\times\{y_+\}) = \mathbb{T}\times\{y_+\}$. If this extension is not possible, the condition of the preservation of the boundary components can be written as $\exists\varepsilon > 0$ such that $T(\mathbb{T}\times(y_-, y_- + \varepsilon)) \subset \mathbb{T}\times(y_-, \bar{y})$ and $T(\mathbb{T}\times(y_+ - \varepsilon, y_+)) \subset \mathbb{T}\times(\bar{y}, y_+)$, where $\bar{y} = (y_+ + y_-)/2$.

REMARK 1.1.2. We say that T twists to the right if $\widetilde{T}_1$ is a monotonically increasing function of y and T twists to the left if $\widetilde{T}_1$ decreases monotonically with y.

Observe that the inverse of an area-preserving twist map is also an area-preserving map, but with the twist twisting to the opposite direction. Therefore, and since a twist map is a diffeomorphism, with no loss of generality, we will be working with twist maps twisting to the right while the results will apply to any twist map.

It is worth to remark that the set of twist maps is not a group under the composition. The reason is that the composition of two twist maps can make some points rotate so much that their second iterates could violate the twist condition.

DEFINITION 1.1.3. The *twist interval* of T is the set of numbers $\alpha \in \mathbb{R}$ for which there exists $\varepsilon > 0$ such that if $(\widetilde{s}, y) \in \mathbb{R}\times(y_-, y_- + \varepsilon)$ then $\widetilde{T}_1(\widetilde{s}, y) - \widetilde{s} \leq \alpha$ and if $(\widetilde{s}, y) \in \mathbb{R}\times(y_+ - \varepsilon, y_+)$ then $\widetilde{T}_1(\widetilde{s}, y) - \widetilde{s} \geq \alpha$. It is denoted by (ω_-, ω_+).

Observe that the twist interval is defined as a set. But, any number between two numbers belonging to the twist interval also belongs to it and therefore it is indeed an interval.

Also, the interval is well defined up to integer translation so we will define it in the interval $(0, 1)$.

As before, if the map can be continuously extended to the closed cylinder, the twist interval is $\left(\lim_{n\to\infty}(\widetilde{T}_1^n(\widetilde{s}, y_-) - \widetilde{s})/n, \lim_{n\to\infty}(\widetilde{T}_1^n(\widetilde{s}, y_+) - \widetilde{s})/n\right)$. These limits of the interval are related to the rotation number of the restriction of the diffeomorphism on both cylinder covers. In Section 1.5, more information on the rotation number of a map on a circle can be found.

If T is a twist map with lift $\widetilde{T}$ and $\partial_2\widetilde{T}_1$ is bounded away from 0, then any sufficiently small $\mathcal{C}^1$-perturbations of $\widetilde{T}$ is also a twist map. Furthermore, the endpoints of the twist interval depend continuously on the perturbation.

A particular type of area-preserving twist maps are of the following form.

DEFINITION 1.1.4. An area-preserving twist map is called *integrable* if it is of the form $T(s, y) = (s + g(y), y)$.

Integrable twist maps leave circles $\mathbb{T}\times\{y\}$ invariant and rotate them by $g(y)$, which has to be a monotone function. They are *foliated* by rotational invariant circles (as we explain in the next section, invariant curves not contractible to a point are called this way). Also, we can observe that for each rational value of g we have a continuous family of periodic orbits separated by circles with irrational rotation number. The twist interval is $(\lim_{y\to y_-} g(y), \lim_{y\to y_+} g(y))$.

1.2. Rotational invariant circles

DEFINITION 1.2.1. A *rotational invariant circle* (RIC) is a closed loop Υ homotopically non trivial such that $T(\Upsilon) = \Upsilon$.

A rotational invariant circle divides the cylinder into two invariant regions. To see this, consider A the region of the cylinder below the RIC and consider its image, $T(A)$. Since T is a continuous map, $T(A)$ is a connected component. Since the RIC is invariant and the boundaries are preserved, $T(A)$ must have the same boundaries as A and since the map is bijective, $T(A) = A$. Therefore, RICs are barriers to motion, they separe the phase space. In fact, a similar argument can be applied to any invariant closed curve, no matter if it is contractible to a point or not.

Birkhoff showed that any invariant set U that looks like "half a cylinder" has a graph-like boundary. Formally this result is written as follows.

THEOREM 1.2.1 (Birkhoff Theorem, [**20**, §IV]). *Let T be an area-preserving twist map on the cylinder C. Let U be an open invariant set homeomorphic to the cylinder such that there exist $a, b \in Y$, $a < b$ satisfying $\mathbb{T} \times (y_-, a) \subset U \subset \mathbb{T} \times (y_-, b)$. Then, the boundary of U is the graph $\{(s, \gamma(s))\}$ of some continuous function $\gamma : \mathbb{T} \to (a, b)$.*

REMARK 1.2.1. In particular, the theorem implies that any invariant set U looking like "half a cylinder" can not have "whorls" on its boundary. See Figure 1.1.

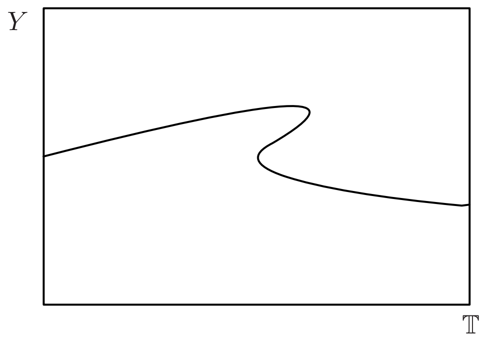

FIG. 1.1. The boundary drawn can not be a graph and therefore the region below it is not invariant.

This result leads to some others which are often useful arguments when looking for RICs or its properties. We will now mention some.

COROLLARY 1.2.2. *Any RIC is a graph.*

PROOF. We can consider U the region below the RIC, which we have already seen that is invariant. Applying Birkhoff Theorem 1.2.1 we obtain that the RIC is a graph. $\square$

COROLLARY 1.2.3 (Lipschitz corollary). *The function γ is not only continuous but also Lipschitz.*

PROOF. Let $v = (0, \delta y)$ be a vertical vector at point (s, y) and let $(\delta s_1, \delta y_1)$ be its image, $(\delta s_1, \delta y_1) = DT(s, y)v$. The slope of $DT(s, y)v$ is $S = \delta y_1/\delta s_1$, $\delta y_1 = \partial_2 T_2(s, y)$, $\delta s_1 = \partial_2 T_1(s, y)$. Since T is a twist map, δs_1 has a positive lower bound and there exists a maximum value for S, S_+. Similarly, we can use that the inverse T^{-1} is also a twist map to find a lower bound for S, S_-.

Therefore, we have found $S_- \leq (\gamma(s) - \gamma(s'))/(s - s') \leq S_+$ for any $s, s' \in \mathbb{T}$ and so γ is a Lipschitz function. □

COROLLARY 1.2.4 (Confinement corollary). *Suppose all the orbits of the points in $\{(s, y) \in C, y < a\}$ stay below some circle $\mathbb{T} \times \{b\}$. Then, there exists a RIC between $y = a$ and $y = b$.*

PROOF. To obtain this result, we construct a suitable set U satisfying the hypothesis of Birkhoff Theorem 1.2.1.

Consider the union of the orbits of the points below $y = a$. This set is invariant but it may have holes and may not satisfy requirements for the set U in the Birkhoff Theorem. Among the complement set, there is a connected component, V, which contains all the points above $y = b$. Take as set U the complement set to V: it is invariant and all its points are below $y = b$. Then we can apply Birkhoff Theorem and proof the statement. □

COROLLARY 1.2.5 (Non existence criterion). *If there exists an orbit that is as close as we want from both cylinder covers, $\mathbb{T} \times \{y_-\}$ and $\mathbb{T} \times \{y_+\}$, there can not exist any RIC.*

Restricting the map T to the RIC, one obtains some interesting results on the restricted dynamics. These results are extracted from the study of homeomorphisms or diffeomorphisms on the circle. Since we also mention results about periodic orbits on the restricted dynamics, we have stated some of these results not in the next section but in Section 1.5.

1.3. Generating function and variational formulation

We will show that any twist map verifies a Lagrangian variational principle. This variational formulation is very useful to deduce properties or different types of orbits. We will see these applications in the next subsections.

Consider the area-preserving twist map $T : (s, y) \mapsto (s_1, y_1)$. We claim that there exists a function $H(s, s_1)$ such that

$$\left\{ \begin{array}{rcl} y & = & -\partial_1 H(s, s_1), \\ y_1 & = & \partial_2 H(s, s_1). \end{array} \right. \tag{1}$$

In fact we define it this way:

DEFINITION 1.3.1. Let $\widetilde{T}$ be a lift of T. If $(\widetilde{s}, \widetilde{s}_1) \in \mathbb{R}^2$ is such that $\widetilde{T}(\{\widetilde{s}\} \times Y) \cap (\{\widetilde{s}_1\} \times Y) \neq \emptyset$, then, we denote by $H(\widetilde{s}, \widetilde{s}_1)$ the area of the region located to the right and under $\widetilde{T}(\{\widetilde{s}\} \times Y)$, to the left of $\{\widetilde{s}_1\} \times Y$ and above $\mathbb{R} \times \{y_-\}$. Function $H : (\widetilde{s}, \widetilde{s}_1) \mapsto H(\widetilde{s}, \widetilde{s}_1)$ is called the *generating function* of T.

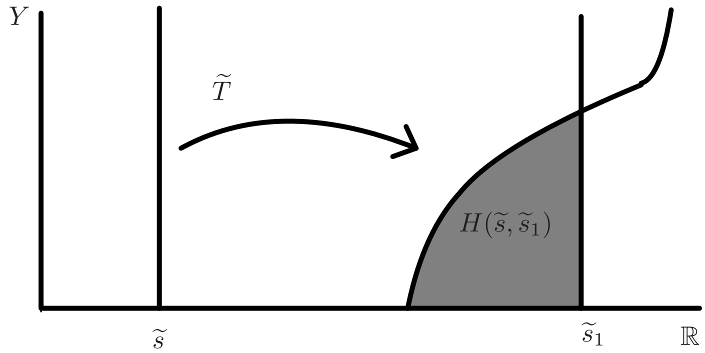

FIG. 1.2. The generating function $H(\widetilde{s}, \widetilde{s}_1)$ is the area colored in grey. It lies to the right and under $\widetilde{T}(\{\widetilde{s}\} \times Y)$ and to the left of $\{\widetilde{s}_1\} \times Y$.

REMARK 1.3.1. If the twist map twisted to the left, $H(\widetilde{s}, \widetilde{s}_1)$ would be instead defined as the area located to the left and under $\widetilde{T}(\{\widetilde{s}\} \times Y)$, to the right of $\{\widetilde{s}_1\} \times Y$ and above $\mathbb{R} \times \{y_-\}$.

REMARK 1.3.2. The twist condition implies that the intersection set $\widetilde{T}(\{\widetilde{s}\} \times Y) \cap \{\widetilde{s}_1\} \times Y$ is either void or is a single point, $(\widetilde{s}_1, f_1(\widetilde{s}, \widetilde{s}_1))$, for some uniquely defined f_1. Moreover, note that $H(\widetilde{s}+k, \widetilde{s}_1+k) = H(\widetilde{s}, \widetilde{s}_1)$ since we know that $\widetilde{T}(\widetilde{s}+k, y) = \widetilde{T}(\widetilde{s}, y)+(k, 0)$. Therefore, function H can be defined on the quotient space $\mathbb{R}\times\mathbb{R}/\sim$, where $(s, s_1) \sim (t, t_1)$ if and only if $t = s + k$ and $t_1 = s_1 + k$ for some $k \in \mathbb{Z}$.

If there exists $(\widetilde{s}_1, f_1(\widetilde{s}, \widetilde{s}_1)) \in \widetilde{T}(\{\widetilde{s}\}\times Y) \cap (\{\widetilde{s}_1\} \times Y)$ we can define a function f_0 as $\widetilde{T}^{-1}(\widetilde{s}_1, f_1(\widetilde{s}, \widetilde{s}_1)) = (\widetilde{s}, f_0(\widetilde{s}, \widetilde{s}_1))$. Observe that we are saying that $\mathrm{graph}(f_1(\widetilde{s}, \cdot)) = \widetilde{T}(\{\widetilde{s}\} \times Y)$ and $\mathrm{graph}(f_0(\cdot, \widetilde{s}_1)) = \widetilde{T}^{-1}(\{\widetilde{s}_1\} \times Y)$.

From the definition of $H(\widetilde{s}, \widetilde{s}_1)$ as an area, we have $H(\widetilde{s}, \widetilde{s}) = 0$ and

$$H(\widetilde{s}, \widetilde{s}_1) = \int_{\widetilde{s}}^{\widetilde{s}_1} f_1(\widetilde{s}, \xi)\,\mathrm{d}\xi. \tag{2}$$

Since T is area-preserving, the area of the preimage is the same and therefore, we also have

$$H(\widetilde{s}, \widetilde{s}_1) = \int_{\widetilde{s}}^{\widetilde{s}_1} f_0(\xi, \widetilde{s}_1)\,\mathrm{d}\xi. \tag{3}$$

So if we have $(\widetilde{s}_1, y_1) = \widetilde{T}(\widetilde{s}, y)$ then $f_1(\widetilde{s}, \widetilde{s}_1) = y_1$ and $f_0(\widetilde{s}, \widetilde{s}_1) = y$ and equations (2) and (3) gives us the relation (1) since

$$\begin{cases} y &= f_0(\widetilde{s}, \widetilde{s}_1) = -\partial_1 H(\widetilde{s}, \widetilde{s}_1), \\ y_1 &= f_1(\widetilde{s}, \widetilde{s}_1) = \partial_2 H(\widetilde{s}, \widetilde{s}_1), \end{cases}$$

In fact, we can say something more.

PROPOSITION 1.3.1 ([**14**, p. 342]). *The generating function H determines the dynamics uniquely.*

PROOF. We want to determine $\widetilde{T}_1(\widetilde{s}, y)$ and $\widetilde{T}_2(\widetilde{s}, y)$ from $H(\widetilde{s}, \widetilde{s}_1)$. We will apply the Implicit Function Theorem to

$$0 \quad = \quad F(\widetilde{s}, \widetilde{s}_1, y, y_1) := \begin{pmatrix} \partial_2 H(\widetilde{s}, \widetilde{s}_1) - y_1 \\ \partial_1 H(\widetilde{s}, \widetilde{s}_1) + y \end{pmatrix}.$$

We need that $\det(D_{\widetilde{s}_1, y_1} F) \neq 0$,

$$\det(D_{\widetilde{s}_1, y_1} F) = \det \begin{pmatrix} \partial_{11} H(\widetilde{s}, \widetilde{s}_1) & -1 \\ \partial_{12} H(\widetilde{s}, \widetilde{s}_1) & 0 \end{pmatrix} = \partial_{12} H(\widetilde{s}, \widetilde{s}_1).$$

We know that $\partial_2 H(\widetilde{s}, \widetilde{s}_1) = f_1(\widetilde{s}, \widetilde{s}_1)$ and from the twist property we can deduce that, once fixed $\widetilde{s}_1$, $f_1(\cdot, \widetilde{s}_1)$ is a decreasing function (see Figure 1.3). Therefore, at any point, we have $\partial_{12} H(\widetilde{s}, \widetilde{s}_1) < 0$ and we can determine $\widetilde{T}(\widetilde{s}, y) = (\widetilde{s}_1, y_1)$. $\square$

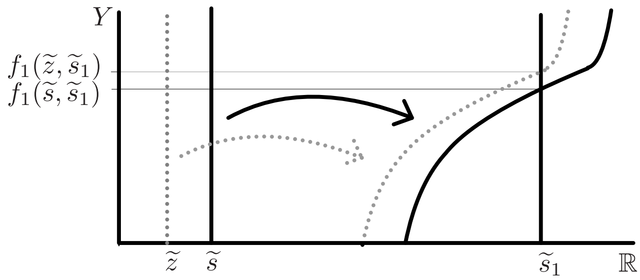

FIG. 1.3. Once fixed $\{\widetilde{s}_1\}$, we apply T to sets $\{\widetilde{z}\} \times Y$ and $\{\widetilde{s}\} \times Y$, with $\widetilde{z} < \widetilde{s}$. The intersection point of both sets with the line $\{\widetilde{s}_1\} \times Y$ is the definition of $f_1(\widetilde{z}, \widetilde{s}_1)$ and $f_1(\widetilde{s}, \widetilde{s}_1)$ respectively. Thus, we can observe that f_1 is a decreasing function with respect to the first coordinate.

Thanks to the generating function we will be able to define a functional whose stationary points are orbits of the map. The following proposition gives a local result.

PROPOSITION 1.3.2 ([**14**, Proposition 9.3.4., p. 354]). *Suppose $\widetilde{s}_0$ is a critical point of $\widetilde{s} \mapsto H(\widetilde{s}_{-1}, \widetilde{s}) + H(\widetilde{s}, \widetilde{s}_1)$. Then, there exist $y_{-1}, y_0, y_1 \in (y_-, y_+)$ such that $\widetilde{T}(\widetilde{s}_{-1}, y_{-1}) = (\widetilde{s}_0, y_0)$ and $\widetilde{T}(\widetilde{s}_0, y_0) = (\widetilde{s}_1, y_1)$.*

REMARK 1.3.3 (About notation). Henceforth, the images and preimages of an initial point $(\widetilde{s}_0, y_0)$ will be denoted, as done in the previous proposition, by $(\widetilde{s}_n, y_n) := \widetilde{T}^n(\widetilde{s}_0, y_0)$ and $(\widetilde{s}_{-n}, y_{-n}) = \widetilde{T}^{-n}(\widetilde{s}_0, y_0)$, respectively. Therefore, an orbit will be described as $\{\ldots, (\widetilde{s}_{-1}, y_{-1}), (\widetilde{s}_0, y_0), (\widetilde{s}_1, y_1), \ldots\}$.

This proposition gives a local result: a critical point $\widetilde{s}_0$ of a functional depending on two other fixed values $\widetilde{s}_{-1}$ and $\widetilde{s}_1$ is in one-to-one correspondence with the point in C with angular coordinate s_0 such that its preimage and image of have angular coordinates s_{-1} and s_1 respectively. Note that this process resembles to the one followed with Lagrangian flows and the role of H is the same of the Hamiltonian function but in a discrete setting.

Proof of Proposition 1.3.2. Since $\widetilde{s}_0$ is a critical point,

$$0 = \partial_{\widetilde{s}} \left(H(\widetilde{s}_{-1}, \widetilde{s}) + H(\widetilde{s}, \widetilde{s}_1) \right)|_{\widetilde{s}=\widetilde{s}_0} = \partial_2 H(\widetilde{s}_{-1}, \widetilde{s})|_{\widetilde{s}=\widetilde{s}_0} + \partial_1 H(\widetilde{s}, \widetilde{s}_1)|_{\widetilde{s}=\widetilde{s}_0},$$

which implies

$$f_1(\widetilde{s}_{-1}, \widetilde{s}_0) = f_0(\widetilde{s}_0, \widetilde{s}_1).$$

Therefore, $(\widetilde{s}_0, f_1(\widetilde{s}_{-1}, \widetilde{s}_0)) = (\widetilde{s}_0, f_0(\widetilde{s}_0, \widetilde{s}_1))$ and this point, which we redefine as $(\widetilde{s}_0, y_0)$, belongs to $\widetilde{T}(\{\widetilde{s}_{-1}\} \times Y) \cap \widetilde{T}^{-1}(\{\widetilde{s}_1\} \times Y)$. Then, taking $y_{-1} = f_0(\widetilde{s}_{-1}, \widetilde{s}_0)$ and $y_1 = f_1(\widetilde{s}_0, \widetilde{s}_1)$, the statement is satisfied. $\square$

Remark 1.3.4. The twist property implies that $f_1(\widetilde{s}_{-1}, \cdot)$ is a monotone increasing function and $f_0(\cdot, \widetilde{s}_1)$ is a monotone decreasing function. Thus, $\widetilde{s}_0$ is unique. Moreover, it is a minimum, since $\partial^2_{\widetilde{s}} \left(H(\widetilde{s}_{-1}, \widetilde{s}) + H(\widetilde{s}, \widetilde{s}_1) \right) = \partial_2 f_1(\widetilde{s}_{-1}, \widetilde{s}) - \partial_1 f_0(\widetilde{s}, \widetilde{s}_1) > 0$.

This last result can be extended to orbits segments larger than a point, its preimage and its image.

Definition 1.3.2. Fixed $k \in \mathbb{Z}$ and $q \in \mathbb{N}$, $q \geq 2$, we define the *action functional*

$$W(\widetilde{s}_k, \widetilde{s}_{k+1}, \ldots, \widetilde{s}_{k+q}) := \sum_{t=k}^{k+q-1} H(\widetilde{s}_t, \widetilde{s}_{t+1}). \tag{4}$$

Definition 1.3.3. An *orbit segment* is a configuration $\{\widetilde{s}_k, \widetilde{s}_{k+1}, \ldots, \widetilde{s}_{k+q}\}$ that is a stationary point of the action holding $\widetilde{s}_k$ and $\widetilde{s}_{k+q}$, which are fixed.

We must impose that the variation is equal to zero, $\delta W = 0$ and, by the last proposition, we obtain the equations

$$f_1(\widetilde{s}_{k+t-1}, \widetilde{s}_{k+t}) = f_0(\widetilde{s}_{k+t}, \widetilde{s}_{k+t+1}) \quad 0 < t < q. \tag{5}$$

We define $y_{k+t} = f_1(\widetilde{s}_{k+t-1}, \widetilde{s}_{k+t})$ for $t = 1, \ldots, q$ and $y_k = f_0(\widetilde{s}_k, \widetilde{s}_{k+1})$. And this orbit segment is in one-to-one correspondence with the orbit segment $\{(\widetilde{s}_k, y_k)$, $(\widetilde{s}_{k+1}, y_{k+1})$, $\ldots$, $(\widetilde{s}_{k+q}, y_{k+q})\}$ of $\widetilde{T}$ on $\widetilde{C}$.

1.4. Periodic orbits

Following the concepts introduced in the last section, we will characterize the periodic orbits of T on C and we will deduce some properties.

Definition 1.4.1. Let $p < q$, $p, q \in \mathbb{N}$. A (p,q)-*periodic orbit* of T on C is an orbit $\{\ldots, (s_0, y_0), (s_1, y_1), \ldots\}$ such that

$$\begin{cases} \widetilde{s}_q &= \widetilde{s}_0 + p \\ y_q &= y_0. \end{cases}$$

Note that (p, q)-periodic orbits are in correspondence with the critical points of the (p, q)-periodic action

$$W^{(p,q)}(\widetilde{s}_0, \widetilde{s}_1, \ldots, \widetilde{s}_{q-1}) = H(\widetilde{s}_0, \widetilde{s}_1) + \cdots + H(\widetilde{s}_{q-1}, \widetilde{s}_0 + p). \tag{6}$$

Effectively, we then have the same equations, (5), for $0 < t < q$ and the variation with respect to $\widetilde{s}_0$ gives equation

$$f_1(\widetilde{s}_0, \widetilde{s}_1) = f_0(\widetilde{s}_{q-1}, \widetilde{s}_q),$$

which is equivalent to the periodicity condition $y_q = y_0$.

Given the set of (p, q)-periodic orbits, there exists a special subset.

Definition 1.4.2. A (p, q)-*Birkhoff periodic orbit*, or a (p, q)-*monotone periodic orbit*, is a (p, q)-periodic orbit, $\{\ldots, (s_0, y_0), (s_1, y_1), \ldots\}$, such that, for any n, n', m and $m' \in \mathbb{Z}$,

$$\widetilde{s}_n + m < \widetilde{s}_{n'} + m' \Rightarrow \widetilde{s}_{n+1} + m < \widetilde{s}_{n'+1} + m',$$

where $\widetilde{s}_{j+1}$ is the lifted angular coordinate of $\widetilde{T}(\widetilde{s}_j, y_j)$ for $j = n, n'$. Observe that the Birkhoff periodic orbits are the ones which have the angular coordinate ordered as a simple rotation on the circle.

We will now state a very important result concerning to the existence of periodic orbits.

Theorem 1.4.1 (Poincaré-Birkhoff Theorem, [**20**, §VI]). *There exist at least two (p, q)-Birkhoff periodic orbits for any (p, q) such that p/q belongs to the twist interval.*

We do not pretend to prove this theorem rigorously. Nevertheless, it seems interesting to see some of its flavour since the two existent orbits are very different in essence.

It can be proved that there exists a first (p, q)-monotone periodic orbit for any (p, q) such that p/q belongs to the twist interval. The proof is done by studying the minima of the functional $W^{(p,q)}$. These minima are not unique: if we translate by $n \in \mathbb{Z}$ any minimum $\{\widetilde{s}_0, \ldots, \widetilde{s}_{q-1}\}$, we obtain the same value of $W^{(p,q)}$ and therefore another minimum.

To obtain the second (p, q)-Birkhoff periodic orbit, we shall introduce the following concepts.

Definition 1.4.3. An orbit segment $S := \{\widetilde{s}_k, \widetilde{s}_{k+1} \ldots \widetilde{s}_{k+q}\}$ is *minimizing* if for any variation with fixed end points $\widetilde{s}_k$ and $\widetilde{s}_{k+q}$,

$$\Xi = \{\widetilde{s}_k, \widetilde{s}_{k+1} + \delta\widetilde{s}_{k+1}, \ldots, \widetilde{s}_{k+q-1} + \delta\widetilde{s}_{k+q-1}, \widetilde{s}_{k+q}\},$$

we have

$$W(\Xi) - W(S) \geq 0.$$

Definition 1.4.4. If every finite segment of an orbit is minimizing, then the orbit is *minimizing*.

The first orbit obtained was one minimizing the periodic action. But if p and q are coprime we have the following result.

THEOREM 1.4.2 ([**20**, §VI.]). *For p and q coprime, the periodic extension of the configuration minimizing the periodic action $W^{(p,q)}$ is a minimizing orbit.*

Observe that this also implies that if the configuration $\{\widetilde{s}_0, \ldots, \widetilde{s}_{q-1}\}$ minimizes $W^{(p,q)}$ with p, q coprimes, then, $\{\widetilde{s}_0, \ldots, \widetilde{s}_{q-1}, \widetilde{s}_0 + p, \ldots, \widetilde{s}_{q-1} + p, \ldots, \widetilde{s}_{q-1} + np\}$ also minimizes $W^{(np,nq)}$.

Finally, it can be shown that the translates $\{\widetilde{\xi}_t, \widetilde{\xi}_t, = \widetilde{s}_{t+k} + j\}_t$ of a minimizing orbit $\{\widetilde{s}_t\}_t$ are also minimizing. Then, the existence of a minimum of $W^{(p,q)}$ implies the existence of many minima and between these points there must be other critical points. These critical points give rise to a minimax (p, q)-monotone orbit.

REMARK 1.4.1. Following these steps, the (p', q')-orbits guaranteed are the same as the (p, q)-orbits, with $p' = np$, $q' = nq$ and $\gcd(p', q') = n$, for some $n \in \mathbb{N}$. The existence of non-Birkhoff (p', q')-periodic orbits is not guaranteed.

1.5. Rotation number of twist maps

1.5.1. Circle diffeomorphisms and rotation number. If there exists a RIC, we can specifically study the dynamics of the diffeomorphism restricted to this curve.

Let $F : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism. Let $\pi : \mathbb{R} \to \mathbb{T}$ be the natural projection and $\widetilde{F} : \mathbb{R} \to \mathbb{R}$ be a lift, that is $F \circ \pi = \pi \circ \widetilde{F}$.

DEFINITION 1.5.1. Let $\rho(\widetilde{F}) := \lim_{|n|\to\infty}(\widetilde{F}^n(\widetilde{x}) - \widetilde{x})/n$, $\widetilde{x} \in \mathbb{R}$. Then, the *rotation number* of F is $\rho(F) := \pi(\rho(\widetilde{F}))$.

REMARK 1.5.1. In particular, with this definition, we are affirming that $\rho(\widetilde{F})$ exists for any $\widetilde{x} \in \mathbb{R}$ and it does not depend on it. Also from the definition, we observe that $\rho(\widetilde{F})$ is well defined up to an integer, otherwise the rotation number would not be correctly defined. We are omitting the proof of these remarks.

PROPOSITION 1.5.1. *Let $G : \mathbb{T} \to \mathbb{T}$ be an orientation-preserving homeomorphism. Then, $\rho(G^{-1} \circ F \circ G) = \rho(F)$. In other words, the rotation number is a topological invariant.*

The dynamics of a diffeomorphism with a rational rotation number and other with an irrational rotation number are very different. For the rational case, we state the following result.

PROPOSITION 1.5.2. *F has a periodic point if and only if $\rho(F) \in \mathbb{Q}$. Indeed, let $p, q \in \mathbb{N}$. If $\rho(F) = p/q$, there exists at least one (p, q)-orbit and viceversa.*

All the periodic orbits on the circle have their lifted coordinate monotonically increasing. That is, an orbit $\{\ldots, x_0, \ldots\}$ satisfies: $\widetilde{x}_q = \widetilde{x}_0 + p$ and also $\widetilde{x}_i < \widetilde{x}_{i+1}$ for all $i \in \mathbb{Z}$. This behaviour is analogous to the behaviour on the lift of the angular coordinate for (p, q)-Birkhoff periodic orbits on area-preserving twist maps.

Remark 1.5.2. Let Υ be a RIC of an area-preserving twist map T. Assume the rotation number of the map restricted to the RIC is rational, $\rho(T|_\Upsilon) = p/q$. Since a RIC is parameterized as the graph of a certain Lipschitz function defined on the angular coordinates, the (p, q)-periodic orbit on the RIC given by the previous proposition is a (p, q)-periodic orbit of the map T. Using the previous remark, this (p, q)-periodic orbit is indeed a Birkhoff periodic orbit.

To state an important result for homeomorphisms with an irrational rotation number, we first need the following definition.

Definition 1.5.2. F is *transitive* if there exists $x \in \mathbb{T}$ such that $\{F^n(x)\}_{n\in\mathbb{Z}}$ is a dense orbit in $\mathbb{T}$.

Proposition 1.5.3. *Let $\rho(F) \in \mathbb{R}\backslash\mathbb{Q}$. If F is transitive, then F is topologically conjugate to the rigid rotation by angle $\rho(F)$.*

If we add some more differentiability, we can state the following theorem.

Theorem 1.5.4 (Denjoy Theorem, [**14**]). *Let $F : \mathbb{T} \to \mathbb{T}$ be a $\mathcal{C}^2$ orientation-preserving diffeomorphism with $\rho(F) \in \mathbb{R}\backslash\mathbb{Q}$. Then F is topologically conjugate to a rigid rotation by angle $\rho(F)$.*

Remark 1.5.3. In particular, Denjoy Theorem can be used on sufficiently smooth RICs with irrational rotation numbers.

If the map is analytic and the rotation number is Diophantine (see relation (23) in Section 3.2), we have a stronger result.

Theorem 1.5.5 (Arnol'd, [**5**]). *Let $F : \mathbb{T} \to \mathbb{T}$ be a $\mathcal{C}^\omega$ area-preserving twist map with a Diophantine rotation number $\rho(F)$. Then F is $\mathcal{C}^\omega$ conjugate to a rigid rotation by angle $\rho(F)$.*

Further information on circle homeomorphisms and diffeomorphisms and on rotation numbers can be found in [**14**, §11,12].

1.5.2. Rotation number of twist maps. We would like to generalize the concept of the rotation number to area-preserving twist maps.

Given $T : C \to C$ an area-preserving twist map, $\widetilde{T} : \widetilde{C} \to \widetilde{C}$ its lift, $pr_1 : \widetilde{C} \to \mathbb{R}$ the projection on the lifted angular coordinate and $\pi : \mathbb{R} \to \mathbb{T}$ the projection on the quotient space, we would like to define "$\rho(T)$" as

$$\pi\left(\lim_{|n|\to\infty} \frac{pr_1(\widetilde{T}^n(\widetilde{s}, y)) - pr_1(\widetilde{s}, y)}{n}\right), \quad \text{for } (\widetilde{s}, y) \in \widetilde{C}. \tag{7}$$

However, this limit often does not exist and even if it does, it may depend on the chosen point $(s, y) \in C$. Of course, for two points on the same orbit, the limit is the same. Thus, in twist maps we will associate rotation numbers to concrete orbits. Recalling the definition given in Section 1.1 of the twist interval, for those orbits in C such that the limit (7) exists, the rotation number must belong to the twist interval.

From the previous subsection it is also clear that all orbits on a same RIC will have the same rotation number.

The rotation number on twist maps is not defined only on orbits on RICs but also in (p, q)-periodic orbits.

1.5.3. Rotation number of (p, q)-periodic orbits. Consider a (p, q)-periodic orbit of an area-preserving twist map, namely $\{\ldots, (s_0, y_0), (s_1, y_1), \ldots\}$. We try to compute limit (7) for this orbit. This is,

$$\rho = \pi \left(\lim_{|n| \to \infty} \frac{\widetilde{s}_n - \widetilde{s}_0}{n} \right).$$

For any n, n can be expressed as $n = k(n)q + r(n)$, $0 \leq r(n) < q$. Then, from periodicity, $\widetilde{s}_n = \widetilde{s}_{r(n)} + k(n)p$, and

$$\begin{aligned} \lim_{|k| \to \infty} \frac{\widetilde{s}_{k(n)q+r(n)} - \widetilde{s}_0}{k(n)q + r(n)} &= \lim_{|k| \to \infty} \frac{\widetilde{s}_{r(n)} + k(n)p - \widetilde{s}_0}{k(n)q + r(n)} \\ &= \lim_{|k| \to \infty} \frac{k(n)p}{k(n)q + r(n)} + \lim_{|k| \to \infty} \frac{s_{r(n)} - \widetilde{s}_0}{k(n)q + r(n)} = p/q \end{aligned}$$

Since $p/q \in (0, 1)$, the projection is already done and the rotation number for a (p, q)-orbit is $\rho = p/q$.

Chapter 2
Billiards

2.1. The model

In this section, we will introduce billiards and we will see that they are particular cases of the area-preserving twist maps introduced in the previous chapter.

Let us consider the motion of a point mass inside a convex bounded region Ω in the plane with a smooth boundary $\partial\Omega$. The orbits of such motion consist of straight line segments inside Ω which are joined at the boundary points according to the rule "the angle of reflection is equal to the one of incidence". The speed of motion is constant and the energy is conserved. Therefore, the motion is completely determined by the sequence of boundary points at which bounces occur.

Convenient coordinates are the Birkhoff coordinates (s, θ), defined as follows. Let $\partial\Omega = M(s)$, M a parameterization in the arc-length parameter s on the counter-clockwise direction. Bounce position can then be determined in terms of s. Let ℓ be the total length of the boundary, then s is cyclic, $s \in \mathbb{R}/\ell\mathbb{Z}$. The direction of motion is measured by the angle θ between the tangent to the boundary at the impact point and the trajectory. Since movement can only be inwards, $\theta \in [0, \pi]$. If we require that Ω be strictly convex, we restrict θ to the open interval $(0, \pi)$.

Let $T : \mathbb{R}/\ell\mathbb{Z} \times (0, \pi) \to \mathbb{R}/\ell\mathbb{Z} \times (0, \pi)$, $(s, \theta) \mapsto (s_1, \theta_1) = T(s, \theta) = (S(s, \theta), \Theta(s, \theta))$ be the map we have described. It is known as the *billiard map* and its construction is shown at Figure 2.1.

Henceforth, to simplify notation, we will denote by $\mathbb{T}$ the quotient space $\mathbb{R}/\ell\mathbb{Z}$.

2.1.1. The twist property on billiards. It can be observed that, for any fixed $s = s_0$, the lifted function $\widetilde{S}(s_0, \theta)$ is a monotone function of θ: $S(s_0, \theta)$ moves from s_0 to $s_0 + \ell$ as θ goes from 0 to π. More specifically one can obtain the concrete value of $\partial_\theta \widetilde{S}$ as follows.

Consider a point (s, θ) and its image (s_1, θ_1). Consider a slight modification of this last impact point $s_1 + \Delta s_1$ obtained when adding $\Delta\theta$ to the angle θ. Finally, consider the triangle obtained when linking points s, s_1 and $s_1 + \Delta s_1$ at the boundary. This

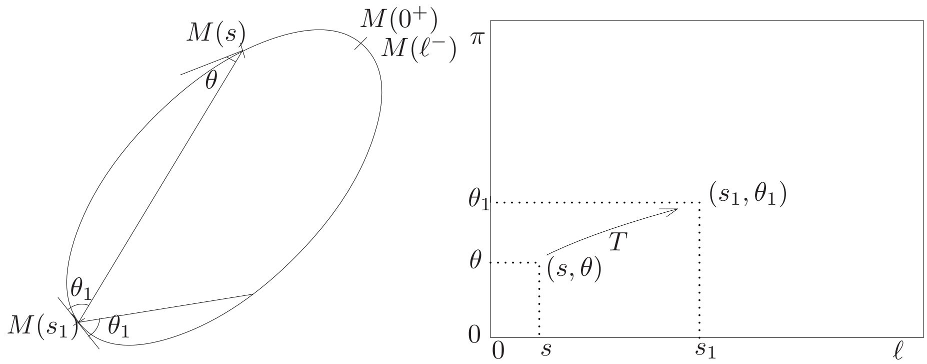

FIG. 2.1. At left, the billiard boundary and a particular trajectory drawn. As described, the incidence angle at an impact point equals to the reflection one and the billiard map is defined this way. At right, the points characterizing the same trajectory at the phase space.

triangle has angles $\Delta\theta$ at s, α at $s_1+\Delta s_1$ and $(\pi-\Delta\theta-\alpha)$ at s_1. This configuration is shown at Figure 2.2.

Then, the sinus law gives relation

$$\frac{||M(s_1)-M(s)||}{\sin\alpha} = \frac{||M(s_1+\Delta s_1)-M(s_1)||}{\sin\Delta\theta}.$$

We can obtain $\partial_\theta \widetilde{S}$ as

$$\partial_\theta \widetilde{S} = \lim_{\Delta\theta\to 0}\frac{\Delta s_1}{\Delta\theta}.$$

It is clear that when $\Delta\theta \to 0$, we have $\sin\Delta\theta = \Delta\theta + \mathcal{O}_3(\Delta\theta)$. We also have $\alpha \to \theta_1$. And, using Taylor formula, we obtain $||M(s_1+\Delta s_1)-M(s_1)|| = ||\Delta s_1 \vec{t}(s_1) + \mathcal{O}_2(\Delta s_1)|| = \Delta s_1 + \mathcal{O}_2(\Delta s_1)$, where $\vec{t}(s_1)$ is the unit tangent vector to the curve at s_1. Then, rewriting the sinus law with these approximations, we finally have

$$\partial_\theta \widetilde{S} = \lim_{\Delta\theta\to 0}\frac{\Delta s_1}{\Delta\theta} = \frac{||M(s_1)-M(s)||}{\sin\theta_1} > 0.$$

So the billiard map has the *twist property.*

2.1.2. Billiards are area-preserving maps. Billiard maps have also a generating function. The function measures the length between two boundary points as a function of parameters s and s_1. The generating function is

$$H(s,s_1) = ||M(s_1)-M(s)||. \tag{8}$$

Then, since $\partial_s H^2(s,s_1) = 2H(s,s_1)\partial_s H(s,s_1)$, we obtain

$$\begin{aligned} \partial_s H(s,s_1) &= \frac{\partial_s H^2(s,s_1)}{2H(s,s_1)} = \frac{\partial_s ||M(s_1)-M(s)||^2}{2||M(s_1)-M(s)||} \\ &= -\left\langle M'(s), \frac{M(s_1)-M(s)}{||M(s_1)-M(s)||}\right\rangle = -\cos\theta, \end{aligned} \tag{9}$$

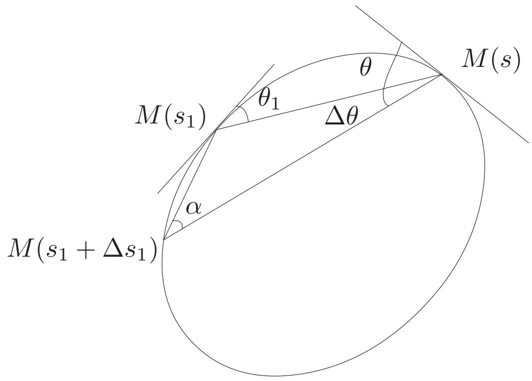

FIG. 2.2. Configuration and notation used for the computation of $\partial_\theta \widetilde{S}$.

and analogously,

$$\begin{aligned} \partial_{s_1} H(s,s_1) &= \frac{\partial_{s_1} H^2(s,s_1)}{2H(s,s_1)} = \frac{\partial_{s_1} ||M(s_1)-M(s)||^2}{2||M(s_1)-M(s)||} \\ &= \left\langle M'(s_1), \frac{M(s_1)-M(s)}{||M(s_1)-M(s)||} \right\rangle = \cos\theta_1, \end{aligned} \tag{10}$$

where we have used the definition of angles θ and θ_1 to deduce the formulae.

We will now consider coordinates (s,r) for the billiard, $r=-\cos\theta$ and the billiard map $T(s,r)=(S(s,r),R(s,r))$ and we will prove that the billiard map, in these coordinates, preserves area. We will call them canonical coordinates. If we define $\bar{H}(s,r):=H(s,S(s,r))$, we have

$$\partial_s \bar{H}(s,r) = \partial_1 H(s,S(s,r)) + \partial_2 H(s,S(s,r))\partial_s S(s,r) = -r + R(s,r)\partial_s S(s,r),$$
$$\partial_r \bar{H}(s,r) = \partial_2 H(s,S(s,r))\partial_r S(s,r) = R(s,r)\partial_r S(s,r).$$

We compute $\partial_r\partial_s\bar{H}$ and $\partial_s\partial_r\bar{H}$,

$$\left\{ \begin{array}{lcl} \partial_r\partial_s\bar{H} &=& \partial_r(\partial_s\bar{H}) = \partial_r\left(-r+R\partial_s S\right) = -1+\partial_r R\partial_s S + R\partial_r\partial_s S \\ \partial_s\partial_r\bar{H} &=& \partial_s(\partial_r\bar{H}) = \partial_s\left(R\partial_r S\right) = \partial_s R\partial_r S + R\partial_s\partial_r S. \end{array} \right.$$

Therefore, combining $\partial_r\partial_s\bar{H} = \partial_s\partial_r\bar{H}$ with the above equalities, we get

$$-1+\partial_r R\partial_s S \quad = \quad \partial_s R\partial_r S.$$

And we have obtained the area-preservating condition for the billiard map

$$\det(DT(s,r)) = \partial_r R\partial_s S - \partial_s R\partial_r S = 1. \tag{11}$$

In the Birkhoff coordinates, the billiard map preserves the area element $\sin\theta\,\mathrm{d}s\,\mathrm{d}\theta$, since $\mathrm{d}r=\sin\theta\mathrm{d}\theta$.

Henceforth, we will be working indifferently with the billiard map defined in Birkhoff coordinates, $T:\mathbb{R}/\ell\mathbb{Z}\times(0,\pi)\to\mathbb{R}/\ell\mathbb{Z}\times(0,\pi)$, $(s,\theta)\mapsto(s_1,\theta_1)=T(s,\theta)$, or in these new ones, $T:\mathbb{R}/\ell\mathbb{Z}\times(-1,1)\to\mathbb{R}/\ell\mathbb{Z}\times(-1,1)$, $(s,r)\mapsto(s_1,r_1)=T(s,r)$.

2.1.3. The billiard map preserves orientation. The determinant of $\mathrm{DT(s,r)}$ is positive as it can be seen in (11).

2.1.4. Rigid boundary conditions for the billiard map. First we remark that, for θ small, the trajectory direction is almost parallel to the tangent vector of the curve $M(s)$. Therefore, next impact s_1 at the boundary is very close to the last one s, but located forward counterclockwise. As for the new angle, θ_1, since the variation on the curve is smooth and s_1 will be so close to s, θ_1 will be also close to θ. The same happens in the clockwise direction, when θ tends to its supremum value, π.

Therefore, billiard maps can be continuously extended to the boundary of the cylinder as the identity map. We will indifferently use the open domain, $\mathbb{T} \times (0, \pi)$, or the closed one, $\mathbb{T} \times [0, \pi]$, for the map T.

For the lift $\widetilde{T}$ we have

$$\lim_{\theta \to 0} \widetilde{T}(\widetilde{s}, \theta) = (\widetilde{s}, 0), \quad \lim_{\theta \to \pi} \widetilde{T}(\widetilde{s}, \theta) = (\widetilde{s} + \ell, \pi).$$

Hence, the rigid boundary frequencies are $\omega_- = 0$ and $\omega_+ = 1$ and the twist interval is $(0, 1)$.

In this section we have seen that billiard maps satisfy all the conditions required to be an area-preserving twist map. Hence we will apply the results in the previous chapter for an area-preserving twist map to the billiard map, emphazising the geometric properties of the latter.

2.2. Properties

2.2.1. Billiard differentiability. The definition of the generating function in the billiard case permits us to state that if the parameterization of the curve $M : s \mapsto M(s)$ is a $\mathcal{C}^k$ curve, then the billiard map is a $\mathcal{C}^{k-1}$ map: if $M \in \mathcal{C}^k$, it is clear that $H \in \mathcal{C}^k$ and as we have seen before, applying the Implicit Function Theorem, we obtain $T_1, T_2 \in \mathcal{C}^{k-1}$.

2.2.2. Twisting clockwise and counterclockwise. Any trajectory on the billiard can be traveled in both directions. Therefore, each billiard trajectory traveled clockwise is in one-to-one correspondence with one traveled counterclockwise. We have a symmetry on the phase space: the orbit of a point (s, θ) is symmetric with respect to line $\theta = \pi/2$ to the one of the point $(s, \pi - \theta)$. Then, it is common to restrict the study of the orbits close to the boundaries to the ones close to the lower boundary, $\theta = 0$, knowing that same results apply to the upper ones.

2.2.3. Periodic orbits on the billiard. From the definition of periodic orbits on area-preserving twist maps, a (p, q)-periodic orbit in the billiard is one that has the following form on the universal cover of phase space, $\widetilde{C}$

$$\begin{cases} \widetilde{s}_q &= \widetilde{s}_0 + p\ell \\ \theta_q &= \theta_0. \end{cases}$$

Note that we have adapted the condition on $\widetilde{s}$ since we are working with angular coordinates defined on $\mathbb{R}/\ell\mathbb{Z}$ instead of $\mathbb{R}/\mathbb{Z}$.

If we look at the billiard table, after q iterates, we arrive at same point, s_0 and we depart with the same direction θ_0. Therefore, after q iterates, we have formed a closed polygon which will be repeated forever. Conversely, the role of p is indicating the number of turns inside $\partial\Omega$ that have been done until the closing of the polygon. These turns are always counterclockwise.

The symmetry mentioned on Subsection 2.2.2 has the following consequence when applied to (p, q)-periodic orbits: a (p, q) counterclockwise periodic orbit becomes a $(q - p, q)$-periodic orbit when it is traveled clockwise. Thus, we can always assume that $p \leq q/2$.

Recall, from Section 1.4, that due to Poincaré-Birkhoff Theorem, there exist at least two (p, q)-periodic orbits. These two orbits are Birkhoff, which means that the angular coordinates of the points of the orbit lifted to $\mathbb{R}\times[0, \pi]$ are monotonically increasing. At Figure 2.3 different periodic orbits, Birkhoff and non-Birkhoff, can be seen.

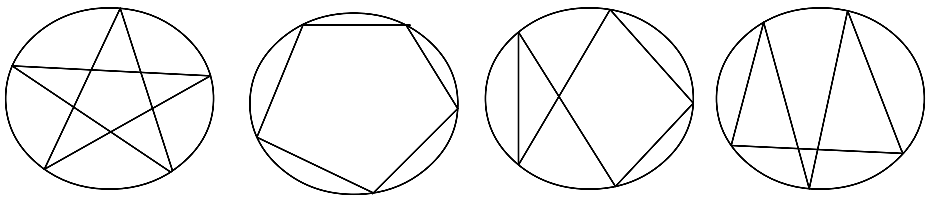

FIG. 2.3. All these orbits have period 5. First two orbits are Birkhoff periodic orbits since its angular coordinate behaves like a rigit rotation. First figure represents a $(2, 5)$-Birkhoff periodic orbit while the second one is a $(1, 5)$-Birkhoff periodic orbit. The two figures on the right have non-Birkhoff periodic orbits, the one on the edge is a $(2, 5)$-periodic orbits while the other is a $(3, 5)$-periodic orbit.

2.2.4. Geometric description of the 2-periodic orbits on the billiard. As we have already seen in Chapter 1, there exist at least 2 orbits of type $(1, 2)$ for any area-preserving twist map such that its rotation interval contains the value $1/2$. We have somehow reasoned that one of the orbits was obtained as a minimum of the action $W^{(1,2)}$ and the other was found as the result of a minimax principle. Here, we will see the geometric translation of this result.

First we need to introduce some notations and definitions. Given an angle θ, consider all the possible lines with slope $\tan\theta$. From this set of lines, consider only the lines that have a non void intersection with the billiard boundary, $\partial\Omega$. Finally, from this subset consider the only two lines that are tangent to the billiard curve at all the intersection points. We define $l(\theta)$ as the distance between these two lines.

DEFINITION 2.2.1. The *diameter* of Ω is defined as $d := \max\{l(\theta), \theta \in (0, \pi)\}$. It coincides with the maximum distance between two points in $\partial\Omega$.

DEFINITION 2.2.2. The *width* of Ω is defined as $w := \min\{l(\theta), \theta \in (0, \pi)\}$.

Both definitions are illustrated in Figure 2.4. As an example, on an elliptic billiard, the diameter coincides with the long axis of the ellipse and the width coincides with the short one.

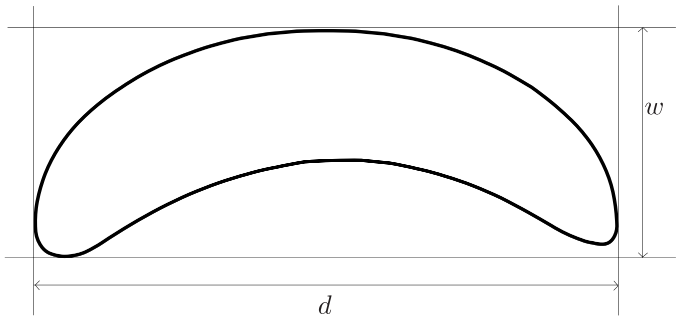

FIG. 2.4. Example of the width and the diameter on a billiard table.

We can now state the following proposition.

PROPOSITION 2.2.1 ([**14**, Proposition 9.2.1., p. 345]). *The billiard associated to a strictly convex $\mathcal{C}^k$-curve $M(s)$, $k \geq 1$ has at least two distinct period-two orbits which are described as follows: for one of them, the distance between the corresponding boundary points is the diameter of Ω, for the other is the width of Ω.*

PROOF. Consider the generating function $H(s, s_1)$ defined before. H is well defined on the torus $\mathbb{T} \times \mathbb{T}$ and since M is differentiable, we have that, for $s_1 \neq s$, H is also $\mathcal{C}^k$. Observe that for $s_1 = s$ we are in a stationary point ($\theta \in \{0, \pi\}$) and we are omitting these cases.

Function $H(s, s_1)$ is 0 on the diagonal and positive elsewhere. Therefore, it must attain a maximum d at some point (s^*, s_1^*), $s^* \neq s_1^*$. Observe that the two points s^* and s_1^* are the ones characterizing the diameter. Since (s^*, s_1^*) is a critical point, we must have $\partial_1 H(s^*, s_1^*) = \partial_2 H(s^*, s_1^*) = 0$. Then, from (9) and (10), we obtain the conditions $\theta^* = \pi/2$ and $\theta_1^* = \pi/2$, that is, we have indeed found a $(1, 2)$-periodic orbit.

Now consider the segment with vertices s and $s_1 = g(s)$, where $g : \mathbb{R}/\ell\mathbb{Z} \to \mathbb{R}/\ell\mathbb{Z}$ is defined by $s_1 = g(s)$ if the billiard map linking points s and s_1 has the form $(s_1, \theta_1) = T(s, \theta) = (s_1, \pi - \theta)$. Observe that g is chosen in such a way that we are imposing that the incidence-reflection angle at s_1, that is θ_1, is the opposite to the one at s, θ. Also, observe that the minimal length of the segments with end points s and $g(s)$ is the width of Ω.

If we restrict H to points of the form $(s, g(s))$, H is bounded from below by a positive number. Thus, it attains a positive minimum (the one we said was in one-to-one correspondence with the width of Ω). Since the curve is strictly convex, we can parameterize $H(s, g(s))$ by a differentiable absolute angle α. Note that

$$\begin{aligned}\partial_\alpha H(s(\alpha), s_1(\alpha)) &= \partial_1 H(s(\alpha), s_1(\alpha))\partial_\alpha s + \partial_2 H(s(\alpha), s_1(\alpha)))\partial_\alpha s_1 \\ &= \cos\theta\,(\partial_\alpha s + \partial_\alpha s_1)\,.\end{aligned}$$

And since we are looking for the positive minimum, we need to impose $\cos\theta = 0$ and then $\theta = \pi/2 = \theta_1$ which again leads to a $(1, 2)$-periodic orbit. □

2.2.5. Elliptic and hyperbolic orbits. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be an area-preserving map. The linearization of the map around a point is a simple tool to obtain information about the behaviour of the map.

Given a fixed point, p, $T(p) = p$, the next result concerning linearization depends on the following classification. According to $\mathrm{Spec}(DT(p)) = \{\lambda, \lambda^{-1}\}$, we say that

- The fixed point p is *parabolic* if both eigenvalues coincide. Thus, it is parabolic if and only if $\lambda = \lambda^{-1} = 1$ or $\lambda = \lambda^{-1} = -1$.
- The fixed point is *elliptic* if $\lambda \neq \lambda^{-1}$ and $|\lambda| = |\lambda^{-1}| = 1$.
- The fixed point is *hyperbolic without refllection* if $|\lambda|, |\lambda^{-1}| \neq 1$ and $\lambda, \lambda^{-1} > 0$.
- The fixed point is *hyperbolic with reflection* if $|\lambda|, |\lambda^{-1}| \neq 1$ and $\lambda, \lambda^{-1} < 0$.

The Hartman-Grobman Theorem, [**14**], states that the map T is topologically conjugated to its linearization, $DT(p)$, in a neighbourhood of a hyperbolic (with or without reflection) fixed point p. Therefore, hyperbolic points are always unstable. For the elliptic case, the fixed point of the linearized system is surrounded by closed invariant circles which ensure the linear stability of the elliptic point. However, in general, there is no guarantee that this behaviour is inherited by the map T.

This same discussion can be applied to periodic orbits on the billiard. Let T be a billiard map T and $\{(s_1, \theta_1), \ldots, (s_q, \theta_q)\}$ a (p, q)-periodic orbit. Then, any point (s_i, θ_i) with $i \in \{1, \ldots, q\}$ is a fixed point of the map T^q.

Definition 2.2.3. We say that a (p, q)-periodic orbit, $\{(s_1, \theta_1), \ldots, (s_q, \theta_q)\}$, is an *elliptic orbit* if, for $i \in \{1, \ldots, q\}$, (s_i, θ_i) is an elliptic fixed point of the area-preserving map T^q.

Definition 2.2.4. Conversely, we say that the (p, q)-periodic orbit is an *hyperbolic orbit* if, for $i \in \{1, \ldots, q\}$, (s_i, θ_i) is a hyperbolic (with or without reflection) fixed point of T^q.

Next, we only consider $(1, 2)$-periodic orbits. The $(1, 2)$-periodic orbit corresponding to the diameter of the billiard map is always hyperbolic, while the one corresponding to the width can be either elliptic, hyperbolic or parabolic.

As it is explained in [**23**], for $(1, 2)$-periodic orbits a geometric condition is sufficient to decide. More concretely, let s_1 and s_2 be the impact points of the $(1, 2)$-periodic orbit on the boundary $\partial\Omega$ parameterized by $M : \mathbb{T} \to \mathbb{R}^2$. Then $H(s_1, s_2)$ is the length of the chord from s_1 to s_2. Let $\kappa : \mathbb{T} \to \mathbb{R}$ the function giving the curvature at each point on the boundary. Then, the $(1, 2)$-periodic orbit is hyperbolic if and only if the condition

$$g(s_1, s_2) := H(s_1, s_2)(\kappa(s_1) + \kappa(s_2)) > 4 \tag{12}$$

holds. Moreover, if the value $g(s_1, s_2)$ obtained is equal to 4 the orbit is parabolic and for $g(s_1, s_2) < 4$ the $(1, 2)$-periodic orbit is elliptic. We apply this formula to elliptic billiards at Section 2.5.

2.3. Convex caustics and rotational invariant circles

Definition 2.3.1. A curve Γ such that a billiard trajectory is tangent to it after every reflection at the billiard boundary $\partial\Omega$ is called a *caustic*.

Definition 2.3.2. A smooth closed convex caustic curve Γ lying inside the billiard table Ω will be called a *convex caustic* (see Figure 2.5).

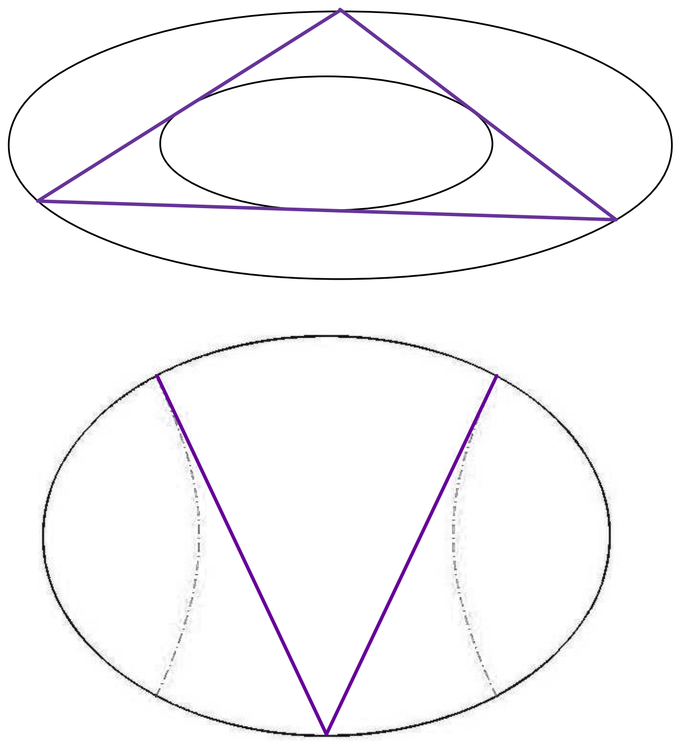

Fig. 2.5. The first figure shows a convex caustic and the second one a nonconvex caustic. In this example, both trajectories are periodic. The convex caustic is tangent to a $(1, 3)$-periodic orbit while the second one is tangent to a $(2, 4)$-periodic orbit.

Convex caustics are related to rotational invariant circles in the following way. Let Γ be a strictly convex smooth caustic of a billiard table with smooth boundary $\partial\Omega$.

Then, the billiard map $T : \mathbb{T} \times (0, \pi) \to \mathbb{T} \times (0, \pi)$ has two smooth RICs, $\Upsilon^{\pm} = \operatorname{graph} \theta^{\pm} \in \mathbb{T} \times (0, \pi)$.

The functions $\theta^{\pm} : \mathbb{R}/\ell\mathbb{Z} \to (0, \pi)$ give the angles $\theta^{+}(s)$ and $\theta^{-}(s)$ determined by the two tangent lines to the caustic Γ at each point $M(s) \in \partial\Omega$. In particular, we obtain that $\theta^{-}(s) = \pi - \theta^{+}(s)$. Geometrically, the two RICs obtained correspond to travelling the billiard trajectory clockwise, Υ^{+}, and counterclockwise, Υ^{-}. This two-to-one correspondence can be seen as another consequence of the existent symmetry on the phase space that we mentioned in Subsection 2.2.2. See [**15**] for the proof of this relation.

From this correspondence, and since RICs can not intersect, we obtain $0 < \theta^{-}(s) \leq \pi/2 \leq \theta^{+}(s) < \pi$. Thus, if a billiard trajectory contains bounces with arbitrary small angles reflection and other bounces with angles of reflection arbitrary close to π no RIC exists and no caustic either. This last argument is related to the non existence criterion that we have already commented in Section 1.2. Some more results on the existence and nonexistence of caustics can be found on [**1**], [**19**] or [**13**].

A convex caustic Γ can be characterized as follows. Given a point $N \in \partial\Omega$ we have tangents NM and NM_1 from N to Γ.

Definition 2.3.3. Let Q be the quantity defined as $Q = |NM| + |NM_1| - \widehat{MM_1}$, where $\widehat{MM_1}$ is the arc-length of Γ between M and M_1. Q is independent of N and it is called the *Lazutkin invariant*.

The map $M \mapsto M_1$ is a diffeomorphism from Γ to itself. Since it is an homeomorphism of $\mathbb{T}$, we can associate a rotation number to each caustic: $\eta = \eta(\Gamma)$. The rotation number is a topological invariant and if we have $\eta \notin \mathbb{Q}$ and the map $M \mapsto M_1$ sufficiently smooth, we have seen that Denjoy Theorem affirms that the map is topologically conjugate to a rotation $R_\eta : \Gamma \to \Gamma$, $\xi \mapsto \xi + \eta$. If we travel counterclockwise the same caustic, the rotation number is $1 - \eta$. Therefore, we fix $\eta < 1/2$. In fact, this rotation number η coincides with the one corresponding to the orbits on the RIC Υ^{-}.

Applying what we have seen in Section 1.5 to this concrete setting, any convex caustic Γ with a rational rotation number, $\eta(\Gamma) = p/q$ has a (p, q)-periodic orbit. It may happen that there exists a caustic with rational rotation number $\eta = p/q$ completely foliated by (p, q)-periodic orbits.

Definition 2.3.4. Let $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$ and $p < q/2$. A (p, q)-*resonant (convex) caustic* Γ is a (convex) caustic such that all billiard trajectories tangent to Γ give rise to closed polygons with the same number of turns around $\partial\Omega$, p, and the same number of sides, q.

Note that resonant caustics are very degenerate. Recall that for any general strictly convex and sufficiently smooth billiard table Ω, we can affirm that there exists at least two (p, q)-periodic orbits (recall Birkhoff Theorem 1.2.1). Besides, when we have a convex (p, q)-resonant caustic, we can guarantee the existence of a continuous family of (p, q)-periodic orbits.

If we recall the comment on the two-to-one correspondence between RICs and a caustic, we find that points on Υ^- belong to (p,q)-periodic points while points on Υ^+ belong to $(q-p,q)$-periodic orbits. Figure 2.6 is an example of $(1,3)$-resonant caustic on an elliptic billiard.

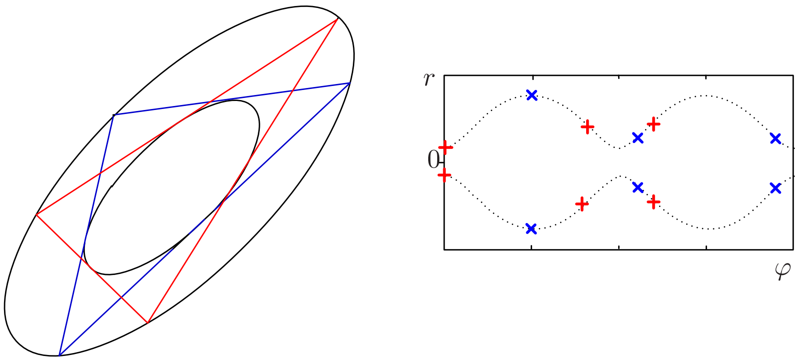

FIG. 2.6. At right, a $(1,3)$-resonant caustic on an elliptic billiard is shown with two $(1,3)$-periodic orbits. The coordinates used here to represent the phase space are not (s,θ) but (φ,r), where φ is such that the parameterization is $M(\varphi) = (a\cos\varphi, b\sin\varphi)$ and $r = ||M'(\varphi)||\cos\theta$. If we follow both trajectories counterclockwise and we mark the points on the phase space, all the points are on the curve below $r=0$, which is a RIC, while the other points we have marked are the ones that would appear if we traveled the billiard clockwise and lie on the symmetric RIC.

The existence of resonant convex caustics is a rare phenomenon. Nevertheless, the following theorem guarantees the existence of resonant caustics in a concrete setting.

THEOREM 2.3.1 (Poncelet's Porism [**7**]). *If $\partial\Omega$ is an ellipse, any caustic with a rational rotation number is resonant.*

As we will see in Section 4.3, (p,q)-resonant caustics can be destroyed by arbitrary small perturbations of the billiard boundary $\partial\Omega$, while caustics with "very irrational numbers" do persist. We will state the result in a more formal way and also define in a better way the concept "very irrational numbers" in Section 4.2.

2.4. Billiards inside a circle

We will explicitly find the map T for a billiard map in the circle and observe that it is an integrable map.

Let (s,θ) be a point on the phase space. Consider the next point on the trajectory, (s_1,θ_1) and the triangle with vertices s_1, s and the center of the circle, O. This triangle is isosceles and therefore angles at s, $\pi/2-\theta$, and at s_1, $\pi/2-\theta_1$, must coincide, leading to $\theta_1=\theta$. Also, we obtain that the angle at the vertex O is 2θ and the arc-length parameterization of the circle gives $s_1 = s + 2\theta$. Figure 2.7 illustrates this argument.

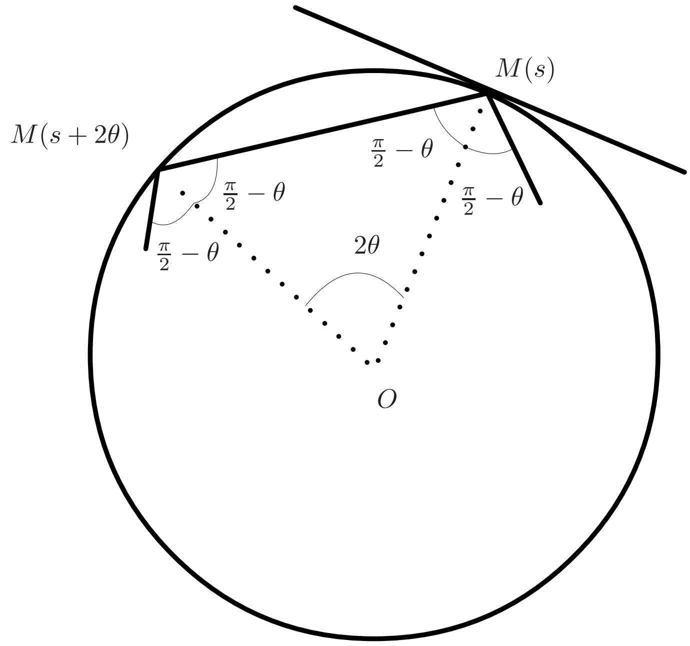

FIG. 2.7. The billiard map inside a circle.

So, given a point (s, θ) on the phase space, the billiard map $T(s, \theta) = (s + 2\theta, \theta)$, which has the form of an integrable map.

The generating function only depends on the difference of the angular coordinates. Let R be the radius of the circumference, the distance between s and s_1 is $H(s, s_1) = 2R\sin\theta = 2R\sin((s_1 - s)/2)$. Usually R is set to 1.

Inside the circle, we can guarantee not only the existence of at least two (p, q)-Birkhoff periodic orbits but the existence of a continuous family of regular polygons of type (p, q) inscribed in the circle $\partial\Omega$. Departing from one of the two (p, q)-periodic orbit given by the Poincaré Birkhoff Theorem, we rotate the polygon formed by the (p, q)-periodic orbit with respect the center of the circle. The envelope of the rotation of all these (p, q)-periodic trajectories delimits a (p, q)-resonant caustic (from Poncelet's Porism). Observe that, since all the (p, q)-orbits are obtained by rotating a (p, q)-Birkhoff periodic orbit, all of them are Birkhoff orbits and have the same length. This same result can be obtained using the following argument. Since every orbit is contained in a RIC, using Remark 1.5.2, we know that all the periodic orbits inside the circle are Birkhoff.

Figure 2.8 illustrates some $(1, 3)$-periodic orbits, its resonant caustic and the RICs associated to it.

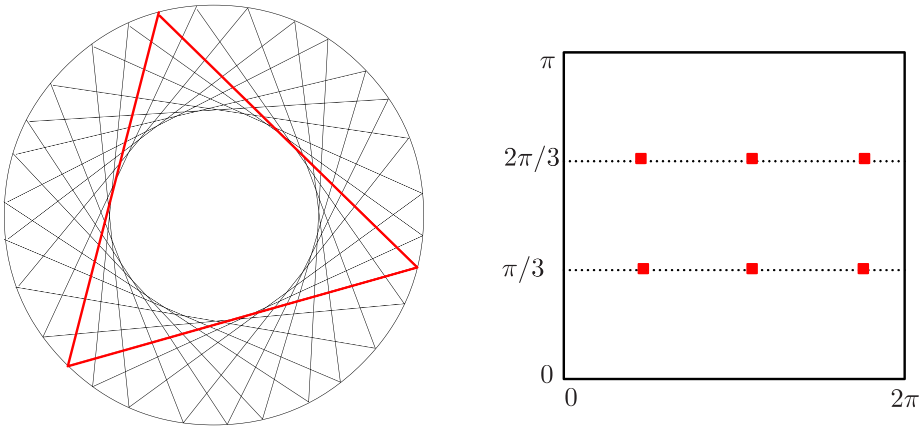

Fig. 2.8. At left, some $(1,3)$-periodic orbits. All the trajectories have the same incidence angle θ at each impact point. The map is integrable and the resonant RICs are $\theta =$ constant on the phase space, in particular $\theta = \pi/3$ when the trajectories are traveled counterclockwise and $\theta = 2\pi/3$ when they are traveled clockwise.

2.5. Elliptic billiards

Let $\partial\Omega$ be an ellipse. Without loss of generality, we can consider by a translation and a similarity that, in cartesian coordinates, the boundary can be expressed as

$$\partial\Omega = \left\{ (x, y) \in \mathbb{R}^2 \ : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\},$$

with $a > b > 0$. Thus, the foci are located at $(c, 0)$ and $(-c, 0)$, where $c^2 = a^2 - b^2$.

We will not be taking the arc-length parameter s to parameterize the boundary. Instead, we will be choosing the parameterization given by $M(\varphi) = (a\cos\varphi, b\sin\varphi)$, which is the most natural way to parameterize an ellipse. Thus, the preserved measure is $||M'(\varphi)|| \sin\theta \mathrm{d}\varphi\mathrm{d}\theta$.

The elliptic billiard is not an integrable map as we have defined before but it is *Liouville integrable.*

Definition 2.5.1. Let T be an area-preserving map. If there exists a non-constant function $I(\varphi, \theta) : \mathbb{T} \times (0, \pi) \to \mathbb{R}$ such that it is a first integral (equivalently, $I \circ T = I$) we say that T is *Liouville integrable.*

The first integral is $\lambda^2(\varphi, \theta) = (a^2 - b^2)\sin^2\varphi - ||M'(\varphi)||^2 \cos^2\theta + b^2$ (see [**2**]). The existence of a first integral implies that the phase space is foliated by invariant curves, $\lambda^2 =$ constant. In particular, if the curve $\lambda^2 =$ constant is not contractible to a point, we have found a RIC. Actually, from the symmetry of the problem, we have found two RICs. The phase space foliated by curves $\lambda^2 =$ constant can be seen in Figure 2.9.

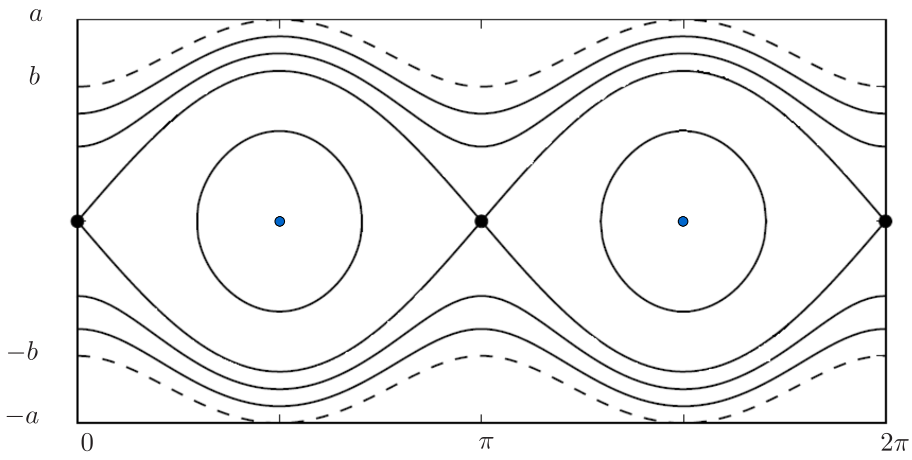

FIG. 2.9. Portrait phase space of an elliptic billiard. Some curves corresponding to λ^2 = constant are shown. The coordinates used on the phase space are (φ, r), where $r = ||M'(\varphi)|| \cos\theta$. In these coordinates, the phase space looks simpler. However, the domain is not the whole annulus, but the region between the dashed lines. This figure has been taken from [**2**]; thanks to R. Ramírez-Ros and P. Sánchez Casas.

As we have seen in Section 2.3, where we have found a one-to-two correspondence between caustics and RICs, this foliation allows us to affirm that any trajectory such that $0 < \lambda^2 < b^2$ has a convex caustic. Also, if this caustic has a rational rotation number p/q, we have found that it is resonant (as we already knew from Poncelet's Porism). Recalling Remark 1.5.2, all these (p, q)-periodic orbits are Birkhoff.

It can be proved that any convex caustic Γ on the elliptic billiard is a confocal ellipse to $\partial\Omega$. Indeed, the convex caustics can be characterized by the first integral λ^2. Given $\lambda \in (0, b)$, the corresponding convex caustic is

$$\Gamma = C_\lambda = \left\{ (x, y) \in \mathbb{R}^2 \ : \ \frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1 \right\}.$$

Observe that for $\lambda = 0$ we obtain the boundary, $C_0 = \partial\Omega$, and the rotation number is $\rho = \rho(0) = 0$. Instead, for $\lambda = b$ the ellipse obtained is degenerate, it is the segment linking the two foci. The trajectory is then the (1,2)-periodic orbit corresponding to the diameter. Therefore, for $\lambda = b$, $\rho(b) = 1/2$. The rotation number is a monotonically increasing function and therefore, for any $p, q \in \mathbb{N}$ such that $\gcd(p, q) = 1$ and $p < q/2$, there exists a convex (p, q)-resonant caustic.

Also, it can be proved that the trajectories such that $b^2 < \lambda^2 < a^2$ are in correspondence with nonconvex caustics which are hyperbolas with the same foci as $\partial\Omega$. From Poncelet's Porism, we know that all the trajectories tangent to hyperbolas with a rational rotation number, p/q, are (p, q)-periodic orbits. None of these periodic orbits are Birkhoff. However, by looking at the phase space we can find some other coordinates with which these (p,q)-periodic orbits from resonant caustics are Birkhoff. To argue it, we prefer to first introduce the following observations.

As we have seen in Section 2.2.5, on any strictly convex billiard, we can assure that at least one $(1,2)$-periodic orbit is hyperbolic, the one associated to the diameter. The $(1,2)$-periodic orbit related to the width of the ellipse has impact points on the boundary $(0,b)$ and $(0,-b)$. Therefore the chord length is $2b$. Using the parameterization given at this section, the two points correspond to impact points $\varphi = \pi/2$ and $\varphi = 3\pi/2$ respectively and the curvature obtained at each impact point is $\kappa(\pi/2) = \kappa(3\pi/2) = b/a^2$. Thus, $2b(b/a^2 + b/a^2) = 4b^2/a^2 < 4$ and this $(1,2)$-periodic orbit is elliptic, see condition (12).

Now, we justify which coordinates choose so that periodic orbits on contractible invariant curves become Birkhoff. We pick an easy example. Consider the $(2,4)$-orbit on Figure 2.10. Taking a look to the phase space, we see that the angular coordinates do not behave as a rigid rotation on the circle. However, if by symmetry we move the two iterates of the orbits that make turns to the second elliptic point, p_2 and p_4, to the first one becoming p_2' and p_4', we see that the four iterates act like a rigid rotation around the first elliptic point. This idea can be rigorously formalized using the Birkhoff normal form presented in Section 3.1 below.

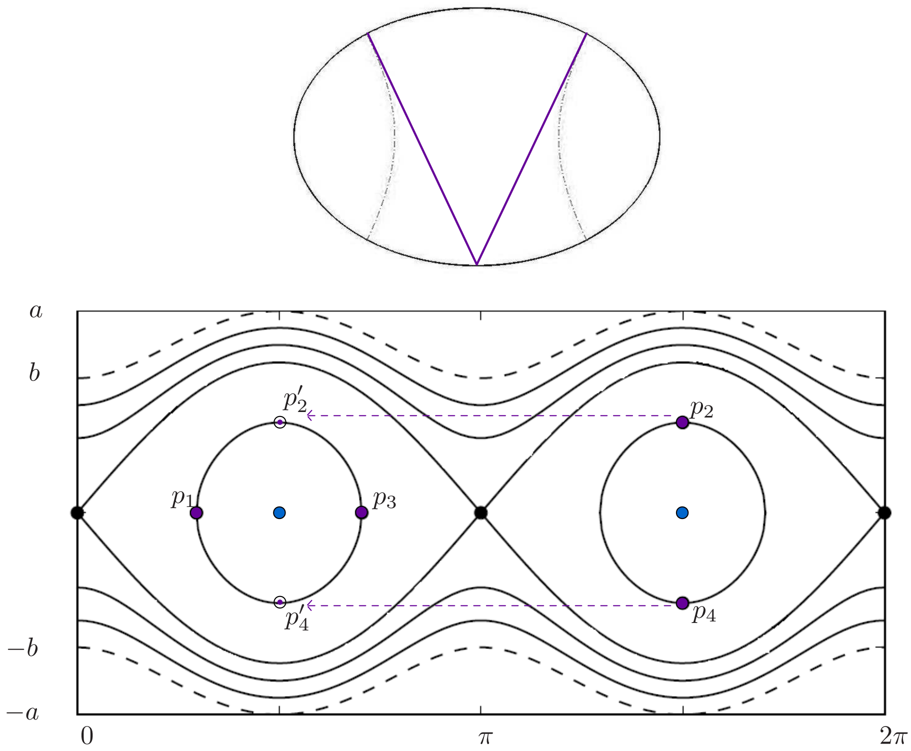

FIG. 2.10. The non-Birkhoff $(2, 4)$-periodic orbit is said to be a $(1, 4)$-Birkhoff when looking at the rotation around one of the two elliptic fixed points on the phase space.

Chapter 3
General tools

3.1. Birkhoff normal form

Normal forms theory consists on writing a map near an invariant object in some new coordinates such that the expression for the map in these new coordinates is simpler. One possible way to achieve this simpler form is by means of a sequence of changes of coordinates, each one of them cancelling some terms in the expansons of the map. This procedure does not need to be convergent. Even if it is divergent, the knowledge of the normal form up to a certain finite order gives important information about the qualitative behaviour of the map.

We will be restricting ourselves to the two dimensional case. Here we quote some results in [**25**, §23].

Let $T : (x, y) \mapsto (x_1, y_1)$ be an area-preserving analytic map defined near a fixed point which, without loss of generality, we will assume to be the origin. We will also assume that the linear terms have already been brought to a normal form. Then our initial map has the following form,

$$x_1 = T_1(x, y) = \lambda x + \sum_{k>1} T_{1k}, \quad y_1 = T_2(x, y) = \mu y + \sum_{k>1} T_{2k}, \tag{13}$$

where T_{ik} are homogeneous polynomial in (x, y) of degree k, for $i = 1, 2$.

As we have seen in Subsection 2.2.5, according to $\mathrm{Spec}(DT(0)) = \{\lambda, \mu\}$, the origin can be classified as parabolic, as hyperbolic with or without reflection or as elliptic. Henceforth, we will not consider the parabolic case.

We want to determine a change of variables $C : (\xi, \eta) \mapsto (x, y)$ such that the map T in the new coordinates, that is $N := C^{-1}TC$, is as simple as possible. Since the linear part of (13) is already in normal form, linear terms of the coordinate transformation correspond to the identity. Thus, we look for a nonlinear transformation of the form

$$x = C_1(\xi, \eta) = \xi + \sum_{k>1} C_{1k}, \quad y = C_2(\xi, \eta) = \eta + \sum_{k>1} C_{2k}, \tag{14}$$

with C_{ik} homogeneous polynomial in (x, y) of degree k, for $i = 1, 2$.

The simplest map we would like to achieve as normal form would be $N(\xi,\eta) = (\lambda\xi, \mu\eta)$ which would imply to cancel all terms of order greater than one. Let us see why this is not possible.

Relation $N = C^{-1}TC$ is equivalent to $CN = TC$ and we can compare the coefficients of the series. Observe that

$$CN = TC \quad \Leftrightarrow \quad \left\{ \begin{array}{l} C_1(\lambda\xi,\mu\eta) = T_1(C_1(\xi,\eta), C_2(\xi,\eta)) \\ C_2(\lambda\xi,\mu\eta) = T_2(C_1(\xi,\eta), C_2(\xi,\eta)). \end{array} \right. \tag{15}$$

It is easy too see that the linear terms coincide when inserting series from (13) and (14) into (15). Assume all the coefficients of all the terms of degree less than k agree in (15) and we have determined polynomials C_{1l} and C_{2l} for $l < k$. Equations (15) lead to

$$C_{1k}(\lambda\xi,\mu\eta) = \lambda C_{1k}(\xi,\eta) + \ldots, \quad C_{2k}(\lambda\xi,\mu\eta) = \mu C_{2k}(\xi,\eta) + \ldots \tag{16}$$

where the terms not written down explicitly are homogeneous polynomials of degree k whose coefficients have already been determined. Writing $C_{1k}(\xi,\eta) = \sum_{l=0}^{k} a_l \xi^{k-l}\eta^l$ and $C_{2k}(\xi,\eta) = \sum_{l=0}^{k} b_l \xi^{k-l}\eta^l$, we have

$$\left\{ \begin{array}{lcl} C_{1k}(\lambda\xi,\mu\eta) - \lambda C_{1k}(\xi,\eta) & = & \sum_{l=0}^{k} a_l(\lambda^{k-l}\mu^l - \lambda)\xi^{k-l}\eta^l \\ C_{2k}(\lambda\xi,\mu\eta) - \mu C_{2k}(\xi,\eta) & = & \sum_{l=0}^{k} b_l(\lambda^{k-l}\mu^l - \mu)\xi^{k-l}\eta^l. \end{array} \right. \tag{17}$$

Using (17) into (16), one can see that coefficients a_l and b_l can only be determined if factors $(\lambda^{k-l}\mu^l - \mu)$ and $(\lambda^{k-l}\mu^l - \lambda)$ are all different from 0.

Since our map T is area-preserving, we have relation $\lambda\mu = 1$ and therefore $\lambda^{l+1}\mu^l - \lambda = 0$ and $\lambda^l\mu^{l+1} - \mu = 0$ for any l. So it is clear we can not obtain a normal form as simple as we have proposed, $N(\xi,\eta) = (\lambda\xi,\mu\eta)$.

The simplest expression we may achieve is a normal form of type

$$N(\xi,\eta) = (u\xi, v\eta), \quad u = \sum_{k\geq 0} \alpha_{2k}(\xi\eta)^k, \quad v = \sum_{k\geq 0} \beta_{2k}(\xi\eta)^k. \tag{18}$$

If λ is not a root of unity and the equations

$$\begin{array}{l} \partial_\xi C_1 - \partial_\eta C_2 = \sigma(\xi,\eta) \\ \partial_\xi C_1 \partial_\eta C_2 - \partial_\xi C_2 \partial_\eta C_1 - 1 = \tau(\xi,\eta) - 1 \end{array} \tag{19}$$

are series not containing powers of $\omega = \xi\eta$ alone, then, there exists a unique formal substitution C of type (14) that brings a map like (13) into the normal form (18). It is shown that C is then an area-preserving map and we also obtain the formal relation $uv = 1$. Moreover, this last condition is not only necessary but also sufficient for (13) to be area-preserving. From this condition, one can observe that $\xi_1\eta_1 = \xi\eta$ and therefore the product $\xi\eta$ is a first integral.

With some additional hypotheses, the normal form can still be reduced a little bit more. We assume the initial map (13) real and, again, λ is not a root of the unity.

If the origin is a hyperbolic point without reflection, there exists a unique real power series,

$$w \quad = \quad \sum_{k=0}^{\infty} \gamma_k(\xi\eta)^k, \quad \gamma_0 \text{ such that } \lambda = e^{\gamma_0}, \tag{20}$$

such that $u = e^w,\ v = e^{-w}$, and the normal form becomes

$$\xi_1 = e^w \xi, \quad \eta_1 = e^{-w}\eta.$$

If the origin is a hyperbolic point with reflection, there exists a unique real power series w of the same form of equation (20) such that $u = -e^w,\ v = -e^{-w}$, and the normal form becomes

$$\xi_1 = -e^w \xi, \quad \eta_1 = -e^{-w}\eta.$$

For the elliptic case we can also find a unique real power series w of the same form as the hyperbolic case (20) but with $\gamma_0 \in (-\pi, \pi)$ such that $\lambda = e^{i\gamma_0}$ and such that $u = e^{iw},\ v = e^{-iw}$, and the normal form is then

$$\xi_1 = e^{iw} \xi, \quad \eta_1 = e^{-iw}\eta.$$

To express this normal form in terms of real variables, we can apply the following linear transformation

$$\xi = r + is, \quad \eta = r - is, \quad \xi_1 = r_1 + is_1, \quad \eta_1 = r_1 - is_1$$

and finally obtain

$$r_1 = r\cos w - s\sin w, \quad s_1 = r\sin w + s\cos w, \quad w = \sum_{k=0}^{\infty} \gamma_k (r^2 + s^2)^k, \tag{21}$$

where $\gamma_k,\ k \geq 0$, are the *Birkhoff coefficients*.

If there exists a non-zero Birkhoff coefficient, this normal form is an integrable twist map, as we see at Subsection 3.2.1.

Relaxing conditions, particularly, requiring that $\partial_\xi C_1 - 1$ and $\partial_\eta C_2 - 1$ do not contain powers of $\omega = \xi\eta$ instead of asking for equations (19) to not be series containing powers of $\omega = \xi\eta$ alone, we find a unique substitution C which is no longer area-preserving.

Once the series are computed in a formal sense, one can look for convergence of these series. It can be shown that in the hyperbolic case the series $C_1(\xi, \eta)$ and $C_2(\xi, \eta)$ converge in some neighbourhood of the origin. In the elliptic case, in general one has divergence. It can be shown that in some cases convergence can occur but there is no general method to determine whether there exists convergence or divergence.

3.2. Moser's Twist Theorem

Moser's Twist Theorem belongs to the KAM (Kolmogorov-Arnol'd-Moser) theory. This theory is the most efficient tool when dealing with RICs with "very irrational" rotation numbers, which, as we will see, are defined more precisely as Diophantine numbers. Here, we will obtain RICs on a perturbative setting of an initially integrable area-preserving twist map.

Let T be an integrable area-preserving twist map

$$\begin{array}{rccc} T: & \mathbb{T} \times [a_0, b_0] & \to & \mathbb{T} \times [a_0, b_0] \\ & (s, r) & \mapsto & (s + \alpha(r), r) \end{array}$$

twisting to the right, that is $\alpha'(r) > 0$ for all $r \in [a_0, b_0]$. Observe that every circle $\mathbb{T} \times \{r\}$ is a RIC. We want to study what happens to the RICs if we add some perturbative terms to T. Precisely, we want to prove the existence of infinitely many RICs.

Consider $A : (s, r) \mapsto (s + \alpha(r) + f(s, r), r + g(s, r))$, with f, g small perturbations. Consider f, g, α real analytic functions, 2π-periodic in s. In order to ensure the existence of a RIC, the smallness condition in f and g is not sufficient. For example, consider $g \equiv \delta$, δ small but constant, then r will increase monotonically and never close. A sufficient hypothesis to ensure this existence is the intersection property.

Definition 3.2.1. The map A satisfies the *intersection property* if for any $\Gamma :=$ graph v, with $v : s \mapsto \gamma(s)$, Γ intersects its image, $\Gamma \cap A(\Gamma) \neq \emptyset$.

Before stating the theorem, we simplify the setting. Consider the change of variables $(s, r) \mapsto (x, y) = (s, \alpha(r)/\gamma)$, with $\gamma = |\alpha(b_0) - \alpha(a_0)|$. In the new variables, the map is

$$\left\{ \begin{array}{rcl} x_1 & = & x + \gamma y + \bar{f}(x, y), \\ y_1 & = & y + \bar{g}(x, y), \end{array} \right. \tag{22}$$

where $\bar{f}$ and $\bar{g}$ still real analytic and 2π-periodic with respect to x.

Observe $y \in [\alpha(a_0)/\gamma, \alpha(b_0)/\gamma] = [a, b]$, which is an interval of length 1. We can impose, restricting ourselves to a narrower annulus, $\gamma \leq 1$.

Since $\bar{f}, \bar{g}$ are real analytic functions, they can be extended to a complex domain of the form $D = \{(x, y) \in \mathbb{C}^2, |\Im x| < r_0, y \in D'\}$, where D' is a complex neighbourhood of $[a, b]$. We may take $0 < r_0 \leq 1$. Finally, we will assume that map A satisfies the intersection property.

Definition 3.2.2. Given $c_0 > 0$ and $\mu > 2$, let

$$\mathcal{D}(c_0, \mu) := \left\{ \omega \in \mathbb{R} : \left| \frac{\omega q}{2\pi} - p \right| < \frac{c_0}{q^\mu}, \forall p \in \mathbb{Z}, q \in \mathbb{N} \right\} \tag{23}$$

and $\mathcal{D}(\mu) := \cup_{c_0 > 0} \mathcal{D}(c_0, \mu)$. The numbers $\omega \in \mathcal{D}(\mu)$ are called *Diophantine.*

Observe that the Diophantine condition on (23) implies that ω is sufficiently away of any rational number p/q. The set $\mathcal{D}(\mu)$ has full measure in $\mathbb{R}$ for any $\mu > 2$. Further information on Diophantine numbers can be found in [**20**, §III. A] or [**4**].

Theorem 3.2.1 (Moser's Twist Theorem, [**25**]). *Under these hypotheses, for any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon, D) > 0$, δ not depending on γ, such that for $|\bar{f}| + |\bar{g}| \leq \gamma\delta$ in D, the map A has a RIC that can be parameterized as*

$$\left\{ \begin{array}{l} x = \xi + u(\xi) \\ y = v(\xi) \end{array} \right. \quad \textit{with } u, v \textit{ real analytic and } 2\pi\textit{-periodic in } \xi, |\Im \xi| < r_0/2$$

and such that the restriction of the application to the curve is the translation $\xi \mapsto \xi + \omega$ for some Diophantine rotation number ω. Also, functions u and v satisfy

$$|u| + |v - \gamma^{-1}\omega| < \varepsilon.$$

In fact, there exists a RIC with rotation number ω for any $\omega \in \mathcal{D}(c_0, \mu) \cap [\gamma(a + s_0), \gamma(b - s_0)]$, for some $c_0 > 0$, $0 < s_0 < 1/4$ and $\mu > 2$.

The invariant curve found is related to the initial invariant curve of the integrable map with a rotation number ω. Since ω is just chosen in order to satisfy the Diophantine condition (23), we can ensure that any RIC that has a Diophantine rotation number persists.

3.2.1. Stability criterion for area-preserving maps around an elliptic fixed point. We know that under suitable hypotheses on the eigenvalues, we can express an area-preserving map near an elliptic fixed point in Birkhoff coordinates (21) as

$$\left\{ \begin{array}{lcl} u_1 & = & u\cos w - v\sin w + \mathcal{O}_{2l+2} \\ v_1 & = & u\sin w + v\cos w + \mathcal{O}_{2l+2}, \end{array} \right. \tag{24}$$

where $w = \gamma_0 + \gamma_l(u^2+v^2)^l$, γ_l is the first non-zero Birkhoff coefficient of the Birkhoff normal form, $\gamma_l > 0$, $l > 0$ and $\mathcal{O}_{2l+2}$ represents a power series in u and v containing terms of order greater or equal that $2l+2$.

We will show that for any $0 < \varepsilon < \varepsilon_0$, with ε_0 sufficiently small, the disk $\mathcal{D} = \{(u,v) \text{ s.t. } u^2+v^2 < \varepsilon^2\}$ contains an invariant curve surrounding $(u,v)=0$. This curve acts as a barrier for the dynamics and therefore we can deduce that the elliptic point is stable.

If we introduce polar coordinates, θ and r, as

$$u = \varepsilon r^{1/2l}\cos\theta, \quad v = \varepsilon r^{1/2l}\sin\theta,$$

then, the equations (24) turn into

$$\left\{ \begin{array}{lcl} \varepsilon r_1^{1/2l}\cos\theta_1 & = & \varepsilon r^{1/2l}\cos\theta\cos w - \varepsilon r^{1/2l}\sin\theta\sin w + \mathcal{O}_{2l+2} \\ \varepsilon r_1^{1/2l}\sin\theta_1 & = & \varepsilon r^{1/2l}\cos\theta\sin w + \varepsilon r^{1/2l}\sin\theta\cos w + \mathcal{O}_{2l+2}, \end{array} \right.$$

which we can rewrite, using trigonometric relations, in coordinates (θ, r) as

$$\left\{ \begin{array}{lcl} \theta_1 & = & \theta + w + \mathcal{O}_{2l+1} \\ r_1 & = & r + \mathcal{O}_{2l+1}, \end{array} \right.$$

Taking into account $w = \gamma_0 + \gamma_l(u^2+v^2)^l = \gamma_0 + \gamma_l\varepsilon^{2l}r$, we can still do another change of variables, considering $x = \theta$ and $r = y + \gamma_0/(\gamma_l\varepsilon^{2l})$ and we get the following map,

$$\left\{ \begin{array}{lcl} x_1 & = & x + \gamma y + f(x,y) \\ y_1 & = & y + g(x,y), \end{array} \right. \tag{25}$$

where $\gamma = \gamma_l\varepsilon^{2l}$ and $f(x,y)$ and $g(x,y)$ contain the terms in ε.

It is clear that we can apply Moser's Twist Theorem to map (25). The functions f and g are real analytic and therefore can be extended to a complex domain. Also, the variable x is 2π periodic and we can consider $0 < y < 1$. The intersection property is easily deduced from the area-preserving property. Last condition to be checked is $|f| + |g| < \gamma\delta(\varepsilon)$ and we have

$$\frac{|f|+|g|}{\gamma} = \frac{\mathcal{O}(\varepsilon^{2l+1})}{\gamma_l\varepsilon^{2l}} = \mathcal{O}(\varepsilon).$$

The same ideas used in this example to find stability around an elliptic fixed point are used by Kamphorst and Pinto-de-Carvalho in [**21**]. There, for strictly convex billiards, the stability of the elliptic (1,2)-periodic orbits is studied by explicitly computing the first Birkhoff coefficient. It only depends on the first derivatives of the curvature of the boundary at the impact points of the (1,2)-periodic orbits and also the distance between them. Then, for a given strictly convex billiard, if the first Birkhoff coefficient is nonzero, one can assure stability of the (1,2)-periodic elliptic orbit.

3.3. Melnikov potential for perturbations of area-preserving twist maps

Moser's Twist Theorem requires Diophantine rotation numbers for the invariant curves. If we have a resonant curve, it will be eventually destroyed under perturbation. The tool used to study the perturbed map is the Melnikov potential.

Melnikov methods are commonly used for computing the splitting of separatrices in maps and flows (see Section 4.1). A less common application of the Melnikov method is the one used for studying the perturbation on the surroundings of a resonant curve of an integrable twist map. This is the one we present in this section. The geometric idea behind this method can be found in [**6**, §VI.] and [**5**, §20], and it was developed and used at [**24**] and [**22**].

Let $T : \mathbb{T}\times[0,\pi] \to \mathbb{T}\times[0,\pi]$ be an area-preserving twist map. Let H be its generating function. Consider $\Upsilon_0^{(p,q)}$ a (p,q)-resonant RIC. By the Birkhoff Theorem 1.2.1, $\Upsilon_0^{(p,q)} = \operatorname{graph} v := \{(s, v(s)), s \in \mathbb{R}/\ell\mathbb{Z}\}$.

Consider $T_\varepsilon = T + \mathcal{O}(\varepsilon)$ to be a perturbation of the area-preserving twist map T and $\widetilde{T}_\varepsilon$ to be its lift. Finally, let $H_\varepsilon = H + \varepsilon H_1 + \mathcal{O}(\varepsilon^2)$ be the perturbation on the generating function.

There exists a couple of *radial curves*, $\Upsilon_\varepsilon = \operatorname{graph} v_\varepsilon$ and $\Upsilon_\varepsilon^* = \operatorname{graph} v_\varepsilon^*$ close to the initial curve, $\Upsilon_0^{(p,q)}$, such that the first graph is vertically projected onto the second one after q iterations of map T_ε.

LEMMA 3.3.1 ([**22**]). *There exist a constant $\eta > 0$ and two smooth functions $v_\varepsilon, v_\varepsilon^* : \mathbb{T} \to [0,\pi]$ defined for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, $\varepsilon_0 > 0$, such that*

i. *$v_\varepsilon(s) = v(s) + \mathcal{O}(\varepsilon)$ and $v_\varepsilon^*(s) = v(s) + \mathcal{O}(\varepsilon)$, uniformly in $s \in \mathbb{T}$;*
ii. *$T_\varepsilon^q(s, v_\varepsilon(s)) = (s, v_\varepsilon^*(s))$ for all $s \in \mathbb{T}$; and*
iii. *$\widetilde{\Upsilon}_\varepsilon = \operatorname{graph} \widetilde{v}_\varepsilon = \{(\widetilde{s}, \theta) : |\theta - \widetilde{v}(\widetilde{s})| < \eta$ and $pr_1 \widetilde{T}_\varepsilon^q(\widetilde{s}, \theta) = \widetilde{s} + p\ell\}$, where $\widetilde{\Upsilon}_\varepsilon$ denotes the lift of Υ_ε obtained when considering the lift $\widetilde{v}_\varepsilon$ of the function v_ε and pr_1 is the projection onto the first coordinate, $pr_1 : \mathbb{R}\times(0,\pi) \to \mathbb{R}, (\widetilde{s}, \theta) \mapsto \widetilde{s}$.*

From this result, one can easily extract the following conclusion.

COROLLARY 3.3.2 ([**22**]). *The intersection of both radial curves contains all the (p,q)-periodic orbits of T_ε close to the former RIC $\Upsilon_0^{(p,q)}$. Also, the (p,q)-resonant RIC persists if and only if both curves coincide everywhere.*

Thus, we need to quantify the separation between graphs Υ_ε and Υ_ε^*.

LEMMA 3.3.3 ([**22**]). $v_\varepsilon^*(s) - v_\varepsilon(s) = \left(W_\varepsilon^{(p,q)}\right)'(s)$*, where* $W_\varepsilon^{(p,q)} : \mathbb{T} \to \mathbb{R}$ *is a function whose lift is*

$$\widetilde{W}_\varepsilon^{(p,q)} = \sum_{j=0}^{q-1} H_\varepsilon(\bar{s}_j(\widetilde{s},\varepsilon), \bar{s}_{j+1}(\widetilde{s},\varepsilon)), \quad \bar{s}_j(\widetilde{s},\varepsilon) := \widetilde{T}_{\varepsilon,1}^j(\widetilde{s}, v_\varepsilon(s)).$$

COROLLARY 3.3.4 ([**22**]). *The unperturbed RIC persists if and only if* $\left(W_\varepsilon^{(p,q)}\right)'(s) \equiv 0$.

DEFINITION 3.3.1. The *subharmonic potential* of the (p,q)-resonant RIC $\Upsilon_0^{(p,q)}$ is the function $W_\varepsilon^{(p,q)} : \mathbb{T} \to \mathbb{R}$.

As any Melnikov method, it is usual to center the interest in the low order terms of the perturbative potential. Consider the expansion of the subharmonic function, $W_\varepsilon^{(p,q)}(s) = W_0^{(p,q)}(s) + \varepsilon W_1^{(p,q)}(s) + \mathcal{O}(\varepsilon^2)$. The zero-order term of this expansion vanishes since

$$\left(W_0^{(p,q)}\right)'(s) = v_0^*(s) - v_0(s) = v(s) - v(s) \equiv 0. \tag{26}$$

DEFINITION 3.3.2. The first order term of the subharmonic potential, $W_1^{(p,q)}$, is called the *subharmonic Melnikov potential* of the (p,q)-resonant RIC $\Upsilon_0^{(p,q)}$ for the perturbation T_ε.

The previous results lead to

COROLLARY 3.3.5 ([**22**]). *If the subharmonic Melnikov potential is not constant, then the (p,q)-resonant RIC $\Upsilon_0^{(p,q)}$ does not persist under the perturbation T_ε.*

And finally, it can be proved that the subharmonic Melnikov potential can be defined in a much simpler way.

PROPOSITION 3.3.6 ([**22**]). *The lift* $\widetilde{W}_1^{(p,q)}(\widetilde{s})$ *is*

$$\widetilde{W}_1^{(p,q)}(\widetilde{s}) = \sum_{j=0}^{q-1} H_1(\widetilde{s}_j, \widetilde{s}_{j+1}), \tag{27}$$

where $\widetilde{s}_j := T^j(\widetilde{s}, v(s))$ *and* $H(\widetilde{s}, \widetilde{s}_1) = H_0(\widetilde{s}, \widetilde{s}_1) + \varepsilon H_1(\widetilde{s}, \widetilde{s}_1) + \mathcal{O}(\varepsilon^2)$.

Chapter 4
Specific results of perturbative theory

4.1. An example of exponentially small phenomena: upper bound of the splitting of invariant curves

In this section we give a result on exponentially small bounds. The problem presented is not directly related to the one we are interested in. Yet, since we will be looking for exponentially small bounds, this example helps to understand the behaviour of certain singular systems.

Before presenting the setting and results, we briefly introduce some basic definitions. Let $T : U \to U$, $U = \mathring{U} \subset \mathbb{R}^2$, be an area-preserving diffeomorphism. An hyperbolic fixed point is also called a *saddle point*.

By the Hadamard-Perron Theorem [**14**, §6], a saddle point p_0 has one-dimensional *stable* and *unstable manifolds*, respectively, for δ sufficiently small,

$$
\begin{aligned}
W^s_{\mathrm{loc}}(p_0) &:= \{p \in U \ : \ \|T^n(p) - p_0\| < \delta \text{ for } n \geq 0\},\\
W^u_{\mathrm{loc}}(p_0) &:= \{p \in U \ : \ \|T^n(p) - p_0\| < \delta \text{ for } n \leq 0\}.
\end{aligned}
$$

These local manifolds can be infinitely continued with the help of the iterates of T giving rise to the *stable and unstable invariant manifolds*

$$
\begin{aligned}
W^s(p_0) &:= \{p \in U \ : \ \lim_{n\to\infty} \|T^n(p) - p_0\| = 0\} = \bigcup_{n\in\mathbb{Z}} T^n(W^s_{\mathrm{loc}}(p_0)),\\
W^u(p_0) &:= \{p \in U \ : \ \lim_{n\to\infty} \|T^n(p) - p_0\| = 0\} = \bigcup_{n\in\mathbb{Z}} T^n(W^u_{\mathrm{loc}}(p_0)).
\end{aligned}
$$

Each invariant manifold has two branches departing from (or arriving to) the saddle point. These branches depart from (or arrive to) the saddle point according to the direction given by the eigenvectors related to the eigenvalues of the linearized system around the saddle point. It is clear that a branch from the stable invariant manifold of a saddle point cannot intersect to another stable branch of the same saddle point or another. The same is established for the unstable branches.

However, a stable branch can intersect with an unstable one. The points on the intersection are called *homoclinic points* when both branches correspond to the invariant manifolds of the same saddle point or *heteroclinic points* otherwise. It is clear that the orbits of homoclinic (heteroclinic) points contain only homoclinic (heteroclinic) points. Thus, we refer to them as *homoclinic (heteroclinic) orbits*.

If a branch of the unstable manifold coincides with a branch of the stable manifold of the same saddle point, we say that this invariant curve is a *separatrix*, which we usually note by Γ. A separatrix has a fish-shape form since both backward and forward iterates of any point in the separatrix tend to the saddle point. The term separatrix is due to the fact that dynamics inside and outside this fish-shaped invariant curve have different behaviours.

Let $T_h : \mathbb{R}^2 \to \mathbb{R}^2$, $h > 0$, be an area-preserving map with a separatrix Γ_h to a saddle point p_0. Suppose $\mathrm{Spec}(DT_h(p_0)) = \{e^{+h}, e^{-h}\}$. Observe that this saddle point is weakly hyperbolic if h is small. Generically, the separatrix splits in a transverse way when introducing perturbations. Here, we consider a perturbation depending on a parameter ε, $T_{h,\varepsilon} = T_h + \mathcal{O}(\varepsilon)$.

We proceed to measure the splitting using a Melnikov method. Suppose we are able to compute the $\mathcal{O}(\varepsilon)$-term of a certain splitting quantity S (for example, distance between the invariant manifolds), so that $S = S(h,\varepsilon) = \varepsilon S_1(h) + \mathcal{O}(\varepsilon^2)$ for any fixed $h > 0$. But, is it always the first term of this expansion the one dominating? What can we say about S when both h and ε are small? When h is no longer fixed, the first term may not be the most important. Then, Melnikov methods fail since one has to know the asymptotic behaviour of all the terms in the expansion (very difficult) and not only the first ones. Thus, another approach has to be taken.

Next theorem gives an upper bound for the size of the splitting in a singular case.

THEOREM 4.1.1 (Fontich-Simó, [**11**]). *Let $F_h : \mathbb{R}^2 \to \mathbb{R}^2$, $h > 0$, be a diffeomorphism such that:*

- $F_h = \mathrm{Id} + hF_1 + h^\beta F_2$, *where* $F_1 : \mathbb{R}^2 \to \mathbb{R}^2$, $F_2 : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{R}^2$, $\beta > 0$ *and* $F_1(0,0) = F_2(0,0,h) = (0,0)$.
- $\mathrm{Spec}\, DF_h(0,0) = \{\lambda, \lambda^{-1}\}$, *where* $\lambda = 1 + h + \mathcal{O}(h^2)$.
- *The system* $z' = F_1(z)$ *has an homoclinic loop to the origin,* $\gamma(t)$.
- F_h *is analytic in a complex neighbourhood of* $\gamma(t)$, $\{|\Im t| < d^*\}$.

Then, for any $d \in (0, d_*)$, *there exists* $C > 0$ *such that the distance between the invariant manifolds is less or equal than* $C\mathrm{e}^{-2\pi d/h}$ *as* h *tends to* 0^+.

4.2. Existence of caustics near the billiard boundary

Next, we present a direct application of Moser's Twist Theorem. This result goes back to Lazutkin [**17**].

If we recall the notation used in Section 2.3, a caustic Γ can be charaterized by the invariant η, which is the rotation number of the diffeomorphism from Γ to itself defined by $M \mapsto M_1$, where M and M_1 are the points on Γ that draw tangents to a point N on $\partial\Omega$. We denote the caustic defined by the rotation number η as $\Gamma(\eta)$.

Consider the Cantor set

$$\mathcal{D} = \mathcal{D}_{\eta_*,c_0,\mu} := (0,\eta_*) \cap \mathcal{D}(c_0,\mu),$$

where $\mathcal{D}(c_0,\mu)$ is the set of Diophantine numbers defined in (23), and $\eta_* \ll 1$. Note that $0 \in \bar{\mathcal{D}}$. Also recall that if a caustic Γ has a rotation number belonging to this set, $\eta(\Gamma) \in \mathcal{D}$, then the rotation number is poorly approximated by rational numbers.

Theorem 4.2.1 (Lazutkin, [**17**]). *Let Ω be a sufficiently smooth and strictly convex billiard table. Then, there exists a Cantor family of smooth convex caustics $\{\Gamma_\eta\}_{\eta\in\mathcal{D}} \in \Omega$, with $\lim_{\eta\to 0^+}\Gamma_\eta = \partial\Omega$, whose union has positive area.*

Recalling the relations and notation between convex caustics and RICs seen at Section 2.3, the theorem can be reformulated as follows.

Theorem 4.2.2. *The billiard map associated to a sufficiently smooth and strictly convex billiard table has two collections of RICs, $\{\Upsilon_\eta^\pm\}_{\eta\in\mathcal{D}} \in \mathbb{T}\times(0,\pi)$, with $\lim_{\eta\to 0^+}\Upsilon_\eta^- = \mathbb{T}\times\{0\}$ and $\lim_{\eta\to 0^+}\Upsilon_\eta^+ = \mathbb{T}\times\{\pi\}$, whose union has positive area.*

Remark 4.2.1. After Douady, the minimum smoothness required is that the boundary $\partial\Omega$ is at least $\mathcal{C}^6$, [**10**].

To prove this result recall that the billiard map T can be extended to its boundary, where it becomes the identity map. Therefore, when θ tends to 0, the point (s,θ) tends to be stationary for all s. We want to find a family of invariant closed curves near this stationary curve which will be $\{\Upsilon_\eta^-\}_{\eta\in\mathcal{D}}$. The other family, $\{\Upsilon_\eta^+\}_{\eta\in\mathcal{D}}$, is directly obtained by the billiard symmetry.

First, we denote by $\vec{x}(s)$ the arc-length parameterization of $\partial\Omega$, $\vec{t}(s)$ its tangent unit vector and $\vec{n}(s)$ the unit normal vector, $\vec{n}(s) = \rho(s)\vec{t}'$, $\rho(s)$ the radius of curvature of $\partial\Omega$.

In a neighbourhood of the stationary curve $\theta = 0$, we can write T as

$$\left\{\begin{array}{lcl} s_1 & = & s + 2\rho\theta + (4/3)\rho\rho'\theta^2 + \left((2/3)\rho^2\rho'' + (4/9)\rho\rho'^2\right)\theta^3 + \mathcal{O}(\theta^4) \\ \theta_1 & = & \theta - (2/3)\rho'\theta^2 + \left(-(2/3)\rho\rho'' + (4/9)\rho'^2\right)\theta^3 + \mathcal{O}(\theta^4), \end{array}\right. \tag{28}$$

where we have not explicitly written the dependence of the curvature radius on the parameter s, $\rho = \rho(s)$. The explicit derivation of (28) is placed in Appendix A.

We can then introduce the coordinates $\xi(s,\theta) \in \mathbb{R}/\mathbb{Z}$ and $\eta = \eta(s,\theta) > 0$ given by

$$\xi = C\int_0^s \rho^{-2/3}(s)ds, \quad \eta = 4C\rho^{1/3}(s)\sin(\theta/2),$$

with $C^{-1} = \int_0^L \rho^{-2/3}(s)\mathrm{d}s$. These coordinates are well-defined for small incidence angles θ. Observe that $\eta(s,0) \equiv 0$ and also that $0 < \theta \ll 1$ if and only if $0 < \eta \ll 1$.

In these new coordinates, for η sufficiently small, $\eta < \eta_*$, the billiard map becomes

$$\begin{cases} \xi_1 &= \xi + \eta + \mathcal{O}(\eta^3) &= \xi + \eta + \eta^3 f(\xi, \eta) \\ \eta_1 &= \eta + \mathcal{O}(\eta^4) &= \eta + \eta^4 g(\xi, \eta), \end{cases}$$

This setting is suitable to apply Moser's Twist Theorem (recall Equation (22)). For $f, g \equiv 0$ all curves $\eta =$ constant are invariant. For f, g small perturbations, if the intersection property holds, which it does because $\mathrm{d}m = \text{constant}\,|\xi|\mathrm{d}\xi\mathrm{d}\eta$ is preserved, the curves with η satisfying a Diophantine condition, that is $\eta \in \mathcal{D}$, persist after the perturbation.

4.3. Break-up of caustics

In the previous section, we have given a criterion for the existence of caustics in the neighbourhod of the boundary $\partial\Omega$. This criterion was based on a KAM theorem applied on a neighbourhood of the stationary curve $\theta = 0$. Hence, the obtained caustics have Diophantine rotation numbers.

Nevertheless, we are interested in periodic trajectories. When the envelope of a set of periodic trajectories defines a caustic, this caustic has a rational rotation number. Therefore, the previous theorem can not be used.

Let $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$ and $p < q/2$. Recall, from Section 2.5, that elliptic billiards are Liouville integrable. Therefore any (p, q)-periodic orbit is on an invariant curve. Following Poncelet's Porism, these curves are foliated by (p, q)-periodic orbits. The invariant curves not contractible to a point (the RICs) are in two-to-one correspondence with the resonant convex caustics, which are confocal ellipses to the billiard boundary.

The results we will give refer to circular and elliptic billiards. We will see that, contrary to the RICs with Diophantine rotation numbers, the RICs with rational rotation numbers do not persist under generic perturbations and therefore the resonant caustics are destroyed too. Recall that there will be at least two (p, q)-periodic orbits after the perturbation but the resonant caustic structure will not persist.

First we state a result focused on circular billiards. To introduce it, we need some notation.

Consider, with out loss of generality, that the circle is centered at the origin and has radius r_0. We parameterize the circular table by $M_0 : \mathbb{T} \to \partial\Omega \in \mathbb{R}^2$, $s \mapsto r_0 \vec{n}_s$, where $\vec{n}_s = (\cos s, \sin s)$.

We consider the generic perturbation $M_\varepsilon : \mathbb{T} \to \partial\Omega_\varepsilon$, $s \mapsto r_\varepsilon(s)\vec{n}_s$, where

$$r_\varepsilon(s) = r_0 + \varepsilon r_1(s) + \mathcal{O}(\varepsilon^2).$$

THEOREM 4.3.1 ([**24**]). *Let $\sum_{j\in\mathbb{Z}} \widehat{r}_1^j \mathrm{e}^{\mathrm{i}\,js}$ be the Fourier expansion of $r_1(s)$. Let p and q be any integers, with $p < q/2$ and $\gcd(p, q) = 1$. If there exists some $j \in q\mathbb{Z}\backslash\{0\}$, the (p, q)-resonant caustic does not persist under this perturbation.*

The following result refers to elliptic billiards. As we have seen in Section 2.5, we parameterize the unperturbed elliptic boundary by $M_0 : (\varphi) = (c\cosh\mu_0\cos\varphi, y = c\sinh\mu_0\sin\varphi)$, $c = \sqrt{a^2 - b^2}$ and μ_0 is such that $a = c\cosh\mu_0$ and $b = c\sinh\mu_0$. The advantage of this parameterization is that if we consider coordinates (μ, φ) instead of only φ we can parameterize any conic with the same foci than $\partial\Omega$. Any confocal ellipse is determine by $\mu =$ constant while we can consider any hyperbola by making $\varphi =$ constant. As told, $\mu = \mu_0$ parameterizes $\partial\Omega$.

The perturbation $\partial\Omega_\varepsilon$ considered is similar to the one considered in the circular billiard. The perturbation affects to μ_0. Instead of being constant, it is substituted by a smooth function $\mu_\varepsilon(\varphi) = \mu_0 + \varepsilon\mu_1(\varphi) + \mathcal{O}(\varepsilon^2)$. Hence, the parameterization is $M_\varepsilon(\varphi) = (c\cosh(\mu_\varepsilon(\varphi))\cos\varphi, c\sinh(\mu_\varepsilon(\varphi))\sin\varphi)$.

THEOREM 4.3.2 ([**22**]). *Let $\mu_1(\varphi)$ be a 2π-periodic entire function. If $\mu_1(\varphi)$ is not constant (respectively, $\mu_1(\varphi)$ is not π-antiperiodic), then none of the convex (p, q)-resonant caustic ellipses with odd q (respectively, even q) persist under the perturbation $\mu_\varepsilon(\varphi)$.*

Both theorems rely on the study of the Melnikov potential for these perturbations.

4.4. Length spectrum of convex domains

Consider $p, q \in \mathbb{Z}$, $\gcd(p, q) = 1, p \leq q/2$. Let $\Gamma^{(p,q)}$ be the set of all the (p, q)-orbits at Ω. Let $\mathcal{L}^{(p,q)}$ be the set of lengths of the orbits on $\Gamma^{(p,q)}$.

DEFINITION 4.4.1. The set of numbers $\mathcal{L} = \ell\mathbb{N} \cup \bigcup_{(p,q)} \mathcal{L}^{(p,q)}$ is called the *length spectrum of the domain Ω.*

We also denote by $L^{(p,q)} = \sup\mathcal{L}^{(p,q)}$ and $l^{(p,q)} = \inf\mathcal{L}^{(p,q)}$. Finally, $\Delta^{(p,q)}$ will denote the difference on these two quantities, $\Delta^{(p,q)} = L^{(p,q)} - l^{(p,q)}$.

In our work, we want to study the asymptotic behaviour of these three quantities. In this field, two results are worth to be mentioned. The first one deals with orbits with a fixed number of turns inside $\partial\Omega$, this is a fixed p, and with a large period, q.

THEOREM 4.4.1 (Marvizi and Melrose, [**18**]). *If the billiard table Ω is smooth and strictly convex, then*

(1) *There exist coefficients $c_{p,j}$ such that*
$$L^{(p,q)}, l^{(p,q)} \asymp c_{p,0} + \sum_{j\geq 1} c_{p,j} q^{-2j} \qquad as \quad q \to \infty.$$
Coefficient $c_{p,0} = p\ell$ and coefficient $c_{p,1}$ is always negative.
(2) *There exist some "universal" functions $g_j : \mathbb{R}^+ \times \mathbb{R}^{j-1}$ such that*
$$c_j := c_{1,j} = \int_0^\ell g_j\left(\kappa(s), \kappa''(s), \ldots, \kappa^{(2j-2)(s)}\right), \quad j \geq 1,$$
where $\kappa(s)$ is the curvature of $\partial\Omega$ parameterized by the arc-length parameter.
(3) $\lim_{q\to\infty} q^k \Delta^{(p,q)} = 0$ *for all $k > 0$. Equivalently, $\Delta^{(p,q)} = \mathcal{O}(q^{-\infty})$.*

The next result is valid for (p, q)-periodic orbits close to the (1,2)-periodic elliptic orbit on a billiard table with axial symmetries. We give the asymptotic behaviour of these orbits as p/q tends to $1/2$. Recall that the definition of an elliptic periodic orbit has been given at Subsection 2.2.5.

THEOREM 4.4.2 (Colin de Verdière, [**8**]). *Let Ω be a smooth convex domain of $\mathbb{R}^2$, symmetric with regard to both axis of coordinates. Assume that its shortest $(1,2)$-periodic billiard trajectory is elliptic and $\partial\Omega$ is strictly convex at its endpoints. Consider in $\Gamma^{(p,q)}$ only the subset of (p,q)-periodic orbits which are close to the $(1,2)$-periodic elliptic orbit. From this subset, we redefine $\mathcal{L}^{(p,q)}$, $l^{(p,q)}$, $L^{(p,q)}$ and $\Delta^{(p,q)}$ as before. Then,*

$$\Delta^{(p,q)} = \mathcal{O}(q^{-\infty}) \quad \textit{as} \quad p/q \to 1/2.$$

This theorem is obtained through the study of the Birkhoff normal form of the billiard map around its $(1,2)$-periodic orbit. The non-resonance condition is necessary to compute this normal form up to an arbitrary order.

Part 2

Goals and first results

Chapter 5
Discussion of problems to treat

5.1. Introduction to the problems

In Dynamical Systems, and more concretely, in the analytic context, the following principle is generally assumed. Consider a quantity Δ which tends asymptotically to zero as a certain parameter, q, tends to infinity and its asymptotic behaviour is faster than any order, that is $\Delta = \mathcal{O}(q^{-\infty})$. Then, quantity Δ is exponentially small in parameter q.

This principle is not proved in general but we have already given an example in Section 4.1 where it applies: the distance between manifolds of close to the identity area-preserving analytic maps is exponentially small in the characteristic exponent.

We have seen the results by Marvizi and Melrose and by Colin de Verdière on the length spectrum which predicted an asymptotic behaviour beyond all orders in different cases. We recall the notation and the results found.

Consider $p, q \in \mathbb{N}$, $\gcd(p, q) = 1$, $p \leq q/2$. Let $\Gamma^{(p,q)}$ be the set of all the (p, q)-orbits at the convex domain Ω. Let $\mathcal{L}^{(p,q)}$ be the set of lengths of the orbits on $\Gamma^{(p,q)}$. The length spectrum of the domain Ω is $\mathcal{L} = \ell\mathbb{N} \cup \bigcup_{(p,q)} \mathcal{L}^{(p,q)}$ and the difference between the maximum and the minimum length among the orbits on the set $\Gamma^{(p,q)}$ is denoted by $\Delta^{(p,q)}$.

On the one hand, the set $\Gamma^{(1,q)}$ approaches to the billiard boundary as q tends to infinity. According to Marvizi and Melrose [**18**], $\Delta^{(1,q)}$ turns out to be of order $\mathcal{O}(q^{-\infty})$.

On the other hand, consider only the (p, q)-periodic trajectories close to the $(1, 2)$-periodic elliptic orbits in billiards which are symmetric with respect to the width and diameter. Then, Colin de Verdière [**8**] found out that the same result holds, $\Delta^{(p,q)} = \mathcal{O}(q^{-\infty})$.

These results, together with the assumption of the principle mentioned above, lead us to search for exponentially small asymptotic behaviours of the difference of lengths among the set of (p, q)-periodic orbits under certain settings (for example the setting given by Marvizi and Melrose or the one by Colin de Verdière).

Yet, there exists another setting where to look for an exponential smallness of the difference on the lengths of the (p, q)-periodic orbits of a billiard.

Consider an elliptic or circular billiard. We have seen that the billiard maps on these type of billiards are Liouville integrable and the trajectories are contained in invariant curves. We center ourselves on those curves which are RICs. Recall that, at a circular billiard, all the invariant curves are RICs while at an elliptic billiard, RICs are associated to convex caustics.

The billiard map can be restricted to each RIC and a rotation number can be assigned to each RIC. Applying KAM theory to those RICs with Diophantine rotation numbers, we know that the RICs will persist under small perturbations and, thus, their caustics will persist too.

Typically, as we have seen in Section 4.3, RICs with a rational rotation number p/q will break down making the (p, q)-resonant caustic disappear. Although we will not have a continuous family of (p, q)-periodic orbits after the perturbation, according to Poincaré-Birkhoff Theorem, there will exist at least two (p, q)-periodic orbits. However, we will no longer be able to guarantee that $\Delta^{(p,q)} = 0$ on the set of persisting (p, q)-periodic orbits.

In analogy to Greene's criterion [**12**], we believe that the difference $\Delta^{(p,q)}$ tend exponentially to zero as p/q tends to the Diophantine rotation number of a RIC. Similar problems are studied in [**9**, Theorem 1.6].

5.2. Our approach

The subject presented is quite large and goes beyond the scope of a Master Thesis. In this work, we have just initiated the computations on the asymptotic behaviour of $\Delta^{(p,q)}$ in some particular cases which we detail below.

We are interested in measuring the behaviour among all the elements on $\mathcal{L}^{(p,q)}$ and, more concretely, in finding the asymptotic growth or order for the function $\Delta^{(p,q)}$ as p/q tends to some specific rotation number whether it is 0 and we are approaching to the billiard boundary or it is an irrational number.

We take a look to Liouville integrable billiards, circular and elliptic billiards. Consider $p, q \in \mathbb{N}$, $p < q/2$, and $\gcd(p, q) = 1$, there always exists a (p, q)-resonant caustic in correspondence to a RIC with rotation number p/q and its symmetric, a RIC with rotation $(q - p)/p$.

Henceforth, we slightly modify definitions on $\Gamma^{(p,q)}$, so that the only (p, q)-periodic orbits considered on the set $\Gamma^{(p,q)}$ are the ones tangent to a convex caustic, leaving the (p, q)-periodic trajectories tangent to hyperbolas apart.

Recalling that the generating function of a billiard map is the length between two points on the billiard boundary, we can observe that the length of a (p, q)-trajectory coincides with the definition of the subharmonic potential of the (p, q)-resonant RIC defined in Section 3.3. Thus,

$$\Delta^{(p,q)} = \max\{W_\varepsilon^{(p,q)}(s) \ : \ s \in \mathbb{T}\} - \min\{W_\varepsilon^{(p,q)}(s) \ : \ s \in \mathbb{T}\}.$$

We write the subharmonic potential as an expansion on the perturbation parameter:

$$W_\varepsilon^{(p,q)}(s) = W_0^{(p,q)}(s) + \varepsilon W_1^{(p,q)}(s) + \mathcal{O}(\varepsilon^2).$$

Since the term $W_0^{(p,q)}(s)$ is also constant with respect to variable s (see equation (26)), we can write the difference $\Delta^{(p,q)}$ as

$$\Delta^{(p,q)} = \varepsilon \Delta_1^{(p,q)} + \mathcal{O}(\varepsilon^2),$$

where $\Delta_1^{(p,q)} := \max_{\mathbb{T}} W_1^{(p,q)} - \min_{\mathbb{T}} W_1^{(p,q)}$.

As it is usual with Melnikov methods, we will study first the behaviour of the low order terms. Just as we discussed in Section 4.1, if q is not a fixed value, the low order terms of this expansion may not be the dominant ones. Thus, if we want to find asymptotic results on $\Delta^{(p,q)}$ as p/q tends to a concrete number rotation number, we can not ignore the higher order terms of the expansion. Therefore, to find an exponentially small bound, we have to deal with all the terms $\Delta_j^{(p,q)}$ for all $j > 0$.

Despite this observation, our approach is focused on the Melnikov prediction for the first order term, $\Delta_1^{(p,q)}$ on circular and elliptic billiards.

We present the results in the next chapter.

Chapter 6
First results

6.1. Circular tables

We consider a circular billiard table, with boundary $\partial\Omega$. Without loss of generality, we can assume

$$\partial\Omega := \left\{(x,y) \in \mathbb{R}^2 \ : \ x^2+y^2=1\right\}.$$

As we have seen in Section 2.4, the billiard map is defined as $T : \mathbb{T}\times[0,\pi] \to \mathbb{T}\times[0,\pi]$, $(s,\theta)\mapsto(s+2\theta,\theta)$, which is an integrable twist map. Therefore, following Section 1.1, for all $p,q\in\mathbb{N}$, $p<q/2$, the (p,q)-periodic orbits are set on a RIC foliated by (p,q)-periodic orbits. Geometrically, given any regular (p,q)-polygon inscribed in the circle, we obtain the rest of (p,q)-periodic orbits by rotating it with respect to the origin. Thus, $\Delta^{(p,q)}=0$.

Our aim is to measure the behaviour on the lengths among all these (p,q)-periodic orbits on a perturbed circular table, $\partial\Omega_\varepsilon$. We want to study the difference of lengths among $\Gamma^{(p,q)}$, that is $\Delta^{(p,q)}$. In particular, we will study the first order term, $\Delta_1^{(p,q)}$.

We parameterize the circular boundary by the arc-length parameter, that is $M_0 : \mathbb{T}\to\partial\Omega, s\mapsto\vec{n}_s := (\cos s,\sin s)$. The generic perturbation we consider is the one already considered in Section 4.3. The perturbative parameter ε affects to the distance to the origin at each direction. The perturbed circular boundary is described as follows

$$\begin{array}{rccl} M_\varepsilon : & \mathbb{T} & \to & \partial\Omega_\varepsilon \\ & s & \mapsto & r_\varepsilon(s)\vec{n}_s = (r_\varepsilon(s)\cos s, r_\varepsilon(s)\sin s), \end{array} \tag{29}$$

where $r_\varepsilon(s) = 1+\varepsilon r_1(s)+\mathcal{O}(\varepsilon^2)$.

As we have seen in Section 3.3, the subharmonic Melnikov potential can be computed as

$$W_1^{(p,q)}(s) = \sum_{j=0}^{q-1} H_1(s_j,s_{j+1}),$$

where s_j is the first coordinate at the j-th iteration of the unperturbed billiard map T at a point $(s,v(s))$ on the RIC $\Upsilon^{(p,q)} = \operatorname{graph} v$ with rotation number p/q.

From the study of the subharmonic Melnikov potential for the circular table and the generic perturbations introduced above, and as it is computed in [**24**], the previous formula for the subharmonic Melnikov potential can be written as

$$W_1^{(p,q)}(s) = 2\sin(\pi p/q)\sum_{k=0}^{q-1} r_1(s+2\pi kp/q) = 2q\sin(\pi p/q)\sum_{j\in q\mathbb{Z}} \widehat{r}_1^{\,j} \mathrm{e}^{\mathrm{i}\,js}, \tag{30}$$

where $\widehat{r}_1^{\,j}$ are the Fourier coefficients for function $r_1(s)$. That is, $r_1(s) = \sum_{j\in\mathbb{Z}} \widehat{r}_1^{\,j}\mathrm{e}^{\mathrm{i}\,js}$.

Now, we can state an exponentially small upper bound for

$$\Delta_1^{(p,q)} = \max_{\mathbb{T}} W_1^{(p,q)} - \min_{\mathbb{T}} W_1^{(p,q)}.$$

Proposition 6.1.1. *Let $r_1(s)$ be a 2π-periodic analytic function in the strip $\mathcal{S} = \{s\in\mathbb{C} \;:\; |\Im s| < \rho\}$, for some $\rho > 0$. Given any $\rho' \in (0,\rho)$, there exists a constant $C = C(\rho')$ such that $\Delta_1^{(p,q)} \le C\mathrm{e}^{-\rho' q}$, if $q \gg 1$.*

Proof. From equation (30) we obtain that $W_1^{(p,q)}$ is $2\pi/q$-periodic. Thus, we can write $W_1^{(p,q)}(s)$ as its expansion in Fourier series,

$$W_1^{(p,q)}(s) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{\mathrm{i}\,nqs},$$

where

$$c_n = \frac{q}{2\pi}\int_0^{2\pi/q} W_1^{(p,q)}(t)\mathrm{e}^{-\mathrm{i}\,nqt}\mathrm{d}t.$$

The constant term c_0 does not affect $\Delta_1^{(p,q)}$. Thus, we want to find an exponentially small upper bound for the coefficients c_n with $n \neq 0$.

Since $r_1(s+2\pi k/q)$ is analytic in $|\Im s| < \rho$, for any $k\in\mathbb{Z}$, so is $W_1^{(p,q)}(s)$. Note that $W_1^{(p,q)}(s)$ is bounded on any imaginary strip $|\Im s| \le \rho' < \rho$.

Then for any closed path γ in the strip $|\Im s| < \rho$, we have

$$\int_\gamma W_1^{(p,q)}(t)\mathrm{e}^{-\mathrm{i}\,nqt}dt = 0.$$

To calculate c_n, $n < 0$, consider the rectangle with vertices 0, $2\pi/q$, $\mathrm{i}\,\rho' + 2\pi/q$, and $\mathrm{i}\,\rho'$, with $\rho' < \rho$. We then have

$$0 \;=\; \left(\int_0^{2\pi/q} + \int_{2\pi/q}^{\mathrm{i}\,\rho'+2\pi/q} + \int_{\mathrm{i}\,\rho'+2\pi/q}^{\mathrm{i}\,\rho'} + \int_{\mathrm{i}\,\rho'}^{0}\right) W_1^{(p,q)}(t)\mathrm{e}^{-\mathrm{i}\,nqt}\mathrm{d}t.$$

By periodicity, the second integral cancels with the fourth. Then the first integral equals to the opposite of the third one. Thus,

$$\begin{aligned}\frac{2\pi}{q}c_n &= \int_{\mathrm{i}\,\rho'}^{\mathrm{i}\,\rho'+2\pi/q} W_1^{(p,q)}(t)\mathrm{e}^{-\mathrm{i}\,nqt}\mathrm{d}t \\ &= \int_0^{2\pi/q} W_1^{(p,q)}(x+\mathrm{i}\,\rho')\mathrm{e}^{-\mathrm{i}\,nqx}\mathrm{e}^{n\rho' q}\mathrm{d}x.\end{aligned}$$

From equation (30),

$$\max_{|\Im s|\le\rho'}|W_1^{(p,q)}(s)|\le 2q\sin(2\pi p/q)\max_{|\Im s|\le\rho'}|r_1|.$$

Since the factor $q\sin(2\pi p/q)$ is bounded as q tends to infinity, we can find an upper bound $K_{\rho'}$ not depending on q. Thus, $\max_{|\Im s|\le\rho'}|W_1^{(p,q)}(s)| < K_{\rho'}$ for $q\gg 1$. Hence, we obtain, for $n<0$,

$$c_n < \frac{q}{2\pi}\mathrm{e}^{-|n|\rho' q}\int_0^{2\pi/q}K_{\rho'}\cdot 1dt = K_{\rho'}\mathrm{e}^{-|n|\rho' q}.$$

In fact, the same bound can be obtained for c_n, $n>0$, by taking the rectangle with vertices 0, $2\pi/q$, $-\mathrm{i}\,\rho'+2\pi/q$ and $-\mathrm{i}\,\rho'$.

Taking back the expression for $W_1^{(p,q)}(s)$, we have, for all $s\in\mathbb{R}$,

$$\begin{aligned}\left|W_1^{(p,q)}(s)-c_0\right| &= \left|\sum_{n\neq 0}c_n\mathrm{e}^{\mathrm{i}\,nqs}\right|\le\sum_{n>0}2K_{\rho'}\mathrm{e}^{-n\rho' q}\\ &\le 2K_{\rho'}\frac{\mathrm{e}^{-\rho' q}}{1-\mathrm{e}^{-\rho' q}}\le 4K_{\rho'}\mathrm{e}^{-\rho' q},\end{aligned}$$

where the last inequality is true by taking q such that $\rho' q>\ln 2$.

Finally,

$$\begin{aligned}\Delta_1^{(p,q)} &= \max_{\mathbb{T}}W_1^{(p,q)}-\min_{\mathbb{T}}W_1^{(p,q)}\le\left|\max_{\mathbb{T}}W_1^{(p,q)}-c_0\right|+\left|c_0-\min_{\mathbb{T}}W_1^{(p,q)}\right|\\ &\le 8K_{\rho'}\mathrm{e}^{-\rho' q}=C\mathrm{e}^{-\rho' q},\end{aligned}$$

where $C=8K_{\rho'}$. □

With this proposition, we have guaranteed that $\Delta_1^{(p,q)}$ is exponentially small for perturbations of the circular billiard of the form (29). In the next proposition, we give an example of perturbations for which the same exponentially small behaviour holds as a lower bound.

PROPOSITION 6.1.2. *Let* $W_1^{(p,q)}(s)=2q\sin(\pi p/q)\sum_{j\in q\mathbb{Z}}\widehat{r}_1^{\,j}\mathrm{e}^{\mathrm{i}\,js}$, *where there exist* $\underline{\alpha},\overline{\alpha}\in\mathbb{R}^+$ *and* $m\in\mathbb{N}$ *such that*

$$\underline{\alpha}|j|^m\mathrm{e}^{-|j|\rho}\le|\widehat{r}_1^{\,j}|\le\overline{\alpha}|j|^m\mathrm{e}^{-|j|\rho},\qquad\forall j\in\mathbb{Z}.$$

Then, $\Delta_1^{(p,q)}\ge Kq^m\mathrm{e}^{-\rho q}$, *for some* $K\in\mathbb{R}^+$.

PROOF. Since $r_1(s)$ is real analytic, its Fourier coefficients verify that $\widehat{r}_1^{\,-j}=\overline{\widehat{r}_1^{\,j}}$ for any $j\in\mathbb{Z}$, where $\overline{\widehat{r}_1^{\,j}}$ denotes the conjugate of the coefficient $\widehat{r}_1^{\,j}$.

Writing $\widehat{r}_1^{\,j}=\widehat{a}_1^{\,j}+\mathrm{i}\,\widehat{b}_1^{\,j}$, with $\widehat{a}_1^{\,j},\ \widehat{b}_1^{\,j}\in\mathbb{R}$, we have

$$\begin{aligned}W_1^{(p,q)}(s) &= 2q\sin(\pi p/q)\sum_{j\in q\mathbb{Z}}(\widehat{a}_1^{\,j}+\mathrm{i}\,\widehat{b}_1^{\,j})\mathrm{e}^{\mathrm{i}\,js}\\ &= 2q\sin(\pi p/q)\left(\widehat{a}_1^{\,0}+2\widehat{a}_1^{\,q}\cos(qs)-2\widehat{b}_1^{\,q}\sin(qs)+\sum_{|j|>q,j\in q\mathbb{Z}}\widehat{r}_1^{\,j}\mathrm{e}^{\mathrm{i}\,js}\right).\end{aligned}$$

We are now going to find an upper bound for $\sum_{|j|>q, j\in q\mathbb{Z}} \widehat{r}_1^j \mathrm{e}^{\mathrm{i}\,js}$. From the hypotheses of the proposition, we have

$$\left|\sum_{|j|>q, j\in q\mathbb{Z}} \widehat{r}_1^j \mathrm{e}^{\mathrm{i}\,js}\right| \leq \overline{\alpha} \sum_{|j|>q, j\in q\mathbb{Z}} |j|^m \mathrm{e}^{-|j|\rho} |\mathrm{e}^{\mathrm{i}\,js}| \leq 2q^m \overline{\alpha} \sum_{j\geq 2} j^m \mathrm{e}^{-j\rho q}.$$

Consider the function $f(x) = x^m \mathrm{e}^{-x\rho q}$. It tends to zero as x tends to infinity, $f(0) = 0$ and it is positive for x positive. Its unique maximum on $\mathbb{R}^+$ is at $x^* = m/(\rho q)$. Then,

$$\left|\sum_{|j|>q, j\in q\mathbb{Z}} \widehat{r}_1^j \mathrm{e}^{\mathrm{i}\,js}\right| \leq 2q^m \overline{\alpha} \left(f(\max\{2, x^*\}) + \int_2^\infty f(x) \mathrm{d}x \right).$$

On the one hand,

$$f(\max\{2, x^*\}) = \begin{cases} 2^m \mathrm{e}^{-2\rho q} & \text{if } 2\rho q \geq m \\ (m/(\rho q))^m \mathrm{e}^{-m} & \text{if } 2\rho q < m. \end{cases} \tag{31}$$

Observe that, since m and ρ are initially fixed, for q sufficiently large, we have

$$f(\max\{2, x^*\}) = f(2) = \mathcal{O}(\mathrm{e}^{-2\rho q}).$$

On the other hand,

$$\int_2^\infty f(x) dx = \int_2^\infty x^m \mathrm{e}^{-x\rho q} \mathrm{d}x.$$

If we use the upper incomplete Gamma function, $\Gamma(s, x) = \int_x^\infty t^{s-1} \mathrm{e}^{-t} \mathrm{d}t$, we obtain

$$\int_2^\infty x^m \mathrm{e}^{-x\rho q} \mathrm{d}x = (\rho q)^{-(m+1)} \int_{2\rho q}^\infty t^m \mathrm{e}^{-t} \mathrm{d}t = (\rho q)^{-(m+1)} \Gamma(m+1, 2\rho q).$$

Using [**3**, §6.5.32], we obtain

$$\int_2^\infty x^m \mathrm{e}^{-\rho q x} \mathrm{d}x \asymp 2^m \mathrm{e}^{-2\rho q}/(\rho q) = \mathcal{O}(q^{-1} \mathrm{e}^{-2\rho q}), \qquad \text{as } q \to \infty.$$

Therefore, for q sufficiently large, we are able to obtain

$$\left|\sum_{|j|>q, j\in q\mathbb{Z}} \widehat{r}_1^j \mathrm{e}^{\mathrm{i}\,js}\right| \leq \mathcal{O}(q^m \mathrm{e}^{-2\rho q}).$$

Up to now, we know that, for all $s \in \mathbb{T}$,

$$W_1^{(p,q)}(s) \leq 2q \sin(\pi p/q) \left(\left(\widehat{a}_0^q + 2\widehat{a}_1^q \cos(qs) - 2\widehat{b}_1^q \sin(qs) \right) + \mathcal{O}(q^m \mathrm{e}^{-2\rho q}) \right).$$

Next, we want to find an exponentially small lower bound for $\Delta_1^{(p,q)}$. We define the function

$$f(s_1, s_2) := W_1^{(p,q)}(s_1) - W_1^{(p,q)}(s_2), \quad s_1, s_2 \in \mathbb{T}.$$

Hence, $\Delta_1^{(p,q)} \geq f(s_1,s_2)$ for all $s_1, s_2 \in \mathbb{T}$. Let $f(s_1,s_2) = 2q\sin(\pi p/q)\,(g(s_1,s_2)+h(s_1,s_2))$, where

$$\begin{aligned} h(s_1,s_2) &:= \sum_{|j|>q, j\in q\mathbb{Z}} \widehat{r}_1^j(\mathrm{e}^{\mathrm{i}\,js_1} - \mathrm{e}^{\mathrm{i}\,js_2}) = \mathcal{O}(q^m\mathrm{e}^{-2\rho q}), \\ g(s_1,s_2) &:= 2\left(\widehat{a}_1^q(\cos(qs_1)-\cos(qs_2)) - \widehat{b}_1^q(\sin(qs_1)-\sin(qs_2))\right). \end{aligned}$$

From the formulae of the sum of cosinus and sinus we obtain that

$$g(s_1,s_2) = 4\left|\sin\left(q(s_1-s_2)/2\right)\right| \left|\widehat{a}_1^q \sin\left(q(s_1+s_2)/2\right) + \widehat{b}_1^q \cos\left(q(s_1+s_2)/2\right)\right|.$$

Observe that there exist s_1^1 and s_2^1 such that

$$\left|\sin\left(q(s_1^1-s_2^1)/2\right)\right| = 1, \quad \left|\cos\left(q(s_1^1+s_2^1)/2\right)\right| = 1, \ \left|\sin\left(q(s_1^1+s_2^1)/2\right)\right| = 0.$$

Then,

$$g(s_1^1,s_2^1) = 4|\widehat{b}_1^q|.$$

And there also exist s_1^2 and s_2^2 such that

$$\left|\sin\left(q(s_1^2-s_2^2)/2\right)\right| = 1, \quad \left|\cos\left(q(s_1^2+s_2^2)/2\right)\right| = 0, \ \left|\sin\left(q(s_1^2+s_2^2)/2\right)\right| = 1.$$

Then,

$$g(s_1^2,s_2^2) = 4|\widehat{a}_1^q|.$$

Consider $s_*, s^* \in \mathbb{T}$ such that $g(s^*,s_*) \geq g(s_1,s_2)$ for all $s_1,s_2 \in \mathbb{T}$. Thus, $g(s^*,s_*) \geq 4\max\{\widehat{a}_1^q, \widehat{b}_1^q\}$. Since

$$\max\{\widehat{a}_1^q, \widehat{b}_1^q\} \geq \sqrt{\left((\widehat{a}_1^q)^2 + (\widehat{b}_1^q)^2\right)/2} = |\widehat{r}_1^q|/\sqrt{2} \geq \underline{\alpha} q^m \mathrm{e}^{-\rho q}/\sqrt{2},$$

we have

$$g(s^*,s_*) \geq 2\sqrt{2}\underline{\alpha} q^m \mathrm{e}^{-\rho q} = \mathcal{O}(q^m\mathrm{e}^{-\rho q}).$$

Thus, the term which dominates $f(s_1,s_2)$ is $g(s_1,s_2)$ and

$$\Delta_1^{(p,q)} \geq f(s^*,s_*) \geq 4q\sin(\pi p/q)\underline{\alpha} q^m \mathrm{e}^{-\rho q}.$$

Finally, as q tends to infinity, $\pi p/q$ tends to zero. Thus, we use inequality $2x/\pi \leq \sin x$ for $x \in [0,\pi/2]$ to find that $2p \leq q\sin(\pi p/q)$ for q large enough. The final lower bound for $q \gg 1$ is

$$\Delta_1^{(p,q)} \geq Kq^m\mathrm{e}^{-\rho q}, \qquad K := 8\underline{\alpha} p.$$

□

The hypotheses required for the proposition are not as strict as one might think. For example, the Fourier coefficients of the function

$$r_1(s) = \frac{1}{1-2\mu\cos s + \cos^2 s}, \qquad \mu \in (0,1),$$

are $\widehat{r}_1^j = \mu^{|j|}$. Since $\mu \in (0,1)$, we can write $\mu = \mathrm{e}^{-\rho}$, for some $\rho \in \mathbb{R}^+$. Then, $\underline{\alpha}|j|^m\mathrm{e}^{-|j|\rho} \leq \widehat{r}_1^j \leq \overline{\alpha}|j|^m\mathrm{e}^{-|j|\rho}$, by taking $m = 0$ and $\underline{\alpha} \leq 1 \leq \overline{\alpha}$.

It is sufficient to take derivatives of the function above to obtain examples satisfying the bounds of Proposition 6.1.2 with $m \neq 0$. Indeed,

$$g_m(s) := \frac{\mathrm{d}^m}{\mathrm{d}s^m} r_1(s) = \sum \widehat{g}_m^j \mathrm{e}^{\mathrm{i}\, js}, \quad \widehat{g}_m^j = \widehat{r}_1^j |j|^m.$$

Also, observe that since $\widehat{r}_1^j$ must be in a certain interval, slight perturbations on the Fourier coefficients of functions satisfying the hypotheses still give rise to other functions satisfying the hypotheses.

6.2. Elliptic tables

We consider the elliptic billiard table

$$\partial\Omega := C_0 = \left\{ (x,y) \in \mathbb{R}^2 \ : \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \right\}, \qquad a > b > 0.$$

For all $p, q \in \mathbb{N}$, $p < q/2$, the (p,q)-periodic orbits set on a RIC give rise to the same (p,q)-resonant convex caustic. This convex caustic is a confocal ellipse which can be expressed as

$$C_\lambda = \left\{ (x,y) \in \mathbb{R}^2 \ : \ \frac{x^2}{a^2 - \lambda^2} + \frac{y^2}{b^2 - \lambda^2} = 1 \right\}$$

for some $0 < \lambda < b$. The parameter λ has a one-to-one correspondence with the rotation number ρ. From Poncelet's Porism, we know that all (p,q)-periodic trajectories have the same length.

Our aim is to measure the behaviour of the lengths of these (p,q)-periodic orbits on a perturbed elliptic table. We want to study the difference of perturbed lengths $\Delta^{(p,q)}$. As a first approximation, we will study the first order term $\Delta_1^{(p,q)}$.

We parameterize the unperturbed elliptic boundary by

$$M_0(\varphi) = (c \cosh \mu_0 \cos \varphi, c \sinh \mu_0 \sin \varphi),$$

where $c = \sqrt{a^2 - b^2}$ and μ_0 is such that $a = c \cosh \mu_0$ and $b = c \sinh \mu_0$, while the generic perturbation is parameterized as

$$M_\varepsilon(\varphi) = (c \cosh(\mu_\varepsilon(\varphi)) \cos \varphi, c \sinh(\mu_\varepsilon(\varphi)) \sin \varphi), \tag{32}$$

with $\mu_\varepsilon(\varphi) = \mu_0 + \varepsilon \mu_1(\varphi) + \mathcal{O}(\varepsilon^2)$.

We have seen in Section 3.3, that the subharmonic Melnikov potential can be computed as

$$W_1^{(p,q)}(\varphi) = \sum_{j=0}^{q-1} H_1(\varphi_j, \varphi_{j+1}),$$

where φ_j is the first coordinate at the j-th iteration of the unperturbed billiard map T at a point $(\varphi, v(\varphi))$ on the RIC $\Upsilon^{(p,q)} = \operatorname{graph} v$ with rotation number p/q.

Following [**22**], the previous formula for the subharmonic Melnikov potential can be written as

$$W_1^{(p,q)}(\varphi) = 2\lambda \sum_{j=0}^{q-1} \mu_1(\varphi_j).$$

The angular dynamics $\varphi_j \mapsto \varphi_{j+1}$ becomes a rigid rotation using a suitable variable t. To define this new variable, we use Jacobian elliptic functions. The reader can find more information on these functions and the properties we use on Appendix B.

For a fixed $p, q \in \mathbb{N}$, $p < q/2$, $\gcd(p, q) = 1$, we have assigned a RIC with rotation number $\rho = p/q$ and a convex caustic C_λ, $0 < \lambda < b$ on the unperturbed billiard table with boundary C_0. Let $k^2 = (a^2 - b^2)/(a^2 - \lambda^2)$. Observe that $k \in (0, 1)$. Thus, we can take k the modulus and consider $K = K(k) = \int_0^{\pi/2}(1 - k^2 \sin^2 \phi)^{-1/2} \mathrm{d}\phi$ the complete elliptic integral of the first kind. Finally the change of variables is given by $\varphi = \operatorname{am} t = \operatorname{am}(t, k)$ or, equivalently,

$$t = \int_0^{\varphi} (1 - k^2 \sin^2 \phi)^{-1/2} \mathrm{d}\phi.$$

The elliptic sinus and cosinus are defined by means of these variables, t and φ, and the relations $\operatorname{sn} t = \sin \varphi$ and $\operatorname{cn} t = \cos \varphi$. Next proposition characterizes the rigid angular rotation followed by any trajectory in this new variable t.

PROPOSITION 6.2.1 ([**7**]). *Let $q_j = (a \cos \varphi_j, b \sin \varphi_j) = (a\operatorname{cn} t_j, b\operatorname{sn} t_j)$ be the j-th impact point on C_0 of a trajectory with caustic C_λ. Then, $t_{j+1} = t_j + \delta$, where*

$$\frac{\delta}{2} = \int_0^{\vartheta/2} (1 - k^2 \sin^2 \phi)^{-1/2} \mathrm{d}\phi$$

with $\vartheta = 2 \arcsin(\lambda/b) \in (0, \pi)$.

The modulus k is the eccentricity of the caustic C_λ. Besides,

$$q\delta = 4Kp. \tag{33}$$

This relation can be interpreted geometrically. When the trajectory makes a turn around C_λ, the old variable $\widetilde{\varphi}$ increases 2π, while looking at the definition of t and K, $\widetilde{t}$ increases $4K$. Also, while variable $\widetilde{\varphi}$ moves forward about p/q at each step, $\widetilde{t}$ moves forward δ at each iteration.

Up to now, we have obtained a formula for the subharmonic Melnikov potential under any generic perturbation of the form (32) using the new variable t. The result in these coordinates is

$$\widehat{W}_1^{(p,q)}(t) = 2\lambda \sum_{j=0}^{q-1} \widehat{\mu}_1(t + j\delta),$$

where $\widehat{\mu}_1(t) := \mu_1(\operatorname{am} t)$ and $\widehat{W}_1^{(p,q)}(t) := W_1^{(p,q)}(\operatorname{am} t)$.

Next, we proceed to study the behaviour of $\Delta_1^{(p,q)}$ under two concrete perturbations, defined by $\mu_1(\varphi) = \cos^r \varphi$, with $r = 1, 2$. To state this result, we first need the following notations and a new closed expression for the subharmonic Melnikov potential in the variable t.

We can observe from the definition of the complete elliptic integral of the first kind that it is an increasing function such that $\lim_{k\to 0^+} K(k) = \pi/2$ and $\lim_{k\to 1^-} K(k) = \infty$. Then, we also obtain that $K'(k) := K\left(\sqrt{1-k^2}\right)$ is a decreasing function with $\lim_{k\to 0^+} K'(k) = \infty$ and $\lim_{k\to 1^-} K'(k) = \pi/2$. Thus, the function $K'(k)/K(k)$ is a positive decreasing function such that

$$\lim_{k\to 0^+} K'(k)/K(k) = \infty \quad \text{and} \quad \lim_{k\to 1^-} K'(k)/K(k) = 0.$$

Therefore, for any $k \in (0,1)$ and any integer $q \geq 2$, there exists a unique $k_q \in (0,k)$ such that

$$\frac{K'(k_q)}{K(k_q)} = q\frac{K'(k)}{K(k)}. \tag{34}$$

For short, we will use the notations $K' := K'(k)$, $K := K(k)$, $K'_q := K'(k_q)$ and $K_q := K(k_q)$.

PROPOSITION 6.2.2. *Given $0 < \lambda < b$, let $W_1^{(p,q)}(\varphi)$ be the subharmonic Melnikov potential of the caustic C_λ for the perturbed ellipse given by (32) with $\mu_1(\varphi) = \cos^r(\varphi)$, $r = 1, 2$. Let $\Delta_1^{(p,q)} = \max_{\mathbb{T}} W_1^{(p,q)} - \min_{\mathbb{T}} W_1^{(p,q)}$. Then,*

$$\Delta_1^{(p,q)} = \begin{cases} 4\lambda k_q K'_q/(kK'), & r = 1 \text{ and odd } q, \\ 0, & r = 1 \text{ and even } q, \\ 2\lambda\left(k_q K'_q/(kK')\right)^2, & r = 2 \text{ and odd } q, \\ 4\lambda\left(k_{q/2} K'_{q/2}/(kK')\right)^2, & r = 2 \text{ and even } q. \end{cases} \tag{35}$$

PROOF. We proof each case separately. First, we study the case $r = 1$ and q odd. We know that

$$W_1^{(p,q)}(\varphi) = 2\lambda \sum_{j=0}^{q-1} \cos\varphi_j. \tag{36}$$

Introducing the change of coordinate above and the definition of the elliptic cosinus, we have

$$\widehat{W}_1^{(p,q)}(t) = 2\lambda\sum_{j=0}^{q-1}\cos(\operatorname{am} t_j) = 2\lambda\sum_{j=0}^{q-1}\operatorname{cn}(t_j,k) = 2\lambda\sum_{j=0}^{q-1}\operatorname{cn}(t+j\delta,k).$$

First, we claim that

$$\widehat{W}_1^{(p,q)}(t) = 2\lambda\frac{k_q K'_q}{kK'}\operatorname{cn}(K'_q t/K', k_q). \tag{37}$$

The proof of this claim is the following.

First, $\widehat{W}_1^{(p,q)}$ is $4Kp/q$-periodic since $\operatorname{cn}(t+q\delta,k) = \operatorname{cn}(t+q(4Kp/q),k) = \operatorname{cn}(t+4Kp,k) = \operatorname{cn}(t,k)$. It is also $4K$-periodic since so are all its terms. Thus, since $\gcd(p,q) = 1$ and q is odd, we find that $\gcd(4Kp/q, 4K) = 4K/q$ and $\widehat{W}_1^{(p,q)}$ is also $4K/q$-periodic. Since all $\operatorname{cn}(t+j\delta,k)$ are also $4K'\mathrm{i}$-periodic and both periods are not linearly dependent, function $\widehat{W}_1^{(p,q)}$ is an elliptic function. Its poles are set at points $t = r(2K/q) + (1+2s)K'\mathrm{i}$, for $r,s \in \mathbb{Z}$ and the residues are the same ones of the elliptic cosinus multiplied by the factor 2λ, $2\lambda(-1)^{r+s+1}\mathrm{i}/k$.

Now we take k_q as defined in (34). Observe that function $\mathrm{cn}\,(u, k_q)$ has periods $4K_q$ and $4K'_q\mathrm{i}$ and its residues are set at points $u = r(2K_q) + (1+2s)K'_q\mathrm{i}$ for $r, s \in \mathbb{Z}$ and have value $(-1)^{r+s+1}\mathrm{i}\,/k_q$.

If we use the change of variable $u = K'_q t/K' = qK_q t/K$, the function $\mathrm{cn}\,(K'_q t/K_q, k_q)$ has periods $4K_qK/(qK_q) = 4K/q$ and $4K'_q\mathrm{i}\,K'/K'_q = 4K'\mathrm{i}$. Analogously, the simple poles are set, in terms of t at points $t = (K'/K'_q)(r(2K_q) + (1+2s)K'_q\mathrm{i}\,) = r(2K/q) + (1+2s)K'\mathrm{i}$ for $r, s \in \mathbb{Z}$.

Both functions, $\widehat{W}_1^{(p,q)}(t)$ and $\mathrm{cn}\,(K'_q t/K_q, k_q)$, have the same periods and have the poles located at same places. If we rescale both functions so that the residues are the same, we will obtain an equality between them. In particular, if we consider function $kK'\widehat{W}_1^{(p,q)}$ the residues become

$$\mathrm{Res}(kK'\widehat{W}_1^{(p,q)}(t); 2rK/q + (1+2s)\mathrm{i}\,K') = kK'2\lambda\frac{(-1)^{r+s+1}\mathrm{i}}{k} = 2\lambda K'(-1)^{r+s+1}\mathrm{i}\,.$$

Repeating the same process using the factor $2\lambda k_q K'_q$ on function $\mathrm{cn}\,(K'_q t/K', k_q)$, the new residues at the simple poles are

$$\mathrm{Res}(2\lambda k_q K'_q \mathrm{cn}\,(K'_q t/K', k_q); r(2K/q) + (1+2s)K'\mathrm{i}\,) = 2\lambda K'(-1)^{r+s+1}\mathrm{i}\,.$$

Thus, we have found two elliptic functions with the same period, location of the poles and values of their principal part (since all poles are simple, we only need to compare their residues). Hence, their difference must be a constant. Yet, both vanish at point $t = K/q$. This proves formula (37).

Next, since the elliptic cosinus oscillates between 1 and -1, from (37), the value of $\Delta_1^{(p,q)}$ is

$$\Delta_1^{(p,q)} = \max_{\mathbb{R}} W_1^{(p,q)} - \min_{\mathbb{R}} W_1^{(p,q)} = \frac{2\lambda k_q K'_q}{kK'} - \frac{2\lambda k_q K'_q(-1)}{kK'} = 4\lambda\frac{k_q K'_q}{kK'}.$$

Next, let us consider $r = 1$ and q even. Observe that, if q is even, the (p, q)-trajectories inscribed in C_0 are symmetric with respect to the origin and we obtain $\varphi_{j+q/2} = \varphi_j + \pi$. If we derive the subharmonic Melnikov potential $W_1^{(p,q)}$ at (36) we obtain

$$\left(W_1^{(p,q)}\right)'(\varphi) = 2\lambda\sum_{j=0}^{q/2-1}(\sin\varphi_j + \sin(\varphi_j+\pi))\frac{\mathrm{d}\varphi_j}{\mathrm{d}\varphi} = 0.$$

Therefore, $W_1^{(p,q)}$ is constant and $\Delta_1^{(p,q)} = 0$.

We now assume $r = 2$ and q odd. As in case $r = 1$ and q odd, comparing periods, the location of the poles and the principal parts, we find that

$$\widehat{W}_1^{(p,q)}(t) = 2\lambda\sum_{j=0}^{q-1}\mathrm{cn}^2(t + j\delta, k) = 2\lambda\left(\frac{k_q K_q}{kK'}\right)^2 \mathrm{cn}^2(K'_q t/K', k_q) + \text{constant}\,.$$

Then, $\Delta_1^{(p,q)} = 2\lambda\left(k_q K'_q/kK'\right)^2(1-0) = 2\lambda\left(k_q K'_q/kK'\right)^2$.

Finally, we consider $r = 2$ and q even. Since $\mathrm{cn}^2(t, k)$ is $2K$-periodic, if we also consider relation (33), we obtain that $\mathrm{cn}^2(t + (q/2)\delta, k) = \mathrm{cn}^2(t + 2Km, k) =$

$\mathrm{cn}^2(t,k)$. Thus,

$$\sum_{j=0}^{q-1} \mathrm{cn}^2(t+j\delta,k) = 2\sum_{j=0}^{q/2-1} \mathrm{cn}^2(t+j\delta,k).$$

Using the same result as the previous case, we obtain

$$\widehat{W}_1^{(p,q)}(t) = 2\lambda\sum_{j=0}^{q-1} \mathrm{cn}^2(t+j\delta,k) = 2\left(2\lambda\left(\frac{k_{q/2}K'_{q/2}}{kK'}\right)^2 \mathrm{cn}^2(K'_{q/2}t/K', k_{q/2}) + \text{constant}\right).$$

And the corresponding first term for the maximum difference of the lengths is

$$\Delta_1^{(p,q)} = 4\lambda\left(\frac{k_{q/2}K_{q/2}}{kK'}\right)^2.$$

□

Proposition 6.2.2 provides a closed expression for $\Delta_1^{(p,q)}$ using the properties of the elliptic functions. In what follows, we compute the asymptotic behaviour of this expression for the different perturbations being studied, $\mu_1(\varphi) = \cos^r(\varphi)$, $r = 1, 2$, as q tends to infinity and as the rotation number p/q asymptotically tends to either a fixed irrational number or 0.

A commonly used function associated to the elliptic functions is the *nome*, $q(k)$. We use it in the next propositions and it is defined as

$$q(k) := \exp\left(-\frac{\pi K'(k)}{K(k)}\right).$$

Notation $q(k)$ for the nome is standard, pay attention when distinguishing $q(k)$ from the period q of the (p,q)-periodic orbits.

PROPOSITION 6.2.3. *Let $\rho_* \in (0,1/2) \notin \mathbb{Q}$, let $(p_l,q_l)_l$ be a sequence such that $p_l < q_l/2$, $\lim_{l\to\infty} p_l/q_l = \rho_*$, $\gcd(p_l,q_l) = 1$ and q_l is odd. Let $\Delta_1^{(p_l,q_l)}$ be the difference between the maximum and the minimum of the subharmonic Melnikov potential for (p_l,q_l)-orbits of the perturbed ellipse $\partial\Omega_\varepsilon = \{\mu = \mu_0 + \varepsilon\cos\varphi + \mathcal{O}(\varepsilon^2)\}$. Then, $\Delta_1^{(p_l,q_l)}$ is exponentially small in q_l. More concretely, its asymptotic behaviour is*

$$\Delta_1^{(p_l,q_l)} \asymp \frac{8\pi\lambda_*}{k_*K_*} q_l q_*^{q_l/2},$$

where $\lambda_* \in (0,b)$ characterizes the convex caustic C_{λ_*} such that its rotation number is ρ_*, $k_*^2 := (a^2-b^2)/(a^2-\lambda_*^2)$, $K_* := K(k_*)$, and $q_* := q(k_*)$.

PROOF. We want to find constants $c_1 \neq 0$, $c_2 \in \mathbb{R}$ and $c_3 > 0$ such that

$$\Delta_1^{(p_l,q_l)} \asymp c_1 q_l^{c_2} \mathrm{e}^{-c_3 q_l} \quad \text{as} \quad p_l/q_l \to \rho_*. \tag{38}$$

By relation (34) and Proposition 6.2.2,

$$\Delta_1^{(p_l,q_l)} = 4\lambda q_l \frac{k_{q_l}K_{q_l}}{kK}. \tag{39}$$

Since $p_l/q_l \to \rho_*$, the (p_l, q_l)-caustic, C_{λ_l}, tends to the caustic with rotation number ρ_*, C_{λ_*}, and therefore $k \to k_* = k(\lambda_*)$. Also, since $\rho_* \in (0, 1/2)$, $\lambda_* \in (0, b)$ and $k_* \in (0, 1)$. Thus,

$$\frac{K'(k)}{K(k)} \to \frac{K'(k_*)}{K(k_*)} \in (0, \infty). \tag{40}$$

Since $q_l \to \infty$, relations (34) and (40) imply that $K'_{q_l}/K_{q_l} \to \infty$. Therefore,

$$K'_{q_l} \to \infty, \quad k_{q_l} \to 0, \quad K_{q_l} \to \pi/2. \tag{41}$$

Now, we try to find an expression for k_q. From [**3**, §17.3.16], we obtain

$$k_{q_l} = 4 \exp\left(\frac{-\pi K'_{q_l}}{2K_{q_l}}\right) \exp\left(\frac{-\pi L\left(\sqrt{1-k_{q_l}^2}\right)}{2K_{q_l}}\right),$$

where $L(k)$ is an auxiliary function (defined in [**3**, §17.3.14]) such that it tends to 0 as k tends to 1 (it can be seen at [**3**, p.612]). Observe that, from (41), we obtain that L tends to 0 as l tends to infinity. Then, using again relation (34) and (41) for K_{q_l}, as $l \to \infty$, we have

$$\begin{aligned} k_{q_l} &= 4 \exp\left(\frac{-\pi K'_{q_l}}{2K_{q_l}}\right) \exp\left(\frac{-\pi L\left(\sqrt{1-k_{q_l}^2}\right)}{2K_{q_l}}\right) \\ &\asymp 4 \exp\left(\frac{-\pi q_l K'_*}{2K_*}\right) = 4q_*^{q_l/2}, \end{aligned} \tag{42}$$

Putting together (39), (41) and (42) we obtain

$$\Delta_1^{(p_l,q_l)} \asymp \frac{8\pi\lambda_*}{k_* K_*} q_l q_*^{q_l/2}.$$

□

Observe that we have found an asymptotic behaviour for any sequence $\{(p_l, q_l)\}_l$, with q_l odd for all l, with (p_l, q_l)-periodic orbits tending to a caustic with an irrational rotation number. The difference on the first order term $\Delta_1^{(p_l,q_l)}$ for sequences $\{(p_l, q_l)\}_l$, with q_l even for all l, is equal to 0 as it follows from Proposition 6.2.2. Thus, for any $\{(p_l, q_l)\}_l$ sequence satisfying $\gcd(p_l, q_l) = 1$, $p_l/q_l \to \rho_*$, $p_l < q_l/2$ the asymptotic behaviour is different for the terms with an even q_l and the terms with an odd q_l.

This oscillation on the asymptotic behaviour of $\Delta_1^{(p_l,q_l)}$ comes directly from the different formulation of $\Delta_1^{(p,q)}$ according to the parity of q at Proposition 6.2.2. The oscillation also appears when $r = 2$ and also when we tend to the boundary ($\rho_* = 0$) as we will see in the next propositions.

Proposition 6.2.4. *Let $\rho_* \in (0, 1/2) \notin \mathbb{Q}$ and let $(p_l, q_l)_l$ be a sequence such that $p_l < q_l/2$, $\lim_{l\to\infty} p_l/q_l = \rho_*$, and $gcd(p_l, q_l) = 1$. Consider the perturbed ellipse $\partial\Omega_\varepsilon = \{\mu = \mu_0 + \varepsilon \cos^2 \varphi + \mathcal{O}(\varepsilon^2)\}$. Then, $\Delta_1^{(p_l,q_l)}$ is exponentially small in q_l.*

More concretely, if q_l is even, the asymptotic behaviour on $\Delta_1^{(p_l,q_l)}$ is

$$\Delta_1^{(p_l,q_l)} \asymp \frac{4\pi^2\lambda_*}{k_*^2 K_*^2} q_l^2 q_*^{q_l/2}$$

and, if q_l is odd, the asymptotic behaviour on $\Delta_1^{(p_l,q_l)}$ is

$$\Delta_1^{(p_l,q_l)} \asymp \frac{8\pi^2\lambda_*}{k_*^2 K_*^2} q_l^2 q_*^{q_l},$$

where λ_*, k_*, K_* and q_* are defined as in Proposition 6.2.3.

Proof. Consider all the terms of the sequence $(p_l, q_l)_l$ such that q_l is even. All these terms form a subsequence which we redenote by $(p_l, q_l)_l$. In Proposition 6.2.2, we have seen

$$\Delta_1^{(p_l,q_l)} = 4\lambda \left(\frac{k_{q_l/2} K'_{q_l/2}}{kK'} \right)^2.$$

Proceeding in the same way as in Proposition 6.2.3, we obtain

$$\begin{aligned} k_{q_l/2} &\asymp 4\exp\left(\frac{-\pi K'_*}{K_*}\right)^{q_l/4} = 4q_*^{q_l/4}, \\ K_{q_l/2} &\asymp \pi/2. \end{aligned}$$

And finally,

$$\begin{aligned} \Delta_1^{(p_l,q_l)} &= 4\lambda \left(\frac{k_{q_l/2} K'_{q_l/2}}{kK'} \right)^2 = 4\lambda \left(\frac{q_l}{2} \frac{k_{q_l/2} K_{q_l/2}}{kK} \right)^2 \\ &\asymp 4\lambda_* \left(\frac{q_l}{2} \frac{4q_*^{q_l/4} \frac{\pi}{2}}{k_* K_*} \right)^2 = \frac{4\pi^2\lambda_*}{k_*^2 K_*^2} q_l^2 q_*^{q_l/2}. \end{aligned}$$

Now we consider the subsequence containing only q_l odd. We denote it again by $(p_l, q_l)_l$. As we have seen in Proposition 6.2.2, we know $\Delta_1^{(p_l,q_l)} = 2\lambda \left(k_{q_l} K'_{q_l}/kK'\right)^2$. The asymptotic behaviour of k_{q_l} and K_{q_l} is again the same as Proposition 6.2.3. Then,

$$\begin{aligned} \Delta_1^{(p_l,q_l)} &= 2\lambda \left(q_l \frac{k_{q_l} K_{q_l}}{kK} \right)^2 \\ &\asymp \frac{2\lambda_*}{k_*^2 K_*^2} q_l^2 \left(4q_*^{q_l/2} \frac{\pi}{2} \right)^2 = \frac{8\pi^2\lambda_*}{k_*^2 K_*^2} q_l^2 q_*^{q_l}. \end{aligned}$$

□

Proposition 6.2.5. *Consider the perturbed ellipse $\partial\Omega_\varepsilon^r = \{\mu = \mu_0 + \varepsilon \cos^r \varphi + \mathcal{O}(\varepsilon^2)\}$, $r = 1, 2$. Then, $\Delta_1^{(1,q)}$ is exponentially small in q. More concretely, there exists a constant $c = c(a, b) > 0$ such that the asymptotic behaviour of $\Delta_1^{(1,q)}$ is*

$$\Delta_1^{(1,q)} \asymp \begin{cases} (8\pi/ck_* K_*) q_*^{q/2}, & r = 1,\ q \text{ odd}, \\ 0, & r = 1,\ q \text{ even}, \\ (8\pi^2/k_*^2 K_*^2) q q_*^q, & r = 2,\ q \text{ odd}, \\ (4\pi^2/ck_*^2 K_*^2) q q_*^{q/2}, & r = 2,\ q \text{ even}, \end{cases}$$

where k_*, K_* and q_* are defined as in Proposition 6.2.3.

PROOF. First, observe that in all these cases we tend to the rotation number $\rho_* = 0$. Then $\lambda_* = 0$ and $k_*^2 = 1 - b^2/a^2 \neq 0$. Thus, $K'_*/K_* \neq 0$ and we obtain the same asymptotic results for k_q, K_q and K'_q as (41).

The claim follows using the same arguments in the proofs of Propositions 6.2.3 and 6.2.4 but, since λ_* tends asymptotically to zero, it must be substituted by its asymptotic behaviour. From [**2**, Proposition 10], $\rho(\lambda) = c\lambda + \mathcal{O}(\lambda^3)$ as $\lambda \to 0^+$, with $c = c(a,b) > 0$. If we take $\rho(\lambda) = 1/q$ and invert the function, we obtain $\lambda = \rho^{-1}(1/q) = 1/(cq) + \mathcal{O}(q^{-3})$. □

Observe that there also exists the oscillation on the asymptotic behaviour of $\Delta_1^{(1,q)}$ according to the parity of the terms q.

Summarizing up, in this section we have been able to prove, for a particular type of perturbed elliptic tables, that the difference on the lengths of the (p,q)-orbits is exponentially small in the period q as the trajectories described tend to the boundary (they are $(1,q)$-periodic orbits) and also as the (p,q)-trajectories tend to trajectories with a concrete irrational rotation number.

Conclusions and further problems

In this work, we have gathered the main definitions and known results about the length spectrum on analytic strictly convex billiards.

More concretely, we have first given an overview on area-preserving twist maps and on billiards. We have also reviewed some general tools to deal with perturbations of these maps. In particular, the Melnikov method is the one we have used on this work.

The initial results obtained have dealt with two special cases of strictly convex tables, the circular and the elliptic ones. These tables give rise to Liouville integrable maps whose (p, q)-periodic orbits are on resonant RICs so that all have the same length. When perturbing the billiard boundary, the RICs break up and the maximum difference on the subharmonic Melnikov potential for different (p, q)-periodic orbits is equal to $\Delta^{(p,q)}$. We have been able to compute the asymptotic behaviour for the first order term of $\Delta^{(p,q)}$ as an expansion on the perturbative parameter, which happens to be exponentially small. This first step is not sufficient; bounding the first order term is not enough when the period q is no longer fixed but tending to infinity.

From the reviewed results on the length spectrum by Marvizi and Melrose [**18**] and Colin de Verdière [**8**], and also taking into account the application of the KAM theory to invariant curves with Diophantine rotation number [**25**], we have argued that we expect an exponentially small behaviour on the maximum difference of lengths of the (p, q)-periodic orbits in several settings. These ideas are summarized as the following three conjectures.

Conjecture 1. *Let Ω be an analytic strictly convex table. Let $\Delta^{(1,q)}$ be the difference of lengths between the longest and the shortest $(1, q)$-periodic orbits approaching to the boundary as q tends to infinity. Then, $\Delta^{(1,q)}$ is exponentially small in q. That is,*

$$\sup \left\{ t \in \mathbb{R} : \lim_{q \to \infty} \mathrm{e}^{tq} \Delta^{(1,q)} = 0 \right\} > 0.$$

Conjecture 2. *Let Ω be an analytic strictly convex table axisymetric with respect the two axis of coordinates. Let $p, q \in \mathbb{N}$, $p < q/2$, $\gcd(p, q) = 1$. Let $\Delta^{(p,q)}$ be the difference of lengths between the longest and the shortest (p, q)-periodic orbits approaching to the elliptic $(1, 2)$-periodic orbit on the billiard. Then, $\Delta^{(p,q)}$ is* *exponentially small in q. That is,*

$$\sup\left\{t \in \mathbb{R} : \lim_{q\to\infty} \mathrm{e}^{tq}\Delta^{(p,q)} = 0\right\} > 0.$$

CONJECTURE 3. *Let Ω be an analytic strictly convex table. Let $p, q \in \mathbb{N}$, $p < q/2$, $\gcd(p, q) = 1$. Let $\Delta^{(p,q)}$ be the difference of lengths between the longest and the shortest (p, q)-periodic orbits approaching to a RIC with a Diophantine rotation number. Then, $\Delta^{(p,q)}$ is exponentially small in q. That is,*

$$\sup\left\{t \in \mathbb{R} : \lim_{q\to\infty} \mathrm{e}^{tq}\Delta^{(p,q)} = 0\right\} > 0.$$

The future work focusses on proving these conjectures. This has to be done by other methods than the ones used in this memoir. On generic analytic strictly convex billiards, there will not be a (p, q)-resonant RIC. Also, the Melnikov approach fails. Other techniques such as using a certain normal form around the invariant object must be considered. The invariant object will be the static billiard boundary, the elliptic $(1, 2)$-periodic orbit or a RIC with a Diophantine rotation number according to each setting.

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Appendix A
Billiard map for a small incidence-reflexion angle

If the incidence-reflection angle, θ, is close to 0 we can find the following approximation for the billiard map T.

For $0 < \theta \ll 1$, given any $k > 1$, there exist some coefficients $\alpha_1(s),\cdots,\alpha_k(s)$, $\beta_2(s)$, $\cdots$, $\beta_k(s)$ that

$$\left\{\begin{array}{lcl} s_1 & = & s + \sum_{j=1}^{k} \alpha_j(s)\theta^j + \mathcal{O}(\theta^{k+1}), \\ \theta_1 & = & \theta + \sum_{j=2}^{k} \beta_j(s)\theta^j + \mathcal{O}(\theta^{k+1}). \end{array}\right. \tag{43}$$

Besides, the error terms are uniform in $s \in \mathbb{T}$.

In what follows, we will find coefficients $\alpha_1(s)$, $\alpha_2(s)$, $\alpha_3(s)$, $\beta_2(s)$ and $\beta_3(s)$ satisfying our claim at equation (28). This result appears in the literature, see [**17**] or [**26**, p. 44], but we have not been able to find its derivation.

We will be using the orthonormal basis formed by the unit tangent vector and the unit normal vector at the generic point s,

$$\left\{\begin{array}{lcl} \vec{t}(s) & = & \vec{x}'(s) \\ \vec{n}(s) & = & \rho(s)\vec{t}'(s) \end{array}\right.$$

$$\|\vec{t}(s)\| = \|\vec{n}(s)\| = 1,$$

where $\rho(s)$ is the radius of curvature and we recall that s is the arc-length parameter. For the sake of simplicity, we will be omitting the dependence on the arc-length parameter s of the curve $\vec{x}$, the normal vector, $\vec{n}$, the tangent vector, $\vec{t}$, the radius of curvature, ρ, and the coefficients $(\alpha_j)_j$ and $(\beta_j)_j$. We will specify it only if the function is applied to a point other than s.

Recall that a billiard map $T : (s, \theta) \mapsto (s_1, \theta_1)$ is defined according the following equations

$$s_1 \text{ is such that } \quad \vec{x}(s_1) = \vec{x} + \lambda(\vec{t}\cos\theta + \vec{n}\sin\theta), \tag{44}$$

$$\theta_1 \text{ is such that } \quad \cos\theta_1 = \langle \vec{t}\cos\theta + \vec{n}\sin\theta, \vec{t}(s_1)\rangle. \tag{45}$$

Before starting the computations, we want to observe some preliminary results.

(i) Vector $(\vec{t}\cos\theta + \vec{n}\sin\theta)$ is a unit vector,

$$\begin{aligned}\|\vec{t}\cos\theta + \vec{n}\sin\theta\|^2 &= \|\vec{t}\|^2\cos^2\theta + 2\|\langle\vec{t},\vec{n}\rangle\|\cos\theta\sin\theta + \|\vec{n}\|^2\sin^2\theta\\ &= 1\cdot\cos^2\theta + 0 + 1\cdot\sin^2\theta = 1.\end{aligned}$$

(ii) $\vec{n}' = -(1/\rho)\vec{t}$.
It can be deduced from the following arguments.
First, since the base $\{\vec{t},\vec{n}\}$ is orthonormal, we can take derivates in the equation $0 = \langle\vec{n},\vec{t}\rangle$ to obtain

$$0 = \langle\vec{n}',\vec{t}\rangle + \langle\vec{n},\vec{t}'\rangle = \langle\vec{n}',\vec{t}\rangle + \frac{1}{\rho}.$$

Then, $\vec{n}' = -(1/\rho)\vec{t} + b\vec{n}$, where $b = \langle\vec{n},\vec{n}'\rangle$.
Observe that, since $\vec{n}$ is unitary, $0 = \partial_s\langle\vec{n},\vec{n}\rangle = 2\langle\vec{n},\vec{n}'\rangle$ and we obtain $b = 0$.
With this result, we are able to calculate any derivative of the curve $\vec{x}$.

(iii) We will be using that $\theta = 0$ is a stationary curve and we will initially apply Taylor at $\vec{x}(s_1)$ around point s.
We know that for θ close to 0, $\Delta s := s_1 - s$ is also close to 0 and then

$$\vec{x}(s+\Delta s) = \vec{x} + \vec{x}'\cdot\Delta s + \vec{x}''\cdot\frac{(\Delta s)^2}{2} + \mathcal{O}_3,$$

where $\mathcal{O}_j$ refers to $\mathcal{O}((\Delta s)^j)$ and we will use the same abbreviation in the following computations.

We will first work on equation (44) to find the expression for s_1. First, when taking modulus at both sides, we can obtain the unknown quantity $\lambda > 0$,

$$\|\vec{x}(s+\Delta s) - \vec{x}\| = \lambda\|\vec{t}\cos\theta + \vec{n}\sin\theta\| = \lambda.$$

And using the Taylor suggested at (iii), we obtain

$$\lambda = \left\|\vec{x}'\Delta s + \vec{x}''\frac{(\Delta s)^2}{2} + \vec{x}'''\frac{(\Delta s)^3}{3!} + \mathcal{O}_4\right\|.$$

From observation (ii), we can calculate as much derivatives as we need. These are the ones we will be using when deducing expansions for s_1 and θ_1.

$$\begin{array}{rcl}\vec{x}' &=& \vec{t},\\ \vec{x}'' &=& -\frac{1}{\rho}\vec{n},\\ \vec{x}''' &=& \frac{\rho'}{\rho^2}\vec{n} - \frac{1}{\rho^2}\vec{t},\\ \vec{x}^{(4)} &=& \frac{3\rho'}{\rho^3}\vec{t} + \frac{2(\rho')^2-\rho\rho''-1}{\rho^3}\vec{n} \text{ and}\\ \vec{x}^{(5)} &=& \frac{-11(\rho')^2+4\rho\rho''+1}{\rho^4}\vec{t} + b_5\vec{n},\end{array} \tag{46}$$

where b_5 is a function depending on s.

We can then proceed on calculating λ up to order 4 in Δs

$$\begin{aligned}\lambda &= \|\vec{x}(s+\Delta s) - \vec{x}\| = \Delta s\left\|\vec{t}\left(1 - \frac{1}{6\rho^2}(\Delta s)^2 + \frac{3\rho'}{24\rho^3}(\Delta s)^3 + \mathcal{O}_4\right) + \right.\\ &\quad \left. +\vec{n}\left(\frac{1}{2\rho}\Delta s - \frac{\rho'}{6\rho^2}(\Delta s)^2 + \frac{2(\rho')^2 - \rho\rho'' - 1}{24\rho^3}(\Delta s)^3 + \mathcal{O}_4\right)\right\|\\ &= \Delta s\sqrt{1 - \frac{1}{12\rho^2}(\Delta s)^2 + \frac{\rho'}{12\rho^3}(\Delta s)^3 + \mathcal{O}_4}.\end{aligned}$$

where we have used that $\vec{n}$ is orthogonal to $\vec{t}$. Using the Taylor expansion of the function $\sqrt{1+x}$ near $x = 0$, we finally obtain

$$\lambda = \Delta s \left(1 - \frac{(\Delta s)^2}{24\rho^2} + \frac{\rho'(\Delta s)^3}{24\rho^3} + \mathcal{O}_4\right).$$

We will now recall again the initial equation (44) and match the relations on the normal direction looking for the coefficients α_j, $j = 1, 2, 3$, that allow us to write $s_1 - s = \Delta s = \alpha_1\theta + \alpha_2\theta^2 + \alpha_3\theta^3 + F(s,\theta)\theta^4$.

On the one hand,

$$\langle \vec{n}, \lambda(\vec{t}\cos\theta + \vec{n}\sin\theta)\rangle = \lambda\left(\theta - \frac{\theta^3}{3!} + \mathcal{O}(\theta^5)\right).$$

On the other hand,

$$\langle \vec{n}, \vec{x}(s+\Delta s) - \vec{x}\rangle = \Delta s\left(\frac{1}{2\rho}\Delta s - \frac{\rho'}{6\rho^2}(\Delta s)^2 + \frac{2(\rho')^2 - \rho\rho'' - 1}{24\rho^3}(\Delta s)^3 + \mathcal{O}_4\right).$$

Making the substitution $\Delta s = \alpha_1\theta + \alpha_2\theta^2 + \alpha_3\theta^3 + F(s,\theta)\theta^4$, we obtain the relations

$$\begin{aligned}
\theta: &\quad \frac{\alpha_1}{2\rho} = 1 \qquad \Rightarrow \qquad \alpha_1 = 2\rho,\\
\theta^2: &\quad \frac{\alpha_2}{2\rho} - \frac{\rho'\alpha_1^2}{6\rho^2} = 0 \qquad \Rightarrow \qquad \alpha_2 = \frac{4}{3}\rho\rho', \quad \text{and}\\
\theta^3: &\quad \frac{-\alpha_1^2}{24\rho^2} - \frac{1}{6} = \frac{\alpha_3}{2\rho} - \frac{\rho'}{6\rho^2}2\alpha_1\alpha_2 + \frac{2(\rho')^2 - \rho\rho'' - 1}{24\rho^3}\alpha_1^3\\
&\qquad \Rightarrow \alpha_3 = \frac{4}{9}\rho(\rho')^2 + \frac{2}{3}\rho^2\rho''.
\end{aligned}$$

Therefore, we have found the following expansion for s_1

$$s_1 = s + 2\rho\theta + \frac{4}{3}\rho\rho'\theta^2 + \left(\frac{4}{9}\rho(\rho')^2 + \frac{2}{3}\rho^2\rho''\right)\theta^3 + \mathcal{O}(\theta^4).$$

We will now work on the equation defining θ_1, equation (45). We will proceed in a similar way. First, we will find the expansion for $\vec{t}(s_1)$ around s. Since we already know Δs as an expansion of θ we will have the expression $\langle \vec{t}\cos\theta + \vec{n}\sin\theta, \vec{t}(s_1)\rangle$ as an expansion on powers of θ. On the other hand, since θ_1 is small for θ small, we will expand $\cos\theta_1$ around $\theta_1 = 0$. Finally, we will impose that θ_1 can be written as a power series of θ and we will match both equations.

Since $\vec{t} = \vec{x}'$, we have $\vec{t}^{(i)} = \vec{x}^{(i+1)}$. Then, using (46), we obtain

$$\begin{aligned}
\vec{t}(s+\Delta s) = \vec{t}&\left(1 - \frac{1}{\rho^2}\frac{(\Delta s)^2}{2} + \frac{3\rho'}{\rho^3}\frac{(\Delta s)^3}{6} + \frac{4\rho\rho'' + 1 - 11(\rho')^2}{\rho^4}\frac{(\Delta s)^4}{24} + \mathcal{O}_5\right)\\
&+\vec{n}\left(\frac{1}{\rho}\Delta s - \frac{\rho'}{\rho^2}\frac{(\Delta s)^2}{2} + \frac{2(\rho')^2 - \rho\rho'' - 1}{\rho^3}\frac{(\Delta s)^3}{6} + \mathcal{O}_5\right).
\end{aligned}$$

Then, since

$$\langle \vec{t}\cos\theta + \vec{n}\sin\theta, \vec{t}(s_1)\rangle = \langle \vec{t}, \vec{t}(s_1)\rangle\cos\theta + \langle \vec{n}, \vec{t}(s_1)\rangle\sin\theta,$$

We use the series expansion for s_1 that we have already found and the expansion of the trigonometric function to obtain

$$\begin{array}{lcl} \langle \vec{t}, \vec{t}(s_1) \rangle & = & 1 - 2\theta^2 + (4\rho'\theta^3/3 + (4\rho\rho''/3 + 2/3 - 10(\rho')^2/9)\theta^4 + \mathcal{O}_5 \\ \cos\theta & = & 1 - \theta^2/2 + \theta^4/24 + \mathcal{O}_6, \\ \langle \vec{n}, \vec{t}(s_1) \rangle & = & 2\theta - 2\rho'\theta^2/3 + (-2\rho\rho''/3 + 4(\rho')^2/9 - 4/3)\theta^3 + \mathcal{O}_4, \\ \sin\theta & = & \theta - (\theta^3/6) + \mathcal{O}_5, \end{array}$$

where, now, $\mathcal{O}_j$ refers to $\mathcal{O}(\theta^j)$.

And finally,

$$\langle \vec{t}\cos\theta + \vec{n}\sin\theta, \vec{t}(s_1) \rangle = 1 - \theta^2/2 + 2\rho'\theta^3/3 + \left(2\rho\rho''/3 - 6(\rho')^2/9 + 1/24\right)\theta^4 + \mathcal{O}_5.$$

On the other hand, $\cos(\theta_1) = 1 - \theta_1^2/2 + \theta_1^4/24$.

Since $\theta_1 = \theta + \beta_2\theta^2 + \beta_3\theta^3 + \mathcal{O}_4$,

$$\cos(\theta_1) = 1 - \theta^2/2 - \beta_2\theta^3 + \left(1/24 - \beta_3 - \beta_2^2/2\right)\theta^4 + \mathcal{O}(\theta^5).$$

And we can obtain coefficients β_2 and β_3 from

$$\begin{array}{ll} \theta^0: & 1 = 1, \\ \theta^1: & 0 = 0, \\ \theta^2: & 1/2 = 1/2, \\ \theta^3: & -\beta_2 = (2/3)\rho' \qquad \Rightarrow \qquad \beta_2 = -(2/3)\rho', \\ \theta^4: & -\beta_3 - (2/9)(\rho')^2 + 1/24 = (2/3)\rho\rho'' - (6/9)(\rho')^2 + 1/24 \\ & \qquad\qquad \Rightarrow \quad \beta_3 = -(2/3)\rho\rho'' + (4/9)(\rho')^2. \end{array}$$

And we have found

$$\theta_1 = \theta - 2\rho'\theta^2/3 + \left(-2\rho\rho''/3 + 4(\rho)'^2/9\right)\theta^3 + \mathcal{O}_4.$$

As it can be seen, this process can be done up to any k we want on the formula (43). We have computed the coefficients up to $k = 3$ and proved equations (28).

Appendix B
Elliptic functions

In this appendix we state the main results on elliptic functions that we have used in the work. Books [**3**, §16.] and [**27**, §XX, XXII] are the main sources.

Definition B.1. Let $f : \mathbb{C} \to \mathbb{C}$ be a periodic function with periods $2\omega_1$ and $2\omega_2$, that is,

$$f(z) = f(z + 2\omega_1) = f(z + 2\omega_2) \qquad \forall z \in \mathbb{C}.$$

If ω_1, $\omega_2 \in \mathbb{C}$ are such that $\omega_1/\omega_2 \notin \mathbb{R}$, we say that f is a *doubly-periodic function*.

Definition B.2. If $f : \mathbb{C} \to \mathbb{C}$ is a doubly-periodic function which is holomorphic in $\mathbb{C}$ but in a set of poles, we say that f is an *elliptic function*.

Definition B.3. Let f be an elliptic function with periods $2\omega_1$ and $2\omega_2$, a *fundamental period-parallelogram* is a parallelogram with vertices z, $z+2\omega_1$, $z+2\omega_2$ and $z + 2\omega_1 + 2\omega_2$, $z \in \mathbb{C}$.

Definition B.4. A *cell* is a fundamental period-parallelogram such that there are neither poles nor zeros on the boundaries of the parallelogram.

Observe that if we know the values of f in a fundamental period-parallelogram, we know the value of f everywhere. We say that $t, t' \in \mathbb{C}$ are *congruent*, if $t' = t+2m\omega_1+2n\omega_2$ for some $m, n \in \mathbb{Z}$. Then, it is clear that for any $t' \in \mathbb{C}$ there always exists t in the fundamental period-parallelogram congruent with t' and applying the doubly-periodicity of f we have $f(t') = f(t)$.

Following the same reasoning, if we specify the set of poles (or zeros) on a cell, we know the position of all the poles (zeros) on the plane. Observe that we are specifying that the parallelogram must be a cell because we omit the problem of having poles (zeros) on the boundaries of the parallelogram. In fact, this is why the concept "cell" is defined.

Definition B.5. The set of poles (zeros) of an elliptic function in a cell is called *an irreducible set* of poles (zeros).

In what follows, we describe some simple properties common to all elliptic functions and we sketch the proofs.

Proposition B.1. *The number of poles of an elliptic function in any cell is finite.*

PROOF. Otherwise, one could construct a sequence of poles in the cell converging to a point which, by construction, would not be isolated. Thus it would be an essential singularity. And this would contradict the fact that all singularities on an elliptic function are poles. □

PROPOSITION B.2. *The number of zeros of an elliptic function in any cell is finite.*

PROOF. We are omitting the case f equal to a constant. We apply a reasoning similar to the previous one to the reciprocal function. If the number of zeros of our function is infinite, the number of poles of the reciprocal function is infinite. Then, there must exist an essential singularity which is also an essential singularity of the initial function, which contradicts the fact that the function is elliptic. □

PROPOSITION B.3. *The sum of the residues of an elliptic function at its poles in any cell is zero.*

PROOF. Let $f(z)$ be an elliptic function and B the boundary of the cell. The sum of residues at its poles inside B is S and we can compute it by

$$\begin{aligned}
2\pi\mathrm{i}\,S &= \int_B f(t)\,\mathrm{d}t = \left(\int_{z+2\omega_2}^{z} + \int_{z}^{z+2\omega_1} + \int_{z+2\omega_1}^{z+2\omega_1+2\omega_2} + \int_{z+2\omega_1+2\omega_2}^{z+2\omega_2}\right) f(t)\,\mathrm{d}t \\
&= \left(\int_{z+2\omega_2}^{z} + \int_{z}^{z+2\omega_1}\right) f(t)\,\mathrm{d}t + \int_{z}^{z+2\omega_2} f(t+2\omega_1)\mathrm{d}t + \int_{z+2\omega_1}^{z} f(t+2\omega_2)\,\mathrm{d}t \\
&= \left(\int_{z}^{z+2\omega_1} (f(t) - f(t+2\omega_1))\,\mathrm{d}t\right) + \left(\int_{z}^{z+2\omega_2} (f(t+2\omega_2) - f(t))\,\mathrm{d}t\right) \\
&= 0.
\end{aligned}$$

□

PROPOSITION B.4. *An elliptic function with no poles in a cell is a constant.*

PROOF. Let $f(z)$ be an elliptic function. If it has no poles, $f(z)$ is analytic. If it is analytic, it is bounded in the cell and on the boundaries. Thus, it is bounded on $\mathbb{C}$ and by Liouville Theorem, $f(z)$ is constant. □

PROPOSITION B.5. *Two elliptic functions with the same periods, poles and principal parts differ only on a constant term.*

PROOF. Let $f(z)$ and $g(z)$ be elliptic functions. Since both functions, have same periods, we can find a cell were both are well defined. Since they share the poles and the principal parts, the difference $f(z)-g(z)$ is analytic in the cell and arguing as the previous proposition, it is constant. □

PROPOSITION B.6. *The sum of the coordinates of an irreducible set of zeros is congruent to the sum of the coordinates of an irreducible set of poles.*

PROOF. Let $f(z)$ be an elliptic function and B the boundary of a cell. We recall the following version of the argument principle [**27**, §6.31]. Let $g(z)$ be an analytic function inside and on B. Then,

$$2\pi \mathrm{i} \int_B g(z)\frac{f'(z)}{f(z)}\mathrm{d}t = \sum_{i=1}^{nz} r_i g(a_i) - \sum_{i=1}^{np} s_i g(b_i), \tag{47}$$

where $\{a_1, \cdots, a_{nz}\}$ are all the different zeros of $f(z)$ in the cell, $\{b_1, \cdots, b_{np}\}$ are all the different poles of $f(z)$ in the cell and r_i and s_i are the corresponding multiplicities.

Taking $g(z) = z$, we are evaluating the difference between the sum of the coordinates of an irreducible set of zeros and the sum of coordinates of an irreducible set of poles. Therefore, we must see that the integral at (47), with $g(z) = z$, is congruent to zero, that is, there exist $m, n \in \mathbb{Z}$ such that

$$\frac{1}{2\pi \mathrm{i}} \int_B t\frac{f'(t)}{f(t)}\,\mathrm{d}t = 2m\omega_1 + 2n\omega_2.$$

Proceeding just as Proposition B.3 we obtain

$$\begin{aligned} \frac{1}{2\pi \mathrm{i}} \int_B t\frac{f'(t)}{f(t)}\,\mathrm{d}t &= = \frac{1}{2\pi \mathrm{i}}\left(-2\omega_2\left[\log(f(t))\right]_z^{z+2\omega_1} + 2\omega_1\left[\log(f(t))\right]_z^{z+2\omega_2}\right) \\ &= \frac{1}{2\pi \mathrm{i}}\left(-2\omega_2(-n)\pi \mathrm{i} + 2\omega_1 \cdot m\pi \mathrm{i}\right) = 2m\omega_1 + 2n\omega_2 \end{aligned}$$

for some $m, n \in \mathbb{Z}$. □

Proposition B.7. *Let f be an elliptic function. The number of zeros of an equation $f(z) = c$ depends only on f and it is equal to the number of poles of f.*

Proof. In order to count the difference between the number of zeros and poles, we apply formula (47), with $g(z) = 1$ and $f(z) - c$. Thus, we have to compute

$$\frac{1}{2\pi \mathrm{i}} \int_B \frac{f'(t)}{f(t) - c}\,\mathrm{d}t.$$

Taking into account that $f'(z) = f'(z + 2\omega_1) = f'(z + 2\omega_2) = f'(z + 2\omega_1 + 2\omega_2)$, we can proceed just as the proof of Proposition B.3 and the integral equals to 0. Then, the number of zeros of funtion $f(z) - c$ is equal to the number of poles of $f(z) - c$. But, any pole of $f(z)$ is a pole of $f(z) - c$ and conversely. □

This result allows us to define the following concepts.

Definition B.6. The *order* of an elliptic function is the number of roots of the equation $f(z) = c$, for any $c \in \mathbb{C}$.

Corollary B.8. *The order of an elliptic function is at least 2.*

Proof. An elliptic function of order 1 would have an single irreducible pole. Therefore its residue could not be zero, contradicting Proposition B.3. □

Therefore, the simplest elliptic functions are of order 2. They can be classified into two classes.

Definition B.7. An elliptic function of order 2 which has a single irreducible double pole is called a *Weierstrassian elliptic function.*

Definition B.8. An elliptic function of order 2 which its irreducible set consists on two simple poles is called a *Jacobian elliptic function.*

Observe that, using Proposition B.3 again, the residues at both poles in the same cell of a Jacobian elliptic function are numerically equal but opposite in sign while the residue at the double irreducible pole on a Weierstrassian elliptic function must be equal to zero.

We are interested in the Jacobian elliptic functions.

B.1. Jacobian elliptic functions

We first define all the Jacobian functions in a different way of the previous one. It can be seen that both ways are equivalent. In fact, we will define them with more properties than before, but it can be proved there are no more Jacobian elliptic functions than the ones defined as below.

We call $k \in (0,1)$ *the modulus.*

The *quarter-periods* are K and $\mathrm{i}\,K'$, where K is

$$K = K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \tag{48}$$

and it is called the *complete elliptic integral of first class* and K' is

$$K' = K'(k) = K(\sqrt{1-k^2}). \tag{49}$$

Proposition B.9. *There exist bijections between k and K, K and K' and also between k and K'.*

We are going to define any Jacobian elliptic functions by describing its poles and double-periods and some more properties that they satisfy.

Consider the parallelogram with vertices $\mathrm{s} = 0$, $\mathrm{c} = K$, $\mathrm{n} = \mathrm{i}\,K'$ and $\mathrm{d} = K + \mathrm{i}\,K'$.

Definition B.9. Let $\mathrm{p}, \mathrm{q} \in \{\mathrm{s}, \mathrm{c}, \mathrm{d}, \mathrm{n}\}$ and $\mathrm{p} \neq \mathrm{q}$. The Jacobian elliptic function $\mathrm{pq}\, u$ is defined by the following properties.

(i) pq has a simple zero at p and a simple pole at q.
(ii) The steps from p to q (clockwise on the parallelogram) is a half-period of $\mathrm{pq}\, u$. Those numbers K, $\mathrm{i}\,K'$, $K + \mathrm{i}\,K'$ which differ from the one being a half-period are only quarter-periods.
(iii) The leading term about $u = 0$ is u, u^{-1} or 1 according as $u = 0$ is a zero, a pole or an ordinary point respectively.

From (48) and (49), it is clear that all the elliptic functions depend on the parameter k. We can make this dependence explicit, when necessary, by using the notation $\mathrm{pq}\,(u,k)$ instead of $\mathrm{pq}\, u$.

Observe that there exist 12 types of Jacobian elliptic functions.

Observe that the Jacobian elliptic functions are indeed elliptic functions in the sense of Definition B.2: we have found doubly-periodic functions (periods $4K$ and $4\mathrm{i}\,K'$) analytic except for a set of poles. As we have stated before, there are as much poles as zeros for any Jacobian function $\mathrm{pq}\,u$.

Note that we can choose the fundamental period-parallelogram to be s, $\mathrm{s}+4K$, $\mathrm{s}+4\mathrm{i}\,K'$ and $\mathrm{s}+4K+4\mathrm{i}\,K'$. It is not a cell because poles or zeros can be found on its boundaries. Nevertheless, we can infer from this definition that on each cell there will be two poles and two zeros, and using Proposition B.7, we have that Jacobian elliptic functions are of order two.

Another property of the Jacobian elliptic functions that can be observed from the definition is the following.

Corollary B.10. *Functions* $\mathrm{pq}\,u$ *with a pole or zero at the origin (that is* $\mathrm{p}=\mathrm{s}$ *or* $\mathrm{q}=\mathrm{s}$*) are odd. All the other functions* $\mathrm{pq}\,u$ *are even.*

Still, there exists another way of defining the Jacobian elliptic functions. This last way is related to a certain type of integrals.

Definition B.10. Let $k\in(0,1)$. The *incomplete elliptic integral of the first kind*, which is

$$u=\int_0^{\varphi}\frac{\mathrm{d}\phi}{\sqrt{1-k^2\sin^2\phi}},\qquad \varphi\in[0,2\pi).$$

There exists a local bijection between u and φ.

Definition B.11. $\varphi=\varphi(u)$ is called the *amplitude*, $\varphi=\mathrm{am}\,u$.

Then, the following definitions coincide with Definition (B.9).

Definition B.12. The *elliptic sinus* is $\mathrm{sn}\,u=\sin\varphi$.

Definition B.13. The *elliptic cosinus* is $\mathrm{cn}\,u=\cos\varphi$.

Definition B.14. $\mathrm{dn}\,u=\sqrt{1-k^2\sin^2\varphi}$.

Observe that with these definitions we obtain that $u=K$ if and only if $\varphi=\pi/2$ and that its name, "quarter-period", is convenient for the definition of the elliptic sinus and cosinus.

Corollary B.11. *We obtain* $\mathrm{sn}\,(u,0)=\sin u$*,* $\mathrm{cn}\,(u,0)=\cos u$ *and* $\mathrm{dn}\,(u,0)=1$. *And we also obtain* $\mathrm{sn}\,(u,1)=\tanh u$*,* $\mathrm{cn}\,(u,1)=\mathrm{sech}\,u$ *and* $\mathrm{dn}\,(u,1)=\mathrm{sech}\,u$*.*

Next, we remark the positions of the poles and zeros of functions $\mathrm{sn}\,u$ and $\mathrm{cn}\,u$.

B.2. Elliptic sinus

- The elliptic sinus, according to its definition, has periods $2\mathrm{i}\,K'$, $4(K+\mathrm{i}\,K')$ and $4K$.

- It has a pole at $\mathrm{n} = \mathrm{i}\,K'$. The function $\operatorname{sn} u$ is $2\mathrm{i}\,K'$-periodic and $2K$-antiperiodic (it follows from its definition). It is also of order two, and thus, the set of poles of $\operatorname{sn} u$ is
$$\mathrm{i}\,K' + 2K\mathbb{Z} + 2\mathrm{i}\,K'\mathbb{Z}.$$
- Analogously, and since $u = \mathrm{s} = 0$ is a zero of $\operatorname{sn} u$, its zeros are found in the set
$$2K\mathbb{Z} + 2\mathrm{i}\,K'\mathbb{Z}.$$
- Since the elliptic sinus is a function of order two, it has two simple poles in each cell. The residues on a cell must be zero (see Proposition B.3) and, thus, the residue on one of the poles is the opposite to the other pole of the same cell. Since the residue at pole $q = \mathrm{i}\,K'$ is $\operatorname{Res}(\operatorname{sn} u, \mathrm{i}\,K') = 1/k$, all the poles congruent to it have the same residue. The others have just the opposite residue. Thus, a pole at position $2rK + (2s+1)\mathrm{i}\,K'$ has a residue
$$\operatorname{Res}(\operatorname{sn} u; 2rK + (2s+1)\mathrm{i}\,K') = \frac{(-1)^r}{k}.$$

B.3. Elliptic cosinus

- The elliptic cosinus, according to its definition, has periods $2(K + \mathrm{i}\,K')$, $4K$ and $4\mathrm{i}\,K'$.
- It has a pole at $\mathrm{n} = \mathrm{i}\,K'$. It is $2(K + \mathrm{i}\,K')$-periodic and of order two. Thus, any pole of $\operatorname{cn} u$ belongs to the set
$$\mathrm{i}\,K' + 2K\mathbb{Z} + 2\mathrm{i}\,K'\mathbb{Z}.$$
- Analogously, since $u = \mathrm{c} = K$ is a zero of $\operatorname{cn} u$, its zeros are found in the set
$$K + 2K\mathbb{Z} + 2\mathrm{i}\,K'\mathbb{Z}.$$
- Just as we reasoned before to obtain the residues at poles of $\operatorname{sn} u$, we obtain the residues at poles of $\operatorname{cn} u$,
$$\operatorname{Res}(\operatorname{cn} u; 2rK + (2s+1)\mathrm{i}\,K') = \frac{(-1)^{r+s+1}\mathrm{i}}{k}.$$