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**Extremal problems in Fourier analysis, Whitney's theorem, and the interpolation of data**

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# Extremal problems in Fourier analysis, Whitney's theorem, and the interpolation of data

**by**

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This thesis is dedicated to my parents, and to all of the teachers I've had throughout my life, especially my high school math teachers Charlotte Hennessy and Brian Lundell.

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**Extremal problems in Fourier analysis, Whitney's theorem, and the interpolation of data**

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This dissertation deals with three problems in interpolation theory. The first two, the Beurling-Selberg box minorant problem and Turán's extremal problem, are optimization problems involving constrained interpolation by bandlimited functions. The Beurling-Selberg box minorant problem is a higher dimensional version of Selberg's minorant problem for the interval. We study the problem of minorizing the indicator function of the unit cube $Q_d = [-1, 1]^d$ by a function bandlimited to $Q_d$. We show that there exists a dimension $d^* \leq 710$ such that if $d > d^*$ then there do not exist $d$-dimensional minorants. We also construct the first non-trivial minorants for dimensions $2, 3, 4$, and 5. Next, we show how to compute upper and lower bounds for the value of Turán's extremal problem by solving finite dimensional linear programs. The problem depends on a convex body $K$; our bounds have been used to compute the sharpest known upper bound in the case in which $K$ is the 3 dimensional $\ell_1$ ball. The third problem we study concerns the interpolation

of data by $C^m$ functions. We give a new proof of the Brudnyi-Shvartsman-Fefferman finiteness principle for $C^{m-1,1}(\mathbb{R}^d)$ functions. We hope that this proof will lead to practical algorithms for $C^m$ interpolation.

# Table of Contents

# Chapter 1

# Introduction

This dissertation is concerned with two areas of interpolation theory: the constrained interpolation of data by bandlimited functions and the interpolation of data by smooth functions.

In Chapters 2 and 3 we consider, respectively, the Beurling-Selberg box minorant problem and Turán's extremal problem, both of which fall into the first category listed above. Since these problems involve finding a constrained, bandlimited interpolant for a set of data we will refer to them as CBI problems. The general problem of finding a bandlimited interpolant for a set of data, i.e. the unconstrained version of the general CBI problem, is well understood. See, for example, the classical work of Shannon [61] and Landau [52]; many others have made important contributions. Additionally, there are specific CBI problems which are well understood. For some examples see the work of Beurling [3], Carneiro et al. [16, 18, 19], Cohn and Elkies [28], Donoho and Logan [32], Montgomery [55], Siegel [62], Selberg [60], and Vaaler et al [46, 50, 64]. There is, however, no theory which attempts to give a unified treatment of CBI problems. For this reason, at the end of Chapter 3 we discuss the connection between the linear programming bounds proved in Chapters 2 and 3 and how similar bounds can be proved for a larger class of problems.

Chapter 2 contains joint work with Noam Elkies (Harvard), Felipe Goncalves (Hausdorff Center for Mathematics, Germany), and Michael Kelly (Center for Communications Research, Princeton). The results are taken from [23]. The Beurling-Selberg box minorant problem is the problem of maximizing the integral of a function which minorizes (i.e. is pointwise less than or equal to) the indicator function of the unit cube and is bandlimited to the unit cube. Selberg solved this problem in one dimension in the 70s [60]. Our first result shows that Selberg's method fails to produce functions with nonnegative integral in dimension two or higher; we give the first construction of such minorants in dimensions 2 through 5. Additionally, our work shows that there is a finite dimension $d^* \leq 710$ such that if $d > d^*$ there exist no nonnegative $d$-dimensional minorants. This last result is proved by bounding a related CBI optimization problem using the linear programming bounds mentioned above. Finally, we discuss the implications of this work and future directions.

Chapter 3 includes results from joint work with Elahesadat Naghib (Princeton) and Robert Vanderbei (Princeton) contained in [57]. Turán's problem for a given convex body $K$ asks for the maximum integral of a function which is supported in $K$, takes the value one at the origin, and is positive definite (i.e. has nonnegative Fourier transform). Our work shows how to compute upper and lower bounds for the optimal value of this problem by solving a finite dimensional linear program. There is a well known conjecture [59] that the extremal value is $\text{vol}(K)/2^d$. In [57], we use our bounds to verify the conjectured value for the 3 dimensional $\ell^1$ ball to within a multiplicative factor of 1.0060. Finally, we give an alternate proof of the linear programming bounds in Chapter 2 to emphasize the connection to the proof of the upper bound in Chapter 3 and discuss how it can be generalized.

The second area of interpolation theory we are interested in, smooth interpolation of data, is the subject of Chapter 4. This is a classical subject and dates back to work of Whitney. In the last 15 years, Charles Fefferman has established a research program with the aim of creating a constructive theoretical framework for these problems and then using this work to develop algorithms which can be practically implemented on a computer. Much progress has been made toward achieving these two goals. Fefferman [33, 34, 35] has developed a theory for, among many other things, determining whether a function on an arbitrary set in $\mathbb{R}^d$ admits a $C^m$ extension and, if it does, constructing an interpolant. He and Klartag [38, 37, 36] have developed very efficient algorithms for implementing this construction on a computer. There is, however, an obstacle to using these results for practical applications.

One would hope that a function which interpolates a set of data does not impose extra information. A way to quantify this is by requiring that, for a given norm, the norm of the computed interpolant is not much bigger than the smallest norm of any interpolant. To motivate this idea, consider why we minimize the $\ell_2$ distance when computing a least squares approximation. The Fefferman-Klartag algorithm mentioned above computes a function which has norm within a constant multiple $C^{\#}$ (depending on $m$ and $d$) of the optimal norm. Unfortunately, $C^{\#}$, which is inherited from something called the finiteness principle for $C^{m-1,1}(\mathbb{R}^d)$, is extremely large for most values of $d$ and $m$.

The content of Chapter 4 is a new proof of the finiteness principle for $C^{m-1,1}(\mathbb{R}^d)$ with the goal of lowering $C^{\#}$. This is joint work with Abraham Frei-Pearson (University of Texas at Austin), Arie Israel (University of Texas at Austin), and Boaz Klartag (Weizmann Institute of Science, Israel) and the results are taken from [24]. One of the reasons that $C^{\#}$ is so large is that Fefferman's proof is carried out via an induction with roughly $2^D$ steps, where $D = \binom{d+m-1}{d}$. Each step of the induction increases the constant $C^{\#}$ by a multiplicative factor. Our proof is by induction on a different quantity, which we call the complexity of the underlying set. Roughly speaking, the complexity of a set $E$ is the number of times the geometry of $E$ changes significantly as we adjust the scale at which we view $E$. There is evidence that this quantity is much smaller than $2^D$ and could therefore lead to lower constants and, eventually, to practical algorithms.

# Chapter 2

# The Beurling-Selberg Box Minorant Problem

## 2.1 Introduction

In the 1970s, in order to prove the large sieve inequality, Selberg [60] introduced a pair of bandlimited functions. Suppose $I \subset \mathbb{R}$ is an interval of finite length, $\delta > 0$, and use $\mathbf{1}_I(x)$ to denote the indicator function of $I$. Then Selberg's functions $M(x)$ and $m(x)$ satisfy the following properties.

*(i)* $\widehat{M}(\xi) = \widehat{m}(\xi)$ if $|\xi| > \delta$

*(ii)* $m(x) \leq \mathbf{1}_I(x) \leq M(x)$ for all $x \in \mathbb{R}$

*(iii)* $\int_{-\infty}^{\infty} (M(x) - \mathbf{1}_I(x))\, dx = \int_{-\infty}^{\infty} (\mathbf{1}_I(x) - m(x))\, dx = \delta^{-1}$

Among all functions satisfying *(i)* and *(ii)* above, Selberg's functions minimize the integrals appearing in *(iii)* if and only if $\delta \cdot \text{length}(I) \in \mathbb{Z}$. Littman [54] has computed the extremal functions for the cases $\delta \text{length} \notin \mathbb{Z}$.

Originally, Selberg was motivated to construct his one dimensional extremal functions to prove a sharp form of the large sieve. His functions and their generalizations have since become part of the standard arsenal in analytic number theory and have a number of applications in fields ranging from probability, dynamical systems, optics, combinatorics, sampling theory, and beyond.

For a non-exhaustive list see [10, 11, 12, 13, 14, 15, 17, 20, 21, 22, 25, 40, 53, 56] and the references therein.

In recent years higher dimensional analogues of Selberg's extremal function and related constructions have proven to be important in the recent studies of Diophantine inequalities [2, 26, 49, 50], visibility problems and quasicrystals [1, 49], and sphere packings[1] [30, 27, 28, 29, 31, 66]. See also [9] for related constructions recently used in signal processing. Since Selberg's original construction of his box minorants there has been some progress on the Beurling-Selberg problem in higher dimensions [2, 4, 16, 18, 42]. In particular, in [50] Holt and Vaaler initiate the study of a variant of Question 1 in which the boxes are replaced by Euclidean balls. They are actually able to establish extremal results in some cases. A complete solution to Question 1 for balls can be found in [41]. There seems to be a consensus among experts that despite four decades of progress on Beurling-Selberg problems, box minorants are poorly understood. This sentiment was recently raised in [44]. This motivates us to consider the following the question.

**Question 1.** *Does there exist a function $F : \mathbb{R}^d \to \mathbb{R}$ such that:*

*(i)* $F(\boldsymbol{x}) \leq \mathbf{1}_{[-1,1]^d}(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \mathbb{R}^d$*;*

*(ii)* *the Fourier transform of* $F(\boldsymbol{x})$ *is supported in the box* $[-\delta, \delta]^d$*;*

[1]The extremal problems considered for sphere packings differ from the problems that we consider here. Instead of the admissible functions being band-limited, their Fourier transforms are only required to be non-negative.

*(iii)* $\int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x} > 0$*?*

Basic considerations will lead the reader to surmise that the existence of such a function depends on the size of $\delta$. If $\delta$ is very large, then such a function will surely exist. On the other hand, if $\delta$ is very small, then no such function ought to exist. When $d = 1$ the above question was settled by Selberg [60, 64] who showed that there is a positive answer to Question 1 if and only if $\delta > \frac{1}{2}$. From here it is not difficult to show that when $d > 1$, Question 1 has a negative answer whenever $\delta \leq \frac{1}{2}$ (see Lemma 14). When $d$ is large it is unknown how small $\delta$ may be for Question 1 to admit a positive answer. The best result in this direction is due to Selberg who proved that when $d > 1$ and $\delta > d - \frac{1}{2}$, then Question 1 has a positive answer. Selberg never published his construction, but he did communicate it to Vaaler and Mongtomery (personal communication). His construction has since appeared several times in the literature, see for instance [47, 48, 49]. More details about Selberg's (and also Montgomery's) construction can be found in Section 2.7.

The following is the main theorem of this paper.

**Theorem 2.** *If $d > 717$ and $\delta = 1$ then Question 1 has a negative answer. In contrast, Question 1 has a positive answer for $\delta = 1$ in dimension $d = 1, 2, 3, 4, 5$.*

The proof of this, as well as our other main results, are based on a detailed analysis of the following extremal problem and a novel technique to bound the objective of this infinite dimensional linear program by the objective of a finite dimensional program (Theorems 4 and 8).

**Question 3.** *For every integer $d \geq 1$ determine the value of the quantity*

$$\nu(N) = \sup \int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x}, \tag{2.1}$$

*where the supremum is taken over functions $F \in L^1(\mathbb{R}^d)$ such that:*

*(i)* $\widehat{F}(\boldsymbol{\xi})$ *is supported in* $[-1,1]^d$*;*

*(ii)* $F(\boldsymbol{x}) \leq \mathbf{1}_{[-1,1]^d}(\boldsymbol{x})$ *for (almost) every* $\boldsymbol{x} \in \mathbb{R}^d$*.*

We show that the admissible minorants are given by an interpolation formula for bandlimited functions similar to the classical Shannon sampling theorem [61]. We then use this formula to demonstrate that the only admissible minorant with non-negative integral that interpolates the indicator function $\mathbf{1}_{[-1,1]^d}(\boldsymbol{x})$ at the integer lattice $\mathbb{Z}^d \setminus \{\mathbf{0}\}$ is the identically zero function. We also define an auxiliary quantity $\Delta(d)$ in (2.8), similar to $\nu(d)$, and derive a functional inequality, which ultimately implies that $\nu(d)$ vanishes for finite $d$.

We hope that the contributions of this research will help reveal why the box minorant problem is so difficult and move us closer to understanding these enigmatic objects.

## 2.2 Main Results

In this section we give some definitions and state the main results of the present article. A function $F(\boldsymbol{x})$ satisfying conditions (i) and (ii) of Question

3 will be called admissible for $\nu(d)$ (or $\nu(d)$-admissible) and if it achieves equality in (2.1), then it is said to be extremal.

An indispensable tool in our investigation is the Poisson summation formula. If $G : \mathbb{R}^d \to \mathbb{R}$ is "sufficiently nice" (see [63] for a precise statement of when the formula holds), $\Lambda$ is a full rank lattice in $\mathbb{R}^d$ of covolume $|\Lambda|$, and $\Lambda^*$ is the corresponding dual lattice[2], then the Poisson summation formula is the assertion that

$$\sum_{\boldsymbol{\lambda}\in\Lambda} G(\boldsymbol{x}+\lambda) = \frac{1}{|\Lambda|}\sum_{\boldsymbol{u}\in\Lambda^*} \widehat{G}(\boldsymbol{u})e^{2\pi i\boldsymbol{u}\cdot\boldsymbol{x}}, \tag{2.2}$$

for every $\boldsymbol{x} \in \mathbb{R}^d$.

If $F(\boldsymbol{x})$ is a $\nu(d)$-admissible function, then it follows from Proposition 13 that the Poisson summation formula may be applied to $F(\boldsymbol{x})$. That is, $\nu(d)$-admissible functions are "sufficiently nice." Thus, upon applying Poisson summation (2.9) to $F(\boldsymbol{x})$ we find that

$$\widehat{F}(\mathbf{0}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \widehat{F}(\boldsymbol{n}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} F(\boldsymbol{n}) \le F(\mathbf{0}).$$

Thus we have the fundamental inequality

$$\widehat{F}(\mathbf{0}) \le F(\mathbf{0}). \tag{2.3}$$

Evidently there is equality in (2.3) if, and only if, $F(\boldsymbol{n}) = 0$ for each non-zero $\boldsymbol{n} \in \mathbb{Z}^d$. If $d = 1$ then, by using the interpolation formula (2.10), Selberg was

[2] That is, $\Lambda^* = \{\boldsymbol{u} \in \mathbb{R}^d : \boldsymbol{u}\cdot\boldsymbol{\lambda} \in \mathbb{Z} \text{ for all } \boldsymbol{\lambda} \in \Lambda\}$.

able to show (see [60, 64]) that $\nu(1) = 1$ and that

$$\frac{\sin^2 \pi x}{(\pi x)^2(1-x^2)}$$

is an extremal function (this is not the unique extremal function). We also note that the Fourier transform of the above function is non-negative, supported in $|x| \leq 1$ and equal to

$$1 - |x| + \frac{\sin(2\pi|x|)}{2\pi}.$$

Therefore, Selberg's function is also extremal for the Cohn and Elkies [28] linear programming bounds for sphere packings in dimension 1 (again not unique).

A more refined version of the inequality (2.3) can be obtained by taking a weighted average of the Poisson summation formula on grids. More precisely, suppose that $\Lambda \subset \mathbb{R}^d$ is a full-rank lattice, $y_1, ..., y_L \in \mathbb{R}^d$, and $\omega_1, ..., \omega_L \geq 0$. By repeatedly applying (2.2) and interchanging the order of summation, we find that

$$\sum_{\ell=1}^{L} \omega_\ell \sum_{\boldsymbol{\lambda} \in \Lambda} F(\boldsymbol{\lambda} + \boldsymbol{y}_\ell) = \frac{1}{|\Lambda|} \sum_{\boldsymbol{u} \in \Lambda^*} \widehat{F}(\boldsymbol{u}) \sum_{\ell=1}^{L} \omega_\ell e^{-2\pi i \boldsymbol{u} \cdot \boldsymbol{y}_\ell}. \tag{2.4}$$

Suppose that

$$\sum_{\ell=1}^{L} \omega_\ell e^{-2\pi i \boldsymbol{u} \cdot \boldsymbol{y}_\ell} = 0, \quad \text{if } \boldsymbol{u} \in \Lambda^* \cap Q_d \setminus \{\boldsymbol{0}\}, \tag{2.5}$$

and

$$\sum_{\ell=1}^{L} \omega_\ell = |\Lambda|. \tag{2.6}$$

If $F(\boldsymbol{x})$ is $\nu(d)$-admissible, then (2.4) yields the following strengthening of (2.3):

$$\widehat{F}(\mathbf{0}) \leq \sum_{\ell=1}^{L} \sum_{\substack{\boldsymbol{\lambda} \in \Lambda \\ \|\boldsymbol{\lambda}+\boldsymbol{y}_\ell\|_\infty < 1}} \omega_\ell. \tag{2.7}$$

Since the right hand side of (2.7) is a finite sum that is linear in $\omega_1, ..., \omega_L$, we have the following finite dimensional linear programming bounds for $\nu(d)$.

**Theorem 4.** *Suppose $\boldsymbol{y}_1, ..., \boldsymbol{y}_L \in \mathbb{R}^d$ and that $\Lambda$ is a full rank lattice in $\mathbb{R}^d$ of covolume $|\Lambda|$. Then*

$$\nu(d) \leq \min \sum_{\ell=1}^{L} \sum_{\substack{\lambda \in \Lambda \\ \|\boldsymbol{\lambda}+\boldsymbol{y}_\ell\|_\infty < 1}} \omega_\ell$$

*where the minimum is taken over $\omega_1, ..., \omega_L \geq 0$ satisfying* (2.5) *and* (2.6).

If there exists a lattice $\Lambda$ and vectors $\boldsymbol{y}_1, ..., \boldsymbol{y}_L$ such that extremal function vanishes at the points $\boldsymbol{y}_\ell + \lambda$ for which $||\boldsymbol{y}_\ell + \lambda||_\infty \geq 1$ and equals one at the points for which $||\boldsymbol{y}_\ell + \lambda||_\infty < 1$, then the above inequality is sharp. This is the case in one dimension, with the lattice $\mathbb{Z}$ and the vector $\boldsymbol{y}_1 = 0$. This leads us to conjecture that this also holds in higher dimensions.

**Conjecture 5.** *Assume $\nu(d) > 0$. Then for any $\varepsilon > 0$ there exists a full rank lattice $\Lambda \subset \mathbb{R}^d$, vectors $\boldsymbol{y}_1, ..., \boldsymbol{y}_L \in \mathbb{R}^d$ and numbers $\omega_1, ..., \omega_L \geq 0$ satisfying* (2.5) *and* (2.6) *such that*

$$\nu(d) + \varepsilon > \sum_{\ell=1}^{L} \sum_{\substack{\lambda \in \Lambda \\ \|\boldsymbol{\lambda}+\boldsymbol{y}_\ell\|_\infty < 1}} \omega_\ell.$$

The following theorem compiles some of the basic properties related to the quantity $\nu(d)$, establishing: (1) that extremizers for the quantity $\nu(d)$ do exist, (2) that $\nu(d)$ is a decreasing function of $d$ and, most curiously, (3) that $\nu(d)$ vanishes for finite $d$.

**Theorem 6.** *The following statements hold.*

$(i)$ *For every $d \geq 2$ there exists a $\nu(d)$-admissible function $F(\boldsymbol{x})$ such that*

$$\nu(d) = \int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x}.$$

$(ii)$ *If $\nu(d) > 0$ then $\nu(d+1) < \nu(d)$. In particular, $\nu(2) < 1$.*

$(iii)$ *There exists a critical dimension $d_c$ such that $\nu(d_c) > 0$ and $\nu(d) = 0$ for all $d > d_c$. Moreover,*

$$5 \leq d_c \leq \left\lfloor \frac{k}{1 - \Delta(k)} \right\rfloor.$$

*for any $k \leq d_c$.*

**Remarks.**

$(i)$ Using Theorem 8 we were able to show that $\Delta(2) < .997212$, yielding an upperbound $d_c \leq 717$ (see Table **??**).

$(ii)$ The quantity $\Delta(k)$ appearing in the above theorem is defined in equation (2.8). It follows from Lemma 18 that $k \mapsto k/(1-\Delta(k))$ is non-increasing for $k \leq d_c$, and from Theorem 9 that $\Delta(k) < 1$ for all $k \geq 2$. Thus producing upper bounds for $\Delta(k)$ in higher dimensions will improve the critical dimension $d_c$, however the problem quickly becomes incredibly hard as the dimension increases, demanding a huge amount of computational time to deliver an upper bound strictly less than one. That is why we were only able to produce upper bounds up to dimension 5. Moreover, the above result can only be applied for $k \leq d_c$ and so far we do not known if $\nu(6) > 0$, thus to use the upper bound derived above in a dimension higher than 5, we have also to find a non-trivial minorant in such dimension.

*(iii)* To put this result in context, note that volume of $Q_d$ is growing exponentially, so there is a lot of volume on both the physical and frequency sides. However, every time another dimension gets added, more constraints also get added so it requires a detailed analysis to determine the behavior of $\nu(d)$. Poisson summation, which yields the non-intuitive bound $\nu(d) \leq 1$, already detects this tug-of-war.

*(iv)* Theorem 6 has some parallels in classical asymptotic geometric analysis, and mass concentration in particular. In our first attempts to prove Theorem 6 we tried to employ asymptotic geometric techniques to exploit properties of $Q_d$ but we were not able to uncover a proof. We found it awkward to incorporate the Fourier analytic and one-sided inequality constraints (i.e. (i) and (ii) in the definition of $\nu(d)$) with the standard tool kit of asymptotic geometric analysis. It would be very interesting to see a proof of Theorem 6 based on such techniques.

Our next result shows that Selberg's $\mathbb{Z}^d$-interpolation strategy to build minorants fails in higher dimensions.

**Theorem 7.** *Let $d \geq 2$. Let $F(\boldsymbol{x})$ be an admissible function for $\nu(d)$ and assume that $F(\mathbf{0}) \geq 0$. If $F(\boldsymbol{n}) = 0$ for every $\boldsymbol{n} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, then $F(\boldsymbol{x})$ vanishes identically.*

We are also interested in studying a "scaled-out" version of the $\nu(d)$-problem defined as follows. Let

$$\Delta(d) = \sup_F \int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x}, \tag{2.8}$$

where the supremum if taken among functions $F(\boldsymbol{x})$ such that:

(i) $\widehat{F}(\boldsymbol{\xi})$ is supported in $Q_d$;

(ii) $F(\boldsymbol{x}) \leq 0$ for (almost) every $\boldsymbol{x} \notin Q_d$;

(iii) $F(\mathbf{0}) = 1$;

(iv) $\widehat{F}(\mathbf{0}) > 0$.

We have the following analogue of Theorem 4 for $\Delta(d)$.

**Theorem 8.** *Suppose $y_1 = \mathbf{0}$, $y_2, ..., y_L \in \mathbb{R}^d$ and that $\Lambda$ is a full rank lattice in $\mathbb{R}^d$ of covolume $|\Lambda|$. Then*

$$\Delta(d) \leq \min \omega_1$$

*where the minimum is taken over $\omega_1, ..., \omega_L \geq 0$ satisfying (2.5), (2.6), and for $\ell = 2, .., L$*

$$\omega_\ell = 0 \text{ if } \|\boldsymbol{\lambda} + y_\ell\|_\infty < 1 \text{ for some } \boldsymbol{\lambda} \in \Lambda.$$

The quantity $\Delta(d)$ may not be well defined for some $d$, in this case we define $\Delta(d) = 0$. Lemma 14 shows that if $\Delta(d_0)$ is well-defined for some $d_0$ (that is $\Delta(d_0) > 0$), then it is well defined for all $d \leq d_0$. One can also verify that $\Delta(d) > 0$ if and only if $\nu(d) > 0$ and

$$\nu(d) \leq \Delta(d).$$

Thus, they vanish for the first time at the same dimension. Poisson summation shows that $\Delta(d) \leq 1$ for all $d$ and thus $\Delta(1) = 1$. A priori, the existence of extremizers for the $\Delta(d)$ problem is not guaranteed since an extremizing sequence may blow-up inside the box $Q_d$. The next theorem shows that $\Delta(d)$ behaves similarly to $\nu(d)$ for $d \geq 2$.

**Theorem 9.** *The following statements hold.*

(i) *There exists a constant $B_d \geq 1$, depending only on d, such that if $F(\boldsymbol{x})$ is admissible for the $\Delta(d)$ problem then $F(\boldsymbol{x}) \leq B_d$ for all $x \in Q_d$.*

(ii) *If $\Delta(d) > 0$, then there exists a $\Delta(d)$-admissible function $F(\boldsymbol{x})$ such that*

$$\Delta(d) = \int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x}.$$

(iii) *If $\Delta(d) > 0$, then $\Delta(d+1) < \Delta(d)$. In particular, $\Delta(2) < 1$.*

(iv) *There exists a critical dimension $d_c$ such that $\Delta(d_c) > 0$ and $\Delta(d) = 0$ for all $d > d_c$. Moreover, the same bound holds*

$$5 \leq d_c \leq \left\lfloor \frac{k}{1 - \Delta(k)} \right\rfloor,$$

*for any $k \leq d_c$.*

We now give some explicit lower bounds for the quantity $\nu(d)$ up to dimension $d = 5$ (see Theorem (21)). These are constructed explicitly in Section 2.6.

**Theorem 10.** *We have the following lower bounds for $\nu(d)$:*

- $\nu(2) \geq \frac{63}{64} = 0.984375$,
- $\nu(3) \geq \frac{119}{128} = 0.9296875$,
- $\nu(4) \geq \frac{95}{128} = 0.7421875$,
- $\nu(5) \geq \frac{31}{256} = 0.12109375$.

## 2.3 Preliminaries

In this section we prove some crucial results and recall as well some basic facts about the theory of Paley-Wiener spaces and extremal functions.

For a given function $F : \mathbb{R}^d \to \mathbb{R}$ we define its Fourier transform as

$$\widehat{F}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} F(\boldsymbol{x}) e^{-2\pi i \boldsymbol{x}\cdot\boldsymbol{\xi}} d\boldsymbol{x}.$$

In this paper we will almost always deal with functions $F(\boldsymbol{x})$ that are integrable and whose Fourier transforms are supported in the box

$$Q_d = [-1, 1]^d.$$

For this reason, given $p \in [1, 2]$ we define $PW^p(Q_d)$ as the set of functions $F \in L^p(\mathbb{R}^d)$ such that their Fourier transform is supported in $Q_d$. By Fourier inversion these functions can be identified with analytic functions that extend to $\mathbb{C}^d$ as entire functions. The following is a special case of the generalization of the Paley-Wiener theorem that appears in [63].

**Theorem 11** (Stein, [63]). *Let $p \in [1,2]$ and $F \in L^p(\mathbb{R}^d)$. The following statements are equivalent:*

*(i)* $\operatorname{supp}(\widehat{F}) \subset Q_d$*;*

*(ii)* $F(\boldsymbol{x})$ *is a restriction to* $\mathbb{R}^d$ *of an analytic function defined in* $\mathbb{C}^d$ *with the property that there exists a constant* $C > 0$ *such that*

$$|F(\boldsymbol{x} + i\boldsymbol{y})| \leq C \exp\left[2\pi \sum_{n=1}^{d} |y_n|\right]$$

*for all* $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d$.

**Remark.** In particular this theorem implies that every function $F \in PW^1(Q_d)$ is bounded on $\mathbb{R}^d$, hence $PW^1(Q_d) \subset PW^2(Q_d)$.

**Theorem 12** (Pólya-Plancherel, [58]). *If $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, ...$ is a sequence in $\mathbb{R}^d$ satisfying that $\|\boldsymbol{\xi}_n - \boldsymbol{\xi}_m\|_\infty \geq \varepsilon$ for all $m \neq n$ for some $\varepsilon > 0$ then*

$$\sum_n |F(\boldsymbol{\xi}_n)|^p \leq C(p, \varepsilon) \int_{\mathbb{R}^N} |F(\boldsymbol{\xi})|^p d\boldsymbol{\xi}$$

*for every $F \in PW^p(Q_d)$.*

**Proposition 13** (Poisson Summation for $PW^1(Q_d)$). *For all $F \in PW^1(Q_d)$ and for any $\boldsymbol{t} \in \mathbb{R}^d$ we have*

$$\int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x} = \sum_{\boldsymbol{n} \in \mathbb{Z}^d} F(\boldsymbol{n} + \boldsymbol{t}), \tag{2.9}$$

*where the convergence is absolute and uniform on compact subsets of* $\boldsymbol{t} \in \mathbb{R}^d$*.*

Let $F \in PW^2(Q_d)$. If $\boldsymbol{t} \in \mathbb{C}^{d-k}$, then the function $\boldsymbol{y} \in \mathbb{R}^k \mapsto G_{\boldsymbol{t}}(\boldsymbol{y}) = F(\boldsymbol{y}, \boldsymbol{t})$ is the inverse Fourier transform of the following function

$$\boldsymbol{\xi} \in \mathbb{R}^k \mapsto \int_{Q_{d-k}} \widehat{F}(\boldsymbol{\xi}, \boldsymbol{u}) e^{2\pi i \boldsymbol{t}\cdot\boldsymbol{u}} d\boldsymbol{u}.$$

Since $\widehat{F} \in L^2(\mathbb{R}^d)$, we conclude that the above function has finite $L^2(\mathbb{R}^k)$-norm and as a consequence $G_{\boldsymbol{t}} \in PW^2(Q_k)$. A similar result is valid for $p = 1$, but only for $\nu(d)$-admissible functions.

**Lemma 14.** *Let* $d > k > 0$ *be integers. If* $F(\boldsymbol{x})$ *is* $\nu(d)$*-admissible then the function* $\boldsymbol{y} \in \mathbb{R}^k \mapsto F(\boldsymbol{y}, \boldsymbol{0})$ *with* $\boldsymbol{0} \in \mathbb{R}^{d-k}$ *is* $\nu(k)$*-admissible and*

$$\int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x} \leq \int_{\mathbb{R}^k} F(\boldsymbol{y}, \boldsymbol{0}) d\boldsymbol{y}.$$

*Proof.* We give a proof only for the case $d = 2$ since it will be clear that the general case follows by an adaption of the following argument.

Let $F(x, y)$ be a function admissible for $\nu(2)$ and define $G(x) = F(x, 0)$. By Fourier inversion we obtain that

$$G(x) = \int_{-1}^{1} \left( \int_{-1}^{1} \widehat{F}(s, t) dt \right) e^{2\pi i s x} ds.$$

This shows that $G \in PW^2(Q_1)$. Now, for every $a \in (0, 1)$ define the functions

$$G_a(x) = G((1-a)x) \left( \frac{\sin(a\pi x)}{a\pi x} \right)^2.$$

and

$$F_a(x, y) = F((1-a)x, y) \left( \frac{\sin(a\pi x)}{a\pi x} \right)^2.$$

By an application of Holder's inequality and Theorem 11, we deduce that $G_a \in PW^1(Q_1)$ and $F_a \in PW^1(Q_2)$ for all $a \in (0,1)$. Hence, we can apply Poisson summation to conclude that

$$\begin{aligned}\int_{\mathbb{R}} G_a(x)dx &= \sum_{n\in\mathbb{Z}} G((1-a)n)\left(\frac{\sin(a\pi n)}{a\pi n}\right)^2 \\ &\geq \sum_{(n,m)\in\mathbb{Z}^2} F((1-a)n,m)\left(\frac{\sin(a\pi n)}{a\pi n}\right)^2 \\ &= \int_{\mathbb{R}^2} F((1-a)x,y)\left(\frac{\sin(a\pi x)}{a\pi x}\right)^2 dxdy,\end{aligned}$$

where the above inequality is valid because the function $F(x,y)$ is a minorant of the box $Q_2$. Observing that $G_a(x) \leq \mathbf{1}_{Q_{1/(1-a)}}(x)$ for every $x \in \mathbb{R}$, we can apply Fatou's lemma to conclude that

$$\begin{aligned}\int_{\mathbb{R}}[\mathbf{1}_{Q_1}(x) - G(x)]dx &\leq \liminf_{a\to 0}\int_{\mathbb{R}}[\mathbf{1}_{Q_{1/(1-a)}}(x) - G_a(x)]dx \\ &\leq \int_{\mathbb{R}} \mathbf{1}_{Q_1}(x)dx \\ &\quad - \limsup_{a\to 0}\int_{\mathbb{R}^2} F((1-a)x,y)\left[\frac{\sin(a\pi x)}{a\pi x}\right]^2 dxdy \\ &= \int_{\mathbb{R}} \mathbf{1}_{Q_1}(x)dx - \int_{\mathbb{R}^2} F(x,y)dxdy < \infty.\end{aligned}$$

This conclude the proof. □

We now introduce an interpolation theorem which has proven indispensable throughout our investigations.

**Proposition 15.** *For all $F \in PW^2(Q_d)$ we have*

$$F(\boldsymbol{x}) = \prod_{n=1}^{d}\left(\frac{\sin \pi x_n}{\pi}\right)^2 \sum_{\boldsymbol{n}\in\mathbb{Z}^d}\sum_{\boldsymbol{j}\in\{0,1\}^d} \frac{\partial_{\boldsymbol{j}} F(\boldsymbol{n})}{(\boldsymbol{x}-\boldsymbol{n})^{\boldsymbol{2}-\boldsymbol{j}}} \tag{2.10}$$

*where* $\partial_{\boldsymbol{j}} = \partial_{j_1 \dots j_d}$ *and* $(\boldsymbol{x}-\boldsymbol{n})^{\mathbf{2}-\boldsymbol{j}} = (x_1-n_1)^{2-j_1}\dots(x_d-n_d)^{2-j_d}$ *and the right hand side of* (2.10) *converges uniformly on compact subsets of* $\mathbb{R}^d$*.*

*Proof.* This proposition is proven by induction using Vaaler's result [64, Theorem 9] as the base case and Theorem 12 (Pólya-Plancherel), which guarantees that the sequence $\{F(\boldsymbol{n}) : \boldsymbol{n} \in \mathbb{Z}^d\}$ is square summable for any $F \in PW^2(Q_d)$. Also note that by Fourier inversion $PW^2(Q_d)$ is closed under partial differentiation. □

Finally, the next lemma demonstrates that extremal functions always exist for $\nu(d)$ and other minorization problems.

**Lemma 16.** *Suppose* $G \in L^1(\mathbb{R}^d)$ *is a real valued function. Let* $F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), \dots$ *be a sequence in* $PW^1(Q_N)$ *such that* $F_\ell(\boldsymbol{x}) \le G(\boldsymbol{x})$ *for each* $\boldsymbol{x} \in \mathbb{R}^d$ *and each* $\ell$*. Assume that there exists* $A > 0$ *such that* $\widehat{F}_\ell(\mathbf{0}) \ge -A$ *for each* $\ell$*. Then there exists a subsequence* $F_{\ell_k}(\boldsymbol{x})$ *and a function* $F \in PW^1(Q_d)$ *such that* $F_{\ell_k}(\boldsymbol{x})$ *converges to* $F(\boldsymbol{x})$ *uniformly on compact sets as* $k$ *tends to infinity. In particular, we deduce that* $F(\boldsymbol{x}) \le G(\boldsymbol{x})$ *for each* $\boldsymbol{x} \in \mathbb{R}^d$ *and* $\limsup_{k\to\infty} \widehat{F}_{\ell_k}(\mathbf{0}) \le \widehat{F}(\mathbf{0})$*.*

*Proof.* By the remark after Theorem 11, each $F_\ell \in PW^2(Q_d)$ and we can bound their $L^2(\mathbb{R}^d)$-norm in the following way

$$\|F_\ell\|_2 = \|\widehat{F}_\ell\|_2 \le \mathrm{vol}_d(Q_d)^{1/2}\|\widehat{F}_\ell\|_\infty \le 2^{d/2}\|F_\ell\|_1$$

and

$$\|F_\ell\|_1 \le \|G - F_\ell\|_1 + \|G\|_1 \le 2\|G\|_1 + A. \tag{2.11}$$

Hence the sequence $F_1(\boldsymbol{x}), F_2(\boldsymbol{x}), ...$ is uniformly bounded in $L^2(\mathbb{R}^d)$ and, by the Banach-Alaoglu theorem, we may extract a subsequence (that we still denote by $F_\ell(\boldsymbol{x})$) that converges weakly to a function $F \in PW^2(Q_d)$. By Theorem 11 we can assume that $F(\boldsymbol{x})$ is continuous. By using Fourier inversion we have

$$F_\ell(\boldsymbol{x}) = \int_{Q_d} \widehat{F}_\ell(\boldsymbol{\xi}) e(\boldsymbol{\xi} \cdot \boldsymbol{x}) d\boldsymbol{\xi}.$$

Thus, the weak convergence implies that $F_\ell(\boldsymbol{x}) \to F(\boldsymbol{x})$ point-wise for all $x \in \mathbb{R}^d$. Fourier inversion also shows that $\|F_\ell\|_\infty \leq 2^N \|F_\ell\|_1$. However, we also have

$$|\partial_j F_\ell(\boldsymbol{x})| = 2\pi \left| \int_{Q_d} \xi_j \widehat{F}_\ell(\boldsymbol{\xi}) e(\boldsymbol{\xi} \cdot \boldsymbol{x}) d\boldsymbol{\xi} \right| \leq 2\pi \|F_\ell\|_1 2^d.$$

We can use (2.11) to conclude that $|F_\ell(\boldsymbol{x})| + |\nabla F_\ell(\boldsymbol{x})|$ is uniformly bounded in $\mathbb{R}^d$. We can apply the Ascoli-Arzelà theorem to conclude that, by possibly extracting a further subsequence, $F_\ell(\boldsymbol{x})$ converges to $F(\boldsymbol{x})$ uniformly on compact sets of $\mathbb{R}^d$.

We conclude that $G(\boldsymbol{x}) \geq F(\boldsymbol{x})$ for each $\boldsymbol{x} \in \mathbb{R}^d$. By applying Fatou's lemma to the sequence of functions $G(\boldsymbol{x}) - F_1(\boldsymbol{x}), G(\boldsymbol{x}) - F_2(\boldsymbol{x}), ...$ we find that $F \in L^1(\mathbb{R}^d)$ and

$$\limsup_{\ell \to \infty} \int_{\mathbb{R}^d} F_\ell(\boldsymbol{x}) d\boldsymbol{x} \leq \int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x}.$$

This concludes the lemma. □

## 2.4 Proofs of the Main Results

The next theorem is the cornerstone in the proof of our main results. This theorem is in stark contrast with the one dimensional case. In the one dimensional case Selberg's function interpolates at all lattice points, and is therefore extremal. In two dimensions, on the other hand, if a minorant interpolates everywhere except for possibly the origin, then it is identically zero. This theorem is therefore troublesome because it seems to disallow the possibility of using interpolation (in conjunction with Poisson summation) to prove an extremality result.

**Theorem 17.** *Let $F(x, y)$ be admissible for $\nu(2)$ such that $F(n, m) = 0$ for each non-zero $(n, m) \in \mathbb{Z}^2$ and $F(0, 0) \geq 0$, then $F(x, y)$ vanishes identically.*

*Proof. Step 1.* First we assume that the function $F(x, y)$ is invariant under the symmetries of the square, that is,

$$F(x, y) = F(y, x) = F(|x|, |y|) \tag{2.12}$$

for all $x, y \in \mathbb{R}$. We claim that for any $(m, n) \in \mathbb{Z}^2$ we have:

(a) $\partial_x F(m, n) = 0$ if $(m, n) \neq (\pm 1, 0)$ and $\partial_y F(m, n) = 0$ if $(m, n) \neq (0, \pm 1)$;

(b) $\partial_{xx} F(m, n) = 0$ if $n \neq 0$ and $\partial_{yy} F(m, n) = 0$ if $m \neq 0$;

(c) $\partial_{xy} F(m, n) = 0$ if $n \neq \pm 1$ or $m \neq \pm 1$.

We can apply Theorem 14 to deduce that, for each fixed non-zero integer $n$, the function $x \in \mathbb{R} \mapsto F(x, n)$ is a non-positive function belonging to $PW^1(Q_1)$ that vanishes in the integers, hence identically zero by formula (2.10). Also note that the points $(m, 0)$ for $m \in \mathbb{Z}$ with $|m| > 1$ are local maximums of the function $x \in \mathbb{R} \mapsto F(x, 0)$. These facts in conjunction with the invariance property (2.12) imply items (a) and (b).

Finally, note that a point $(m, n)$ with $|n| > 1$ has to be a local maximum of the function $F(x, y)$. Thus, the Hessian determinant of $F(x, y)$ at such a point has to be non-negative, that is,

$$\text{Hess}_F(m, n) := \partial_{xx}F(m, n)\partial_{yy}F(m, n) - [\partial_{xy}F(m, n)]^2 \geq 0$$

However, by item (b), $\partial_{xx}F(m, n) = 0$ and we conclude that $\partial_{xy}F(m, n) = 0$. This proves item (c) after using again the property (2.12).

*Step 2.* We can now apply formula (2.10) and deduce that $F(x, y)$ has to have the following form:

$$\left(\frac{\sin \pi x}{\pi x}\right)^2 \left(\frac{\sin \pi y}{\pi y}\right)^2 \left\{F(0, 0) + a\frac{x^2}{x^2 - 1} + a\frac{y^2}{y^2 - 1} + b\frac{x^2 y^2}{(x^2 - 1)(y^2 - 1)}\right\},$$

where $a = 2\partial_x F(1, 0)$ and $b = 4\partial_{xy}F(1, 1)$. Denote by $B(x, y)$ the expression in the brackets above and note that it should be non-positive if $|x| \geq 1$ or $|y| \geq 1$. We deduce that

$$F(0, 0) + a + (a + b)\frac{x^2}{x^2 - 1} = B(x, \infty) \leq 0$$

for all real $x$. We conclude that $a + b = 0$, $F(0, 0) \leq -a$ and

$$B(x, y) = F(0, 0) + a\left[1 - \frac{1}{(x^2 - 1)(y^2 - 1)}\right].$$

For each $t > 0$, the set of points $(x, y) \in \mathbb{R}^2 \backslash Q_2$ such that $(x^2-1)(y^2-1) = 1/t$ is non-empty and $B(x, y) = F(0, 0) + a - at$ at such a point. Therefore $a \geq 0$ and we deduce that $F(0, 0) \leq 0$. We conclude that $F(0, 0) = 0$, which in turn implies that $a = 0$. Thus $F(x, y)$ vanishes identically.

*Step 3.* Now we finish the proof. Let $F(x, y)$ be a $\nu(2)$-admissible function such that $F(0, 0) = \widehat{F}(0, 0) \geq 0$. Define the function

$$G_1(x, y) = \frac{F(x, y) + F(-x, y) + F(x, -y) + F(-x, -y)}{4}.$$

Clearly the following function

$$G_0(x, y) = \frac{G_1(x, y) + G_1(y, x)}{2}$$

is also $\nu(2)$-admissible and $G_0(0, 0) = \widehat{G}_0(0, 0) = F(0, 0) \geq 0$. Moreover, $G_0(x, y)$ satisfies the symmetry property (2.12). By Steps 1 and 2 the function $G_0(x, y)$ must vanish identically. Thus, we obtain that

$$G_1(x, y) = -G_1(y, x).$$

However, since $G_1(x, y)$ is also $\nu(2)$-admissible we conclude that $G_1(x, y)$ is identically zero outside the box $Q_2$, hence it vanishes identically. An analogous argument can be applied to the function $G_2(x, y) = [F(x, y) + F(-x, y)]/2$ to conclude that this function is identically zero outside the box $Q_2$, hence it vanishes identically. Using the same procedure again we finally conclude that $F(x, y)$ vanishes identically and the proof of the theorem is complete. $\square$

### 2.4.1 Proof of Theorem 7

The proof is done via induction and the base case is Theorem 17. Assume that the theorem is proven for some dimension $d \geq 2$. Let $F(\boldsymbol{x}, x_{d+1})$ be a $\nu(d+1)$-admissible function such that $F(\boldsymbol{n}, m) = 0$ for all non-zero $(\boldsymbol{n}, m) \in \mathbb{Z}^{d+1}$. Now, for every fixed $t \in \mathbb{R}$ define $G_t(\boldsymbol{x}) = F(\boldsymbol{x}, t)$. An application of Lemma 14 shows that $G_t \in PW^1(Q_d)$ for all $t \in \mathbb{R}$ and is $\nu(d)$-admissible if $|t| < 1$ and non-positive if $|t| \geq 1$. Moreover, for any fixed non-zero $m \in \mathbb{Z}$ we have $G_m(\boldsymbol{n}) = 0$ for all $\boldsymbol{n} \in \mathbb{Z}^d$, thus by induction we have $G_m \equiv 0$ for every non-zero $m \in \mathbb{Z}$. By symmetry we have $F(\boldsymbol{x}, x_{d+1}) = 0$ if one of its entries is a non-zero integer. We conclude that the $\nu(d)$-admissible function $G_t(\boldsymbol{x})$ satisfies $G_t(\boldsymbol{n}) = 0$ for every non-zero $\boldsymbol{n} \in \mathbb{Z}^d$. By induction again, $G_t \equiv 0$ for all real $t$. This implies that $F \equiv 0$ and this finishes the proof.

### 2.4.2 Proof of Theorem 6

The item $(i)$ is a direct consequence o Lemma 16 while item $(iii)$ is a consequence of Theorem 9 item $(iv)$. It remains to show item $(ii)$.

Clearly by Lemma 14, we have $\nu(d) \geq \nu(d+1)$. Suppose by contradiction that $\nu(d) = \nu(d+1)$. Let $(\boldsymbol{x}, t) \in \mathbb{R}^d \times \mathbb{R} \mapsto F(\boldsymbol{x}, t)$ be an extremal function for $\nu(d+1)$. Let $G_m(\boldsymbol{x}) = F(\boldsymbol{x}, m)$ for each $m \in \mathbb{Z}$. Lemma 14 implies that $G_m(\boldsymbol{x})$ is also admissible for $\nu(d)$ (if $m \neq 0$ then the function is non-positive). By the Poisson summation formula we have for each non-zero

$m \in \mathbb{Z}$

$$\widehat{F}(\mathbf{0}) = \sum_{\boldsymbol{n}\in\mathbb{Z}^d} \sum_{k\in\mathbb{Z}} F(\boldsymbol{n}, k) \leq \sum_{\boldsymbol{n}\in\mathbb{Z}^d} (F(\boldsymbol{n}, m) + F(\boldsymbol{n}, 0)) = \widehat{G}_m(0) + \widehat{G}_0(0). \quad (2.13)$$

By assumption

$$\widehat{G}_0(0) \leq \nu(d) = \nu(d+1) = \widehat{F}(\mathbf{0}) \quad (2.14)$$

Combining (2.13) and (2.14) yields $0 \leq \widehat{G}_m(0)$ for each $m \neq 0$. However, the function $G_m(\boldsymbol{x}) \leq 0$ for each $\boldsymbol{x} \in \mathbb{R}^d$ whenever $m$ is a non-zero integer. Consequently, $G_m(\boldsymbol{x})$ vanishes identically. It follows that $F(\boldsymbol{n}) = 0$ for each non-zero $\boldsymbol{n} \in \mathbb{Z}^{d+1}$. By Theorem 7, $F(\boldsymbol{x})$ vanishes identically. Therefore $\nu(d+1) = \nu(d) = 0$, a contradiction. The theorem is finished.

### 2.4.3 Proof of Theorem 9

First we prove item $(i)$. Assume by contradiction that there exists a sequence of $\Delta(d)$-admissible functions $F_\ell(\boldsymbol{x})$ $\ell = 1, 2, ...$ such that $M_\ell = \max_{x\in Q_d}\{F_\ell(\boldsymbol{x})\}$ converges to $\infty$ when $\ell \to \infty$. Let $G_\ell(\boldsymbol{x}) = F_\ell(\boldsymbol{x})/M_\ell$, and note that $G_\ell(\boldsymbol{x})$ is $\nu(d)$-admissible for all $\ell$. Also let $\boldsymbol{x}_\ell \in Q_d$ be such that $F_\ell(\boldsymbol{x}_\ell) = M_\ell$. We can assume by compactness that $\boldsymbol{x}_\ell \to \boldsymbol{x}_0$. By Lemma 16 we may also assume that there exists a function $G(\boldsymbol{x})$, $\nu(d)$-admissible such that $G_\ell(\boldsymbol{x})$ converges uniformly on compact sets to $G(\boldsymbol{x})$. We also have by Lemma 16 that

$$0 \leq \limsup_{\ell} \int_{\mathbb{R}^d} G_\ell(\boldsymbol{x}) d\boldsymbol{x} \leq \int_{\mathbb{R}^d} G(\boldsymbol{x}) d\boldsymbol{x}.$$

However, $G_\ell(\mathbf{0}) = 1/M_\ell \to 0$ and thus $G(\mathbf{0}) = 0$. By the Poisson summation formula, for any fixed non-zero $\boldsymbol{n} \in \mathbb{Z}^d$ we have

$$0 \le \widehat{G}_\ell(\mathbf{0}) \le 1/M_\ell + G_\ell(\boldsymbol{n}).$$

Thus, we conclude that $G_\ell(\boldsymbol{n}) \to 0$ as $\ell \to \infty$. This, implies that $G(\boldsymbol{n}) = 0$ for all $\boldsymbol{n} \in \mathbb{Z}^d$. By Theorem 7 we conclude that $G(\boldsymbol{x})$ vanishes identically. However, by uniform convergence we have $G(\boldsymbol{x}_0) = 1$, a contradiction. This proves item $(i)$

Item $(ii)$ is a consequence of Lemma 16 in conjunction with item $(i)$. Item $(iii)$ can be proven exactly as in Theorem 6 item $(ii)$, since now we know that extremizers exist. It remains to show the upper bound of item $(iv)$. For this we will show a stronger result.

**Lemma 18.** *The function*

$$\delta(d) = \frac{1 - \Delta(d)}{d}$$

*is non-decreasing. That is, if $\Delta(d) > 0$ and $M < d$ then $\delta(M) \le \delta(d)$.*

*Proof.* For a given $\boldsymbol{n} \in \mathbb{Z}^d$ let $\sigma(\boldsymbol{n})$ denote the quantity of distinct numbers in $\mathbb{Z}^d$ that can be constructed by only permuting the entries in $\boldsymbol{n}$. It is simple to see that if $M < d$, $\boldsymbol{m} \in \mathbb{Z}^M$ is non-zero and $(\boldsymbol{m}, \mathbf{0}) \in \mathbb{Z}^d$ then

$$\sigma(\boldsymbol{m}, \mathbf{0}) \ge (d/M)\sigma(\boldsymbol{m}),$$

and equality is attained if $\boldsymbol{m}$ has only one entry different than zero. Let $\Gamma^d$ be the subset of non-zero $\boldsymbol{n} = (n_1, ..., n_N) \in \mathbb{Z}^d_+$ such that $n_1 \ge n_2 \ge ... \ge n_d \ge 0$

($\mathbb{Z}_+ = \{0, 1, 2, ...\}$). Note that $(\Gamma^M, \mathbf{0}) \subset \Gamma^d$ if $M < d$. Also let $\varepsilon(\boldsymbol{n})$ be the number of non-zero entries in a vector $\boldsymbol{n} \in \Gamma^d$. Now, for a given $d$, let $F_d(\boldsymbol{x})$ be an extremal function for the $\Delta(d)$ problem. We can assume that it is invariant under the symmetries of $Q_d$. Define $G_d(\boldsymbol{y}) = F_d(\boldsymbol{y}, \mathbf{0})$ for every $\boldsymbol{y} \in \mathbb{R}^M$, $M < d$. By Poisson summation we obtain

$$\begin{aligned}\Delta(d) = \widehat{F}_d(\mathbf{0}) &= 1 + \sum_{\boldsymbol{n} \in \Gamma^d} 2^{\varepsilon(\boldsymbol{n})} \sigma(\boldsymbol{n}) F_d(\boldsymbol{n}) \\ &\leq 1 + \sum_{\boldsymbol{m} \in \Gamma^M} 2^{\varepsilon(\boldsymbol{m}, \mathbf{0})} \sigma(\boldsymbol{m}, \mathbf{0}) F_d(\boldsymbol{m}, \mathbf{0}) \\ &= 1 + \sum_{\boldsymbol{m} \in \Gamma^M} 2^{\varepsilon(\boldsymbol{m})} \sigma(\boldsymbol{m}, \mathbf{0}) G_d(\boldsymbol{m}) \\ &\leq 1 + (d/M) \sum_{\boldsymbol{m} \in \Gamma^M} 2^{\varepsilon(\boldsymbol{m})} \sigma(\boldsymbol{m}) G_d(\boldsymbol{m}) = 1 + (d/M)(\widehat{G}_d(0) - 1) \\ &\leq 1 + (d/M)(\Delta(M) - 1).\end{aligned}$$

We conclude that

$$\frac{1 - \Delta(d)}{d} \geq \frac{1 - \Delta(M)}{M},$$

and this finishes the lemma. □

*Proof of Theorem 9 continued.* The previous lemma implies that if $\Delta(d) > 0$ then

$$\frac{1}{d} > \frac{1 - \Delta(d)}{d} = \delta(d) \geq \delta(k) = \frac{1 - \Delta(k)}{k},$$

for any $k \leq d$. We conclude that

$$\frac{k}{1 - \Delta(k)} > d,$$

and this finishes the proof of the theorem. □

**Remark.** Let $k < d \leq d_c$, then

$$\Delta(k) \geq (1 - k/d) + (k/d)\Delta(d).$$

Since the right hand side above is always larger than $\Delta(d)$, this inequality produces better lower bounds for lower dimensions once a lower bound is given for a higher dimension.

## 2.5 Linear Programming Bounds

In this section we'll solve the linear program in Theorem 4 to calculate explicit upper bounds for $\nu(d)$ and $\Delta(d)$ for $d = 2, 3, 4, 5$. First, we will exploit the symmetry of the problem to make the linear program in Theorem 4 less computationally expensive. We'll then describe our strategy for computing upper bounds via this new linear program. Finally, we describe how to modify this strategy to derive rigorous bounds.

Let $R \in \mathbb{Z}_{>1}$. For $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$, define $\cos(\boldsymbol{x}) := \prod_{i=1}^{d} \cos(x_i)$. Let $\mathcal{A} := \{\boldsymbol{x} \in \mathbb{R}^d : 0 < x_1 < \cdots < x_d < R\}$. Let $\Sigma(Q_d)$ denote the group of symmetries of the unit cube. Note that this group has a natural action on $\mathbb{R}^d$ of permuting coordinates and switching signs. The orbit of any point in $\mathcal{A}$ under $\Sigma(Q_d)$ has cardinality $2^d d!$. Let $S_d$ be the symmetric group on $d$ elements. For $\sigma \in S_d$ and $\boldsymbol{x} = (x_1, \ldots, x_d) \in \mathbb{R}^d$ we write $\sigma(\boldsymbol{x}) := \big(x_{\sigma(1)}, \ldots, x_{\sigma(d)}\big)$, i.e. we let $\sigma$ act on the indices of the components of $\boldsymbol{x}$. For $x, y \in \mathbb{R}^d$ we write $xy$ to mean the component-wise product of $x$ and $y$; we'll use $\langle x, y \rangle$ to mean the scalar product.

Note that we solved the linear program with CVX (using the Gurobi solver) in MATLAB. The rational arithmetic was done in Maple. The code will be made available before publication on the first author's website.

### 2.5.1 Reducing the size of the linear program

Recall that Theorem 4 gives a linear program in which we want to optimize a set of weights $\omega_1, \ldots, \omega_L$, where each weight $\omega_i$ corresponds to shifting the periodic summation of the extremal function $F$ by a single point $\boldsymbol{y}_i$ in $\mathbb{R}^d$. If instead we let each weight $\omega_i$ correspond to the shifts by each of the points in $\Sigma(\boldsymbol{y}_i) := \{\boldsymbol{z} : \sigma(\boldsymbol{y}_i) = z \text{ for some } \sigma \in \Sigma(Q_d)\}$, then we can exploit the symmetry of the problem to get the following simplification of Theorem 4.

**Theorem 19.** *Let $\boldsymbol{y}_0 = 0 \in \mathbb{R}^d$ and suppose $\boldsymbol{y}_1, \ldots, \boldsymbol{y}_L \in \mathcal{A}$, $\omega_0, \ldots, \omega_L \in \mathbb{R}^{\geq 0}$ are such that*

*(i)* $\omega_0 + \sum_{i=1}^{L} \omega_i 2^N \sum_{\sigma \in S_d} \cos\left(\frac{2\pi\sigma(\boldsymbol{y}_i)\boldsymbol{n}}{d}\right) = 0$ *for all* $n \in \mathcal{A} \cap \mathbb{Z}^d$ *such that* $0 < ||n||_\infty < R$

*(ii)* $\omega_0 + \sum_{i=1}^{L} \omega_i 2^d d! = R^d$

*Then*

$$\nu(k) \leq \omega_0 + \sum_{\{1 \leq i \leq L : ||\boldsymbol{y}_i||_\infty < 1\}} \omega_i 2^d d!$$

*and, if* $||y_i||_\infty \geq 1$ *for all* $0 < i \leq L$, *then*

$$\Delta(k) \leq \omega_0$$

### 2.5.2 A simple algorithm

In order to use Theorem 19 to compute explicit bounds, we first fix values of $d$ and $R$ and generate a large number of random points $\boldsymbol{y}_i \in \mathcal{A}$. Then we solve the linear program in Theorem 19, store the values of $\boldsymbol{y}_i$ for which $w_i > 0$ (solutions are very sparse due to the relatively small number of constraints), generate a large number of new points and add them to the collection of values, and then repeat this process until the upper bound appears to stabilize.

For larger values of $d$ and $R$ sometimes this method is not good at finding a feasible value of $w$ until $L$ is very large. In this case one can speed things up by first solving the problem by taking the $\boldsymbol{y}$'s to belong to the lattice $(1/S)\mathbb{Z}^d$ for some value of $S > R$; in our experience this always gives a feasible value of $w$ and then the bound can be improved by remembering the nonzero entries of $w$ and the corresponding values of $\boldsymbol{y}$, generating a random set of $\boldsymbol{y}$'s, and iterating as above.

In our experience when we solve the linear program all of the nonzero values of $w_i$ correspond to points $\boldsymbol{y}_i$ which satisfy $||y_i||_\infty > 1$. So, in our experience, this method gives the same bounds on $\Delta(d)$ and $\nu(d)$.

We summarize the upper bounds for $\Delta(d)$ in the following table.

| |
|---|
| 0.9946333 ($d = 2,\ R = 16,\ S = 5$) |
| 0.9849928 ($d = 3,\ R = 10,\ S = 5$) |
| 0.9802947 ($d = 4,\ R = 5,\ S = 6$) |
| 0.9553936 ($d = 5,\ R = 7,\ S = 4$) |

Table 2.1: Upper bounds on $\Delta(d)$ with parameters $R$ and $S$

### 2.5.3 Making the bounds rigorous

Note that the bounds in the previous section were obtained using floating point arithmetic and therefore are not rigorous (the upper bound $\Delta(5) \leq 0.9553936$, if correct within 2 significant digits, would produce $d_c \leq 125$). In this section we will describe how to use rational arithmetic to remedy this. We carry out the computations only for the case $d = 2$, $R = 6$ and leave the other cases for future work. In this case we are able to get a rigorous upper bound of [3]

$$\Delta(2) \leq 0.997212,$$

and this gives a rigorous upper bound of

$$d_c \leq 717.$$

Notice that the linear program in Theorem 19 can be made rational if we choose values of $\boldsymbol{y}_i$ such that $\cos\left(\frac{2\pi n \boldsymbol{y}_i}{R}\right)$ is rational for all values of $n \in \mathbb{Z}^d$. This will happen if and only if $C_i := \cos\left(\frac{2\pi \boldsymbol{y}_i}{R}\right)$ and $S_i := \sin\left(\frac{2\pi \boldsymbol{y}_i}{R}\right)$ are both rational numbers. Moreover, since we don't actually need the values of $\boldsymbol{y}_i$ to

[3]In fact, we get a rigorous upper bound of a rational number slightly smaller than this, but it has too many digits to fit in this paper. The interested reader can find it in the Maple script on the first author's website.

solve the linear program, just the values of cos, instead of generating random points $\boldsymbol{y}_i$ we can generate random rational points on the circle, or equivalently random Pythagorean triples. These will be the values of $S_i$ and $C_i$.

Rather than solve the linear program using rational arithmetic, which is computationally expensive, we solve the problem using floating point arithmetic to identify the nonzero entries of $w_i$ and then use rational arithmetic to solve the resulting linear system. In our experience the number of nonzero entries is always the same as the number of equality constraints. Therefore the only rational arithmetic we have to do is solving a relatively small full rank square linear system.

## 2.6 Explicit Minorants in Low dimensions

We define an auxiliary variational quantity $\lambda(d)$ over a more restrictive set of admissible functions than $\nu(d)$. Let

$$\lambda(d) = \sup \int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x}$$

where the supremum is taken over functions $F(\boldsymbol{x})$ that are admissible for $\nu(d)$ and, in addition, $F(\mathbf{0}) = 1$, and

$$F(\boldsymbol{n}) = 0$$

for each non-zero $\boldsymbol{n} \in \mathbb{Z}^d$ unless $\boldsymbol{n}$ is a "corner" of the box $Q_d$. Here, a corner of the box $Q_d$ is a vector $\boldsymbol{n} \in \partial Q_d \cap \mathbb{Z}^d$ with at least 2 non-zero entries. This definition makes any $k$-dimensional slice of an admissible function for $\lambda(d)$

$(k < d)$ admissible for $\lambda(k)$, which in turn implies that

$$\lambda(d+1) \leq \lambda(d)$$

for all $d$. We note that Selberg's functions (see Appendix) are always admissible for $\lambda(d)$ but have negative integral. Our aim is to mimick Selberg's construction but to incorporate a correction term so that our minorants have positive integral. Notice that by Theorem 6 it is impossible to do this in sufficiently high dimensions.

Making use of the interpolation formula (2.10) we conclude that every function $F(\boldsymbol{x})$ admissible for $\lambda(d)$ has the following useful representation

$$F(\boldsymbol{x}) = S(\boldsymbol{x})P(\boldsymbol{x}) \tag{2.15}$$

where

$$S(\boldsymbol{x}) = \prod_{n=1}^{d} \left( \frac{\sin(\pi x_n)}{\pi x_n (x_n^2 - 1)} \right)^2$$

and $P(\boldsymbol{x})$ is a polynomial such that each variable $x_n$ appearing in its expression has an exponent not greater than 4. Notice that, by Poisson summation, if $F(\boldsymbol{x})$ is admissible for $\lambda(d)$ and is invariant under the symmetries of $Q_d$ then

$$\int_{\mathbb{R}^d} F(\boldsymbol{x}) d\boldsymbol{x} = 1 + \sum_{k=2}^{d} \binom{d}{k} 2^k P(\boldsymbol{u}_k) \tag{2.16}$$

where $\boldsymbol{u}_k = (\overbrace{1, 1, 1, .., 1}^{\text{k times}}, 0, ..., 0)$.

In what follows it will be useful to use a particular family of symmetric functions. For given integers $d \geq k \geq 1$ we define

$$\sigma_{d,k}(\boldsymbol{x}) = \sum_{1 \leq n_1 < n_2 < ... < n_k \leq d} x_{n_1}^2 x_{n_2}^2 ... x_{n_k}^2$$

and

$$\widetilde{\sigma}_{d,k}(\boldsymbol{x}) = \sum_{1 \le n_1 < n_2 < ... < n_k \le d} x_{n_1}^4 x_{n_2}^4 ... x_{n_k}^4.$$

**Lemma 20.** *Let $F_d : \mathbb{R}^d \to \mathbb{R}$ be $\lambda(d)$-admissible function constructed using* (2.15) *with*

$$P_d(\boldsymbol{x}) = \prod_{n=1}^{d}(1 - x_n^2) - \sum_{k=1}^{d} a_k \sigma_{d,k}(\boldsymbol{x}) - \sum_{k=1}^{d} b_k \widetilde{\sigma}_{d,k}(\boldsymbol{x})$$

*and $a_k, b_k \ge 0$ for $k = 1, ..., d$. Let $\boldsymbol{y}$ denote vectors in $\mathbb{R}^{d+1}$. If $d$ is even then the function $F_{d+1} : \mathbb{R}^{d+1} \to \mathbb{R}$ constructed using* (2.15) *with*

$$P_{d+1}(\boldsymbol{y}) = \prod_{n=1}^{d+1}(1 - y_n^2) - \sum_{k=1}^{d} a_k \sigma_{d+1,k}(\boldsymbol{y}) - \sum_{k=1}^{d} b_k \widetilde{\sigma}_{d+1,k}(\boldsymbol{y})$$

*is $\lambda(d+1)$-admissible. If $d$ is odd then the function $F_{d+1} : \mathbb{R}^{d+1} \to \mathbb{R}$ constructed using* (2.15) *with*

$$P_{d+1}(\boldsymbol{y}) = \prod_{n=1}^{d+1}(1 - y_n^2) - \sum_{k=1}^{d} a_k \sigma_{d+1,k}(\boldsymbol{y}) - \sum_{k=1}^{d} b_k \widetilde{\sigma}_{d+1,k}(\boldsymbol{y}) - \delta \sigma_{d+1,d+1}(\boldsymbol{y}),$$

*where $\delta \ge 0$ and*

$$\delta \ge \sup_{|y_n| \ge 1} \frac{\prod_{n=1}^{d+1}(y_n^2 - 1) - \sum_{k=1}^{d} a_k \sigma_{d+1,k}(\boldsymbol{y}) - \sum_{k=1}^{d} b_k \widetilde{\sigma}_{d+1,k}(\boldsymbol{y})}{\sigma_{d+1,d+1}(\boldsymbol{y})}$$

*is a $\lambda(d+1)$-admissible function.*

*Proof.* Assume $d$ is even. If all $y_1, ..., y_{d+1}$ have moduli less than 1 then trivially $P_{d+1}(\boldsymbol{y}) \le 1$. If an even number of the variables $y_1, ..., y_{d+1}$ have moduli less than 1, then also clearly $P_{d+1}(\boldsymbol{y}) \le 0$. If an odd number, but not all, have moduli less than 1, assume for instance $|y_1| < 1$, then we would have $P_{d+1}(\boldsymbol{y}) \leq P_d(y_2, ..., y_{d+1}) \leq 0$.

Assume now that $d$ is odd. Again, if all $y_1, ..., y_{d+1}$ have moduli less than 1 then trivially $P_{d+1}(\boldsymbol{y}) \leq 1$. If an odd number of the variables $y_1, ..., y_{d+1}$ have moduli less than 1, then also clearly $P_{d+1}(\boldsymbol{y}) \leq 0$. If an even number of variables, not non of them, have moduli less than 1, say $|y_1| < 1$, then $P_{d+1}(\boldsymbol{y}) \leq P_d(y_2, ..., y_{d+1}) \leq 0$. If all variables have moduli greater than 1 then by the choice of $\delta$ we have $P_{d+1}(\boldsymbol{y}) \leq 0$. □

Note that $\delta = 1$ always work, but that is often not the best choice since we want to minimize $\delta$ so to make $\widehat{F}_{d+1}(\mathbf{0})$ as large as possible, hence this forces the coefficients $b_k$ being not too small. Also note that in this way $P_d(\boldsymbol{x}) = P_{d+1}(\boldsymbol{x}, 0)$. Using the above lemma we were able to construct admissible functions up to dimension $d = 5$ by starting with a good two dimensional minorant.

**Theorem 21.** *Define the functions* $\mathcal{F}_2(x_1, x_2)$, $\mathcal{F}_3(x_1, x_2, x_3)$, $\mathcal{F}_4(x_1, ..., x_4)$ *and* $\mathcal{F}_5(x_1, ..., x_5)$ *by using representation* (2.15) *and the following polynomials respectively :*

- $P_2(x_1, x_2) = (1 - x_1^2)(1 - x_2^2) - \frac{1}{16}\widetilde{\sigma}_{2,2}(x_1, x_2)$
- $P_3(x_1, x_2, x_3) = \prod_{n=1}^{3}(1 - x_n^2) - \frac{1}{16}\widetilde{\sigma}_{3,2}(x_1, x_2, x_3)$

- $P_4(x_1, ..., x_4) = \prod_{n=1}^{4}(1 - x_n^2) - \frac{3}{4}\sigma_{4,4}(x_1, ..., x_4) - \frac{1}{16}\widetilde{\sigma}_{4,2}(x_1, ..., x_4)$
- $P_5(x_1, ..., x_5) = \prod_{n=1}^{5}(1 - x_n^2) - \frac{3}{4}\sigma_{5,4}(x_1, ..., x_5) - \frac{1}{16}\widetilde{\sigma}_{5,2}(x_1, ..., x_5)$.

*These functions are admissible for* $\lambda(2)$, $\lambda(3)$, $\lambda(4)$ *and* $\lambda(5)$ *respectively and their respective integrals are equal to:* $63/64 = 0.984375$, $119/128 = 0.9296875$, $95/128 = 0.7421975$ *and* $31/256 = 0.12109375$.

*Proof.* The integrals of these functions can be easily calculated using formula (2.16), we prove only their admissibility. We start with $\mathcal{F}_2(\boldsymbol{x})$. Clearly, if $|x_1| > 1 > |x_2|$ then $P_2(x_1, x_2) < 0$. Also, writing $t = |x_1 x_2|$ we obtain

$$\begin{aligned} P_2(x_1, x_2) &= 1 + x_1^2 x_2^2 - x_1^2 - x_2^2 - x_1^4 x_2^4/16 \\ &\leq 1 + x_1^2 x_2^2 - 2|x_1 x_2| - x_1^4 x_2^4/16 \\ &= 1 + t^2 - 2t - t^4/16. \end{aligned}$$

On the other hand, we have

$$1 + t^2 - 2t - t^4/16 = (1 - t)^2 - t^4/16 \tag{2.17}$$

and

$$1 + t^2 - 2t - t^4/16 = (t - 2)^2(4 - 4t - t^2)/16. \tag{2.18}$$

If $|x_1|, |x_2| < 1$ then $0 \leq t < 1$, and by (2.17) we deduce that $P_2(x_1, x_2) < 1$. If $|x_1|, |x_2| > 1$ then $t > 1$, and by (2.18) we deduce that $P_2(x_1, x_2) \leq 0$.

This proves that $\mathcal{F}_2(\boldsymbol{x})$ is $\lambda(2)$-admissible. Lemma 20 shows that $\mathcal{F}_3(\boldsymbol{x})$ is admissible for $\lambda(3)$.

We now deal with $\mathcal{F}_4(\boldsymbol{x})$, which from the proof of Lemma (20) we only need to worry when $|x_1|, |x_2|, |x_3|, |x_4| > 1$. In this case, suppressing the variables, we have

$$P_4 = 1 - \sigma_{4,1} + \sigma_{4,2} - \sigma_{4,3} + \tfrac{1}{4}\sigma_{4,4} - \tfrac{1}{16}\widetilde{\sigma}_{4,2}.$$

Observing that

$$\sigma_{4,2} - \sigma_{4,3} \leq x_1^2x_2^2 + x_3^2x_4^2,$$

we obtain

$$\begin{aligned} 1 - \sigma_{4,1} + \sigma_{4,2} - \sigma_{4,3} - \tfrac{1}{16}\widetilde{\sigma}_{4,2} \leq & -1 + P_2(x_1, x_2) + P_2(x_3, x_4) \\ & - \tfrac{1}{16}[x_1^4x_3^4 + x_1^4x_4^4 + x_2^4x_3^4 + x_2^4x_4^4]. \end{aligned}$$

Since $P_2(x_1, x_2) \leq 0$ and $P_2(x_3, x_4) \leq 0$, we deduce that

$$\begin{aligned} P_4(x_1, ..., x_4) &\leq -1 + \tfrac{1}{4}x_1^2x_2^2x_3^2x_4^2 - \tfrac{1}{16}[x_1^4x_3^4 + x_1^4x_4^4 + x_2^4x_3^4 + x_2^4x_4^4] \\ &\leq -1 + \tfrac{1}{16}(x_1^4 + x_2^4)(x_3^4 + x_4^4) - \tfrac{1}{16}[x_1^4x_3^4 + x_1^4x_4^4 + x_2^4x_3^4 + x_2^4x_4^4] \\ &= -1. \end{aligned}$$

This proves that $\mathcal{F}_4(\boldsymbol{x})$ is admissible for $\lambda(4)$. Lemma 20 shows that $\mathcal{F}_5(\boldsymbol{x})$ is admissible for $\lambda(5)$. □

## 2.7 Selberg and Montomgery's Constructions

In this section we will present the box minorant constructions of Selberg and Montgomery and we will preform some asymptotic analysis on their integrals. In particular we will show in which regimes Selberg's minorant is a better approximate than Montgomery's and visa-versa. The interested readers are encouraged to consult [49, 60, 64] for more on Selberg's functions and [2, 26] for more on Montgomery's functions. Our treatment is by no means exhaustive.

Both constructions begin with the following entire functions

$$K(z) = \left( \frac{\sin \pi z}{\pi z} \right)^2$$

and

$$H(z) = \left\{ \frac{\sin^2 \pi z}{\pi^2} \right\} \left( \sum_{n=-\infty}^{\infty} \frac{\text{sign}(n)}{(z-n)^2} + \frac{2}{z} \right)$$

where

$$\text{sign}(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0. \end{cases}$$

Let $[a_1, b_1], ..., [a_d, b_d] \subset \mathbb{R}$ where $b_n > a_n$ for each $n = 1, ..., d$, and set $B = \prod [a_i, b_i]$. For each $i = 1, ..., d$ define

$$\begin{aligned} V_i(z) &= \tfrac{1}{2} H(z - a_i) + \tfrac{1}{2} H(b_i - z) \\ E_i(z) &= \tfrac{1}{2} K(z - a_i) + \tfrac{1}{2} K(b_i - z) \\ C_i(z) &= V_i(z) + E_i(z) \\ c_i(z) &= V_i(z) - E_i(z). \end{aligned}$$

The following theorem can be deduced from [49, 60, 64].

**Theorem 22** (Selberg)**.** *The function*

$$\mathcal{C}_B(\boldsymbol{x}) = -(N-1)\prod_{i=1}^{d} C_i(x) + \sum_{n=1}^{d} c_n(x) \prod_{m\neq n} C_m(x)$$

*satisfies:*

*(i)* $\widehat{\mathcal{C}}_B(\boldsymbol{\xi}) = 0$ *for each* $\|\boldsymbol{\xi}\|_\infty > 1$*;*

*(ii)* $\mathcal{C}_B \leq \mathbf{1}_B(\boldsymbol{x})$ *for each* $\boldsymbol{x} \in \mathbb{R}^d$*; and*

*(iii)*

$$\begin{aligned}\int_{\mathbb{R}^d} \mathcal{C}_B(\boldsymbol{x}) d\boldsymbol{x} &= -(d-1)\prod_{i=1}^{d}(b_i - a_i + 1) \\ &\quad + \sum_{n=1}^{d}(b_n - a_n - 1)\prod_{m\neq n}(b_m - a_m + 1).\end{aligned}$$

**Corollary 23.** *Let* $B = [-\delta, \delta]^d$*. We have*

$$\int_{\mathbb{R}^d} \mathcal{C}_B(\boldsymbol{x}) d\boldsymbol{x} > 0$$

*if and only if*

$$\delta > d - \frac{1}{2}.$$

*On the other hand, if* $d$ *is fixed, then*

$$\int_{\mathbb{R}^d} \mathcal{C}_B(\boldsymbol{x}) d\boldsymbol{x} = (2\delta)^d - (d-1)(2\delta)^{d-1} + O(\delta^{d-2})$$

*as* $\delta \to \infty$*.*

*Proof.* Setting $a_n = -\delta$ and $b_n = \delta$ it follows from Theorem 22 (iii) that

$$\int_{\mathbb{R}^d} \mathcal{C}_B(\boldsymbol{x})d\boldsymbol{x} = (2\delta+1)^{d-1}(2\delta - (2d-1)).$$

This quantity is positive if and only if $2\delta - (2d-1) > 0$, which occurs if and only if $\delta > d - \frac{1}{2}$. On the other hand,

$$(2\delta+1)^{d-1}(2\delta-(2d-1)) = (2\delta)^d - (2d-1)(2\delta)^{d-1} + O_d(\delta^{d-2})$$

as $\delta \to \infty$. □

The following theorem can be deduced from [26].

**Theorem 24** (Montgomery)**.** *The function*

$$\mathcal{G}_B(\boldsymbol{x}) = \prod_{i=1}^{d} V_i(x) - \prod_{i=1}^{d}(V_i(x) + 2E_i(x)) + \prod_{i=1}^{d}(V_i(x) + E_i(x))$$

*satisfies:*

*(i)* $\widehat{\mathcal{G}}_B(\boldsymbol{\xi}) = 0$ *for each* $\|\boldsymbol{\xi}\|_\infty > 1$*;*

*(ii)* $\mathcal{G}_B \leq \mathbf{1}_B(\boldsymbol{x})$ *for each* $\boldsymbol{x} \in \mathbb{R}^d$*; and*

*(iii)*

$$\int_{\mathbb{R}^d} \mathcal{G}_B(\boldsymbol{x})d\boldsymbol{x} \;=\; \prod_{n=1}^{d}(b_n - a_n) - \prod_{n=1}^{d}(b_n - a_n + 2) + \prod_{n=1}^{d}(b_n - a_n + 1).$$

**Corollary 25.** *Let* $B = [-\delta, \delta]^d$*,* $\epsilon > 0$*, and* $\phi = (1+\sqrt{5})/2$*. We have*

$$\int_{\mathbb{R}^d} \mathcal{G}_B(\boldsymbol{x})d\boldsymbol{x} < 0$$

*if*

$$\delta < \left(\frac{1}{2\log(\phi)} - \epsilon\right) d = (1.039... - \epsilon)\, d$$

*and*

$$\int_{\mathbb{R}^d} \mathcal{G}_{rQ_d}(\boldsymbol{x}) d\boldsymbol{x} > 0$$

*if*

$$\delta > \left(\frac{1}{2\log(\phi)} + \epsilon\right) d = (1.039... + \epsilon)\, d$$

*when $d$ is sufficiently large. When $d$ is fixed and $\delta \to \infty$ we have*

$$\int_{\mathbb{R}^d} \mathcal{G}_B(\boldsymbol{x}) d\boldsymbol{x} = (2\delta)^d - (2\delta)^{d-1} + O(\delta^{d-2}).$$

*Proof.* We will only prove the first statement of the corollary since the second statement is straightforward. Setting $a_n = -\delta$ and $b_n = \delta$ we have by Theorem 24

$$\int_{\mathbb{R}^d} \mathcal{G}_B(\boldsymbol{x}) d\boldsymbol{x} = (2\delta)^d - (2\delta + 2)^d + (2\delta + 1)^d.$$

Since the right hand side remains positive if we divide by $(2\delta)^d$ it suffices to determine when

$$1 - \left(1 + \frac{1}{\delta}\right)^d + \left(1 + \frac{1}{2\delta}\right)^d > 0.$$

Setting $\delta = d/c$ for some $c > 0$ we find that for large $d$

$$1 - \left(1 + \frac{1}{\delta}\right)^d + \left(1 + \frac{1}{2\delta}\right)^d \approx 1 - e^c + e^{c/2}.$$

The equation $1 - e^c + e^{c/2} = 0$ has one real solution, namely $c = 2\log(\phi)$. The function $c \mapsto 1 - e^c + e^{c/2}$ is a decreasing function at $c = 2\log(\phi)$ so if

$c < 2\log(\phi)$ is a constant independent of $d$, then for $d$ sufficiently large we have

$$1 - \left(1 + \frac{1}{\delta}\right)^d + \left(1 + \frac{1}{2\delta}\right)^d > 0.$$

On the other hand, if $c > 2\log(\phi)$ then

$$1 - \left(1 + \frac{1}{\delta}\right)^d + \left(1 + \frac{1}{2\delta}\right)^d < 0.$$

The proof of the first statemnt is complete upon setting $\delta = ((2\log(\phi))^{-1} \pm \epsilon)d$. $\square$

It follows from the above corollaries that Montgomery's minorants are better approximates when $\delta$ is very large compared to $d$, and Selberg's are better when $d$ is large compared to $\delta$.

## 2.8 Future work

While our work on the Beurling-Selberg problem is progress, Question 1 is far from resolved. We state some open problems here.

**Problem 26.** *Find the exact value of* $\nu(2)$. *Better yet, construct the extremal function for* $\nu(2)$.

**Problem 27.** *Find the exact value of the critical dimension* $d_c$, *i.e. the value for which* $\nu(d_c) > 0$ *but* $\nu(d) = 0$ *for all* $d > d_c$.

**Problem 28.** *Construct admissible functions in dimensions higher than five.*

The results of this chapter offer insight into the following uncertainty principle.

**Question 29.** *Suppose $f \in L^1(\mathbb{R}^d)$ satisfies 1. $f(0) \geq 0$, 2. $f(x) \leq 0$ for $x \notin [-1,1]^d$, and 3. $\int f(x)dx > 0$. Define*

$$B(f) := \inf\{\delta > 0 : \widehat{f}(\xi) = 0 \text{ for all } ||\xi||_\infty \geq \delta\}$$

*For a fixed dimension $d$, what can we say about the quantity*

$$\delta^*(d) := \inf B(f)$$

*where the infimum is taken over $f$ satisfying the hypotheses above.*

Uncertainty principles tell us what happens to a function's Fourier transform when we localize some information about said function. The most classical uncertainty principle states that the Fourier transform of a compactly supported function cannot also have compact support. Our bounds on $\Delta(d)$ and the constructions of Selberg and Montgomery discussed in Section 2.7 imply the following bounds on $\delta^*(d)$.

**Theorem 30.** *1. $\delta^*(d) \geq 1$ for all $d > d_c$*

*2. $\delta^*(d) \leq Cd$ for some constant $C > 0$ independent of $d$*

Question 29 is related to an uncertainty principle first studied by Bourgain, Clozel, and Kahane [7]. Their work stems from the observation that if a function $f$ satisfies $f(0) \geq 0$ and $\widehat{f}(0) \geq 0$ then it is not possible for both $f$

and $\widehat{f}$ to be negative outside of an arbitrarily small neighborhood of the origin. They made this observation quantitative by proving the following theorem.

**Theorem 31** ((Bourgain, Clozel, Kahane)). *Let $f : \mathbb{R} \to \mathbb{R}$ be a nonzero, integrable, even function with $\widehat{f} \in L^1(\mathbb{R})$ such that 1. $f(0) \geq 0$, 2. $f(x) \leq 0$ for all $|x| > 1$, and 3. $\int f(x)dx \geq 0$ Define*

$$A(f) := \inf\{\delta > 0 : \widehat{f}(\xi) \leq 0 \textit{ for all } |\xi| > \delta\}$$

*Then* $0.1687 \leq A(f) \leq .41$.

Let us compare and contrast Theorem 31 and Question 29. First, in Question 29, we want to know how small the support of $\widehat{f}$ can be when $f$ satisfies hypotheses (1) - (3). In Theorem 31, the lower bound on $A(f)$ is a bound on the smallest region outside of which $f$ can be nonpositive. This can be viewed as a relaxation of studying the size of the support via $B(f)$. Second, the hypotheses (1) through (3) are nearly identical; only (3) changes slightly. We do not know whether the change in (3) makes a difference. We could have the following situation. Suppose there is an interval of values $I = [\delta_1, \delta_2]$ such that for each $\delta \in I$ there do not exist functions as in Question 29, but there does exist a function $f_\delta$ satisfying the first two hypotheses of Question 29 and for which $\int f_\delta = 0$. In this case, the answer to Question 29 would change if we changed hypothesis (3) to $\inf f(x)dx \geq 0$.

As with Selberg's extremal problem for the interval (Section 2.1), there are many ways to generalize the one-dimensional problem of Theorem 31 to higher dimensions. In Question 29, we replace the interval with the cube. Another natural choice is to replace the interval with the ball. This was done by Gonçalves, Silva, and Steinerberger [43]. Their argument relies heavily on radial symmetry and so is unlikely to be adaptable to Question 29.

# Chapter 3

# Turan's Problem

In spite of its name, Turán's extremal problem was first studied by Siegel [62] in the thirties as a means to sharpen Minkowski's lattice theorem. This problem asks

**Question 32.** *Given a symmetric convex body $K \subset \mathbb{R}^d$, how large can the integral of a continuous, positive-definite function $f$ be if we require $f(0) = 1$ and $f$ to be supported in $K$? We denote the extremal value, i.e. the supremum of the integral of all functions satisfying the conditions above, by $\eta(K)$.*

Note that this is perhaps the simplest example of a CBI optimization problem. Let $\chi_{K/2}$ be the characteristic function of the set $K/2$. Then the function $\left(\chi_{K/2}\right)^2$ satifies the conditions in Question 32 and has integral $\frac{\mathrm{vol}(K)}{2^d}$. Multiple authors [59, 4] have conjectured that this is the extremal function.

**Conjecture 33.**

$$\eta(K) = \frac{vol(K)}{2^d}$$

The goal of this chapter is to prove upper and lower bounds for Turan's problem which can be computed by solving a finite dimensional linear program. This makes it possible to numerically test Conjecture 33; we carry out the computations for the 3-dimensional $\ell_1$ ball in [57], giving the sharpest known bound in this case.

Our upper bound for $\eta(K)$ is a consequence of a Poisson summation technique which can also be used to bound $\Delta(d)$ (see Section 3.4). Roughly speaking, we take the periodic summation of $\widehat{f}$ to get a closely related extremal problem on the torus which is an upper bound for the original problem. This new extremal problem is a linear programming problem with finitely many variables and infinitely many constraints. We reduce it to a finite dimensional linear programming problem simply by imposing finitely many of the constraints.

Our lower bound also comes from studying a related extremal problem on the torus. It differs from the technique used to prove the upper bounds in a couple of ways. First, we arrive at the related problem by taking the periodic summation of $f$ rather than of $\widehat{f}$. Second, the related problem does not directly give a lower bound for $\eta(K)$. Rather, we rely on a quantitative relationship between the extremal values of the two problems established by Gorbachev [45]. We exhibit a method for constructing admissible functions for the problem on the torus, use this to lower bound the extremal value of the problem on the torus, and then use Gorbachev's relationship to arrive at a lower bound on $\eta(K)$.

## 3.1 Preliminaries

We define the Fourier transform of a function $f \in L^1(\mathbb{R}^d)$ by

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \langle \xi, x \rangle} dx$$

We recall Bochner's theorem, which relates positive definiteness of a function to its Fourier transform.

**Theorem 34.** *$f$ is a countinuous positive definite function on $\mathbb{R}^d$ with $f(0) = 1$ if and only if there exists a unique probability measure $\mu$ on $\mathbb{R}^d$ such that*

$$f(x) = \int_{\mathbb{R}^d} e^{2\pi i \langle \xi, x \rangle} d\mu(\xi)$$

For a history of Turán's problem see the excellent survey [59]. We briefly what is known. When $d = 1$, Conjecture 33 is known to be true. When $K$ is an ellipsoid, the Conjecture 33 is known to be true. When $K$ tiles Euclidean space the conjecture is true. Finally, the conjecture is known to be true when $K$ is a spectral domain.

*Remark* 35. Taking the self-convolution of $\chi_{1/2K}$ shows that we always have

$$\eta_{\mathbb{R}^d} \geq \frac{\text{vol}(K)}{2^d}$$

## 3.2 An Upper Bound on $\eta(K)$

Fix an integer $N \geq 1$. Define $\mathcal{L} := \mathbb{Z}^d \cap (N \cdot K)$. We say that a sequence $\{f_\ell\}_{\ell \in \mathcal{L}}$ belongs to the class $\mathcal{A}_U(K)$ if the following conditions hold:

1. $f_0 = 1$

2. $\frac{1}{N^d}\sum_{\ell\in\mathcal{L}} f_\ell e^{2\pi i\langle\xi,\ell\rangle/N} \geq 0$ for all $\xi \in [-N/2, N/2)^d$

We now define the extremal quantity $\eta_U(K)$.

$$\eta_U(K) := \sup_{\{f_\ell\}\in\mathcal{A}_U(N,K)} \frac{1}{N^d}\sum_{\ell\in\mathcal{L}} f_\ell \tag{3.1}$$

The main result of this section is the following theorem.

**Theorem 36.** *For any convex body $K$,*

$$\eta(K) \leq \eta_U(N, K)$$

Note that $\eta_U(K)$ is a linear programming problem with finitely many variables and infinitely many constraints. If we modify the problem by imposing only finitely many of the constraints, we enlarge the search space, and therefore the extremal quantity can only increase. In other words, let $E \subset [-1/2, 1/2)^d$ be a finite set. If we modify the nonnegativity constraint (2) so that we only impose it on the set $E$, the new extremal quantity will be an upper bound for $\eta_U(K)$. This leads to another definition.

A sequence $\{f_\ell\}_{\ell\in\mathcal{L}}$ belongs to the class $\mathcal{A}_U(K, N, E)$ if

1. $f_0 = 1$

2. $\frac{1}{N^d}\sum_{\ell\in\mathcal{L}} f_\ell e^{2\pi i\langle\xi,\ell\rangle/N} \geq 0$ for all $\xi \in E$

$$\eta_U(K,N,E) := \sup_{\{f_\ell\}\in\mathcal{A}_U(K,N,E)} f_0$$

By the above discussion, we have the following corollary of Theorem 36.

**Corollary 37.**

$$\eta(K) \leq \eta_U(K,E)$$

### 3.2.1 Proof of Theorem 36

Let $f \in \mathcal{A}(K)$. The idea of the proof is to show that the sequence $\{f(\ell/N)\} \in \mathcal{A}_U(K)$ and that

$$\int_{\mathbb{R}^d} f(x)dx \leq \frac{1}{N^d}\sum_{\ell\in\mathcal{L}} f_\ell \tag{3.2}$$

These two facts tell us that for every $f \in \mathcal{A}(K)$, there is an element $\{f_\ell\}$ of $\mathcal{A}_U(K)$ for which the objective quantity $\eta_U(K)[\{f_\ell\}]$ is at least as big as the objective quantity $\eta(K)[f]$. This gives the desired result.

Let $g : N\mathbb{T}^d \to \mathbb{R}$ be the periodic summation of $\widehat{f}$ with respect to the lattice $N\mathbb{Z}^d$, i.e.

$$g(\xi) := \sum_{\nu\in\mathbb{Z}^d} \widehat{f}(\xi + \nu\cdot N) \tag{3.3}$$

Note that we can apply the Polya-Plancherel theorem (Theorem 12) to show that this sum is absolutely convergent. We apply the Poisson summation formula (Equation (2.2) in Section 2.2) to see that

$$g(\xi) = \frac{1}{N^d}\sum_{\ell\in\mathbb{Z}^d} f\left(\frac{\ell}{N}\right) e^{2\pi i\langle\xi,\ell\rangle/N} = \frac{1}{N^d}\sum_{\ell\in\mathcal{L}} f\left(\frac{\ell}{N}\right) e^{2\pi i\langle\xi,\ell\rangle/N}$$

We will first show that $\{f(\ell/N)\} \in \mathcal{A}_U(K)$. Since $f \in \mathcal{A}(K)$, we have $f(0) = 1$. Next, note that for any $\xi \in N\mathbb{T}^d$

$$\frac{1}{N^d} \sum_{\ell \in \mathcal{L}} f\left(\frac{\ell}{N}\right) e^{2\pi i \langle \xi, \ell \rangle / N} = g(\xi) \geq 0$$

The fact that $g$ is nonnegative follows from the nonnegativity of $\widehat{f}$. These two observations show that $\{f(\ell/N)\} \in \mathcal{A}_U(K)$.

Finally, we have that

$$\int_{\mathbb{R}^d} f(x)dx = \widehat{f}(0) \leq g(0) = \frac{1}{N^d} \sum_{\ell \in \mathcal{L}} f\left(\frac{\ell}{N}\right) e^{2\pi i \langle \xi, \ell \rangle / N}$$

Here the first equality follows from the definition of the Fourier transform and the inequality comes from the definition of $g$ and the fact that $\widehat{f}$ is nonnegative. This proves (3.2) and therefore the theorem.

## 3.3 A lower bound on $\eta(K)$

Note that $\eta(hK) = h^d \eta(K)$ for any $h > 0$. Therefore, without loss of generality, we can assume that $K \subset [-1/2, 1/2)^2$. We say that a continuous, integrable function $f : [-1/2, 1/2)^d \to \mathbb{R}$ belongs to the class $\mathcal{A}_L(K)$ if the following hold.

1. $f(x) = \sum_{\ell \in \mathbb{Z}^d} \widehat{f}_\ell e^{2\pi i \langle x, \ell \rangle}$, where $\widehat{f}_\ell = \int f(x) e^{-2\pi i \langle x, \ell \rangle} dx$

2. $f(0) = 1$

3. $\widehat{f}_\ell \geq 0$ for all $\ell \in \mathbb{Z}^d$

4. $f(x) = 0$ for $x \in \mathbb{T}^d \backslash K$.

We now define the extremal quantity

$$\eta_L(K) = \sup_{f \in \mathcal{A}_L(K)} \int f(x)dx$$

We will make use of the following theorem of Gorbachev [45].

**Theorem 38.** *For any* $0 < h < 1/4$*,*

$$\frac{\eta_L(hK)}{h^d} - 2^d(1 - (1 - 8h^2 + 12h^3)^d) \leq \eta(K)$$

This theorem tells us that a lower bound on $\eta_L(hK)$ implies a lower bound on $\eta(K)$. To compute a lower bound on $\eta_L(hK)$ we introduce another extremal problem. First, however, we need a bit of notation. Let $M \geq 2$ be an even integer. Then let $\mathcal{M}$ be the evenly spaced grid on $[-1/2, 1/2)^d$ defined by the set of d-tuples $\{-1/2, -(M-1)/(2M), \ldots, (M-2)/(2M), (M-1)/(2M)\}^d$.

The grid $\mathcal{M}$ defines a partition of $[-1/2, 1/2)^d$ into cubes. Each point $k \in \mathcal{M}$ is a corner in exactly $2^d$ cubes. For each $k \in \mathcal{M}$, we let $\mathcal{Q}_k$ be the set of these $2^d$ cubes. We now define a subset $\Delta \subset \mathcal{M}$ by

$$\Delta := \{k \in \mathcal{M} : \exists Q \in \mathcal{Q}_k : Q \not\subseteq K\}$$

We say that a sequence $\{\varphi_\ell\}_{\ell \in \mathcal{M}}$ belongs to the class $\mathcal{A}_L(K, M)$ if the following conditions hold.

1. $\sum_{\ell \in \mathcal{M}} \varphi_\ell e^{2\pi i \langle \ell, k \rangle} \geq 0$ for all $k \in (N \cdot \mathcal{M})$

2. $\varphi_0 = 1$

3. $\varphi_\ell = 0$ for all $\ell \in \Delta$

We think of the sequence $\{\varphi_\ell\}$ as a discrete approximation to a function in $\mathcal{A}_L(K)$. We now define the extremal quantity

$$\eta_L(K, M) = \sup_{\varphi \in \mathcal{A}_L(K,M)} \frac{1}{M^d} \sum_{\ell \in \mathcal{M}} \varphi_\ell$$

**Theorem 39.**

$$\eta_L(K, M) \leq \eta_L(K)$$

We will prove this theorem by showing that the linear interpolant of a sequence of points in $\mathcal{A}_L(K, M)$ belongs to $\mathcal{A}_L(K)$. Let $\{\varphi_\ell\} \in \mathcal{A}_L(K, M)$. Sometimes we will write $\varphi(\ell)$ instead of $\varphi_\ell$ to improve readability. We define the *linear interpolant* of $\{\varphi_\ell\}$ to be the function

$$\Lambda_\varphi(x_1, \ldots, x_d) = \sum_{\varepsilon \in \{0,1\}^d} \varphi\left(\ell + \frac{\epsilon}{M}\right) \prod_{j=1}^{d} L_{\varepsilon_j}(x_j)$$

on $\frac{k_1}{M} \leq x_1 < \frac{k_1+1}{M}, \ldots, \frac{k_d}{M} \leq x_d < \frac{k_d+1}{M}$ with

$$L_\varepsilon(x) := (1 - \varepsilon) + (-1)^{1-\varepsilon} M(x - \lfloor x \rfloor_N)$$

Where we write $\lfloor x \rfloor_M$ to denote the largest rational number of the form $k/M$ smaller than $x$.

Note that

$$\int_{\frac{k_j}{M}}^{\frac{k_j+1}{M}} L_{\varepsilon_j}(x_j)dx_j = \frac{(1-\varepsilon_j)}{M} + (-1)^{1-\varepsilon_j}\left(\frac{1}{2M}\right)$$

We now record a few facts about $\Lambda_\varphi$.

**Lemma 40.** *We have the identity*

$$\int_{[-1/2,1/2)^d} \Lambda_\varphi(x_1,\ldots,x_d)dx =$$

$$= \sum_{\ell\in\mathcal{M}} \sum_{\varepsilon\in\{0,1\}^d} \varphi\left(\ell + \frac{\varepsilon}{M}\right) \prod_{j=1}^{d} \left[\frac{1-\varepsilon_j}{M} + (-1)^{1-\varepsilon_j}\left(\frac{1}{2M}\right)\right]$$

$$= \frac{1}{M^d}\sum_{\ell\in\mathcal{M}} \varphi\left(\ell\right)$$

**Lemma 41.** *Suppose $\{\varphi_\ell\} \in \mathcal{A}_L(K,M)$, and let $\lambda_k$ be the Fourier series coefficients of $\Lambda_\varphi$. Then $\lambda_k \geq 0$ for all $k \in \mathbb{Z}^d$*

The proof of this lemma follows by applying the next lemma to $\Lambda_\varphi$ (using Fubini's theorem to justify the iterated integrals).

**Lemma 42.** *Let $g(y)$ be the function which linearly interpolates between the function values $a\left(\frac{k}{N}\right)$ for $k = 0,\ldots,N-1$. Then*

$$\int_0^1 g(y)e^{-2\pi i n y}dy = \frac{2N}{(2\pi n)^2}\left(1-\cos\left(\frac{2\pi n}{N}\right)\right)\left(\sum_{k=0}^{N-1} a\left(\frac{k}{N}\right)e^{-2\pi i k n/N}\right)$$

To prove Theorem 39, we note that the above lemmas imply that the linear interpolant of any $\{\varphi_\ell\} \in \mathcal{A}_L(K,M)$ belongs to $\mathcal{A}_L(K)$.

## 3.4 Linear programming bounds for CBI problems

In this section we prove an upper bound for the infinite dimensional linear program $\Delta(d)$. This upper bound turns out to be given by solving the dual of the linear program in Theorem 8 (we will not show that this is the case as it is not relevant to the proof, but it is straightforward to check) and is therefore equivalent to that upper bound. We include the proof, however, because it uses the same technique as the proof of Theorem 36. We emphasize here the versatility of this technique for CBI problems. As long as the constraints and the objective are preserved in some form under periodic summation one should be able to use this method. We now return to the setting of Chapter 2. We will prove the following theorem.

**Theorem 43.** *Let $M > 2$ be an integer and let $E \subset [-M/2, M/2)^d \backslash [-1, 1]^d$ be a finite set. Let $\mathcal{L} = \{\ell \in \mathbb{Z}^d : ||\ell||_\infty < M\}$. Define*

$$\Delta_M(d, E) = \max_{\{\widehat{f}_\ell\}_{\ell \in \mathcal{L}}} \widehat{f}_0$$

*subject to*

- $f(x) = \frac{1}{M^d} \sum_{\ell \in \mathbb{Z}^d} \widehat{f}_\ell e^{2\pi i \langle \ell, x \rangle / M}$
- $f(0) \leq 1$
- $f(x) \leq 0$ *for all* $x \in E$

*We then have*

$$\Delta(d) \leq \Delta_M(d, E)$$

This theorem is a corollary of the following lemma.

**Lemma 44.** *Let $M > 2$ be an integer. Define*

$$\Delta_M(d) := \max_{\{\widehat{f}_\ell\}_{\ell \in \mathcal{L}}} \widehat{f}_0$$

*subject to*

- $f(x) = \frac{1}{M} \sum_{\ell \in \mathcal{L}} \widehat{f}_\ell e^{2\pi i \langle x, \ell \rangle / M}$
- $f(x) \leq 0$ *for all* $x \in [-M/2, M/2)^d \backslash [-1, 1]^d$
- $f(0) \leq 1$

*Then we have the upper bound*

$$\Delta(d) \leq \Delta_M(d)$$

*Proof of Theorem 43.* To see that Theorem 43 follows from Lemma 44, note that the linear program $\Delta_M(d, E)$ is identical to the linear program $\Delta_M(d)$ except that we only enforce the constraint $f(x) \leq 0$ on the finite set $E$. Therefore any vector admissible to $\Delta_M(d)$ is also admissible to $\Delta_M(d, E)$, and so

$$\Delta_M(d) \leq \Delta_M(d, E)$$

□

*Proof of Lemma 44.* Let $F$ be an admissible function for $\Delta(d)$. Let $f$ be the periodic summation of $F$ with respect to $M\mathbb{Z}^d$ for some $M > 2$, i.e. $f(x) :=$

$\sum_{n\in\mathbb{Z}^d} F(x+nM)$. As the name suggests, this function is periodic with period $M\mathbb{Z}^d$, i.e. $f(x) = f(x+Mn)$ for all $n \in \mathbb{Z}^d$. Therefore $f$ can be identified with a function on the quotient space $\mathbb{R}^d/(M\mathbb{Z}^d)$; the quotient space can in turn be identified with the $M$-torus $M\mathbb{T}^d$ in the standard way. This is all standard; see, for example, [63].

By the Poisson summation formula (which we can apply by Theorem 13 in Chapter 2) we have

$$f(x) = \frac{1}{M^d}\sum_{\ell\in\mathbb{Z}^d}\widehat{F}\left(\frac{\ell}{M}\right)e^{2\pi i\langle \ell,x\rangle/M} = \frac{1}{M^d}\sum_{||\ell||_\infty<M}\widehat{F}\left(\frac{\ell}{M}\right)e^{2\pi i\langle \ell,x\rangle/M}$$

where the second equality comes from the fact that $F$ is bandlimited. Note that $f$ satisfies a set of constraints related to those satisfied by $\Delta(d)$-admissible functions. Namely, we have

1. $f(0) \leq 1$
2. $f(x) \leq 0$ for all $x \in M\mathbb{T}^d \backslash [-1,1]^d$

We also have that

$$\int_{\mathbb{R}^d} F(x)dx = \sum_{n\in\mathbb{Z}^d}\int_{[0,M)^d} F(x+nM)dx$$

$$= \int_{[0,M)^d}\sum_{n\in\mathbb{Z}^d} F(x+nM)dx = \int_{\mathbb{T}^d} f(x)dx$$

where we've used Fubini's theorem to get the second equality and the definition of $f$ to get the third.

We see, then, that every function $F$ admissible to $\Delta(d)$ gives us a vector $\{\widehat{f}_\ell\}_{\ell \in \mathcal{L}}$ which is admissible to $\Delta_M(d)$ and which satisfies $\int_{\mathbb{R}^d} F(x)dx = \widehat{f}_0$. Therefore

$$\Delta(d) \leq \Delta_M(d)$$

$\square$

# Chapter 4

# A coordinate free proof of the finiteness principle for Whitney's extension theorem

Consider the following problem, first posed by Whitney [69].

**Problem 45.** *Fix an arbitrary subset $E \subset \mathbb{R}^d$ and a function $f : E \to \mathbb{R}$. How can we tell whether there exists a function $F \in C^m(\mathbb{R}^d)$ which extends $f$, i.e. for which $F(x) = f(x)$ for all $x \in E$?*

The first progress on this problem is due to Whitney. He solved the one-dimensional problem using difference quotients. In higher dimensions, difference quotients are not available and therefore new ideas are required. Glaeser [39] solved the case $C^1(\mathbb{R}^d)$; his key idea, the notion of an iterated paratangent bundle, was generalized by Bierston, Milman, and Pawlucki [5] to solve the problem for $C^m(\mathbb{R}^d)$ when $E$ is a subanalytic set. Y. Brudnyi and Shvartsman [8] solved the analogous problem for $C^{1,1}(\mathbb{R}^d)$ by establishing something called a finiteness principle. They conjectured the following finiteness principle for $C^{m-1,1}(\mathbb{R}^d)$, which was then proved by Fefferman [33]

**Theorem 46** (Brudnyi-Shvartsman-Fefferman Finiteness Principle)**.** *Given integers $m, d \geq 1$, there exist $k^\#$ and $C^\#$, depending only on $m$ and $d$, for* *which the following holds. Let $f : E \to \mathbb{R}$ be a function on a finite set $E \subset \mathbb{R}^d$. Suppose that, for every subset $S \subset E$ with $\#(S) \leq k^\#$, there exists a function $F^S \in C^{m-1,1}(\mathbb{R}^d)$ with $||F^S||_{C^{m-1,1}(\mathbb{R}^d)} \leq M$ and $F^S(x) = f(x)$ for all $x \in S$.*

*Then there exists a function $F \in C^{m-1,1}(\mathbb{R}^d)$ with $||F||_{C^{m-1,1}(\mathbb{R}^d)} \leq C^\# M$ and $F(x) = f(x)$ for all $x \in E$.*

Building on this work, Fefferman [34] resolved Problem 45 for arbitrary $E$. He went on to study many generalizations and extensions of Problem 45, including the following.

**Problem 47.** *Fix a finite subset $E \subset \mathbb{R}^d$ and a function $f : E \to \mathbb{R}$. How can we compute a $C^m$ extension of $f$ with norm as small as possible in a reasonable amount of time?*

This problem was solved by Fefferman and Klartag [38, 37, 36]. The solution turns out to be closely related to Theorem 46. To see why, first note that when $E$ is a finite set the $C^m(\mathbb{R}^d)$ and $C^{m-1,1}(\mathbb{R}^d)$ interpolation problems are equivalent. Second, the proof of Theorem 46 is constructive, so it gives a general idea of how to actually compute an interpolant. Third, the norm of this interpolant is actually optimal up to the multiplicative constant $C^\#$. To see this, note that the hypotheses of Theorem 46 imply that the extension $F$ must have norm at least $M$. We now state (a somewhat imprecise version of) their main theorem.

**Theorem 48** (Fefferman, Klartag)**.** *Given $m, d \geq 1$, $E \subset \mathbb{R}^d$ a finite set with*

*$\#(E) = N$, and $f : E \to \mathbb{R}$, there exist algorithms which compute an extension $F \in C^{m-1,1}(\mathbb{R}^d)$ such that*

- *$F(x) = f(x)$ for all $x \in E$*
- *$||F||_{C^{m-1,1}(\mathbb{R}^d)} \leq M$*
- *$M \leq C^{\#} \cdot \inf\{||G||_{C^{m-1,1}(\mathbb{R}^d)} : G = f \text{ on } E\}$*

The algorithms alluded to in this theorem are expected to be optimal with respect to run-time and storage. This theorem has one shortcoming, however, which is that the constant $C^{\#}$ is extremely large for most values of $d$ and $m$. Therefore there is no reason to expect the computed interpolant to be reasonably sized. This constant $C^{\#}$ is inherited from Fefferman's proof of the finiteness principle, which motivates us to look for a new proof.

In [24], A. Frei-Pearson, A. Israel, B. Klartag, and I give a new proof of the finiteness principle. We state our main theorem here. It relies on a quantity $\mathcal{C}(E)$ called the complexity of the set $E$, which we will discuss later in this section.

**Theorem 49.** *Given integers $m, d \geq 1$, there exist $\lambda_1, \lambda_2 \geq 1$ determined by $m$ and $d$ such that the following holds. Fix a finite set $E \subset \mathbb{R}^d$ and a function $f : E \to \mathbb{R}$. Set $k^{\#} = 2^{\lambda_1 \mathcal{C}(E)}$ and $C^{\#} = 2^{\lambda_2 \mathcal{C}(E)}$. Suppose that, for every subset $S \subset E$ with $\#(S) \leq k^{\#}$ there exists a function $F^S \in C^{m-1,1}(\mathbb{R}^d)$ with $||F^S||_{C^{m-1,1}(\mathbb{R}^d)} \leq 1$ and $F^S(x) = f(x)$ for all $x \in S$.*

*Then there exists a function* $F \in C^{m-1,1}(\mathbb{R}^d)$ *with* $||F||_{C^{m-1,1}(\mathbb{R}^d)} \leq C^{\#}1$ *and* $F(x) = f(x)$ *for all* $x \in E$*.*

This proof is the content of this chapter. Later in this section, we will compare the constants given by our proof to the constant $C^{\#}$ above. First, however, we give an overview of our proof and compare it with Fefferman's.

We begin with the observation that in order to prove Theorem 46, it suffices to prove the following local version.

**Theorem 50** (Local Finiteness Principle)**.** *Given a set* $E \subset \mathbb{R}^d$*, a function* $f : E \to \mathbb{R}$*, and a ball* $B \subset \mathbb{R}^d$*, there exists constant* $k^{\#}$ *and* $C^{\#}$ *depending only on* $m$ *and* $d$ *such that the following holds. If for every subset* $S \subset E$ *with* $\#(S) \leq k^{\#}$ *there exists a function* $F^S \in C^{m-1,1}(\mathbb{R}^d)$ *with* $F^S = f$ *on* $S$ *and* $||F^S|| \leq M$ *then there exists a function* $F \in C^{m-1,1}(\mathbb{R}^d)$ *with* $F = f$ *on* $E \cap B$ *and* $||F|| \leq C^{\#}M$*.*

To see that this implies Theorem 46, let $B_0$ be a ball which contains $E$ and apply Theorem 50 to get an extension which agrees with $f$ on $E \cap B_0 = E$.

Both our proof and Fefferman's proof are based on proving this local version of the theorem by induction on a quantity that measures the difficulty of the local interpolation problem. Our proofs differ, however, in the ways that we measure difficulty.

Note the points of $E$ impose constraints on the local interpolation problem. The more constraints, the more careful we have to be in finding an interpolant and therefore the more difficult the associated local problem. To make this precise, Fefferman assigns each problem a label $\mathcal{A}$, which is an element of the power set $2^{\mathcal{M}}$. This label tells us in which coordinate directions of the vector space $\mathcal{P}$ of $(m-1)$st degree polynomials we have constraints. He then puts an order relation on the set $2^{\mathcal{M}}$ to measure the difficulty. The "easiest" problem, which is assigned the label $\mathcal{M}$, is the smallest element with respect to this order relation. The largest element is the empty set, which corresponds to a problem with no label.

Through a multiscale analysis which relies on an elaborate decomposition scheme, Fefferman shows that we can always decompose a local extension problem into easier problems on smaller balls. Since the problems get strictly easier, and since there are a finite number of labels, this process will eventually terminate. This is a serious oversimplification of the proof, but it gives the correct order of magnitude, i.e. $2^D$ where $D = \dim(\mathcal{P}) = \binom{d+m-1}{d}$, for the number of steps in the induction. This implies the bound $C^{\#} \leq c_1 2^{c_2 2^D}$, where $c_1$ and $c_2$ are constants depending on $m$ and $d$.

Our local extension theorem is proved by induction on a different quantity. Instead of ordering the local problems in a way that depends on the label $\mathcal{A}$, we order them in a way that depends on how many times the label changes. More specifically, we observe that the property of having some label $\mathcal{A}$ is equivalent to a transversality condition between a set $\sigma$ which depends only on $E$ and a dilation and translation invariant subspace depending on $\mathcal{A}$. We induct on a quantity which we call the complexity of the problem. It measures how many times the transversality conditions change as the scale of the problem decreases. The hope is that this number is much smaller than $2^D$ and therefore gives an improved constant $C^\#$ in Theorem 46.

In its current stage, our argument emphasizes compactness arguments and algebraic methods. For this reason, some of the constants are either non-explicit or depend poorly on $m$ and $d$. In particular, we prove that the complexity is always bounded by a constant $K_0$ which is not explicit. By the use of more direct methods (which will lengthen the proofs), it is possible to obtain $K_0 = \exp(\exp(\gamma D))$ for some constant $\gamma$ depending on $m$ and $d$. While this implies a worse bound for $C^\#$ then Fefferman's proof, we conjecture that it is far from optimal. In fact, evidence suggests that it is possible to take $K_0$ to be an explicit polynomial function of $D$.

One reason for thinking this is that it is true in every case we have been able to calculate so far. For example, when $E$ is an equispaced $N \times N$ grid in $\mathbb{R}^2$, for $N \gg m$, then $\mathcal{C}(E) \approx \log(m)$, whereas $p(E) \approx 2^{m^2}$. We will discuss ideas we have for improving these bounds in Section 4.9.

We now give an outline of this chapter. In Section 4.1, we introduce notation and prove some preliminary lemmas. Section 4.2 introduces the notion of transversality, and section 4.3 introduces the notion of complexity. In Section 4.4 set up the local interpolation problem. In Section 4.5 we setup the induction, and in Section 4.6 we prove the main decomposition lemma, which is instrumental in carrying out the induction. Section 4.7 is a very technical section in which we prove compatibility of the local extensions. Finally, in Section 4.8 we put all of this together to complete the proof of the local finiteness principle.

## 4.1 Notation, definitions, and preliminary lemmas

Given a convex domain $G \subset \mathbb{R}^n$ with nonempty interior, we let $C^{m-1,1}(G)$ denote the space of real-valued functions $F : G \to \mathbb{R}$ whose $(m-1)$-st order partial derivatives are Lipschitz continuous. Define a seminorm on $C^{m-1,1}(G)$ by

$$\|F\|_{C^{m-1,1}(G)} := \sup_{x,y\in G} \left( \sum_{|\alpha|=m-1} \frac{(\partial^\alpha F(x) - \partial^\alpha F(y))^2}{|x-y|^2} \right)^{\frac{1}{2}}, \quad F \in C^{m-1,1}(G).$$

The seminorm on $C^{m-1,1}(\mathbb{R}^n)$ is abbreviated by $\|F\| := \|F\|_{C^{m-1,1}(\mathbb{R}^n)}$.

Let $\mathcal{P}$ be the space of polynomials of degree at most $m-1$ in $n$ real variables. Let us review some of the structure and basic properties of $\mathcal{P}$. First, $\mathcal{P}$ is a vector space of dimension $D := \#\{\alpha \in \mathbb{Z}^n_{\geq 0} : |\alpha| \leq m-1\}$. For $x \in \mathbb{R}^n$, define an inner product on $\mathcal{P}$:

$$\langle P, Q\rangle_x := \sum_{|\alpha|\leq m-1} (\frac{1}{\alpha!})\partial^\alpha P(x) \cdot \partial^\alpha Q(x),$$

where $\alpha! = \prod_{i=1}^n \alpha_i!$ and we also set $x^\alpha = \prod_{i=1}^n x_i^{\alpha_i}$. If $P(z) = \sum_{|\alpha|\leq m-1} a_\alpha \cdot (z-x)^\alpha$ and $Q(z) = \sum_{|\alpha|\leq m-1} b_\alpha \cdot (z-x)^\alpha$, then $\langle P, Q\rangle_x = \sum_{|\alpha|\leq m-1} \alpha! \cdot a_\alpha b_\alpha$. Therefore, the inner product space $(\mathcal{P}, \langle\cdot,\cdot\rangle_x)$ admits an orthonormal basis of monomials $\{\sqrt{\alpha!} \cdot (z-x)^\alpha\}_{|\alpha|\leq m-1}$. We define a norm on $\mathcal{P}$ by $|P|_x := \sqrt{\langle P, P\rangle_x}$.

We define translation operators $T_h : \mathcal{P} \to \mathcal{P}$ (for $h \in \mathbb{R}^n$) by $T_h(P)(z) := P(z-h)$, and dilation operators $\tau_{x,\delta} : \mathcal{P} \to \mathcal{P}$ (for $(x,\delta) \in$

$\mathbb{R}^n \times (0, \infty))$ by $\tau_{x,\delta}(P)(z) := \delta^{-m} P(x + \delta \cdot (z - x))$. The dilation operators lead us to define a scaled inner product on $\mathcal{P}$: For $(x, \delta) \in \mathbb{R}^n \times (0, \infty)$, let

$$\langle P, Q \rangle_{x,\delta} := \langle \tau_{x,\delta}(P), \tau_{x,\delta}(Q) \rangle_x \qquad (P, Q \in \mathcal{P}),$$

and the corresponding scaled norm is denoted by $|P|_{x,\delta} := \sqrt{\langle P, P \rangle_{x,\delta}}$. The unit ball associated to this norm is the subset

$$\mathcal{B}_{x,\delta} := \left\{ P : \ |P|_{x,\delta} = \left( \sum_{|\alpha| \le m-1} (\frac{1}{\alpha!})(\delta^{|\alpha|-m} \cdot \partial^\alpha P(x))^2 \right)^{\frac{1}{2}} \le 1 \right\} \subset \mathcal{P}.$$

We write $\langle \cdot, \cdot \rangle$ and $|\cdot|$ to denote the "standard" inner product $\langle \cdot, \cdot \rangle_{0,1}$ and norm $|\cdot|_{0,1}$ on $\mathcal{P}$, and $\mathcal{B} = \mathcal{B}_{0,1}$ for the corresponding unit ball.

Given $\Omega \subset \mathcal{P}$, $P_0 \in \mathcal{P}$, and $r \in \mathbb{R}$, let $r\Omega := \{rP : P \in \Omega\}$ and $P_0 + \Omega := \{P_0 + P : P \in \Omega\}$. For future use, we record below a few identities and inequalities which connect the dilation and translation operators with the scaled inner products, norms, and balls.

(a) (i) $T_{h_1} \circ T_{h_2} = T_{h_1 + h_2}$.

(ii) $\tau_{x,\delta_1} \circ \tau_{x,\delta_2} = \tau_{x,\delta_1 \cdot \delta_2}$.

(iii) $T_h \circ \tau_{x,\delta} = \tau_{x+h,\delta} \circ T_h$.

(b) (i) $\langle \tau_{x,\rho}(P), \tau_{x,\rho}(Q) \rangle_{x,\delta} = \langle P, Q \rangle_{x,\delta \cdot \rho}$.

(ii) $|\tau_{x,\rho}(P)|_{x,\delta} = |P|_{x,\delta \cdot \rho}$.

(iii) $\tau_{x,\rho} \mathcal{B}_{x,\delta} = \mathcal{B}_{x,\delta/\rho}$.

(c) (i) $\langle T_h(P), T_h(Q) \rangle_{x,\delta} = \langle P, Q \rangle_{x-h,\delta}$.

(ii) $|T_h(P)|_{x,\delta} = |P|_{x-h,\delta}$.

(iii) $T_h \mathcal{B}_{x,\delta} = \mathcal{B}_{x+h,\delta}$.

Furthermore, for any $\delta \geq \rho > 0$,

$$\begin{cases} (\rho/\delta)^m \cdot |P|_{x,\rho} \leq |P|_{x,\delta} \leq (\rho/\delta) \cdot |P|_{x,\rho}, \text{ and hence} \\ (\delta/\rho) \cdot \mathcal{B}_{x,\rho} \subset \mathcal{B}_{x,\delta} \subset (\delta/\rho)^m \cdot \mathcal{B}_{x,\rho}. \end{cases} \tag{4.1}$$

Let $J_x F \in \mathcal{P}$ denote the $(m-1)$-jet of a function $F \in C^{m-1,1}(\mathbb{R}^n)$ at $x$, namely, the Taylor polynomial

$$(J_x F)(z) := \sum_{|\alpha| \leq m-1} (\frac{1}{\alpha!}) \partial^\alpha F(x) \cdot (z-x)^\alpha \quad (z \in \mathbb{R}^n).$$

The importance of the norms $|\cdot|_{x,\delta}$ on $\mathcal{P}$ stems from the Taylor and Whitney theorems. According to Taylor's theorem, if $F \in C^{m-1,1}(G)$, where $G$ is any convex domain in $\mathbb{R}^n$ with nonempty interior, then

$$|\partial^\beta (F - J_y F)(x)| \leq C \cdot \|F\|_{C^{m-1,1}(G)} \cdot |x-y|^{m-|\beta|}, \quad \text{for } x, y \in G,\ |\beta| \leq m-1.$$

This is easily seen to imply

$$\begin{cases} |J_x F - J_y F|_{x,\delta} \leq C_T \|F\|_{C^{m-1,1}(G)}, \text{ or equivalently} \\ J_x F - J_y F \in C_T \|F\|_{C^{m-1,1}(G)} \cdot \mathcal{B}_{x,\delta} \quad \text{for } x, y \in G,\ \delta \geq |x-y|, \end{cases} \tag{4.2}$$

where $C_T = C_T(m,n)$ is a constant determined by $m$ and $n$. Therefore the norm $|\cdot|_{x,\delta}$ may be used to describe the compatibility conditions on the $(m-1)$-jets of a $C^{m-1,1}$ function at two points $x, y$ in $\mathbb{R}^n$, whenever $|x-y| \leq \delta$. The conditions in (4.2) capture the essence of the concept of a $C^{m-1,1}$ function in the following sense: Whitney's theorem [68] states that whenever $E \subset \mathbb{R}^n$ is an arbitrary set, $M > 0$, and $\{P_x\}_{x \in E}$ is a collection of polynomials with

$$|P_x - P_y|_{x,\delta} \leq M \qquad \text{for } x, y \in E,\ \delta = |x-y|, \tag{4.3}$$

then there exists a $C^{m-1,1}$ function $F : \mathbb{R}^n \to \mathbb{R}$ with $\|F\| \leq CM$ and $J_x F = P_x$ for all $x \in E$. As usual, $C$ is a constant depending solely on $m$ and $n$.

The vector space of $(m-1)$-jets is a ring, denoted by $\mathcal{P}_x$, equipped with the product $\odot_x$ (indexed by a basepoint $x \in \mathbb{R}^n$) defined by $P \odot_x Q = J_x(P \cdot Q)$. The product and translation/dilation operators are related by

$$\begin{cases} \tau_{x,\delta}\left(P \odot_x Q\right) = \delta^m \cdot \tau_{x,\delta}(P) \odot_x \tau_{x,\delta}(Q), & \\ T_h\left(P \odot_x Q\right) = T_h(P) \odot_{x+h} T_h(Q) & \text{for } x, h \in \mathbb{R}^n,\ \delta > 0. \end{cases} \tag{4.4}$$

The following lemma, taken verbatim from [37, section 12], summarizes a few basic properties of the product and norms introduced above. See the proof of Lemma 1 in [37, section 12] for a direct argument that leads to explicit constants. Our argument below emphasizes the rôle of rescaling and compactness.

**Lemma 51.** *Let $x, y \in \mathbb{R}^n$ and $\delta, \rho > 0$. Assume that $|x - y| \leq \rho \leq \delta$. Then for any $P, Q \in \mathcal{P}$,*

*(i)* $|P|_{y,\rho} \leq \widetilde{C}|P|_{x,\rho}$.

*(ii)* $|P \odot_x Q|_{x,\rho} \leq \widetilde{C}\delta^m |P|_{x,\delta}|Q|_{x,\rho}$.

*(iii)* $|(P \odot_y Q) - (P \odot_x Q)|_{x,\rho} \leq \widetilde{C}\delta^m |P|_{x,\delta}|Q|_{x,\delta}$.

*Here, $\widetilde{C} > 0$ is a constant depending solely on $m$ and $n$.*

*Proof.* The main step is to use (4.4) and observe that by translating and rescaling, we may reduce matters to the case $x = 0$ and $\rho = 1$. Next, note that it suffices to prove the lemma for non-zero polynomials $P$ and $Q$. Normalizing, we assume that $|P|_{0,1} = |Q|_{0,1} = 1$.

In order to prove (i), observe that the space of all relevant parameters is compact, since $|y| \leq 1$ and $|P|_{0,1} = 1$. The left-hand side of (i) is a continuous function on this space of parameters, hence the maximum is attained, and yields the constant $\widetilde{C}$ on the right-hand side. In order to prove (ii), observe that the left-hand side in (ii) is bounded from above by a constant $\widetilde{C}$ by compactness, while

$$\delta^m |P|_{0,\delta} \geq |P|_{0,1} = 1$$

for any $\delta \geq 1$, according to (4.1). Hence (ii) holds true as well. In order to prove (iii), it is more convenient to rescale so that $\delta = 1$, rather than $\rho = 1$. We may still assume that $|P|_{0,1} = |Q|_{0,1} = 1$. Consider the unit ball $B = \{x \in \mathbb{R}^n : |x| \leq 1\}$ and the function $F(x) = P(x)Q(x)$. Yet another compactness argument yields that $\|F\|_{C^{m-1,1}(B)} \leq C_0$ for a constant $C_0$ determined by $m$ and $n$. From Taylor's theorem, rendered above as (4.2),

$$|(P \odot_y Q) - (P \odot_0 Q)|_{0,\rho} = |J_y F - J_0 F|_{0,\rho} \leq C_T \cdot C_0,$$

and the lemma is proven. □

If $|x - y| \leq \lambda\delta$ for some $\lambda \geq 1$, then we have the inequality

$$|P|_{y,\delta} \leq \widetilde{C} \lambda^{m-1} |P|_{x,\delta}, \tag{4.5}$$

or the equivalent inclusion $\mathcal{B}_{x,\delta} \subset \widetilde{C} \lambda^{m-1} \mathcal{B}_{y,\delta}$. Indeed, this follows from (4.1) and Lemma 51:

$$|P|_{y,\delta} \leq \lambda^m |P|_{y,\lambda\delta} \leq \widetilde{C}\lambda^m |P|_{x,\lambda\delta} \leq \widetilde{C}\lambda^{m-1}|P|_{x,\delta}.$$

Furthermore, if $\theta \in C^{m-1,1}(\mathbb{R}^n)$ is supported on a ball $B \subset \mathbb{R}^n$, then

$$|J_x(\theta)|_{x,\mathrm{diam}(B)} \leq C_T\|\theta\| \quad (x \in \mathbb{R}^n). \tag{4.6}$$

Indeed, this inequality is trivial if $x \in \mathbb{R}^n \setminus B$, as then $J_x(\theta) = 0$. Fix $x_0 \in \partial B$. Then $J_{x_0}(\theta) = 0$. As $|x - x_0| \leq \mathrm{diam}(B)$ for any $x \in B$, we may apply Taylor's theorem (rendered as (4.2)) and obtain $|J_x(\theta)|_{x,\mathrm{diam}(B)} = |J_x(\theta) - J_{x_0}(\theta)|_{x,\mathrm{diam}(B)} \leq C_T\|\theta\|$, which yields (4.6).

We next give a more general form of Lemma 51(iii) involving products of up to three polynomials which are allowed to vary from point to point.

**Lemma 52.** *Fix polynomials $P_x, Q_x, R_x$ and $P_y, Q_y, R_y$ in $\mathcal{P}$, for $|x - y| \leq \rho \leq \delta$. Suppose that $P_x, P_y \in M_0\mathcal{B}_{x,\delta}$, $Q_x, Q_y \in M_1\mathcal{B}_{x,\delta}$, and $R_x, R_y \in M_2\mathcal{B}_{x,\delta}$. Also suppose that $P_x - P_y \in M_0\mathcal{B}_{x,\rho}$, $Q_x - Q_y \in M_1\mathcal{B}_{x,\rho}$, and $R_x - R_y \in M_2\mathcal{B}_{x,\rho}$. Then*

$$|P_x \odot_x Q_x \odot_x R_x - P_y \odot_y Q_y \odot_y R_y|_{x,\rho} \leq C\delta^{2m} M_0 M_1 M_2,$$

*where $C$ is a constant determined by $m$ and $n$.*

*Proof.* In view of (4.4), we may assume that $\delta = 1$. By renormalizing, we may assume $M_0 = M_1 = M_2 = 1$. Then all six polynomials belong to $\mathcal{B}_{x,1}$, and the three differences $P_x - P_y$, $Q_x - Q_y$, and $R_x - R_y$ belong to $\mathcal{B}_{x,\rho}$. The letter $x$ appears five times in the expression $P_x \odot_x Q_x \odot_x R_x$, and we will change these five $x$'s to five $y$'s one by one. We first apply Lemma 51(ii) three times and replace $R_x$, $Q_x$, and $P_x$ by $R_y$, $Q_y$, and $P_y$, in that respective order, as follows:

$$|P_x \odot_x Q_x \odot_x R_x - P_y \odot_x Q_y \odot_x R_y|_{x,\rho} \leq C.$$

This step also requires the bounds $|P_x \odot_x Q_x|_{x,1} \leq C$, $|P_x \odot_x R_y|_{x,1} \leq C$, and $|Q_y \odot_x R_y|_{x,1} \leq C$, which are all consequences of Lemma 51(ii). Next we apply Lemma 51(iii) twice, and deduce that

$$|P_y \odot_x Q_y \odot_x R_y - P_y \odot_y Q_y \odot_y R_y|_{x,\rho} \leq C.$$

This step requires the bounds $|P_y \odot_x Q_y|_{x,1} \leq C$ and $|Q_y \odot_y R_y|_{x,1} \leq C$, which follow from Lemma 51(ii) and, for the second inequality, also Lemma 51(iii). This concludes the proof of the lemma. □

*Remark* 53. We can obtain a version of Lemma 52 also for products of two polynomials. Notice that $1 \in \delta^{-m}\mathcal{B}_{x,\delta}$ for any $\delta > 0$. Thus, by taking $P_x = P_y = 1$, under the hypotheses of Lemma 52, $|Q_x \odot_x R_x - Q_y \odot_y R_y|_{x,\rho} \leq C\delta^m M_1 M_2$.

Finally, we state a few elementary facts from convex geometry. A convex set $\Omega$ in a finite-dimensional vector space $\mathcal{V}$ is said to be *symmetric* if $P \in \Omega \implies -P \in \Omega$. If $A$, $K$, and $T$ are symmetric convex sets then

$$K \subset T \implies (A + K) \cap T \subset (A \cap 2T) + K, \tag{4.7}$$

and also if $K$ is bounded then

$$K \subset T + K/3 \implies K \subset 2T. \tag{4.8}$$

To prove (4.7), pick $x \in (A+K)\cap T$. Then $x = a+k$ with $a \in A$ and $k \in K$. It suffices to show that $a \in 2T$. This holds since $a = x - k \in T - K \subset 2T$. Next observe that the condition $K \subset T + K/3$ implies $\sup_{x\in K} f(x) \le \sup_{x\in T} f(x) + \frac{1}{3}\sup_{x\in K} f(x)$ for any linear functional $f : \mathcal{V} \to \mathbb{R}$. If $K$ is bounded, this implies $\frac{2}{3}\sup_{x\in K} f(x) \le \sup_{x\in T} f(x)$. From the Hahn-Banach theorem, $K$ is contained in the closure of $\frac{3}{2}T$, and therefore $K \subset 2T$.

### 4.1.1 Taylor polynomials of functions with prescribed values.

Fix a finite set $E \subset \mathbb{R}^n$ and function $f : E \to \mathbb{R}$ satisfying the hypothesis of Theorem 49. That is, we assume that for some natural number $k^\# \in \mathbb{N}$, the following holds:

$$\mathcal{FH}(k^\#)\begin{cases} \text{For all } S \subset E \text{ with } \#(S) \le k^\# \text{ there exists } F^S \in C^{m-1,1}(\mathbb{R}^n) \\ \quad \text{with } F^S = f \text{ on } S \text{ and } \|F^S\| \le 1. \end{cases} \tag{4.9}$$

We call $\mathcal{FH}(k^\#)$ the *finiteness hypothesis* and $k^\#$ the *finiteness constant*. We aim to construct a function $F \in C^{m-1,1}(\mathbb{R}^n)$ satisfying $F = f$ on $E$ and $\|F\| \le C^\#$ for a suitable constant $C^\# \ge 1$. We first introduce a family of convex subsets of $\mathcal{P}$ that contain information on the Taylor polynomials of extensions associated to subsets of $E$:

$$\Gamma_S(x, f, M) := \{J_x F : F \in C^{m-1,1}(\mathbb{R}^n),\ F = f \text{ on } S,\ \|F\| \le M\},$$
$$\text{for } S \subset E,\ x \in \mathbb{R}^n,\ f : E \to \mathbb{R},\ M > 0.$$

We also denote $\Gamma(x, f, M) := \Gamma_E(x, f, M)$. Notice that $\Gamma_S(x, f, M)$ is nonempty if and only if there exists an extension of the restricted function

$f|_S$ with $C^{m-1,1}$ seminorm at most $M$. Therefore the finiteness hypothesis $\mathcal{FH}(k^\#)$ is equivalent to the condition that $\Gamma_S(x,f,1) \neq \emptyset$ for all $S \subset E$ with $\#(S) \leq k^\#$. Now, for $\ell \in \mathbb{Z}_{\geq 0}$ we define

$$\begin{aligned} \Gamma_\ell(x,f,M) := \{P \in \mathcal{P} : \forall S \subset E,\ \#(S) \leq (D+1)^\ell,\ \exists F^S \in C^{m-1,1}(\mathbb{R}^n), \\ F^S = f \text{ on } S,\ J_x F^S = P,\ \|F^S\| \leq M\}; \end{aligned}$$

here, recall that $D = \dim \mathcal{P}$. In other words, an element of $\Gamma_\ell(x,f,M)$ is simultaneously the jet of a solution to any extension problem associated to a subset $S \subset E$ of cardinality at most $(D+1)^\ell$. The sets denoted by $\Gamma_\ell$ arise in the analysis in [33] as a tool to demonstrate that the set $\Gamma(x,f,M)$ is nonempty – if we can accomplish this, we will have demonstrated the existence of an extension of $f$ with $C^{m-1,1}$ seminorm at most $M$. Finally we note that

$$\Gamma_\ell(x,f,M) = \bigcap_{S \subset E,\ \#(S) \leq (D+1)^\ell} \Gamma_S(x,f,M). \tag{4.10}$$

Given $x \in \mathbb{R}^n$ and $S \subset E$, let

$$\sigma(x,S) := \{J_x \varphi : \varphi \in C^{m-1,1}(\mathbb{R}^n),\ \varphi = 0 \text{ on } S,\ \|\varphi\| \leq 1\},$$

and given $\ell \in \mathbb{Z}_{\geq 0}$, let

$$\sigma_\ell(x) = \bigcap_{S \subset E,\ \#(S) \leq (D+1)^\ell} \sigma(x,S). \tag{4.11}$$

We also denote $\sigma(x) := \sigma(x,E)$.

Note that $\sigma(x)$ and $\sigma_\ell(x)$ are symmetric convex subsets of $\mathcal{P}$, whereas $\Gamma(x, f, M)$ and $\Gamma_\ell(x, f, M)$ are only convex. By a straightforward application of the Arzela-Ascoli theorem one can show that $\sigma(x)$, $\sigma_\ell(x)$, $\Gamma(x, f, M)$, and $\Gamma_\ell(x, f, M)$ are closed. Finally, we observe the relationships $\sigma(x, S) = \Gamma_S(x, 0, 1)$, $\sigma_\ell(x) = \Gamma_\ell(x, 0, 1)$, and $\sigma(x) = \Gamma(x, 0, 1)$.

**Lemma 54** (Relationship between $\Gamma_\ell$ and $\sigma_\ell$)**.** *For any $\ell \in \mathbb{Z}_{\geq 0}$,*

$$\Gamma_\ell(x, f, M/2) + (M/2)\sigma_\ell(x) \subset \Gamma_\ell(x, f, M), \quad \textit{and}$$

$$\Gamma_\ell(x, f, M) - \Gamma_\ell(x, f, M) \subset 2M\sigma_\ell(x).$$

*Proof.* By definition we have $\Gamma_S(x, f, M/2) + (M/2)\sigma(x, S) \subset \Gamma_S(x, f, M)$ and $\Gamma_S(x, f, M) - \Gamma_S(x, f, M) \subset 2M\sigma(x, S)$. The conclusion of the lemma then follows from the definition of $\Gamma_\ell$ and $\sigma_\ell$ in (4.10) and (4.11). □

*Remark* 55. Lemma 54 implies that $P_x + \frac{M}{2} \cdot \sigma_\ell(x) \subset \Gamma_\ell(x, f, M) \subset P_x + 2M \cdot \sigma_\ell(x)$, for any $P_x \in \Gamma_\ell(x, f, M/2)$. Later on we will be concerned with the geometry of the sets $\Gamma_\ell(x, f, M)$ at various points $x \in \mathbb{R}^n$. Lemma 54 implies that it is sufficient to understand the geometry of the sets $\sigma_\ell(x)$ (which depend on fewer parameters and are therefore more manageable).

Recall the translation and scaling transformations $T_h$ and $\tau_{x,\delta}$ on $\mathcal{P}$. With a slight abuse of notation, we also denote the transformations $T_h$ and $\tau_{x,\delta}$ on $\mathbb{R}^n$ given by

$$T_h(y) = y + h, \quad \tau_{x,\delta}(y) = x + \delta \cdot (y - x) \qquad (x, y, h \in \mathbb{R}^n, \delta > 0).$$

Then,

$$\sigma(T_h(y), T_h(S)) = T_h\left\{\sigma(y, S)\right\}, \text{ and } \sigma(\tau_{x,\delta}(y), \tau_{x,\delta}(S)) = \tau_{x,\delta}\left\{\sigma(y, S)\right\}, \tag{4.12}$$

for any $x, y, h \in \mathbb{R}^n$, $\delta > 0$, and $S \subset \mathbb{R}^n$, as may be verified directly. Here in our notation, if $T : \mathbb{R}^n \to \mathbb{R}^n$ then $T(S) = \{T(y) : y \in S\}$.

In the next lemma we establish two important properties of the sets $\Gamma_\ell(x, f, M)$. We show that the finiteness hypothesis $\mathcal{FH}(k^\#)$ (see (4.9)) implies that $\Gamma_\ell(x, f, M)$ is non-empty if $\ell$ and $k^\#$ are suitably related and if $M \geq 1$. We also show that the mappings $x \mapsto \Gamma_\ell(x, f, M)$ are "quasicontinuous" in a sense to be made precise below.

**Lemma 56.** *If $x \in \mathbb{R}^n$, $(D+1)^{\ell+1} \leq k^\#$, and $M \geq 1$, then*

$$\mathcal{FH}(k^\#) \implies \Gamma_\ell(x, f, M) \neq \emptyset. \tag{4.13}$$

*If $x, y \in \mathbb{R}^n$, $\ell \geq 1$, $\delta \geq |x - y|$, and $M > 0$, then*

$$\Gamma_\ell(x, f, M) \subset \Gamma_{\ell-1}(x, f, M) + C_T M \cdot \mathcal{B}_{x,\delta} \tag{4.14}$$

*and*

$$\sigma_\ell(x) \subset \sigma_{\ell-1}(x) + C_T \cdot \mathcal{B}_{x,\delta}, \tag{4.15}$$

*where $C_T$ is the constant in* (4.2).

*Proof.* We first show that the finiteness hypothesis with constant $k^\# \geq (D+1)^{\ell+1}$ implies the intersection of the sets in (4.10) is nonempty for $M = 1$. As $\Gamma(x, f, M) \supset \Gamma(x, f, 1)$ for $M \geq 1$, the implication (4.13) will then follow. By Helly's theorem and the fact that $\dim \mathcal{P} = D$, it suffices to show that the intersection of any $(D+1)$-element subcollection is nonempty. Fix $S_1, \cdots, S_{D+1} \subset E$ with $\#(S_i) \leq (D+1)^\ell$. Let $S := S_1 \cup \cdots \cup S_{D+1}$. Note that $\Gamma_{S_1}(x, f, 1) \cap \cdots \cap \Gamma_{S_{D+1}}(x, f, 1) \supset \Gamma_S(x, f, 1)$. Furthermore, $\#(S) \leq (D+1) \cdot (D+1)^\ell \leq k^\#$, and so $\Gamma_S(x, f, 1) \neq \emptyset$ by the finiteness hypothesis $\mathcal{FH}(k^\#)$. This finishes the proof of (4.13).

To prove (4.14) and (4.15) we reproduce the proof of [34, Lemma 5.6]. Note (4.15) is a special case of (4.14), as $\sigma_\ell(x) = \Gamma_\ell(x, 0, 1)$. So it suffices to prove (4.14). Given $P \in \Gamma_\ell(x, f, M)$, we will find $Q \in \Gamma_{\ell-1}(y, f, M)$ with

$$|P - Q|_{x,\delta} \leq C_T M. \tag{4.16}$$

For a subset $S \subset E$, consider

$$\mathcal{K}(S) := \left\{ J_y F : F \in C^{m-1,1}(\mathbb{R}^n),\ F = f \text{ on } S,\ \|F\| \leq M,\ J_x F = P \right\}.$$

Then $\mathcal{K}(S) \subset \mathcal{P}$ is convex, and according to (4.2),

$$\mathcal{K}(S) \subset P + C_T M \cdot \mathcal{B}_{x,\delta}. \tag{4.17}$$

Note that $\mathcal{K}(S) \neq \emptyset$ whenever $\#(S) \leq (D+1)^\ell$, due to the fact that $P \in \Gamma_\ell(x, f, M)$. We will show that

$$\emptyset \neq \bigcap_{\substack{S \subset E \\ \#(S) \leq (D+1)^{\ell-1}}} \mathcal{K}(S) \subset \Gamma_{\ell-1}(y, f, M). \tag{4.18}$$

The inclusion on the right-hand side of (4.18) is immediate from the definition of $\Gamma_{\ell-1}(y, f, M)$. All that remains is to show that the intersection of the collection of sets in (4.18) is non-empty. By Helly's theorem it suffices to show that the intersection of any $(D+1)$-element subcollection is nonempty. Thus, pick $S_1, \ldots, S_{D+1} \subset E$ with $\#(S_i) \le (D+1)^{\ell-1}$. Then $S = S_1 \cup \ldots \cup S_{D+1}$ is of cardinality at most $(D+1)(D+1)^{\ell-1} = (D+1)^{\ell}$, and thus $\mathcal{K}(S) \neq \emptyset$. Clearly, $\mathcal{K}(S) \subset \mathcal{K}(S_1) \cap \cdots \cap \mathcal{K}(S_{D+1})$. This finishes the proof of (4.18). Fix a polynomial $Q$ belonging to the intersection in (4.18). According to (4.18), $Q \in \Gamma_{\ell-1}(y, f, M)$. By (4.17), $Q \in \mathcal{K}(\emptyset) \subset P + C_T M \cdot \mathcal{B}_{x,\delta}$, and so $Q - P \in C_T M \cdot \mathcal{B}_{x,\delta}$, giving (4.16). □

**Lemma 57.** *If $x, y \in \mathbb{R}^n$, and $\delta \ge |x-y|$, then $\sigma(x) \subset \sigma(y) + C_T \cdot \mathcal{B}_{x,\delta}$.*

*Proof.* Let $P \in \sigma(x)$. Then there exists $\varphi \in C^{m-1,1}(\mathbb{R}^n)$ with $\varphi = 0$ on $E$, $\|\varphi\| \le 1$, and $J_x\varphi = P$. Let $Q = J_y\varphi$. Then $Q \in \sigma(y)$, and by (4.2) we have $P - Q \in C_T \cdot \mathcal{B}_{x,\delta}$. □

*Remark* 58. By (4.1), $\mathcal{B}_{x,\delta} \subset \delta \cdot \mathcal{B}_{x,1}$ for $\delta \le 1$. Therefore, Lemma 57 implies the mapping $x \mapsto \sigma(x)$ is continuous, where the space of subsets of $\mathcal{P}$ carries the topology induced by the Hausdorff metric with respect to any of the topologically equivalent scaled norms.

**Lemma 59.** *There exists a constant $C \ge 1$ determined by $m$ and $n$ so that, for any ball $B \subset \mathbb{R}^n$ and $z \in \frac{1}{2}B$, we have*

$$\sigma(z, E \cap B) \cap \mathcal{B}_{z,diam(B)} \subset C \cdot \sigma(z, E).$$

*Proof.* Choose a cutoff function $\theta \in C^{m-1,1}(\mathbb{R}^n)$ which is supported on $B$, equal to 1 on $(\frac{1}{2})B$, and satisfies $\|\theta\| \le C \cdot \delta^{-m}$. Fix $z \in (\frac{1}{2})B$ and a polynomial

$P \in \sigma(z, E \cap B) \cap \mathcal{B}_{z,\delta}$. Since $P \in \sigma(z, E \cap B)$ there exists $\varphi \in C^{m-1,1}(\mathbb{R}^n)$ with $\varphi = 0$ on $E \cap B$, $\|\varphi\| \leq 1$, and $J_z(\varphi) = P$. Define $\widetilde{\varphi} = \varphi\theta$. This function clearly vanishes on all of $E$. Since $z$ belongs to the ball $(\frac{1}{2})B$ on which $\theta$ is identically 1, we have $J_z(\widetilde{\varphi}) = J_z(\varphi) = P$. To prove $P \in C\sigma(z, E)$, all that remains is to establish the seminorm bound $\|\widetilde{\varphi}\| \leq C$. As $\widetilde{\varphi}$ vanishes on $\mathbb{R}^n \setminus B$, it suffices to prove $\|\widetilde{\varphi}\|_{C^{m-1,1}(B)} \leq C$. To do so, we will prove that

$$|J_x(\widetilde{\varphi}) - J_y(\widetilde{\varphi})|_{x,\rho} = |J_x(\varphi) \odot_x J_x(\theta) - J_y(\varphi) \odot_y J_y(\theta)|_{x,\rho} \leq C \tag{4.19}$$

for $x, y \in B$ and $\rho = |x - y|$.

To prove this estimate we will apply Lemma 52. According to (4.6), $J_x(\theta) \in C\delta^{-m}\mathcal{B}_{x,\delta}$. On the other hand, by (4.5) and the fact $|x - y| \leq \delta$, also $J_y(\theta) \in C\delta^{-m}\mathcal{B}_{y,\delta} \subset C'\delta^{-m}\mathcal{B}_{x,\delta}$. By Taylor's theorem (in the form (4.2)), $J_x(\theta) - J_y(\theta) \in C\|\theta\|\mathcal{B}_{x,\rho} \subset C\delta^{-m}\mathcal{B}_{x,\rho}$.

Note that $|x - z| \leq \delta$, since $x \in B$ and $z \in (\frac{1}{2})B$. Thus, by Taylor's theorem (see (4.2)) and (4.5), $J_x(\varphi) = (J_x(\varphi) - J_z(\varphi)) + P \in C_T\mathcal{B}_{x,\delta} + \mathcal{B}_{z,\delta} \subset C_T\mathcal{B}_{x,\delta} + \widetilde{C}\mathcal{B}_{x,\delta} \subset C\mathcal{B}_{x,\delta}$. On the other hand, by Taylor's theorem, $J_x(\varphi) - J_y(\varphi) \in C_T\mathcal{B}_{x,\rho}$. We are therefore in a position to apply Lemma 52 (see Remark 53), with $Q_x$, $Q_y$, $R_x$, and $R_y$ picked to be the jets at $x$ and $y$ of $\varphi$ and $\theta$, respectively. This finishes the proof of (4.19). $\square$

### 4.1.2 Whitney convexity

The next definition illustrates an additional important property of the sets $\sigma_\ell(x)$ beyond convexity.

**Definition 60** (Whitney convexity)**.** Given a symmetric convex set $\Omega$ in $\mathcal{P}$, and $x \in \mathbb{R}^n$, the Whitney coefficient of $\Omega$ at $x$ is the infimum over all $R > 0$ such that $(\Omega \cap \mathcal{B}_{x,\delta}) \odot_x \mathcal{B}_{x,\delta} \subset R\delta^m\Omega$ for all $\delta > 0$. Denote the Whitney coefficient of $\Omega$ at $x$ by $w_x(\Omega)$. If no finite $R$ exists, then $w_x(\Omega) = +\infty$. If $w_x(\Omega) < +\infty$ then we say that $\Omega$ is Whitney convex at $x$.

The term "Whitney convexity" was coined by Fefferman [34]. It is a quantitative analog of the concept of an ideal; roughly, a small Whitney coefficient means that $\Omega$ is "close" to an ideal. For example, any ideal $I$ in $\mathcal{P}_x$ is Whitney convex at $x$ with $w_x(I) = 0$.

For $x \in \mathbb{R}^n$, a symmetric convex set $\Omega \subset \mathcal{P}$ and $r \geq 1$, it holds that $w_x(r\Omega) \leq w_x(\Omega)$. One can also check that $w_x(\Omega_1 \cap \Omega_2) \leq \max\{w_x(\Omega_1), w_x(\Omega_2)\}$. Furthermore, it follows from (4.4) that $w_x(\Omega) = w_x(\tau_{x,\delta}(\Omega))$ and $w_x(\Omega) = w_{x+h}(T_h\Omega)$ for any $\delta > 0$.

**Lemma 61.** *For any $z \in \mathbb{R}^n$, the sets $\sigma_\ell(z)$ and $\sigma(z)$ are Whitney convex at $z$ with Whitney coefficient at most $C_0$, for a universal constant $C_0 = C_0(m, n)$.*

*Proof.* Note that $w_x(\sigma_\ell(z)) \leq \max\{w_z(\sigma(x, S)) : S \subset E,\ \#(S) \leq (D+1)^\ell\}$. Hence, it will be sufficient to show that the Whitney coefficient of $\sigma(z, S)$ at $z$ is at most $C$ for any subset $S \subset E$ and $z \in \mathbb{R}^n$, where $C$ is a constant determined by $m$ and $n$. Fix $\delta > 0$, and let $P \in \sigma(z, S) \cap \mathcal{B}_{z,\delta}$ and $\widetilde{P} \in \mathcal{B}_{z,\delta}$. In order to prove the lemma, we need to show that

$$P \odot_z \widetilde{P} \in C\delta^m \sigma(z, S). \tag{4.20}$$

Since $P \in \sigma(z, S)$, there exists $\varphi \in C^{m-1,1}(\mathbb{R}^n)$ with $\varphi = 0$ on $S$, $J_z(\varphi) = P$, and $\|\varphi\| \leq 1$. Fix a $C^\infty$-function $\theta : \mathbb{R}^n \to \mathbb{R}$, which is supported on the ball $B = \{y \in \mathbb{R}^n : |y - z| \leq \frac{\delta}{2}\}$, which equals one in a neighborhood of $z$, and satisfies $\|\theta\| \leq C\delta^{-m}$ for a constant $C$ determined by $m$ and $n$. Since $J_z(\theta) = 1$ and $J_z(\varphi) = P$, we conclude that $J_z(\theta\widetilde{P}\varphi) = 1 \odot_z \widetilde{P} \odot_z P = \widetilde{P} \odot_z P$. In order to establish (4.20) and conclude the proof of the lemma, it therefore suffices to show $J_z(\theta\widetilde{P}\varphi) \in C\delta^m\sigma(z, S)$. Since the function $\theta\widetilde{P}\varphi$ vanishes on $S$ (as does $\varphi$), all that remains is to establish the seminorm bound $\|\theta\widetilde{P}\varphi\| \leq C\delta^m$, and as this function vanishes on $\mathbb{R}^n \setminus B$, it suffices to establish $\|\theta\widetilde{P}\varphi\|_{C^{m-1,1}(B)} \leq C\delta^m$. To that end, we need to show that

$$\left| J_x(\theta) \odot_x \widetilde{P} \odot_x J_x(\varphi) - J_y(\theta) \odot_y \widetilde{P} \odot_y J_y(\varphi) \right|_{x,\rho} \leq C\delta^m, \tag{4.21}$$

for $x, y \in B$ and $\rho = |x - y|$.

We prepare to apply Lemma 52 to prove this estimate.

As in the proof of Lemma 59 (using that $J_z(\varphi) = P \in \mathcal{B}_{z,\delta}$ and $\mathrm{diam}(\{x, y, z\}) \leq \delta = \mathrm{diam}(B)$), and by (4.6), the jets $J_x(\varphi)$, $J_y(\varphi)$ belong to $C\mathcal{B}_{x,\delta}$; and $J_x(\theta)$, $J_y(\theta)$ belong to $C\delta^{-m}\mathcal{B}_{x,\delta}$. Furthermore, $\widetilde{P} \in \mathcal{B}_{z,\delta}$, and hence by (4.5), $\widetilde{P} \in \widetilde{C}\mathcal{B}_{x,\delta}$. Finally, by Taylor's theorem (rendered as (4.2)), $J_x(\varphi) - J_y(\varphi) \in C\mathcal{B}_{x,\rho}$ and $J_x(\theta) - J_y(\theta) \in C\delta^{-m}\mathcal{B}_{x,\rho}$.

We are in a position to apply Lemma 52, with $P_x$, $P_y$, $R_x$, and $R_y$ picked to be the jets at $x$ and $y$ of $\varphi$ and $\theta$, respectively, and with $Q_x = Q_y = \widetilde{P}$. This finishes the proof of the estimate (4.21), and with it the proof of (4.20). $\square$

**Lemma 62.** *If $\Omega$ is Whitney convex at $x$, then span($\Omega$) is an $\odot_x$-ideal in $\mathcal{P}_x$.*

*Proof.* Choose any $R \in (w_x(\Omega), \infty)$. Then $(\Omega \cap \mathcal{B}_{x,\delta}) \odot_x \mathcal{B}_{x,\delta} \subset R\delta^m\Omega$ for all $\delta > 0$, and so

$$\Omega \odot_x \mathcal{P}_x = \bigcup_{\delta>0} (\Omega \cap \mathcal{B}_{x,\delta}) \odot_x \mathcal{B}_{x,\delta} \subset \bigcup_{\delta>0} R\delta^m\Omega = \mathrm{span}(\Omega).$$

Thus, $\mathrm{span}(\Omega) \odot_x \mathcal{P}_x = \bigcup_{r>0} r \cdot \Omega \odot_x \mathcal{P}_x \subset \mathrm{span}(\Omega)$, and hence $\mathrm{span}(\Omega)$ is an $\odot_x$-ideal. □

### 4.1.3 Covering lemmas

This section contains a few elementary covering lemmas that will be used later in the paper.

#### 4.1.3.1 Whitney covers

**Definition 63.** A finite collection of closed balls $\mathcal{W}$ is a *Whitney cover* of a ball $\widehat{B} \subset \mathbb{R}^n$ if (a) $\mathcal{W}$ is a cover of $\widehat{B}$, (b) the collection of third-dilates $\{\frac{1}{3}B : B \in \mathcal{W}\}$ is pairwise disjoint, and (c) $\mathrm{diam}(B_1)/\mathrm{diam}(B_2) \in [1/8, 8]$ for all balls $B_1, B_2 \in \mathcal{W}$ with $\frac{6}{5}B_1 \cap \frac{6}{5}B_2 \neq \emptyset$.

**Lemma 64** (Bounded overlap)**.** *If $\mathcal{W}$ is Whitney cover of $\widehat{B}$ then $\#\{B \in \mathcal{W} : x \in \frac{6}{5}B\} \leq 100^n$ for all $x \in \mathbb{R}^n$.*

*Proof.* We may assume $\mathcal{W}_x := \{B \in \mathcal{W} : x \in \frac{6}{5}B\}$ is nonempty, and fix $B_0 \in \mathcal{W}_x$ of maximal radius. By rescaling, we may assume $\mathrm{diam}(B_0) = 1$. If $B \in \mathcal{W}_x$ then $\frac{6}{5}B \cap \frac{6}{5}B_0 \neq \emptyset$, and so condition (c) of Definition 63 implies that diam$(B) \in [\frac{1}{8}, 1]$; thus, by the triangle inequality, $\frac{1}{3}B \subset (\frac{12}{5} + \frac{1}{3})B_0 = \frac{41}{15}B_0$ for all $B \in \mathcal{W}_x$. Since the collection $\{\frac{1}{3}B\}_{B \in \mathcal{W}}$ is pairwise disjoint, a volume comparison shows that $\#\mathcal{W}_x \leq (24 \cdot \frac{41}{15})^n \leq 100^n$. □

#### 4.1.3.2 Partitions of unity

**Lemma 65** (Existence of partitions of unity)**.** *If $\mathcal{W}$ is a Whitney cover of $\widehat{B} \subset \mathbb{R}^n$, then there exist non-negative $C^\infty$ functions $\theta_B : \widehat{B} \to [0, \infty)$ $(B \in \mathcal{W})$ such that*

1. *$\theta_B = 0$ on $\widehat{B} \setminus \frac{6}{5}B$.*
2. *$|\partial^\alpha \theta_B(x)| \leq C\,diam(B)^{-|\alpha|}$ for all $|\alpha| \leq m$ and $x \in \widehat{B}$.*
3. *$\sum_{B \in \mathcal{W}} \theta_B = 1$ on $\widehat{B}$.*

*Here, $C$ is a constant determined by $m$ and $n$.*

*Proof.* For each $B \in \mathcal{W}$, fix a $C^\infty$ cutoff function $\psi_B : \mathbb{R}^n \to \mathbb{R}$ which is supported on $\frac{6}{5}B$, equals 1 on $B$, and satisfies the natural derivative bounds $|\partial^\alpha \psi_B(x)| \leq C \text{diam}(B)^{-|\alpha|}$ for $x \in \mathbb{R}^n$, $|\alpha| \leq m$. Set $\Psi = \sum_{B \in \mathcal{W}} \psi_B$ and define

$$\theta_B(x) := \psi_B(x)/\Psi(x), \quad x \in \widehat{B}.$$

Each point in $\widehat{B}$ belongs to some $B \in \mathcal{W}$, thus $\Psi \geq 1$ on $\widehat{B}$. Thus $\theta_B \in C^\infty(\widehat{B})$ is well-defined. Property 1 follows because $\psi_B$ is supported on $\frac{6}{5}B$. Furthermore, $\sum_B \theta_B = \sum_B \psi_B/\Psi = 1$ on $\widehat{B}$, yielding property 3.

Property 2 is trivial for $x \in \widehat{B} \setminus \frac{6}{5}B$, as then $J_x(\theta_B) = 0$. Now fix $x \in \frac{6}{5}B \cap \widehat{B}$. If $\psi_{B'}(x) \neq 0$ then $x \in \frac{6}{5}B'$. In particular, $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$, and hence $\mathrm{diam}(B')/\mathrm{diam}(B) \in [\frac{1}{8}, 8]$. Furthermore, by Lemma 64, the cardinality of $\mathcal{W}_x := \{B' : x \in \frac{6}{5}B'\}$ is at most $100^n$. Hence,

$$|\partial^\alpha \Psi(x)| \leq \sum_{B' \in \mathcal{W}_x} |\partial^\alpha \psi_{B'}(x)| \leq \sum_{B' \in \mathcal{W}_x} C\mathrm{diam}(B')^{-|\alpha|} \leq C'\mathrm{diam}(B)^{-|\alpha|}. \tag{4.22}$$

By a repeated application of the quotient rule, and substituting the bounds (4.22) and $|\partial^\alpha \psi_B(x)| \leq C\mathrm{diam}(B)^{-|\alpha|}$, we conclude that $|\partial^\alpha \theta_B(x)| = |\partial^\alpha(\psi_B/\Psi)(x)| \leq C''\mathrm{diam}(B)^{-|\alpha|}$. □

We obtain two additional properties of the partition of unity $\{\theta_B\}$ in Lemma 65. First, by property 2 of Lemma 65 and the definition of the scaled norm $|\cdot|_{x,\delta}$,

$$|J_x(\theta_B)|_{x,\mathrm{diam}(B)} \leq C\mathrm{diam}(B)^{-m} \quad (x \in \widehat{B}). \tag{4.23}$$

By the equivalence of $C^{m-1,1}(\widehat{B})$ and the homogeneous Sobolev space $\dot{W}^{m,\infty}(\widehat{B})$ and by property 2 of Lemma 65,

$$\|\theta_B\|_{C^{m-1,1}(\widehat{B})} \leq C \max_{|\alpha|=m} \|\partial^\alpha \theta_B\|_{L^\infty(\widehat{B})} \leq C\mathrm{diam}(B)^{-m}. \tag{4.24}$$

**Lemma 66** (Gluing lemma)**.** *Fix a Whitney cover $\mathcal{W}$ of $\widehat{B}$, a partition of unity $\{\theta_B\}_{B\in\mathcal{W}}$ as in Lemma 65, and points $x_B \in \frac{6}{5}B$ for each $B \in \mathcal{W}$. Suppose $\{F_B\}_{B\in\mathcal{W}}$ is a collection of functions in $C^{m-1,1}(\mathbb{R}^n)$ with the following properties:*

- $\|F_B\| \leq M_0$.

- $F_B = f$ *on* $E \cap \frac{6}{5}B$.

- $|J_{x_B}F_B - J_{x_{B'}}F_{B'}|_{x_B, diam(B)} \le M_0$ *whenever* $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$.

*Let* $F = \sum_{B \in \mathcal{W}} \theta_B F_B$. *Then* $F \in C^{m-1,1}(\widehat{B})$ *with* $F = f$ *on* $E \cap \widehat{B}$ *and* $\|F\|_{C^{m-1,1}(\widehat{B})} \le CM_0$, *where* $C$ *is a constant determined by* $m$ *and* $n$.

*Proof.* The nonzero terms in the sum $F(x) = \sum_B \theta_B(x)F_B(x)$, $x \in E \cap \widehat{B}$, occur when $x \in \frac{6}{5}B$. By assumption, $F_B(x) = f(x)$ for such $B$. Thus $F(x) = \sum_B \theta_B(x)f(x) = f(x)$. Therefore, $F = f$ on $E \cap \widehat{B}$.

We will now bound the seminorm of $F$. Recall the following well-known characterization: $F \in C^{m-1,1}(\widehat{B})$ if and only if there exists $\epsilon > 0$ and $M \ge 0$ such that for any $x, y \in \widehat{B}$ with $|x - y| \le \epsilon$ and any multiindex $\alpha$ with $|\alpha| = m - 1$, we have $|\partial^\alpha F(x) - \partial^\alpha F(y)| \le M \cdot |x - y|$. Furthermore, the seminorm $\|F\|_{C^{m-1,1}(\widehat{B})}$ is comparable to the least possible $M$ up to constant factors depending on $m$ and $n$. This characterization is an easy consequence of the triangle inequality on $\mathbb{R}^n$. Thus, it suffices to prove that if $|x-y| \le \frac{1}{100}\delta_{\min}$ for $\delta_{\min} := \min_{B \in \mathcal{W}} \mathrm{diam}(B)$, then

$$|J_x(F) - J_y(F)|_{x,\rho} \le CM_0, \text{ for } \rho := |x - y|. \tag{4.25}$$

Fix an arbitrary ball $B_0 \in \mathcal{W}$ with $x \in B_0$. Since $|x - y| \le \frac{1}{100}\mathrm{diam}(B_0)$, we have that both $x$ and $y$ belong to $\frac{6}{5}B_0$. Note that $\sum_B J_x(\theta_B) = \sum_B J_y(\theta_B) = 1$. This lets us write

$$
\begin{aligned}
J_x(F) - J_y(F) = \sum_{B \in \mathcal{W}} \Big[ & (J_x(F_B) - J_x(F_{B_0})) \odot_x J_x(\theta_B) \\
& - (J_y(F_B) - J_y(F_{B_0})) \odot_y J_y(\theta_B) \Big] \\
& + (J_x(F_{B_0}) - J_y(F_{B_0})).
\end{aligned}
$$

The summands in the main sum on the right-hand side are nonzero only if $x \in \frac{6}{5}B$ or $y \in \frac{6}{5}B$. By Lemma 64, there can be at most $2 \cdot 100^n$ many elements $B \in \mathcal{W}$ with this property. Therefore, to prove inequality (4.25) it suffices to show that the $|\cdot|_{x,\rho}$ norm of each summand on the right-hand side is at most $CM_0$. To start, consider the last term and apply Taylor's theorem (in the form (4.2)):

$$
|J_x(F_{B_0}) - J_y(F_{B_0})|_{x,\rho} \le C_T \|F_{B_0}\| \le CM_0.
$$

Next we select a summand in the main sum by fixing an element $B \in \mathcal{W}$ with either $x \in \frac{6}{5}B$ or $y \in \frac{6}{5}B$. In either case, $\frac{6}{5}B \cap \frac{6}{5}B_0 \neq \emptyset$. Let $\delta := \mathrm{diam}(B)$. By condition (c) in the definition of a Whitney cover (see Definition 63), we have $\delta/\mathrm{diam}(B_0) \in [\frac{1}{8}, 8]$. Define four polynomials $P_x = J_x(F_B) - J_x(F_{B_0})$ and $R_x = J_x(\theta_B)$, and similarly $P_y = J_y(F_B) - J_y(F_{B_0})$ and $R_y = J_y(\theta_B)$. We will be finished once we show that

$$
|P_x \odot_x R_x - P_y \odot_y R_y|_{x,\rho} \le CM_0. \tag{4.26}
$$

We will prove (4.26) using Lemma 52 (specifically, the form in Remark 53). Let us verify that the hypotheses of this lemma are satisfied. Using

$|x-y|=\rho$ and Taylor's theorem (see (4.2)),

$$|P_x - P_y|_{x,\rho} \leq |J_x(F_B) - J_y(F_B)|_{x,\rho} + |J_x(F_{B_0}) - J_y(F_{B_0})|_{x,\rho} \tag{4.27}$$

$$\leq C_T \cdot (\|F_B\| + \|F_{B_0}\|) \leq CM_0.$$

Next write $|P_x|_{x,\delta} \leq |P_{x_{B_0}} - P_x|_{x,\delta} + |P_{x_{B_0}}|_{x,\delta}$. As $x \in B_0$ and $x_{B_0} \in \frac{6}{5}B_0$, we have $|x - x_{B_0}| \leq \frac{6}{5}\text{diam}(B_0) \leq 3\delta$. Thus, by (4.1) and following the proof of (4.27), $|P_{x_{B_0}} - P_x|_{x,\delta} \leq 3^m |P_{x_{B_0}} - P_x|_{x,3\delta} \leq C' M_0$. Then by (4.1) and (4.5), the hypothesis in the third bullet point of this lemma, and another application of Taylor's theorem,

$$\begin{aligned}
|P_{x_{B_0}}|_{x,\delta} &\leq |J_{x_B}(F_B) - J_{x_{B_0}}(F_{B_0})|_{x,\delta} + |J_{x_B}(F_B) - J_{x_{B_0}}(F_B)|_{x,\delta} \\
&\leq C|J_{x_B}(F_B) - J_{x_{B_0}}(F_{B_0})|_{x_{B_0},\delta} + C|J_{x_B}(F_B) - J_{x_{B_0}}(F_B)|_{x_{B_0},\delta} \\
&\leq C'|J_{x_B}(F_B) - J_{x_{B_0}}(F_{B_0})|_{x_{B_0},\text{diam}(B_0)} + C'|J_{x_B}(F_B) - J_{x_{B_0}}(F_B)|_{x_{B_0},4\delta} \\
&\leq C'' M_0.
\end{aligned}$$

Here, note we are using that $|x_B - x_{B_0}| \leq \frac{6}{5}\text{diam}(B) + \frac{6}{5}\text{diam}(B_0) \leq 4\delta$ in the final application of Taylor's theorem. In conclusion, $|P_x|_{x,\delta} \leq CM_0$. By the identical argument, $|P_y|_{y,\delta} \leq CM_0$ – then by (4.5), $|P_y|_{x,\delta} \leq C'M_0$.

Next, note the estimate $|R_x - R_y|_{x,\rho} \leq C\delta^{-m}$ is a direct consequence of Taylor's theorem and (4.24). Also, $|R_x|_{x,\delta} \leq C\delta^{-m}$ is a direct consequence of (4.23). Similarly, $|R_y|_{y,\delta} \leq C\delta^{-m}$, and thus by (4.5), $|R_y|_{x,\delta} \leq C'\delta^{-m}$.

We obtain (4.26) by an application of Lemma 52 (see Remark 53), which finishes the proof of the lemma. □

## 4.2 Transversality

Let $(X, \langle\cdot,\cdot\rangle)$ be a real Hilbert space of finite dimension $d := \dim X < \infty$. We denote the norm of $X$ by $|\cdot| = \sqrt{\langle\cdot,\cdot\rangle}$, and let $\mathcal{B}$ be the unit ball of $X$. Let $\mathcal{S}$ denote the set of symmetric, closed, convex subsets of $X$, and let $d_H : \mathcal{S} \times \mathcal{S} \to [0, \infty]$ be the Hausdorff metric, namely,

$$d_H(\Omega_1, \Omega_2) := \inf\{\epsilon > 0 : \Omega_1 \subset \Omega_2 + \epsilon\mathcal{B},\ \Omega_2 \subset \Omega_1 + \epsilon\mathcal{B}\}.$$

Given a set $A \subset X$ and subspace $V \subset X$, let $A/V$ (the *quotient* of $A$ by $V$) be the image of $A$ under the quotient mapping $\pi : X \to X/V$, i.e., $A/V := \{a + V : a \in A\}$.

**Definition 67.** Let $V$ be a linear subspace of $X$, let $\Omega \in \mathcal{S}$, and let $R \geq 1$. We say that $\Omega$ is $R$-transverse to $V$ if (1) $\mathcal{B}/V \subset R \cdot (\Omega \cap \mathcal{B})/V$, and (2) $\Omega \cap V \subset R \cdot \mathcal{B}$.

*Remark* 68. Transversality captures the idea that there is a uniform lower bound on the angle between the subspace $V$ and the "large" vectors of $\Omega$. If $\Omega$ is an ellipsoid in $X$, it is equivalent (modulo multiplicative factors in the constants) to say that the principle axes of $\Omega$ of length at least $R$ make an angle of at least $\frac{1}{R}$ with $V$; furthermore, $\Omega$ will be 1-transverse to the subspace $V$ spanned by the principle axes of $\Omega$ of length at most 1. By approximation with John ellipsoids, this shows that every symmetric, closed, convex set $\Omega \subset X$ is $\sqrt{d}$-transverse to some subspace $V$.

**Lemma 69** (Stability I)**.** *If $\Omega$ is $R$-transverse to $V$, then $\Omega + \lambda\mathcal{B}$ is $(R + 3R^2\lambda)$-transverse to $V$ for any $\lambda > 0$.*

*Proof.* Note that $\mathcal{B}/V \subset R \cdot (\Omega \cap \mathcal{B})/V \subset R \cdot ((\Omega + \lambda\mathcal{B}) \cap \mathcal{B})/V$. All that remains is to show

$$(\Omega + \lambda\mathcal{B}) \cap V \subset (R + 3R^2\lambda)\mathcal{B}.$$

Fix $P \in (\Omega + \lambda\mathcal{B}) \cap V$. Write $P = P_0 + P_1$ with $P_0 \in \Omega$ and $P_1 \in \lambda\mathcal{B}$. By the transversality of $\Omega$ and $V$, we have $\lambda\mathcal{B}/V \subset R\lambda(\Omega \cap \mathcal{B})/V$. Since $P_1 \in \lambda\mathcal{B}$, there exists a polynomial $P_2 \in R\lambda(\Omega \cap \mathcal{B})$ with $P_1/V = P_2/V$ – or rather, $P_1 - P_2 \in V$. Define $\tilde{P} := P - (P_1 - P_2) \in V$. We write $\tilde{P} = P_0 + P_2$, where $P_0 \in \Omega$ and $P_2 \in R\lambda \cdot \Omega$, and thus $\tilde{P} \in (R\lambda + 1) \cdot (\Omega \cap V) \subset (R\lambda + 1) \cdot R\mathcal{B}$, where the second containment is by transversality of $\Omega$ and $V$. Therefore,

$$P = \tilde{P} + P_1 - P_2 \in (R^2\lambda + R)\mathcal{B} + \lambda\mathcal{B} + R\lambda\mathcal{B} \subset (R^2\lambda + R + \lambda + R\lambda)\mathcal{B}.$$

We conclude that $P \in (R + 3R^2\lambda)\mathcal{B}$, which completes the proof of the lemma.

□

**Lemma 70** (Stability II). *Let $\Omega_1, \Omega_2 \in \mathcal{S}$, and let $R \geq 1$. If $\Omega_1$ is $R$-transverse to $V$, then the following holds:*

- *If $d_H(\Omega_1, \Omega_2) \leq \frac{1}{4R}$ then $\Omega_2$ is $4R$-transverse to $V$.*
- *If $d_H(\Omega_1 \cap \tilde{R}\mathcal{B}, \Omega_2 \cap \tilde{R}\mathcal{B}) \leq \frac{1}{4R}$ for any $\tilde{R} \geq 4R$, then $\Omega_2$ is $4R$-transverse to $V$.*

*Proof.* For the proof of the first bullet point, we may suppose $\Omega_1 \subset \Omega_2 + \lambda\mathcal{B}$ and $\Omega_2 \subset \Omega_1 + \lambda\mathcal{B}$ for $\lambda = \frac{1}{3R}$. According to Lemma 69, $\Omega_1 + \lambda\mathcal{B}$ is $2R$-transverse to $V$. Thus,

$$\Omega_2 \cap V \subset (\Omega_1 + \lambda\mathcal{B}) \cap V \subset 2R \cdot \mathcal{B}. \tag{4.28}$$

Also,

$$\mathcal{B}/V \subset R \cdot (\Omega_1 \cap \mathcal{B})\,/V \subset R \cdot ((\Omega_2 + \lambda\mathcal{B}) \cap \mathcal{B})\,/V.$$

By (4.7), $(\Omega_2 + \lambda\mathcal{B}) \cap \mathcal{B} \subset (\Omega_2 \cap 2\mathcal{B}) + \lambda\mathcal{B}$, hence,

$$\mathcal{B}/V \subset R \cdot (\Omega_2 \cap 2\mathcal{B} + \lambda\mathcal{B})/V = R \cdot (\Omega_2 \cap 2\mathcal{B})/V + R\lambda \cdot \mathcal{B}/V.$$

Recall $R\lambda = \frac{1}{3}$, hence $K \subset T + K/3$ for $K = \mathcal{B}/V$ and $T = R \cdot (\Omega_2 \cap 2\mathcal{B})/V$. From (4.8) we conclude that $K \subset 2T$, i.e.,

$$\mathcal{B}/V \subset 2R \cdot (\Omega_2 \cap 2\mathcal{B})/V \subset 4R \cdot (\Omega_2 \cap \mathcal{B})/V. \tag{4.29}$$

From (4.28) and (4.29) we conclude that $\Omega_2$ is $4R$-transverse to $V$.

Note $\Omega_1$ is $R$-transverse to $V$ iff $\Omega_1 \cap \tilde{R}\mathcal{B}$ is $R$-transverse to $V$ (since $\tilde{R} \geq R$), and similarly, $\Omega_2$ is $4R$-transverse to $V$ iff $\Omega_2 \cap \tilde{R}\mathcal{B}$ is $4R$-transverse to $V$ (since $\tilde{R} \geq 4R$). Thus, by applying the first bullet point to the sets $\Omega_1 \cap \tilde{R}\mathcal{B}$ and $\Omega_2 \cap \tilde{R}\mathcal{B}$, we obtain the conclusion in the second bullet point. □

**Lemma 71** (Stability III)**.** *Suppose $\Omega$ is $R$-transverse to $V$, and let If $U : X \to X$ be a unitary transformation. Then $U(\Omega)$ is $R$-transverse to $U(V)$. If additionally $\|U - id\|_{op} \leq \frac{1}{16R^2}$, then $U(\Omega)$ is $4R$-transverse to $V$ and $\Omega$ is $4R$-transverse to $U(V)$.*

*Proof.* Unitary transformations preserve the metric structure of $X$, and in particular, they preserve transversality. If $\|U - id\|_{op} \leq \frac{1}{16R^2}$ then

$$d_H(\Omega \cap 4R\mathcal{B}, U(\Omega) \cap 4R\mathcal{B}) = d_H(\Omega \cap 4R\mathcal{B}, U(\Omega \cap 4R\mathcal{B})) \leq \|U - id\|_{op} \cdot 4R \leq \frac{1}{4R}.$$

Therefore, by Lemma 70, $U(\Omega)$ is $4R$-transverse to $V$. Similarly, $U^{-1}(\Omega)$ is $4R$-transverse to $V$, and thus by the first claim we have that $\Omega$ is $4R$-transverse to $U(V)$. □

### 4.2.1 Transversality in the space of polynomials

**Definition 72.** Given a closed, symmetric, convex set $\Omega \subset \mathcal{P}$, a subspace $V \subset \mathcal{P}$, $R \geq 1$, $x \in \mathbb{R}^n$, and $\delta > 0$, we say that $\Omega$ is $(x, \delta, R)$-transverse to $V$ if $\Omega$ is $R$-transverse to $V$ with respect to the Hilbert space structure $(\mathcal{P}, \langle \cdot, \cdot \rangle_{x,\delta})$, i.e., (1) $\mathcal{B}_{x,\delta}/V \subset R \cdot (\Omega \cap \mathcal{B}_{x,\delta})/V$, and (2) $\Omega \cap V \subset R \cdot \mathcal{B}_{x,\delta}$.

Our next result establishes a few basic properties of transversality in this setting.

**Lemma 73.** *If $\Omega$ is $(x, \delta, R)$-transverse to $V$, then the following holds:*

- *$T_h\Omega$ is $(x + h, \delta, R)$-transverse to $T_hV$.*
- *$\tau_{x,r}\Omega$ is $(x, \delta/r, R)$-transverse to $\tau_{x,r}V$.*
- *If $\delta' \in [\kappa^{-1}\delta, \kappa\delta]$ for some $\kappa \geq 1$, then $\Omega$ is $(x, \delta', \kappa^m R)$-transverse to $V$.*

*Proof.* The proof of the first and second bullet points is easy: Apply $T_h$ and $\tau_{x,r}$ to both sides of (1) and (2) in Definition 72, and use the identities $T_h\mathcal{B}_{x,\delta} = \mathcal{B}_{x+h,\delta}$ and $\tau_{x,r}\mathcal{B}_{x,\delta} = \mathcal{B}_{x,\delta/r}$. The third bullet point follows from the equivalence of the unit balls $\mathcal{B}_{x,\delta} \subset \max\{1, (\delta/\delta')^m\} \cdot \mathcal{B}_{x,\delta'}$ and $\mathcal{B}_{x,\delta'} \subset \max\{1, (\delta'/\delta)^m\} \cdot \mathcal{B}_{x,\delta}$, as well as the property that $A \cap (r \cdot B) \subset r \cdot (A \cap B)$ if $A$ and $B$ are symmetric convex sets, and $r \geq 1$.

□

The continuity of the mapping $x \mapsto \sigma(x)$ can be used to show that the transversality of the set $\sigma(x)$ with respect to a fixed subspace is stable with respect to small perturbations of the basepoint.

**Lemma 74.** *There exists $c_1 = c_1(m, n) > 0$ so that the following holds. Let $V \subset \mathcal{P}$ be a subspace, $x, y \in \mathbb{R}^n$, $\delta > 0$, $R \geq 1$. If $\sigma(x)$ is $(x, \delta, R)$-transverse to $V$ and $|x - y| \leq c_1 \frac{\delta}{R}$ then $\sigma(y)$ is $(y, \delta, 8R)$-transverse to $V$.*

*Proof.* If $c_1 < \frac{1}{4C_T}$, where $C_T$ is the constant in (4.2), then by Lemma 57,

$$\sigma(y) \subset \sigma(x) + C_T \cdot \mathcal{B}_{x, c_1 \cdot (\frac{\delta}{R})} \subset \sigma(x) + C_T \cdot (\frac{c_1}{R}) \cdot \mathcal{B}_{x,\delta} = \sigma(x) + (\frac{1}{4R}) \cdot \mathcal{B}_{x,\delta}.$$

Similarly, $\sigma(x) \subset \sigma(y) + (\frac{1}{4R}) \cdot \mathcal{B}_{x,\delta}$. Thus, $d_H^{x,\delta}(\sigma(x), \sigma(y)) \leq \frac{1}{4R}$, where $d_H^{x,\delta}$ is the Hausdorff distance with respect to the norm $|\cdot|_{x,\delta}$ on $\mathcal{P}$. By Lemma 70, since $\sigma(x)$ is $(x, \delta, R)$-transverse to $V$, we conclude that $\sigma(y)$ is $(x, \delta, 4R)$-transverse to $V$. Since $|x - y| \leq c_1 \delta / R \leq c_1 \delta$, if $c_1 = c_1(m, n)$ is chosen small enough then $(\frac{9}{10}) \cdot \mathcal{B}_{y,\delta} \subset \mathcal{B}_{x,\delta} \subset (\frac{10}{9}) \cdot \mathcal{B}_{y,\delta}$. Substituting these inclusions in conditions (1) and (2) in the definition of transversality, we learn that $\sigma(y)$ is $(y, \delta, 8R)$-transverse to $V$. □

### 4.2.2 Ideals in the ring of polynomials and DTI subspaces

**Definition 75.** A subspace $V \subset \mathcal{P}$ is translation-invariant if $T_h V = V$ for all $h \in \mathbb{R}^n$, and $V$ is dilation-invariant at $x \in \mathbb{R}^n$ if $\tau_{x,\delta} V = V$ for all $\delta > 0$. Say that $V$ is dilation-and-translation-invariant (DTI) if $T_h \tau_{x,\delta} V = V$ for all

$x, h \in \mathbb{R}^n$, $\delta > 0$. We write DTI to denote the collection of all DTI subspaces of $\mathcal{P}$.

*Remark* 76. Equivalently, $V \subset \mathcal{P}$ is translation-invariant if $P \in V, Q \in \mathcal{P} \implies Q(\partial)P \in V$. Since $T_h = \tau_{(1-\delta)^{-1}h,\delta^{-1}} \circ \tau_{0,\delta}$ (for some $\delta > 1$), any translation operator is a composition of dilation operators. Thus, $V$ is DTI if and only if $\tau_{x,\delta}V = V$ for all $(x,\delta) \in \mathbb{R}^n \times (0,\infty)$.

We now illustrate a connection between translation-invariant subspaces and ideals in $\mathcal{P}_x$.

**Lemma 77.** *Let $(x,\delta) \in \mathbb{R}^n \times (0,\infty)$. Let $V^\perp$ be the orthogonal complement of a subspace $V \subset \mathcal{P}$ with respect to the inner product $\langle \cdot,\cdot \rangle_{x,\delta}$. Then $V$ is translation-invariant if and only if $V^\perp$ is an $\odot_x$-ideal in $\mathcal{P}_x$.*

*Proof.* Translating, we may assume that $x = 0$. Rescaling preserves the property of $V$ being translation-invariant, and also of $V^\perp$ being an $\odot_x$-ideal, according to (4.4). Hence we may assume that $\delta = 1$. Note the identity $\langle Q, P\rangle = Q(\partial)(P)(0)$ for any $P, Q \in \mathcal{P}$. Note $\partial^\alpha$ annihilates $\mathcal{P}$ for $|\alpha| \geq m$, and hence $R(\partial)[Q(\partial)P] = (R \odot_0 Q)(\partial)P$ for any $P, Q, R \in \mathcal{P}$. Suppose that $V$ is a translation-invariant subspace, and let $Q \in V^\perp$. Then, for any $h \in \mathbb{R}^n$ and $P \in V$, also $T_h P \in V$ and hence,

$$0 = \langle Q, T_h(P)\rangle = Q(\partial)\left[T_h(P)\right](0) = T_h(Q(\partial)P)(0) = Q(\partial)P(-h).$$

Consequently, $Q(\partial)P = 0$. Thus, for any $R \in \mathcal{P}$, we have $(R \odot_0 Q)(\partial)P = R(\partial)\left[Q(\partial)P\right] = 0$. In particular, $\langle R \odot_0 Q, P\rangle = 0$ for any $P \in V$ and hence $R \odot_0 Q \in V^\perp$. This shows that $V^\perp$ is an $\odot_0$-ideal.

For the other direction, suppose that $V^\perp$ is an $\odot_0$-ideal. Let $P \in V$ and $R \in \mathcal{P}$. Then for any $Q \in V^\perp$,

$$0 = \langle R \odot_0 Q, P\rangle = Q(\partial)\,[R(\partial)P]\,(0) = \langle Q, R(\partial)P\rangle.$$

This means that $R(\partial)P \in (V^\perp)^\perp = V$. Hence $R(\partial)P \in V$ whenever $P \in V$ and $R \in \mathcal{P}$, and consequently the subspace $V$ is translation-invariant. $\square$

We say that two subspaces $V_1, V_2 \subset \mathcal{P}$ are *complementary* if $V_1+V_2 = \mathcal{P}$ and $V_1 \cap V_2 = \{0\}$.

**Lemma 78.** *For any $\odot_0$-ideal $I$ in $\mathcal{P}_0$, there exists $V \in \mathrm{DTI}$ that is complementary to $I$.*

*Proof.* Set $I_* = \lim_{\delta\to 0} \tau_{0,\delta}(I)$ (where the Grassmanian is endowed with the usual topology). Let us first show that this limit exists: Consider the canonical projection $\pi_k : \mathcal{P}_0 \to \mathcal{P}_0^k$ onto the subspace of $k$-homogeneous polynomials $\mathcal{P}_0^k := \mathrm{span}\{z^\alpha : |\alpha| = k\}$, and denote the subspace of $(\geq k)$-homogeneous polynomials $\mathcal{P}_0^{\geq k} := \mathrm{span}\{z^\alpha : |\alpha| \geq k\}$. By Gaussian elimination we can pick a basis $\mathcal{B}_1 := \{P_j^k\}_{1\leq j\leq N_k}^{0\leq k\leq m-1}$ for $I$ in the *block form*: $P_j^k \in \mathcal{P}_0^{\geq k}$, and $\mathcal{B}_0 := \{\pi_k P_j^k\}_{1\leq j\leq N_k}^{0\leq k\leq m-1}$ is linearly independent in $\mathcal{P}_0$. The family $\mathcal{B}_\delta := \{\delta^{m-k}\tau_{0,\delta}(P_j^k)\}_{k,j}$ converges elementwise as $\delta \to 0$ to $\mathcal{B}_0$. Since $\mathcal{B}_\delta$ is a basis for $\tau_{0,\delta}(I)$, and $\mathcal{B}_0$ is a basis for $I_* := \mathrm{span}(\mathcal{B}_0)$, we learn that $\tau_{0,\delta}(I)$ converges to $I_*$, as desired.

The ideals form a closed subset of the Grassmanian, thus $I_*$ is an ideal in the ring $\mathcal{P}_0$. Let $V$ be the orthogonal complement of $I_*$ with respect to the standard inner product on $\mathcal{P}_0$. Observe that $I_*$ is dilation-invariant at $x = 0$, i.e., $\tau_{0,\delta} I_* = I_*$ for all $\delta > 0$. Equivalently, $I_*$ is a direct sum of homogeneous subspaces of $\mathcal{P}_0$, i.e., $I_* = I^0 + \cdots + I^{m-1}$, with $I^k \subset \mathcal{P}_0^k$. But then $V$ is also a direct sum of homogeneous subspaces of $\mathcal{P}_0$, and so $V$ is dilation-invariant at $x = 0$. From Lemma 77, we also know that $V$ is translation-invariant. Thus, $V \in \text{DTI}$. The subspaces $I_*$ and $V$ are complementary and this property is open in $\mathcal{G} \times \mathcal{G}$. By definition of $I_*$ as a limit, $\tau_{0,\delta}(I)$ and $V$ are complementary for some $\delta > 0$. By an application of the isomorphism of vector spaces $\tau_{0,\delta^{-1}}$, we learn that $I$ and $\tau_{0,\delta^{-1}} V$ are complementary. To finish the proof, recall that $V \in \text{DTI}$, and hence $\tau_{0,\delta^{-1}} V = V$. □

Our next result says that every Whitney convex set is transverse to a DTI subspace.

**Lemma 79.** *Given $A \in [1, \infty)$, there exists a constant $R_0 = R_0(A, m, n)$ so that the following holds. Let $\Omega$ be a closed, symmetric, convex subset of $\mathcal{P}$. If $\Omega$ is Whitney convex at $x \in \mathbb{R}^n$ with $w_x(\Omega) \leq A$, and $\delta > 0$, then there exists $V \in \text{DTI}$ such that $\Omega$ is $(x, \delta, R_0)$-transverse to $V$.*

*Proof.* By the second bullet point in Lemma 73, $\Omega$ is $(x, \delta, R)$-transverse to $V$ if and only if $\tau_{x,\delta}\Omega$ is $(x, 1, R)$-transverse to $\tau_{x,\delta} V$. Thus, by the remark following Definition 60, we may rescale and assume that $\delta = 1$. Similarly, by translating we may assume that $x = 0$.

Let $\mathcal{S}$ denote the set of closed, symmetric, convex subsets of $\mathcal{P}$. We endow $\mathcal{S}$ with the topology of local Hausdorff convergence, i.e., $\Omega_j \to \Omega$ iff

$\lim_{j\to\infty} d_H(\Omega_j \cap R\mathcal{B}, \Omega \cap R\mathcal{B}) = 0$ for all $R > 0$ – here, $\mathcal{B} \subset \mathcal{P}$ is the unit ball with respect to the standard norm $|\cdot| = |\cdot|_{0,1}$ on $\mathcal{P}$, and $d_H$ is the Hausdorff distance with respect to this norm. As a consequence of the Blaschke selection theorem (see []), thus endowed, $\mathcal{S}$ is a compact space. Write $\mathcal{G}$ to denote the Grassmanian of all subspaces of $\mathcal{P}$, and $\mathcal{G}_k \subset \mathcal{G}$ the Grassmanian of all $k$-dimensional subspaces. We may identify $\mathcal{G}$ as a compact subspace of $\mathcal{S}$.

For any $(x, \delta) \in \mathbb{R}^n \times (0, \infty)$, the isomorphism $\tau_{x,\delta} : \mathcal{P} \to \mathcal{P}$ induces a continuous mapping on the Grassmanian $\tau_{x,\delta} : \mathcal{G} \to \mathcal{G}$. Thus, $\mathrm{DTI} = \{V \in \mathcal{G} : \tau_{x,\delta} V = V \ \ \forall (x, \delta) \in \mathbb{R}^n \times (0, \infty)\}$ is a closed subset of $\mathcal{G}$, and hence DTI is compact.

The conclusion of the lemma is equivalent to the existence of a constant $R_0 = R_0(A, m, n)$ so that $\phi(\Omega) \leq R_0$ for all $\Omega \in wc_A$, where

$$wc_A := \{\Omega \in \mathcal{S} : \Omega \text{ is Whitney convex at } 0 \text{ with } w_0(\Omega) \leq A\}$$

$$\phi : wc_A \to [0, \infty], \ \ \phi(\Omega) := \inf\{\psi(\Omega, V) : V \in \mathrm{DTI}\}$$

$$\psi : wc_A \times \mathrm{DTI} \to [0, \infty],$$

$$\psi(\Omega, V) := \inf\{R : \Omega \cap V \subset R \cdot \mathcal{B}, \ \ \mathcal{B}/V \subset R \cdot (\Omega \cap \mathcal{B})/V\}.$$

If $\Omega_n \to \Omega$, $\Omega_n \in wc_A$, $\delta > 0$, and $A^* > A$, then

$$(\Omega \cap \mathcal{B}_{0,\delta}) \odot_0 \mathcal{B}_{0,\delta} = \lim_{n\to\infty} (\Omega_n \cap \mathcal{B}_{0,\delta}) \odot_0 \mathcal{B}_{0,\delta} \subset \lim_{n\to\infty} A^* \delta^m \Omega_n = A^* \delta^m \Omega,$$

where we used the continuity of $\odot_0$ on $\mathcal{S} \times \mathcal{S}$. So $wc_A$ is closed, and hence compact. We claim that $\psi$ is upper semicontinuous (usc). Indeed, $\psi = \inf_{R>0} \psi_R$, with $\psi_R = R1_{E_R} + \infty 1_{E_R^c}$ and

$$E_R = \{(\Omega, V) \in \mathcal{S} \times \text{DTI} : \text{for some } R' < R,\ \Omega \cap V \subset R' \cdot \mathcal{B}$$
$$\text{and } \mathcal{B}/V \subset R' \cdot (\Omega \cap \mathcal{B})/V\}.$$

As $E_R$ is open, $\psi_R$ is usc. Hence the same is true of $\psi$, and also of $\phi$.

Since $\phi$ is usc and $wc_A$ is compact, it suffices to show that $\phi(\Omega) < \infty$ for all $\Omega \in wc_A$. Since $\Omega$ is Whitney convex at 0, $I = \text{span}(\Omega)$ is an ideal in $\mathcal{P}_0$ (see Lemma 62). By Lemma 78 there exists a subspace $V \in \text{DTI}$ which is complementary to $I$, i.e., $V \cap I = \{0\}$ and $V + I = \mathcal{P}$. Note that $\text{span}(\Omega + V) = I + V = \mathcal{P}$, and so by convexity, $\Omega + V$ contains a ball $\epsilon\mathcal{B}$ for some $\epsilon > 0$. If $\epsilon\mathcal{B} \subset \Omega + V$, it follows that $\epsilon\mathcal{B}/V \subset \Omega/V$. Thus,

$$\epsilon\mathcal{B}/V \subset \bigcup_{R>0} (\Omega \cap R\mathcal{B})/V.$$

By compactness, there exists an $R > 0$ with $\frac{\epsilon}{2}\mathcal{B}/V \subset (\Omega \cap R\mathcal{B})/V \subset R(\Omega \cap \mathcal{B})/V$. Thus, $\mathcal{B}/V \subset \frac{2R}{\epsilon}(\Omega \cap \mathcal{B})/V$. Combined with $V \cap \Omega \subset V \cap I = \{0\}$, this implies that $\phi(\Omega) \leq \frac{2R}{\epsilon}$. □

For any $x \in \mathbb{R}^n$, the set $\sigma(x) = \sigma(x, E)$ is Whitney convex at $x$ with $w_x(\sigma(x)) \leq C_0$ (see Lemma 61). Let $R_0$ be the constant from Lemma 79 with $A = C_0$. Then

$$\begin{cases} \text{for any finite set } E \subset \mathbb{R}^n, \text{ for any } (x, \delta) \in \mathbb{R}^n \times (0, \infty), \\ \text{there exists } V \in \text{DTI} \text{ such that } \sigma(x) \text{ is } (x, \delta, R_0)\text{-transverse to } V. \end{cases} \tag{4.30}$$

**Constants:** Recall the constant $c_1$ is defined in Lemma 74. We specify constants $R_{\text{label}} \ll R_{\text{med}} \ll R_{\text{big}} \ll R_{\text{huge}}$, $C_*$, and $C_{**}$, defined as follows:

$$\begin{cases} R_{\text{label}} := 8R_0 & R_{\text{med}} := 256DR_{\text{label}} \\ R_{\text{big}} := 10^m R_{\text{med}} & R_{\text{huge}} := 2^{m+3} R_{\text{big}} \\ C_* := 20c_1^{-1}R_{\text{big}} & C_{**} = 1 + 2^m C_T \cdot (1 + R_{\text{label}} \cdot (5C_*)^m). \end{cases} \tag{4.31}$$

**Lemma 80.** *Let $B$ be a closed ball in $\mathbb{R}^n$. There exists $V \in \text{DTI}$ such that $\sigma(z)$ is $(z, C_* diam(B), R_{label})$-transverse to $V$ for all $z \in 100B$.*

*Proof.* Let $x_0 \in \mathbb{R}^n$ be the center of $B$. We apply (4.30) with $x = x_0$ and $\delta = C_*\text{diam}(B)$. Thus, $\sigma(x_0)$ is $(x_0, C_*\text{diam}(B), R_0)$-transverse to some $V \in$ DTI. Let $z \in 100B$ be arbitrary. Then $|z - x_0| \leq 100\text{diam}(B) \leq c_1 \frac{C_*\text{diam}(B)}{R_0}$ (see (4.31)). By Lemma 74, we conclude that $\sigma(z)$ is $(z, C_*\text{diam}(B), 8R_0)$-transverse to $V$.

□

## 4.3 Complexity

Let $l(I)$ and $r(I)$ denote the left and right endpoints of an interval $I \subset \mathbb{R}$. An interval $J$ is *to the left* of an interval $I$, written $J < I$, if either $r(J) < l(I)$ or $r(J) = l(I)$ and $l(J) < l(I)$. Let $X$ be a finite-dimensional Hilbert space with inner product $\langle \cdot, \cdot \rangle_X$, set $d := \dim X < \infty$, and denote the norm and unit ball of $X$ by $|\cdot|_X = \sqrt{\langle \cdot, \cdot \rangle}_X$ and $\mathcal{B} = \{x \in X : |x|_X \leq 1\}$. Let $\Psi : \mathbb{R}^D \to X$ be a coordinate transformation of the form $\Psi(v) = \sum_j v_j e_j$ for an orthonormal basis $\{e_j\}_{1 \leq j \leq d}$ of $X$. Fix $\vec{m} = (m_1, \cdots, m_d) \in \mathbb{Z}^d_{\geq 0}$ and a

1-parameter family of maps $T_\delta : X \to X$ ($\delta > 0$) of the form $T_\delta = \Psi \widetilde{T}_\delta \Psi^{-1}$, where the transformation $\widetilde{T}_\delta : \mathbb{R}^d \to \mathbb{R}^d$ is represented in standard Euclidean coordinates by a diagonal matrix $D_\delta = \mathrm{diag}(\delta^{-m_1}, \cdots, \delta^{-m_d})$.

**Definition 81.** Given a closed, symmetric, convex set $\Omega \subset X$, the complexity of $\Omega$ relative to the dynamical system $\mathcal{X} = (X, T_\delta)_{\delta>0}$ at scale $\delta_0 > 0$ with parameter $R \geq 1$– written $\mathcal{C}_{\mathcal{X},\delta_0,R}(\Omega)$ – is the largest integer $K \geq 1$ such that there exist intervals $I_1 > I_2 > \cdots > I_K$ in $(0, \delta_0]$ and subspaces $V_1, V_2 \cdots, V_K \subset X$, such that $T_{r(I_k)}(\Omega)$ is $R$-transverse to $V_k$, but $T_{l(I_k)}(\Omega)$ is not $256dR$-transverse to $V_k$ for all $k = 1, \cdots, K$. If no such $K$ exists, let $\mathcal{C}_{\mathcal{X},\delta_0,R}(\Omega) := 0$.

**Proposition 82.** *Given $R \geq 1$ and $\vec{m} \in \mathbb{Z}^d_{\geq 0}$, there is a constant $K_0 = K_0(d, \vec{m}, R)$ such that $\mathcal{C}_{\mathcal{X},\delta_0,R}(\Omega) \leq K_0$ for all closed, symmetric, convex sets $\Omega \subset X$ and all $\delta_0 > 0$.*

### 4.3.1 Background on semialgebraic geometry

We review some standard terminology from semialgebraic geometry: A *basic set* is a subset of $\mathbb{R}^d$ defined by a finite number of polynomial inequalities, i.e., $B = \{x \in \mathbb{R}^d : p_i(x) \leq 0,\ q_j(x) < 0\ \forall i \forall j\}$, for polynomials $p_1, \cdots, p_k, q_1, \cdots, q_l$ on $\mathbb{R}^d$. A *semialgebraic set* is a finite union of basic sets. Clearly the class of semialgebraic sets is closed under finite unions/intersections and complements. The celebrated Tarski-Seidenberg theorem on quantifier elimination implies that the class of semialgebraic sets is closed under projections $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$; see [65]. Semialgebraic sets are closely related to *first* *order formulas* over the reals, which are defined by the following elementary rules: (1) If $p$ is a polynomial on $\mathbb{R}^d$, then "$p \leq 0$" and "$p < 0$" are formulas, (2) If $\Phi$ and $\Psi$ are formulas, then "$\Phi$ and $\Psi$", "$\Phi$ or $\Psi$", and "not $\Phi$" are formulas, and (3) If $\Phi$ is a formula and $x$ is a variable of $\Phi$ (ranging in $\mathbb{R}$), then "$\exists x\ \Phi$" and "$\forall x\ \Phi$" are formulas. A formula is *quantifier-free* if it arises only via (1) and (2). The Tarski-Seidenberg theorem states that every formula is equivalent (i.e., has an identical solution set) to a quantifier-free formula. Accordingly, every semialgebraic set coincides with the solution set of a first-order formula, and visa versa. In the next section, we will consider the set $\mathcal{M}^+ \subset \mathbb{R}^{d \times d}$ of all positive-definite $d \times d$ matrices. Notice that $\mathcal{M}^+$ is semialgebraic because it is the solution set of a formula: $\mathcal{M}^+ = \{(a_{ij})_{1 \leq i,j \leq d} : a_{ij} = a_{ji} \text{ for } i,j = 1, \cdots, d \text{ and } \sum_{i,j=1}^d a_{ij} x_i x_j > 0\ \forall x_1, \cdots, \forall x_d\}$. We will need the following theorem which gives an upper bound on the number of connected components of a semialgebraic set.

**Theorem 83** (Corollary 3.6, Chapter 3 of [65])**.** *If $S \subset \mathbb{R}^{k_1+k_2}$ is semialgebraic then there is a natural number $M$ such that for each point $a \in \mathbb{R}^{k_1}$ the fiber $S_a := \{b \in \mathbb{R}^{k_2} : (a,b) \in S\}$ has at most $M$ connected components.*

### 4.3.2 Proof of Proposition 82

Note that $\Psi^{-1} : X \to \mathbb{R}^d$ is a Hilbert space isomorphism, where $\mathbb{R}^d$ is equipped with the standard Euclidean inner product $\langle \cdot, \cdot \rangle$. Thus $\mathcal{C}_{(X,T_\delta),\delta_0,R}(\Omega) = \mathcal{C}_{(\mathbb{R}^d,\widetilde{T}_\delta),\delta_0,R}(\Psi^{-1}(\Omega))$, where $\widetilde{T}_\delta := \Psi^{-1} T_\delta \Psi$. Therefore, we may reduce to the case where $(X, \langle \cdot, \cdot \rangle_X) = (\mathbb{R}^d, \langle \cdot, \cdot \rangle)$ and the transformation

$T_\delta$ on $\mathbb{R}^d$ is represented in Euclidean coordinates by the diagonal matrix $D_\delta = \operatorname{diag}(\delta^{-m_1}, \cdots, \delta^{-m_d})$ (i.e., $T_\delta(x) = D_\delta \cdot x$).

We give a proof by contradiction, for a value of $K_0$ to be determined later. Thus we suppose there is a one-parameter family of linear transformations $T_\delta : \mathbb{R}^d \to \mathbb{R}^d$ of the above form, a closed, symmetric, convex set $\Omega \subset \mathbb{R}^d$, $\delta_0 > 0$, and $R \geq 1$, so that $\mathcal{C}(\Omega|(\mathbb{R}^d, T_\delta)_{\delta>0}, \delta_0, R) \geq K_0 + 1$. Note that $(T_\delta)_{\delta>0}$ satisfies the semigroup properties $T_1 = id$ and $T_{\delta_1\delta_2} = T_{\delta_1} \circ T_{\delta_2}$. Hence, by exchanging $\Omega$ and $T_{\delta_0}(\Omega)$, we may reduce to the case $\delta_0 = 1$. Thus there exist intervals $I_1 > \cdots > I_{K_0+1}$ in $(0,1]$ and subspaces $V_1, \cdots, V_{K_0+1} \subset \mathbb{R}^d$ such that (a) $T_{r(I_k)}(\Omega)$ is $R$-transverse to $V_k$, whereas (b) $T_{l(I_k)}(\Omega)$ is not $256dR$-transverse to $V_k$, for all $1 \leq k \leq K_0 + 1$.

Let $\mathcal{G}$ be the Grassmanian of subspaces of $\mathbb{R}^d$, endowed with the metric

$$d_{\mathcal{G}}(V_1, V_2) := \inf\{\|U - id\| : U : \mathbb{R}^d \to \mathbb{R}^d \text{ unitary},\ U(V_1) = V_2\}.$$

In particular, $d_{\mathcal{G}}(V_1, V_2) < \infty \iff \dim(V_1) = \dim(V_2)$. Let $\epsilon := \frac{1}{2^{12} d R^2}$, and let $\mathcal{N}$ be an $\epsilon$-net in $\mathcal{G}$.

By perturbation, we can approximate $\Omega$ by an ellipsoid $\mathcal{E}$ with similar properties. Let $R_0 := 256dR$. Fix a compact, symmetric, convex set $\tilde{\Omega} \subset \mathbb{R}^d$ with nonempty interior, and

$$\begin{cases} d_H(T_{r(I_k)}(\Omega) \cap R_0\mathcal{B}, T_{r(I_k)}(\tilde{\Omega}) \cap R_0\mathcal{B}) < R_0^{-1}, \\ d_H(T_{l(I_k)}(\Omega) \cap R_0\mathcal{B}, T_{l(I_k)}(\tilde{\Omega}) \cap R_0\mathcal{B}) < R_0^{-1} \text{ for all } 1 \leq k \leq K_0 + 1, \end{cases}$$

where $d_H$ is the Hausdorff metric with respect to the Euclidean norm on $\mathbb{R}^d$. By Lemma 70 and properties (a),(b), $T_{r(I_k)}(\tilde{\Omega})$ is $4R$-transverse to $V_k$, but

$T_{l(I_k)}(\tilde{\Omega})$ is not $64dR$-transverse to $V_k$. If $\mathcal{E}$ is the John ellipsoid of $\tilde{\Omega}$, satisfying $\mathcal{E} \subset \tilde{\Omega} \subset \sqrt{d}\mathcal{E}$, then $T_{r(I_k)}(\mathcal{E})$ is $4\sqrt{d}R$-transverse to $V_k$, but $T_{l(I_k)}(\mathcal{E})$ is not $64\sqrt{d}R$-transverse to $V_k$. Hence, setting $\widehat{R} = 16\sqrt{d}R$,

$$\begin{cases} T_{r(I_k)}(\mathcal{E}) \text{ is } (\frac{1}{4})\widehat{R}\text{-transverse to } V_k, \text{ but} \\ T_{l(I_k)}(\mathcal{E}) \text{ is not } 4\widehat{R}\text{-transverse to } V_k, \text{ for all } 1 \leq k \leq K_0 + 1. \end{cases} \tag{4.32}$$

We identify an ellipsoid in $\mathbb{R}^d$ with a positive-definite $d \times d$ matrix in the usual way: every ellipsoid has the form $\mathcal{E}_A := \{x \in \mathbb{R}^d : \langle Ax, x \rangle \leq 1\}$ for some $A \in \mathcal{M}^+$. Furthermore, every subspace of $X$ has the form $V_C := \text{rowsp}(C)$ for $C \in \mathbb{R}^{d \times d}$, where $\text{rowsp}(C)$ denotes the row space of the matrix $C$. Consider the set

$$S = \{(C, A, \overline{R}, \delta) \in \mathbb{R}^{d^2} \times \mathcal{M}^+ \times [1, \infty) \times (0, \infty) : T_\delta(\mathcal{E}_A) \text{ is } \overline{R}\text{-transverse to } V_C\}.$$

Here, it is useful to note that $T_\delta(\mathcal{E}_A) = \mathcal{E}_{A_\delta}$, with $A_\delta := D_{\delta^{-1}} A D_{\delta^{-1}}$. Then $S$ is a semialgebraic subset of $\mathbb{R}^{2d^2+2}$ because $\mathcal{M}^+$ is semialgebraic and the statement "$T_\delta(\mathcal{E}_A)$ is $\overline{R}$-transverse to $V_C$" is expressable by a first order formula in the variables $(A, C, \delta, \overline{R}) \in \mathbb{R}^{2d^2+2}$.

Consider the ellipsoid $\mathcal{E}$ fixed before and fix an arbitrary subspace $V \subset \mathbb{R}^d$. Write $V = V_C$ and $\mathcal{E} = \mathcal{E}_A$ for $C \in \mathbb{R}^{d^2}$, $A \in \mathcal{M}^+$. By Theorem 83 there exists $M = M(d, \vec{m}) \geq 1$ so that for any $\overline{R} \geq 1$ there exists a set $\Lambda = \Lambda(V_C, \mathcal{E}_A, \overline{R}) \subset (0, \infty)$ with $\#(\Lambda) \leq M$ so that, for any interval $I \subset (0, \infty) \setminus \Lambda$, either $T_\delta(\mathcal{E}_A)$ is $\overline{R}$-transverse to $V_C$ for all $\delta \in I$, or $T_\delta(\mathcal{E}_A)$ is not $\overline{R}$-transverse to $V_C$ for all $\delta \in I$. Set

$$\Lambda_{\text{bad}} := \bigcup_{V \in \mathcal{N}} \Lambda(V, \mathcal{E}, \widehat{R}).$$

Note $\#(\Lambda_{\text{bad}}) \leq \#(\mathcal{N}) \cdot M$, and for any interval $I \subset (0,\infty) \setminus \Lambda_{\text{bad}}$, and for all $V \in \mathcal{N}$,

$(*)$ [$T_\delta(\mathcal{E})$ is $\widehat{R}$-transverse to $V$ for all $\delta \in I$] or [$T_\delta(\mathcal{E})$ is not $\widehat{R}$-transverse to $V$ for all $\delta \in I$].

Set $K_0 := 2 \cdot \#(\mathcal{N}) \cdot M$. Then $K_0 + 1 > 2 \cdot \#(\Lambda_{\text{bad}})$. By definition of the order relation on intervals, at most two of the intervals $I_1 > \cdots > I_{K_0+1}$ can contain a given number $\delta \in \mathbb{R}$. Thus, we can find $k_*$ so that $I_* := I_{k_*}$ is disjoint from $\Lambda_{\text{bad}}$.

Consider the subspace $V_* := V_{k_*}$. By definition of the metric on $\mathcal{G}$ and the fact that $\mathcal{N} \subset \mathcal{G}$ is an $\epsilon$-net, there is a unitary transformation $U : X \to X$ and an element $V \in \mathcal{N}$ with $\|U^{-1} - id\| = \|U - id\| < \epsilon = \frac{1}{2^{12} d R^2} = \frac{1}{16\widehat{R}^2}$ and $U(V_*) = V$. From $(*)$, either $T_\delta(\mathcal{E})$ is $\widehat{R}$-transverse to $V$ for all $\delta \in I_*$, or $T_\delta(\mathcal{E})$ is not $\widehat{R}$-transverse to $V$ for all $\delta \in I_*$. By Lemma 71, either $T_\delta(\mathcal{E})$ is $(\frac{1}{4})\widehat{R}$-transverse to $V_*$ for all $\delta \in I_*$, or $T_\delta(\mathcal{E})$ is not $4\widehat{R}$-transverse to $V_*$ for all $\delta \in I_*$. This contradicts (4.32) for $k = k_*$ and finishes the proof of the proposition.

## 4.4 The Local Main Lemma

**Definition 84.** For $x \in \mathbb{R}^n$, let $\mathcal{P}_x = \mathcal{P}$ be the Hilbert space endowed with the inner product $\langle \cdot, \cdot \rangle_x := \langle \cdot, \cdot \rangle_{x,1}$. Write $\mathcal{X}_x$ for the system $(\mathcal{P}_x, \tau_{x,\delta})_{\delta > 0}$, where the rescaling transformations $\tau_{x,\delta} : \mathcal{P}_x \to \mathcal{P}_x$ ($\delta > 0$) are given by $\tau_{x,\delta}(P)(z) = \delta^{-m} P(x + \delta(z - x))$; note that $\tau_{x,\delta}$ is represented by a diagonal matrix with negative integer powers of $\delta$ along the diagonal with respect to the monomial basis $\{(z-x)^\alpha\}_{|\alpha|\leq m-1}$. Given a ball $B \subset \mathbb{R}^n$ and a finite set $E \subset \mathbb{R}^n$, the local complexity of $E$ on $B$ is the integer-valued quantity

$$\mathcal{C}(E|B) := \sup_{x \in B} \mathcal{C}_{\mathcal{X}_x, C_* \mathrm{diam}(B), R_{\text{label}}}(\sigma(x)).$$

*Remark* 85. We obtain an equivalent formulation of local complexity by inspection of Definition 81: We have $\mathcal{C}(E|B) \geq K$ if and only if there exists $x \in B$ and there exist subspaces $V_1, \cdots, V_K \subset \mathcal{P}$ and intervals $I_1 > I_2 > \cdots > I_K$ in $(0, \mathrm{diam}(B)]$, such that $\tau_{x,r(I_k)}(\sigma(x))$ is $(x, C_*, R_{\text{label}})$-transverse to $V_k$, but $\tau_{x,l(I_k)}(\sigma(x))$ is not $(x, C_*, R_{\text{med}})$-transverse to $V_k$ for all $k = 1, \cdots, K$. Here, $R_{\text{med}} := 256 D R_{\text{label}}$ (see (4.31)).

We have a basic monotonicity property of complexity: $B_1 \subset B_2 \implies \mathcal{C}(E|B_1) \leq \mathcal{C}(E|B_2)$. Furthermore, as a consequence of Proposition 82:

**Corollary 86.** *There exists $K_0 = K_0(m,n)$ such that $\mathcal{C}(E|B) \leq K_0$ for any closed ball $B \subset \mathbb{R}^n$ and finite subset $E \subset \mathbb{R}^n$.*

We are now ready to state the main apparatus that we will use to prove Theorem 49.

**Lemma 87** (Local Main Lemma for $K$)**.** *Let $K \geq -1$. There exist constants $C^\# = C^\#(K) \geq 1$ and $\ell^\# = \ell^\#(K) \in \mathbb{Z}_{\geq 0}$, depending only on $K, m, n$, with the following properties.*

*Let $E \subset \mathbb{R}^n$ be finite and let $B_0 \subset \mathbb{R}^n$ be a closed ball. If $\mathcal{C}(E|5B_0) \leq K$ then the following holds:*

*Local Finiteness Principle on $B_0$:* *Let $f : E \to \mathbb{R}$, $M > 0$, $x_0 \in B_0$, and $P_0 \in \mathcal{P}$, satisfy the following finiteness hypothesis: For all $S \subset E$ with $\#(S) \leq (D+1)^{\ell^\#}$ there exists $F^S \in C^{m-1,1}(\mathbb{R}^n)$ with $F^S = f$ on $S$, $J_{x_0}F^S = P_0$, and $\|F^S\| \leq M$. Then there exists a function $F \in C^{m-1,1}(\mathbb{R}^n)$ with $F = f$ on $E \cap B_0$, $J_{x_0}F = P_0$, and $\|F\| \leq C^\# M$.*

*Remark* 88. Equivalently, the Local Finiteness Principle on $B_0$ states that

$$\Gamma_{\ell^\#}(x_0, f, M) \subset \Gamma_{E \cap B_0}(x_0, f, C^\# M).$$

In particular, by taking $f = 0$ and $M = 1$, we have

$$\sigma_{\ell^\#}(x_0) \subset C^\# \cdot \sigma(x_0, E \cap B_0).$$

Our plan is to prove the Local Main Lemma by induction on the complexity parameter $K$. The condition $\mathcal{C}(E|5B_0) \leq K_0$ in the Local Main Lemma for $K = K_0$ is vacuously true. Therefore the Local Main Lemma for $K = K_0$ implies the Local Finiteness Principle for any ball $B_0$ and finite set $E \subset \mathbb{R}^n$, for universal constants $\ell^\# = \ell^\#(K_0)$ and $C^\# = C^\#(K_0)$ determined only by $m$ and $n$ (recall that $K_0$ itself depends only on $m$ and $n$). Now choose $B_0$ with $E \subset B_0$. The Local Finiteness Principle for this ball states that $\Gamma_{\ell^\#}(x_0, f, M) \subset \Gamma_E(x_0, f, C^\# M)$ for any $M > 0$. Our main result, Theorem 49, now follows easily: By Lemma 56 (see (4.13)), the Finiteness Hypothesis with constant $k^\# = (D+1)^{\ell^\# + 1}$ implies $\Gamma_{\ell^\#}(x_0, f, 1) \neq \emptyset$, and thus $\Gamma_E(x_0, f, C^\#) \neq \emptyset$. In particular, there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ with $F = f$ on $E$ and $\|F\| \leq C^\#$. Therefore, Theorem 49 holds whenever $E \subset \mathbb{R}^n$ is finite.

Now, an easy compactness argument allows one to deduce Theorem 49 for infinite $E$.

The rest of the paper is organized as follows. In section 4.5 we formulate the Main Induction Argument that will be used to prove the Local Main Lemma. In section 4.6 we prove the Main Decomposition Lemma which allows us to pass from a local extension problem on a ball $B_0$ to a collection of easier subproblems on a family of smaller balls $B \subset 5B_0$; this lemma is the main component in the analysis of the induction step. In section 4.7, we state a technical lemma that allows us to control the shape of the set $\sigma_\ell(x)$ at lengthscales which are much coarser than the lengthscales of the balls in the decomposition; we next apply this lemma to enforce mutual consistency for a family of jets that are associated to the local extension problems on the smaller balls. In section 4.8 we will construct a solution to the local extension problem on $B_0$ by gluing together the solutions to the local problems on the smaller balls by means of a partition of unity; the consistency conditions arranged in the previous step will ensure that the individual local extensions are sufficiently compatible, which will allow us to obtain the necessary control on the $C^{m-1,1}$ seminorm of the glued-together function.

## 4.5 The Main Induction Argument I: Setup

We will prove the Local Main Lemma by induction on the complexity parameter $K \in \{-1, 0, \cdots, K_0\}$ – recall, $K_0$ is a finite upper bound on the local complexity of any set. When $K = -1$, the Local Main Lemma is vacuously true (say, for $C^{\#}(-1) = 1$, $\ell^{\#}(-1) = 0$) since complexity is non-negative. This establishes the base case of the induction.

For the induction step, fix $K \in \{0, 1, \cdots, K_0\}$. The induction hypothesis states that the Local Main Lemma for $K - 1$ is valid. Denote the finiteness constants in the Local Main Lemma for $K - 1$ by $\ell_{\text{old}} := \ell^{\#}(K - 1)$ and $C_{\text{old}} := C^{\#}(K - 1)$. Applying the Local Main Lemma to a closed ball of the form $\frac{6}{5}B$, we obtain

$$\begin{aligned}&\text{If } \ x \in (6/5) \cdot B \text{ and } \mathcal{C}(E|6B) \leq K - 1, \text{ then,}\\ &\Gamma_{\ell_{\text{old}}}(x, f, M) \subset \Gamma_{E \cap \frac{6}{5}B}(x, f, C_{\text{old}}M) \text{ for any } f : E \to \mathbb{R}, \ M > 0.\end{aligned} \tag{4.33}$$

(Here we use the formulation of the Local Finiteness Principle in Remark 88.)

Fix a ball $B_0 \subset \mathbb{R}^n$ with $\mathcal{C}(E|5B_0) \leq K$. To prove the Local Main Lemma for $K$, we are required to prove the Local Finiteness Principle (LFP) on $B_0$ for a suitable choice of the constants $\ell^{\#} \in \mathbb{Z}_{\geq 0}$ and $C^{\#} \geq 1$, determined by $m$, $n$, and $K$. Thus, our goal is to prove that $\Gamma_{\ell^{\#}}(x_0, f, M) \subset \Gamma_{E \cap B_0}(x_0, f, C^{\#}M)$ for any $f : E \to \mathbb{R}$, $x_0 \in B_0$, $M > 0$. A rescaling of the form $f \mapsto f/M$ allows us to reduce to the case $M = 1$. If $\#(B_0 \cap E) \leq 1$ then the LFP is true as long as $C^{\#} \geq 1$ and $\ell^{\#} \geq 0$ – indeed,

$$\Gamma_{\ell^{\#}}(x_0, f, 1) \subset \Gamma_0(x_0, f, 1) = \bigcap_{S \subset E, \ \#(S) \leq 1} \Gamma_S(x_0, f, 1) \tag{4.34}$$

$$\subset \Gamma_{E \cap B_0}(x_0, f, 1) \subset \Gamma_{E \cap B_0}(x_0, f, C^{\#}). \tag{4.35}$$

Accordingly, it suffices to assume that

$$\#(B_0 \cap E) \geq 2. \tag{4.36}$$

Under these assumptions, we will prove that for any $x_0 \in B_0$ and $f : E \to \mathbb{R}$,

$$\Gamma_{\ell^\#}(x_0, f, 1) \subset \Gamma_{E \cap B_0}(x_0, f, C^\#). \tag{4.37}$$

## 4.6 The Main Decomposition Lemma

In this section we fix the following data:

- A closed ball $B_0 \subset \mathbb{R}^n$ and a point $x_0 \in B_0$.
- A finite set $E \subset \mathbb{R}^n$ satisfying $\#(E \cap B_0) \geq 2$ and $\mathcal{C}(E|5B_0) \leq K$.
- A function $f : E \to \mathbb{R}$.
- An integer $\ell^\# \in \mathbb{Z}_{\geq 0}$.
- A polynomial $P_0 \in \Gamma_{\ell^\#}(x_0, f, 1)$.

Our plan is to introduce a cover of the ball $2B_0$ which will later be used to decompose the local extension problem on $B_0$ into a family of easier subproblems associated to the elements of the cover.

**Lemma 89** (Main Decomposition Lemma)**.** *Recall the constants* $R_{label} \ll R_{med} \ll R_{big} \ll R_{huge}$*,* $C_*$*, and* $C_{**}$ *defined in* (4.31)*. Given data as above, there exists a subspace* $V \in \mathrm{DTI}$ *such that*
*(a)* $\sigma(x)$ *is* $(x, C_* diam(B_0), R_{label})$*-transverse to* $V$ *for all* $x \in 100B_0$*.*

*There exists a Whitney cover* $\mathcal{W}$ *of* $2B_0$ *such that, for all* $B \in \mathcal{W}$*,*
*(b)* $B \subset 100B_0$ *and* $diam(B) \leq \frac{1}{2} diam(B_0)$*.*

*(c) The subspace $\sigma(x)$ is $(x, C_*\delta, R_{huge})$-transverse to $V$ for all $x \in 8B$, $\delta \in [diam(B), diam(B_0)]$.*
*(d) Either $\#(6B \cap E) \leq 1$ or $\mathcal{C}(E|6B) < K$.*

*For every $B \in \mathcal{W}$ there exists a point $z_B \in \mathbb{R}^n$ and a jet $P_B \in \mathcal{P}$ satisfying*
*(e) $z_B \in \frac{6}{5}B \cap 2B_0$; also, if $x_0 \in \frac{6}{5}B$ then $z_B = x_0$.*
*(f) $P_B \in \Gamma_{\ell\#-1}(z_B, f, C_{**})$ and $P_0 - P_B \in C_{**}\mathcal{B}_{z_B,diam(B_0)}$; also, if $x_0 \in \frac{6}{5}B$ then $P_B = P_0$.*
*(g) $P_0 - P_B \in V$.*

Condition (d) of Lemma 89 and the induction hypothesis allow us to prove a local finiteness principle on the elements of the cover $\mathcal{W}$. That is,

**Lemma 90.** *For every $B \in \mathcal{W}$, the local finiteness principle holds on $\frac{6}{5}B$ with constants $\ell_{old} = \ell^{\#}(K-1) \in \mathbb{Z}_{\geq 0}$ and $C_{old} = C^{\#}(K-1) \geq 1$. That is, $\Gamma_{\ell_{old}}(x, f, M) \subset \Gamma_{E\cap\frac{6}{5}B}(x, f, C_{old}M)$, for all $x \in \frac{6}{5}B$, $M > 0$.*

*Proof.* If $\mathcal{C}(E|6B) < K$ then the result follows from (4.33). On the other hand, if $\#(E \cap 6B) \leq 1$ then the result follows from (4.34). These cases are exhaustive thanks to condition (d). $\square$

### 4.6.1 Proof of the Main Decomposition Lemma

We apply Lemma 80 to pick a subspace $V \in$ DTI such that $\sigma(x)$ is $(x, C_*\text{diam}(B_0), R_{\text{label}})$-transverse to $V$ for all $x \in 100B_0$. This establishes property (a). The construction of $\mathcal{W}$ is based on the following definition:

**Definition 91.** A ball $B \subset 100B_0$ is OK if $\#(B \cap E) \geq 2$ and if there exists $z \in B$ such that $\sigma(z)$ is $(z, C_*\delta, R_{\text{big}})$-transverse to $V$ for all $\delta \in [\text{diam}(B), \text{diam}(B_0)]$.

The OK property is *inclusion monotone* in the sense that if $B \subset B' \subset 100B_0$ and $B$ is OK then $B'$ is OK.

For each $x \in 2B_0$, let $r(x) := \inf\{r > 0 : B(x,r) \subset 100B_0,\ B(x,r) \text{ is OK}\}$. Every subball of $100B_0$ that contains $2B_0$ is OK, so the infimum is well-defined – this also implies $r(x) \leq 2\text{diam}(B_0)$ for all $x \in 2B_0$. If $B \subset 100B_0$ is sufficiently small then $\#(B \cap E) \leq 1$, and so $B$ is not OK – in particular, this shows that $r(x) \geq \Delta := \min\{|x-y| : x, y \in E, x \neq y\} > 0$ for all $x \in 2B_0$. Let $B_x := B(x, \frac{1}{7}r(x))$ for $x \in 2B_0$. Then

$$70B_x \subset 100B_0, \quad \text{for } x \in 2B_0. \tag{4.38}$$

Note that the family of closed balls $\mathcal{W}^* = \{B_x\}_{x \in 2B_0}$ is a cover of $2B_0$.

**Lemma 92.** *If $B \in \mathcal{W}^*$ then $8B$ is OK, and $6B$ is not OK.*

*Proof.* We can write $B = B(x, \frac{1}{7}r(x))$ for some $x \in 2B_0$. According to (4.38), $6B \subset 8B \subset 100B_0$. By definition of $r(x)$ as an infimum and the inclusion monotonicity of the OK property, the result follows. □

We apply the Vitali covering lemma to extract a finite subcover $\mathcal{W} \subset \mathcal{W}^*$ of $B_0$ so that the family of third-dilates $\{\frac{1}{3}B\}_{B \in \mathcal{W}}$ is pairwise disjoint.

**Lemma 93.** *$\mathcal{W}$ is a Whitney cover of $2B_0$.*

*Proof.* We only have to verify condition (c) in the definition of a Whitney cover (see Definition 63). Suppose for sake of contradiction that there exist balls $B_j = B(x_j, r_j) \in \mathcal{W}$ for $j = 1, 2$, with $\frac{6}{5}B_1 \cap \frac{6}{5}B_2 \neq \emptyset$ and $r_1 < \frac{1}{8}r_2$. Since $\frac{6}{5}B_1 \cap \frac{6}{5}B_2 \neq \emptyset$, we have $|x_1 - x_2| \leq \frac{6}{5}r_1 + \frac{6}{5}r_2$. If $z \in 8B_1$ then $|z - x_1| \leq 8r_1$, and therefore

$$|z - x_2| \leq |z - x_1| + |x_1 - x_2| \leq 8r_1 + \frac{6}{5}r_1 + \frac{6}{5}r_2 \leq r_2 + \frac{3}{20}r_2 + \frac{6}{5}r_2 \leq 6r_2.$$

Hence, $8B_1 \subset 6B_2$. By Lemma 92, $8B_1$ is OK. Thus, by inclusion monotonicity, $6B_2$ is OK. But this contradicts Lemma 92. This finishes the proof by contradiction. □

We now establish conditions (b)-(d) in the Main Decomposition Lemma. Fix a ball $B \in \mathcal{W}$.

We will use the following preparatory claim: (PC) If $\#(6B \cap E) \geq 2$ then for all $x \in 6B$ there exists $\delta_x \in [6\mathrm{diam}(B), \mathrm{diam}(B_0)]$ so that $\sigma(x)$ is not $(x, C_*\delta_x, R_{\text{big}})$-transverse to $V$. This follows because $6B$ is not OK.

*Proof of (b):* The inclusion $B \subset 100B_0$ follows from (4.38). For sake of contradiction, suppose that $\mathrm{diam}(B) > \frac{1}{2}\mathrm{diam}(B_0)$. Since $B \cap B_0 \neq \emptyset$, we have $B_0 \subset 5B$. Therefore, $\#(5B \cap E) \geq \#(B_0 \cap E) \geq 2$. Fix a point $x \in B$. Then (PC) implies that the interval $[6\mathrm{diam}(B), \mathrm{diam}(B_0)]$ is nonempty, thus $\mathrm{diam}(B) \leq \frac{1}{6}\mathrm{diam}(B_0)$, which gives the contradiction.

*Proof of (c):* Since $8B$ is OK, $\sigma(z)$ is $(z, C_*\delta, R_{\text{big}})$-transverse to $V$ for some $z \in 8B$ and all $\delta \in [8\mathrm{diam}(B), \mathrm{diam}(B_0)]$. If $x \in 8B$ then $|x - z| \leq 8\mathrm{diam}(B) \leq$

$\frac{c_1}{R_{\text{big}}} \cdot (C_*\delta)$ (see (4.31)), and so, by Lemma 74,

$$\sigma(x) \text{ is } (x, C_*\delta, 8R_{\text{big}})\text{-transverse to } V \text{ for all } \delta \in [8\text{diam}(B), \text{diam}(B_0)].$$

Any number in $[\text{diam}(B), \text{diam}(B_0)]$ differs from a number in $[8\text{diam}(B), \text{diam}(B_0)]$ by a factor of at most 8. Hence, by Lemma 73, $\sigma(x)$ is $(x, C_*\delta, 8^{m+1}R_{\text{big}})$-transverse to $V$ for all $\delta \in [\text{diam}(B), \text{diam}(B_0)]$. Since $R_{\text{huge}} \geq 8^{m+1}R_{\text{big}}$ (see (4.31)), this implies (c).

*Proof of (d):* Suppose that $\#(6B \cap E) \geq 2$ and set $J := \mathcal{C}(E|6B)$. According to the formulation of complexity in Remark 85, there exist intervals $I_1 > I_2 > \cdots > I_J$ in $(0, 6\text{diam}(B)]$, subspaces $V_1, \cdots, V_J \subset \mathcal{P}$, and a point $z \in 6B$, such that

(A) $\tau_{z,r(I_j)}(\sigma(z))$ is $(z, C_*, R_{\text{label}})$-transverse to $V_j$, and

(B) $\tau_{z,l(I_j)}(\sigma(z))$ is not $(z, C_*, R_{\text{med}})$-transverse to $V_j$, for $1 \leq j \leq J$, where $R_{\text{med}} = 256DR_{\text{label}}$.

From $B \cap B_0 \neq \emptyset$ and $\text{diam}(B) \leq \frac{1}{2}\text{diam}(B_0)$ (see (b)) it follows that $6B \subset 5B_0$. Hence, $z \in 5B_0$.

Since $\#(6B \cap E) \geq 2$, the condition (PC) implies that there exists $\delta_z \in [6\text{diam}(B), \text{diam}(B_0)]$ such that

$$\sigma(z) \text{ is not } (z, C_*\delta_z, R_{\text{big}})\text{-transverse to } V. \tag{4.39}$$

We will now establish that (A) and (B) hold for $j = 0$ with $I_0 := [\delta_z, \text{diam}(B_0)]$ and $V_0 := V$. Since $V$ is a DTI subspace, $\tau_{z,l(I_0)}V = V$, and therefore, by rescaling (4.39),

$$\tau_{z,l(I_0)}(\sigma(z)) \text{ is not } (z, C_*, R_{\text{big}})\text{-transverse to } V. \tag{4.40}$$

On the other hand, from property (a) we learn that $\sigma(z)$ is $(z, C_* \mathrm{diam}(B_0), R_{\text{label}})$-transverse to $V$. Therefore, by rescaling,

$$\tau_{z,r(I_0)}(\sigma(z)) \text{ is } (z, C_*, R_{\text{label}})\text{-transverse to } V. \tag{4.41}$$

The conditions (4.40) and (4.41) together imply (A) and (B) for $j = 0$ (recall $R_{\text{big}} \geq R_{\text{med}}$).

Notice that $r(I_1) \leq 6\mathrm{diam}(B) \leq \delta_z = l(I_0)$, thus $I_1 < I_0$. In conclusion, $I_0 > I_1 > \cdots > I_J$ are subintervals of $(0, \mathrm{diam}(B_0)]$.

We produced intervals $I_0 > I_1 > \cdots > I_J$ in $(0, 5\mathrm{diam}(B_0)]$ and subspaces $V_0, \cdots, V_J \subset \mathcal{P}$, so that (A) and (B) hold for $j = 0, 1, \cdots, J$. Since $z \in 5B_0$, by the formulation of complexity in Remark 85, we have $\mathcal{C}(E|5B_0) \geq J + 1$. Since $\mathcal{C}(E|5B_0) \leq K$, this completes the proof of (d).

Finally we will define a collection of points $\{z_B\}_{B \in \mathcal{W}}$ and polynomials $\{P_B\}_{B \in \mathcal{W}}$ and prove properties (e)-(g).

*Proof of (e):* We define the collection $\{z_B\}_{B \in \mathcal{W}}$ to satisfy property (e). For all $B \in \mathcal{W}$ such that $x_0 \in \frac{6}{5}B$ we set $P_B = P_0$. We define $P_B$ for the remaining balls $B \in \mathcal{W}$ in the proof of (f) and (g) below.

*Proofs of (f) and (g):* If $x_0 \in \frac{6}{5}B$ then $z_B = x_0$ and $P_B = P_0$, in which case (f) and (g) are trivially true (note that $P_0 \in \Gamma_{\ell\#}(x_0, f, 1) \subset \Gamma_{\ell\#-1}(x_0, f, 1)$). Suppose instead $x_0 \notin \frac{6}{5}B$. Then $z_B \in \frac{6}{5}B \cap 2B_0$ and so $|x_0 - z_B| \leq 2\mathrm{diam}(B_0)$. By Lemma 56, given that $P_0 \in \Gamma_{\ell\#}(x_0, f, 1)$, we can find $P_B \in \Gamma_{\ell\#-1}(z_B, f, 1)$ with $P_0 - P_B \in C_T \mathcal{B}_{z_B, 2\mathrm{diam}(B_0)} \subset 2^m C_T \mathcal{B}_{z_B, \mathrm{diam}(B_0)}$. We still have to arrange

$P_0 - P_B \in V$ as in (g). Unfortunately, there is no reason for this to be true, and we will have to perturb $P_B$ to arrange this property. This is where we use condition (a), which implies that $\sigma(z_B)$ is $(z_B, 5C_*\mathrm{diam}(B_0), R_{\text{label}})$-transverse to $V$. Therefore,

$$\begin{aligned}\mathcal{B}_{z_B,\mathrm{diam}(B_0)}/V \subset \mathcal{B}_{z_B,5C_*\mathrm{diam}(B_0)}/V \subset R_{\text{label}} \cdot (\sigma(z_B) \cap \mathcal{B}_{z_B,5C_*\mathrm{diam}(B_0)})/V \\ \subset R_{\text{label}} \cdot (\sigma_{\ell\#-1}(z_B) \cap \mathcal{B}_{z_B,5C_*\mathrm{diam}(B_0)})/V.\end{aligned}$$

Since $P_0 - P_B \in 2^m C_T \mathcal{B}_{z_B,\mathrm{diam}(B_0)}$, the last containment implies we can find a bounded correction

$$R_B \in 2^m C_T R_{\text{label}} \cdot (\sigma_{\ell\#-1}(z_B) \cap \mathcal{B}_{z_B,5C_*\mathrm{diam}(B_0)}),$$

so that $R_B/V = (P_0 - P_B)/V$, i.e., $P_0 - P_B - R_B \in V$. Set $\widetilde{P}_B = P_B + R_B$. Then $P_0 - \widetilde{P}_B \in V$ and

$$\widetilde{P}_B \in \Gamma_{\ell\#-1}(z_B, f, 1) + 2^m C_T R_{\text{label}} \sigma_{\ell\#-1}(z_B) \subset \Gamma_{\ell\#-1}(z_B, f, 1 + 2^m C_T R_{\text{label}}).$$

Furthermore,

$$\begin{aligned}P_0 - \widetilde{P}_B = (P_0 - P_B) - R_B \in 2^m C_T \mathcal{B}_{z_B,\mathrm{diam}(B_0)} + 2^m C_T R_{\text{label}} \mathcal{B}_{z_B,5C_*\mathrm{diam}(B_0)} \\ \subset 2^m C_T \cdot (1 + R_{\text{label}} \cdot (5C_*)^m) \cdot \mathcal{B}_{z_B,\mathrm{diam}(B_0)}.\end{aligned}$$

Thus we have proven (f) and (g) for all $B \in \mathcal{W}$ such that $x_0 \notin \frac{6}{5}B$, with $\widetilde{P}_B$ in place of $P_B$, where $C_{**} = 1 + 2^m C_T \cdot (1 + R_{\text{label}} \cdot (5C_*)^m)$. This finishes the proof of Lemma 89.

## 4.7 The Main Induction Argument II

We return to the setting of the Main Induction Argument in section 4.5. Let $\ell_{\text{old}} = \ell^{\#}(K-1)$ and $C_{\text{old}} = C^{\#}(K-1)$ be as in (4.33). We fix data

$(B_0, x_0, E, f)$ as in section 4.5. Recall our goal is to establish the containment (4.37) for a suitable choice of $\ell^{\#} = \ell^{\#}(K)$ and $C^{\#} = C^{\#}(K)$ which will be determined by the end of the proof. We fix a polynomial $P_0 \in \Gamma_{\ell^{\#}}(x_0, f, 1)$, and apply Lemma 89 to the data $(B_0, x_0, E, f, \ell^{\#}, P_0)$. Through this we obtain a Whitney cover $\mathcal{W}$ of $2B_0$, a DTI subspace $V \subset \mathcal{P}$, and two families $\{P_B\}_{B\in\mathcal{W}} \subset \mathcal{P}$ and $\{z_B\}_{B\in\mathcal{W}} \subset \mathbb{R}^n$.

Let $\mathcal{W}_0$ be the collection of all balls $B \in \mathcal{W}$ with $B \cap B_0 \neq \emptyset$. Then $\mathcal{W}_0$ is a Whitney cover of $B_0$.

The main goal of this section is to prove that the family of polynomials $\{P_B\}_{B\in\mathcal{W}}$ are mutually compatible. Specifically, we will prove:

**Lemma 94.** *There exist constants $\overline{\ell} > \ell_{old}$ and $\overline{C} \geq 1$, determined by $m$ and $n$, such that the following holds. If $\ell^{\#} \geq \overline{\ell}$, and $\{P_B\}_{B\in\mathcal{W}}$ is a family of polynomials satisfying the conditions in Lemma 89, then $P_B - P_{B'} \in \overline{C} \cdot \mathcal{B}_{z_B, diam(B)}$ for any $B, B' \in \mathcal{W}_0$ with $(\frac{6}{5})B \cap (\frac{6}{5})B' \neq \emptyset$.*

We will see that Lemma 94 follows easily from the next result.

**Lemma 95.** *There exist $\epsilon^* \in (0,1)$, $\ell^* > \ell_{old}$, and $R^* \geq 1$, depending only on $m$ and $n$, such that the following holds. If $\widehat{B} \in \mathcal{W}_0$ satisfies $diam(\widehat{B}) \leq \epsilon^* diam(B_0)$, and if the subspace $V$ is as in Lemma 89, then $\sigma_{\ell^*}(x)$ is $(x, diam(B), R^*)$-transverse to $V$ for any $B \in \mathcal{W}_0$ and $x \in 6B$.*

Lemma 95 is difficult for subtle reasons: We know from condition (c) of the Main Decomposition Lemma that $\sigma(x)$ is $(x, \mathrm{diam}(B), R)$-transverse to

$V$ for any $B \in \mathcal{W}$ and $x \in 8B$, where $R = R_{\text{huge}} \cdot (6C_*)^m$. But it is not apparent why $V$ would also be transverse to $\sigma_\ell(x)$, which generally can be significantly larger than $\sigma(x)$. The key point in the proof of this proposition is that we are able to use the validity of the Local Finiteness Principle on the balls $B$ in $\mathcal{W}$ to establish a two-sided relationship between the sets $\sigma(x)$ and $\sigma_{\ell^*}(x)$ (for sufficiently large $\ell^*$) as long as we are willing to "blur" these sets at a lengthscale larger than $\mathrm{diam}(B)$. Since transversality is stable under "blurrings" (e.g., see Lemma 69), the result will follow.

The proof of Lemma 95 is the most technical part of the paper. We next explain how Lemma 94 follows from Lemma 95. After this we will establish a preparatory lemma, Lemma 96, and finally give the proof of Lemma 95 in section 4.7.2.

*Proof of Lemma 94.* We fix $\epsilon^* \in (0,1)$ and $\ell^*$ as in Lemma 95, and define $\overline{\ell} = \ell^* + 2$. We consider the following two situations:

- *Case 1:* $\mathrm{diam}(B) > \epsilon^* \mathrm{diam}(B_0)$ for all $B \in \mathcal{W}_0$.
- *Case 2:* There exists $\widehat{B} \in \mathcal{W}_0$ with $\mathrm{diam}(\widehat{B}) \leq \epsilon^* \mathrm{diam}(B_0)$.

Fix $B, B' \in \mathcal{W}_0$ with $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$. In Case 1, by condition (f) in Lemma 89, we have

$$P_B - P_{B'} = (P_B - P_0) + (P_0 - P_{B'}) \in C_{**}\mathcal{B}_{z_B,\mathrm{diam}(B_0)} + C_{**}\mathcal{B}_{z_{B'},\mathrm{diam}(B_0)}.$$

Note that $|z_B - z_{B'}| \leq 2\mathrm{diam}(B_0)$ (recall $z_B, z_{B'} \in 2B_0$), and so by (4.5), $\mathcal{B}_{z_{B'},\mathrm{diam}(B_0)} \subset \widetilde{C}2^{m-1}\mathcal{B}_{z_B,\mathrm{diam}(B_0)}$. As $\mathrm{diam}(B) > \epsilon^* \mathrm{diam}(B_0)$, we have

$\mathcal{B}_{z_B,\mathrm{diam}(B_0)} \subset (\epsilon^*)^{-m}\mathcal{B}_{z_B,\mathrm{diam}(B)}$. When put together, we learn that $P_B - P_{B'} \in C_{**} \cdot (\epsilon^*)^{-m}(1+\widetilde{C}2^{m-1})\mathcal{B}_{z_B,\mathrm{diam}(B)}$, which gives the desired result in this case.

Now suppose that Case 2 holds. By property (g) in Lemma 89, we have

$$P_B - P_{B'} = (P_B - P_0) + (P_0 - P_{B'}) \in V.$$

By property (1) we have $P_{B'} \in \Gamma_{\ell\#-1}(z_{B'}, f, C)$. By Lemma 56, there exists $\tilde{P}_B \in \Gamma_{\ell\#-2}(z_B, f, C)$ with $\tilde{P}_B - P_{B'} \in C' \cdot \mathcal{B}_{z_B,\mathrm{diam}(B)}$. Furthermore, since $\tilde{P}_B \in \Gamma_{\ell\#-2}(z_B, f, C)$ and $P_B \in \Gamma_{\ell\#-1}(z_B, f, C) \subset \Gamma_{\ell\#-2}(z_B, f, C)$, we have

$$\tilde{P}_B - P_B \in 2C \cdot \sigma_{\ell\#-2}(z_B) = 2C \cdot \sigma_{\ell^*}(z_B),$$

where we have used the fact that $\ell^{\#} - 2 \geq \overline{\ell} - 2 = \ell^*$. Thus,

$$P_B - P_{B'} = (P_B - \tilde{P}_B) + (\tilde{P}_B - P_{B'}) \in 2C \cdot \sigma_{\ell^*}(z_B) + C' \cdot \mathcal{B}_{z_B,\mathrm{diam}(B)}$$

and hence

$$P_B - P_{B'} \in (2C \cdot \sigma_{\ell^*}(z_B) + C' \cdot \mathcal{B}_{z_B,\delta_B}) \cap V \subset C'' \cdot (\sigma_{\ell^*}(z_B) + \mathcal{B}_{z_B,\mathrm{diam}(B)}) \cap V.$$

Since $\sigma_{\ell^*}(z_B)$ is $(z_B, \mathrm{diam}(B), R^*)$-transverse to $V$ (see Lemma 95), also $\sigma_{\ell^*}(z_B) + \mathcal{B}_{z_B,\mathrm{diam}(B)}$ is $(z_B, \mathrm{diam}(B), R^{**})$-transverse to $V$, with $R^{**} = R^* + 3 \cdot (R^*)^2$ (see Lemma 69). In particular,

$$(\sigma_{\ell^*}(z_B) + \mathcal{B}_{z_B,\mathrm{diam}(B)}) \cap V \subset R^{**} \cdot \mathcal{B}_{z_B,\mathrm{diam}(B)}.$$

Therefore, $P_B - P_{B'} \in C'' R^{**} \cdot \mathcal{B}_{z_B,\mathrm{diam}(B)}$, which concludes the proof of the lemma.

□

### 4.7.1 Finiteness principles for set unions with weakly controlled constants

Through the use of Lemma 66 and Helly's theorem we will obtain the following result: If a ball $\widehat{B} \subset \mathbb{R}^n$ is covered by a collection of balls each of which satisfies a Local Finiteness Principle, then $\widehat{B}$ satisfies a Local Finiteness Principle with constants that may depend on the cardinality of the cover. We should remark that we lack any control on the cardinality of the cover $\mathcal{W}_0$ of $B_0$, and so this type of result cannot be used to obtain a Local Finiteness Principle on $B_0$ with any control on the constants. This lemma will be used in the next subsection, however, to obtain a local finiteness principle on a family of intermediate balls that are much larger than the balls of the cover, yet small when compared to $B_0$.

**Lemma 96.** *Fix $C_0 \geq 1$ and $\ell_0 \in \mathbb{Z}_{\geq 0}$. Let $\mathcal{W}$ be a Whitney cover of a ball $\widehat{B} \subset \mathbb{R}^n$ with cardinality $N = \#\mathcal{W}$. If the Local Finiteness Principle holds on $\frac{6}{5}B$ with constants $C_0$ and $\ell_0$, for all $B \in \mathcal{W}$, then the Local Finiteness Principle holds on $\widehat{B}$ with constants $C_1$ and $\ell_1 := \ell_0 + \lceil \frac{\log(D \cdot N+1)}{\log(D+1)} \rceil$, where $C_1$ depends only on $C_0$, $m$, and $n$ – in particular, $C_1$ is independent of the cardinality $N$ of the cover.*

*Proof.* Let $f : E \to \mathbb{R}$ and $M > 0$. By assumption, $\Gamma_{\ell_0}(x, f, M) \subset \Gamma_{E \cap \frac{6}{5}B}(x, f, C_0 M)$ for all $x \in \frac{6}{5}B$, $B \in \mathcal{W}$. Fix a point $x_0 \in \widehat{B}$. Our goal is to prove that

$$\Gamma_{\ell_1}(x_0, f, M) \subset \Gamma_{E \cap \widehat{B}}(x_0, f, C_1 M), \tag{4.42}$$

for a constant $C_1 \geq 1$, to be determined later.

For each $B \in \mathcal{W}$, we fix $x_B \in \frac{6}{5}B$. We demand that

$$x_B = x_0 \iff x_0 \in (6/5)B; \tag{4.43}$$

otherwise, $x_B$ is an arbitrary element of $\frac{6}{5}B$.

Fix an arbitrary element $P \in \Gamma_{\ell_1}(x_0, f, M)$. We will define a family of auxiliary convex sets to which we will apply Helly's theorem and obtain the desired conclusion. The convex sets will belong to the vector space $\mathcal{P}^N$ consisting of $N$-tuples of $(m-1)$-st order Taylor polynomials indexed by the elements of the cover $\mathcal{W}$. For each $S \subset E$, define the convex set

$$\begin{aligned}\mathcal{K}(S, M) := \{(J_{x_B} F)_{B \in \mathcal{W}} :& F \in C^{m-1,1}(\mathbb{R}^n),\ \|F\| \leq M, \\ & F = f \text{ on } S,\ J_{x_0} F = P\} \subset \mathcal{P}^N.\end{aligned}$$

If $\#(S) \leq (D+1)^{\ell_1}$ then $P \in \Gamma_{\ell_1}(x_0, f, M) \subset \Gamma_S(x_0, f, M)$. Thus, there exists $F \in C^{m-1,1}(\mathbb{R}^n)$ with $\|F\| \leq M$, $F = f$ on $S$, and $J_{x_0} F = P$. Therefore, $(J_{x_B} F)_{B \in \mathcal{W}} \in \mathcal{K}(S, M)$. In particular, $\mathcal{K}(S, M) \neq \emptyset$ if $\#(S) \leq (D+1)^{\ell_1}$.

If $S_1, \cdots, S_J \subset E$, with $J := \dim(\mathcal{P}^N) + 1 = D \cdot N + 1$, then

$$\bigcap_{j=1}^{J} \mathcal{K}(S_j, M) \supset \mathcal{K}(S, M), \text{ for } S = S_1 \cup \cdots \cup S_J.$$

If furthermore $\#(S_j) \leq (D+1)^{\ell_0}$ for every $j$, then $\#(S) \leq J \cdot (D+1)^{\ell_0} \leq (D+1)^{\ell_1}$, and consequently by the previous remark $\mathcal{K}(S, M) \neq \emptyset$. Thus we have shown

$$\bigcap_{j=1}^{J} \mathcal{K}(S_j, M) \neq \emptyset, \quad \text{if } S_1, \cdots, S_J \subset E,\ J = \dim(\mathcal{P}^N) + 1,$$
$$\text{and } \#(S_j) \leq (D+1)^{\ell_0} \text{ for all } j.$$

Therefore, by Helly's theorem,

$$\mathcal{K} := \bigcap_{\substack{S \subset E \\ \#(S) \leq (D+1)^{\ell_0}}} \mathcal{K}(S, M) \neq \emptyset.$$

Fix an arbitrary element $(P_B)_{B \in \mathcal{W}} \in \mathcal{K}$. By definition of $\mathcal{K}$,

($*$) for any $S \subset E$ with $\#(S) \leq (D+1)^{\ell_0}$ there exists a function $F^S \in C^{m-1,1}(\mathbb{R}^n)$ with $\|F^S\| \leq M$, $F^S = f$ on $S$, $J_{x_0} F^S = P$, and $J_{x_B} F^S = P_B$ for all $B \in \mathcal{W}$. From this condition we will establish the following properties:

(a) $P_B = P$ if $x_0 \in (6/5)B$,

(b) $|P_B - P_{B'}|_{x_B, \mathrm{diam}(B)} \leq CM$ whenever $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$,

(c) for each $B \in \mathcal{W}$ there exists $F_B \in C^{m-1,1}(\mathbb{R}^n)$ such that $\|F_B\| \leq C_0 M$, $F_B = f$ on $E \cap \frac{6}{5}B$, and $J_{x_B} F_B = P_B$.

For the proof of (a) and (b) take $S = \emptyset$ in ($*$). Then $P_B = J_{x_B} F^{\emptyset} = J_{x_0} F^{\emptyset} = P$ whenever $x_0 \in \frac{6}{5}B$ (see (4.43)), which yields (a). For (b), note that $x_B \in \frac{6}{5}B$, $x_{B'} \in \frac{6}{5}B'$, and $\frac{6}{5}B \cap \frac{6}{5}B' \neq \emptyset$, and hence by the definition of Whitney covers, $\mathrm{diam}(B)$ and $\mathrm{diam}(B')$ differ by a factor of at most 8. Thus, $|x_B - x_{B'}| \leq \frac{6}{5}\mathrm{diam}(B) + \frac{6}{5}\mathrm{diam}(B') \leq 11 \mathrm{diam}(B)$. Thus, by (4.1) and Taylor's theorem (see (4.2)),

$$
\begin{aligned}
|P_B - P_{B'}|_{x_B,\mathrm{diam}(B)} &\leq 11^m |P_B - P_{B'}|_{x_B,11\mathrm{diam}(B)} \\
&= 11^m |J_{x_B}F^{\emptyset} - J_{x_{B'}}F^{\emptyset}|_{x_B,11\mathrm{diam}(B)} \\
&\leq 11^m C_T \|F^{\emptyset}\| \leq CM.
\end{aligned}
$$

For the proof of (c), note that $(*)$ implies $P_B \in \Gamma_{\ell_0}(x_B, f, M)$ for each $B \in \mathcal{W}$. By assumption, the Local Finiteness Principle holds on $\frac{6}{5}B$ with constants $C_0$ and $\ell_0$, and therefore $P_B \in \Gamma_{E\cap\frac{6}{5}B}(x_B, f, C_0M)$ for each $B \in \mathcal{W}$. This completes the proof of (c).

Fix a partition of unity $\{\theta_B\}$ adapted to the Whitney cover $\mathcal{W}$ as in Lemma 65, and set $F = \sum_{B\in\mathcal{W}} \theta_B F_B$. By use of properties (b) and (c), we conclude via Lemma 66 that (A) $\|F\|_{C^{m-1,1}(\widehat{B})} \leq CM$ and (B) $F = f$ on $E\cap\widehat{B}$. Since $\mathrm{supp}\theta_B \subset \frac{6}{5}B$, we learn that $J_{x_0}\theta_B = 0$ if $x_0 \notin \frac{6}{5}B$; on the other hand, $J_{x_0}F_B = J_{x_B}F_B = P_B = P$ if $x_0 \in \frac{6}{5}B$ (see (4.43)). Thus, if we compare the following sums term-by-term, we obtain the identity

$$
J_{x_0}F = \sum_{B\in\mathcal{W}} J_{x_0}\theta_B \odot_{x_0} J_{x_0}F_B = \sum_{B\in\mathcal{W}} J_{x_0}\theta_B \odot_{x_0} P.
$$

Recall that $\sum_{B\in\mathcal{W}} \theta_B = 1$ on $\widehat{B}$ and $x_0 \in \widehat{B}$. Thus, $\sum_{B\in\mathcal{W}} J_{x_0}\theta_B = J_{x_0}(1) = 1$. Therefore, (C) $J_{x_0}F = P$. By a standard technique we extend the function $F \in C^{m-1,1}(\widehat{B})$ to a function in $C^{m-1,1}(\mathbb{R}^n)$ with norm bounded by $C\|F\|_{C^{m-1,1}(\widehat{B})} \leq C'M$ – by abuse of notation, we denote this extension by the same symbol $F$. Then (D) $\|F\| \leq C'M$. Furthermore, (B) and (C) continue to hold for this extension. From (B),(C), and (D) we conclude that $P \in \Gamma_{E\cap\widehat{B}}(x_0, f, C'M)$. This finishes the proof of (4.42).

□

### 4.7.2 Proof of Lemma 95

We need to generate an upper containment on $\sigma_\ell(x) \cap V$ for a suitable integer constant $\ell$. Recall from property (c) in Lemma 89, $\sigma(x) \cap V \subset R_{\text{huge}} \cdot \mathcal{B}_{x,\text{diam}(B)}$ for $x \in 8B$ and $B \in \mathcal{W}$. To generate a similar containment for $\sigma_\ell(x) \supset \sigma(x)$ we introduce the idea of "keystone balls" (based on the keystone cubes introduced in [51]) which are elements of the cover for which we may obtain a local finiteness principle on a dilate of the balls by a large constant factor (much larger than the constants $C, C_*, R_{\text{huge}}$, etc.). By an appropriate choice of this factor, we can deduce information about the shape of $\sigma_\ell(x)$ (through the existence of a transverse subspace) on a neighborhood of a keystone ball. This information can then be passed along to the remaining elements of the cover due to the "quasicontinuity" of the sets $\sigma_\ell(x)$ (Lemma 56) and the fact that every ball is close to a keystone ball (as established in Lemma 99).

#### 4.7.2.1 Keystone balls

Let $\epsilon^* \in (0, \frac{1}{300}]$ be a free parameter, which will later be fixed to be a small enough constant determined by $m$ and $n$. In what follows all constants may depend on $m$ and $n$. If a constant depends additionally on $\epsilon^*$ we will be explicit and write it as $C(\epsilon^*)$, $C_0(\epsilon^*)$, etc. Set $A = (3\epsilon^*)^{-\frac{1}{2}} \geq 10$.

By hypothesis of Lemma 95, $\text{diam}(\widehat{B}) \leq \epsilon^* \text{diam}(B_0)$ for some $\widehat{B} \in \mathcal{W}_0$.

**Definition 97.** A ball $B^\# \in \mathcal{W}$ is keystone if $\text{diam}(B) \geq \frac{1}{2}\text{diam}(B^\#)$ for every $B \in \mathcal{W}$ with $B \cap A \cdot B^{\#} \neq \emptyset$. Write $\mathcal{W}^{\#} \subset \mathcal{W}$ to denote the set of all keystone balls.

**Lemma 98.** *For each ball $B \in \mathcal{W}$ there exists a keystone ball $B^{\#} \in \mathcal{W}^{\#}$ with $B^{\#} \subset 3AB$, $dist(B, B^{\#}) \leq 2A diam(B)$, and $diam(B^{\#}) \leq diam(B)$.*

*Proof.* If $B$ is itself keystone, take $B^{\#} = B$ to establish the result. Otherwise, let $B_1 = B$. Since $B_1$ is not keystone there exists $B_2 \in \mathcal{W}$ with $B_2 \cap AB_1 \neq \emptyset$ and $\operatorname{diam}(B_2) < \frac{1}{2}\operatorname{diam}(B_1)$. Similarly, if $B_2$ is not keystone there exists $B_3 \in \mathcal{W}$ with $B_3 \cap AB_2 \neq \emptyset$ and $\operatorname{diam}(B_3) < \frac{1}{2}\operatorname{diam}(B_2)$. We continue to iterate this process. As $\mathcal{W}$ is finite, the process must terminate after finitely many steps. By iteration, there exists a sequence of balls $B_1, B_2, \cdots, B_J \in \mathcal{W}$ with $B_j \cap AB_{j-1} \neq \emptyset$ and $\operatorname{diam}(B_j) < \frac{1}{2}\operatorname{diam}(B_{j-1})$ for all $j$, and with $B_J$ keystone. As $B_j \cap AB_{j-1} \neq \emptyset$ we have $\operatorname{dist}(B_{j-1}, B_j) \leq \frac{A}{2}\operatorname{diam}(B_{j-1})$. Now estimate

$$\begin{aligned}
\operatorname{dist}(B_1, B_J) &\leq \sum_{j=2}^{J} \operatorname{dist}(B_{j-1}, B_j) + \sum_{j=2}^{J-1} \operatorname{diam}(B_j) \\
&\leq (A/2 + 1) \sum_{j=1}^{J} \operatorname{diam}(B_j) \\
&\leq (A+2)\operatorname{diam}(B_1) \leq 2A\operatorname{diam}(B_1).
\end{aligned}$$

Since $\operatorname{diam}(B_J) \leq \operatorname{diam}(B_1)$, we have $B_J \subset (2A+6)B_1 \subset 3AB_1$. We set $B^{\#} = B_J$ and this finishes the proof. □

We define a mapping $\kappa : \mathcal{W}_0 \to \mathcal{W}^{\#}$. By applying Lemma 98, we obtain a keystone ball $\widehat{B}^{\#}$ with $\widehat{B}^{\#} \subset 3A\widehat{B}$ and $\operatorname{diam}(\widehat{B}^{\#}) \leq \operatorname{diam}(\widehat{B})$. For each $B \in \mathcal{W}_0$, we proceed as follows:

- If $\mathrm{diam}(B) > \epsilon^*\mathrm{diam}(B_0)$ ($B$ is *medium-sized*), set $\kappa(B) := \widehat{B}^\#$.

- If $\mathrm{diam}(B) \leq \epsilon^*\mathrm{diam}(B_0)$ ($B$ is *small-sized*), Lemma 98 yields a keystone ball $B^\#$ with $B^\# \subset 3AB$; set $\kappa(B) := B^\#$.

**Lemma 99.** *The mapping $\kappa : \mathcal{W}_0 \to \mathcal{W}^\#$ satisfies the following properties: For any $B \in \mathcal{W}_0$, (a) dist$(B, \kappa(B)) \leq C_4$diam$(B)$, for $C_4 = C_4(\epsilon^*)$, (b) diam$(\kappa(B)) \leq$ diam$(B)$, and (c) $A \cdot \kappa(B) \subset 2B_0$.*

*Proof.* Suppose $B$ is medium-sized. Then $\kappa(B) = \widehat{B}^\#$. Since $\mathrm{diam}(B) > \epsilon^*\mathrm{diam}(B_0)$ and $B \subset B_0$ we have $9(\epsilon^*)^{-1}B \supset B_0 \supset \widehat{B}$; furthermore, $\widehat{B}^\# \subset 3A\widehat{B}$. Thus, $\widehat{B}^\# \subset 27(\epsilon^*)^{-1}AB$, which gives (a) for $C_4 = 27(\epsilon^*)^{-1}A$. Also, $\mathrm{diam}(\widehat{B}^\#) \leq \mathrm{diam}(\widehat{B}) \leq \epsilon^*\mathrm{diam}(B_0) < \mathrm{diam}(B)$, which establishes (b). Finally, since $\widehat{B} \subset B_0$ and $\mathrm{diam}(\widehat{B}) \leq \epsilon^*\mathrm{diam}(B_0)$, we have $A\widehat{B}^\# \subset 3A^2\widehat{B} \subset (1 + 3\epsilon^* A^2)B_0 = 2B_0$, which gives (c).

Now suppose $B$ is small-sized. Then we defined $\kappa(B) = B^\#$, where $B^\#$ is related to $B$ as in Lemma 98. In particular, $\mathrm{dist}(B, B^\#) \leq 2A\mathrm{diam}(B)$ and $\mathrm{diam}(B^\#) \leq \mathrm{diam}(B)$, yielding (a) and (b). Furthermore, $B^\# \subset 3AB$, and from $B \subset B_0$ and $\mathrm{diam}(B) \leq \epsilon^*\mathrm{diam}(B_0)$ we deduce that $AB^\# \subset 3A^2B \subset (1 + 3\epsilon^* A^2)B_0 = 2B_0$, yielding (c). □

This completes the description of the geometric relationship between the balls of $\mathcal{W}_0$ and keystone balls in $\mathcal{W}$. We will next need a lemma about the shape of $\sigma_\ell(z_{B^\#})$ for a keystone ball $B^\#$.

**Lemma 100.** *Let $B^{\#} \in \mathcal{W}$ be a keystone ball with $AB^{\#} \subset 2B_0$. Then there exists an integer constant $\ell(\epsilon^*) > \ell_{old}$, determined by $\epsilon^*$, $m$, and $n$, and a constant $C \geq 1$ determined by $m$ and $n$, so that the Local Finiteness Principle holds on $AB^{\#}$ with constants $C$ and $\ell(\epsilon^*)$, namely, $\Gamma_{\ell(\epsilon^*)}(x, f, M) \subset \Gamma_{E \cap AB^{\#}}(x, f, CM)$ for all $x \in AB^{\#}$ and $M > 0$. In particular, by taking $f = 0$ and $M = 1$, we have $\sigma_{\ell(\epsilon^*)}(x) \subset C \cdot \sigma(x, E \cap AB^{\#})$ for any $x \in AB^{\#}$.*

*Proof.* Let $\mathcal{W}(B^{\#})$ be the collection of all elements of $\mathcal{W}$ that intersect $AB^{\#}$. Since $\mathcal{W}$ is a Whitney cover of $2B_0$ and $AB^{\#} \subset 2B_0$, we have that $\mathcal{W}(B^{\#})$ is a Whitney cover of $AB^{\#}$. The Local Finiteness Principle holds on $\frac{6}{5}B$ for all $B \in \mathcal{W}(B^{\#})$, with constants $C_{\text{old}}$ and $\ell_{\text{old}}$ (see Lemma 90). Therefore, the Local Finiteness Principle holds on $AB^{\#}$ with the constant $C_1$ determined by $m$ and $n$, and the constant $\ell_1 = \ell_{\text{old}} + \lceil \frac{\log(D \cdot N+1)}{\log(D+1)} \rceil$, where $N = \#\mathcal{W}(B^{\#})$; see Lemma 96.

We will estimate $N = \#\mathcal{W}(B^{\#})$ using a volume comparison bound. By the definition of keystone balls, $\text{diam}(B) \geq \frac{1}{2}\text{diam}(B^{\#})$ for all $B \in \mathcal{W}(B^{\#})$ – furthermore, we claim that $\text{diam}(B) \leq 10A\text{diam}(B^{\#})$. We proceed by contradiction. If $\text{diam}(B) > 10A\text{diam}(B^{\#})$ for some $B \in \mathcal{W}(B^{\#})$ then $B \cap AB^{\#} \neq \emptyset$, which implies that $(6/5)B \cap B^{\#} \neq \emptyset$. Then $\text{diam}(B) \leq 8\text{diam}(B^{\#})$ thanks to the definition of a Whitney cover, which gives a contradiction.

For any $B \in \mathcal{W}(B^{\#})$ we have $B \cap AB^{\#} \neq \emptyset$ and $\text{diam}(B) \leq 10A\text{diam}(B^{\#})$, and therefore $B \subset 30AB^{\#}$.

We can estimate the volume of $\Omega := \bigcup_{B \in \mathcal{W}(B^{\#})} \frac{1}{3}B$ in two ways. First, note that $\mathrm{Vol}(\Omega) \leq \mathrm{Vol}(30AB^{\#}) = (30A)^n \mathrm{Vol}(B^{\#})$. Next, using that the collection $\{\frac{1}{3}B\}_{B\in\mathcal{W}}$ is pairwise disjoint, $N = \#\mathcal{W}(B^{\#})$, and $\mathrm{diam}(B) \geq \frac{1}{2}\mathrm{diam}(B^{\#}) \forall B \in \mathcal{W}(B^{\#})$, we have

$$\mathrm{Vol}(\Omega) = \sum_{B\in\mathcal{W}(B^{\#})} 3^{-n}\mathrm{Vol}(B) \geq N6^{-n}\mathrm{Vol}(B^{\#}).$$

Hence, $N \leq (180A)^n \leq 180^n(\epsilon^*)^{-\frac{n}{2}}$. Thus, $\ell_1 \leq \ell(\epsilon^*) := \ell_{\mathrm{old}} + \lceil \frac{\log(D\cdot 180^n(\epsilon^*)^{-\frac{n}{2}}+1)}{\log(D+1)} \rceil$. □

**Lemma 101.** *If the parameter $\epsilon^*$ is picked sufficiently small depending on $m$ and $n$, and if $A = (3\epsilon^*)^{-\frac{1}{2}}$ in the definition of keystone balls, then for any keystone ball $B^{\#} \in \mathcal{W}^{\#}$ such that $AB^{\#} \subset 2B_0$, we have*

$$\sigma_\ell(z_{B^{\#}}) \cap V \subset \mathcal{B}_{z_{B^{\#}}, A\mathrm{diam}(B^{\#})}, \quad \text{for } \ell = \ell(\epsilon^*) > \ell_{old}.$$

*Proof.* By Lemma 100, and Lemma 59 (applied to the ball $AB^{\#}$ and point $z = z_{B^{\#}} \in \frac{1}{2}AB^{\#}$),

$$\begin{aligned}\sigma_{\ell(\epsilon^*)}(z_{B^{\#}}, E) \cap \mathcal{B}_{z_{B^{\#}}, A\mathrm{diam}(B^{\#})} &\subset (C\sigma(z_{B^{\#}}, E \cap AB^{\#})) \cap \mathcal{B}_{z_{B^{\#}}, A\mathrm{diam}(B^{\#})} \\ &\subset C_3\sigma(z_{B^{\#}}, E),\end{aligned}$$

for a constant $C_3$ determined by $m$ and $n$. Dropping the dependence on $E$, we have shown that

$$\sigma_{\ell(\epsilon^*)}(z_{B^{\#}}) \cap \mathcal{B}_{z_{B^{\#}}, A\mathrm{diam}(B^{\#})} \subset C_3\sigma(z_{B^{\#}}). \tag{4.44}$$

By property (c) in Lemma 89, $\sigma(z_{B^{\#}})$ is $(z_{B^{\#}}, C_*\mathrm{diam}(B^{\#}), R_{\mathrm{huge}})$-transverse to $V$. Hence, $\sigma(z_{B^{\#}}) \cap V \subset R_{\mathrm{huge}}\mathcal{B}_{z_{B^{\#}}, C_*\mathrm{diam}(B^{\#})} \subset \widehat{R}\mathcal{B}_{z_{B^{\#}}, \mathrm{diam}(B^{\#})}$, for $\widehat{R} =$

$R_{\text{huge}}(C_*)^m$. Combined with (4.44), this implies

$$\sigma_\ell(z_{B\#}) \cap V \cap \mathcal{B}_{z_{B\#},A\text{diam}(B\#)} \subset C_3\sigma(z_{B\#}) \cap V \subset C_3\widehat{R}\mathcal{B}_{z_{B\#},\text{diam}(B\#)}$$
$$\subset \mathcal{B}_{z_{B\#},C_3\widehat{R}\text{diam}(B\#)}.$$

As long as $A = (3\epsilon^*)^{-\frac{1}{2}} \geq 2C_3\widehat{R}$, this implies $\sigma_\ell(z_{B\#}) \cap V \subset \mathcal{B}_{z_{B\#},C_3\widehat{R}\text{diam}(B\#)} \subset \mathcal{B}_{z_{B\#},A\text{diam}(B\#)}$. □

#### 4.7.2.2 Finishing the proof of Lemma 95

We fix $A = (3\epsilon^*)^{-\frac{1}{2}}$, $\epsilon^* = \epsilon^*(m,n)$, and $\ell = \ell(\epsilon^*) > \ell_{\text{old}}$ via Lemma 101. The constants $\epsilon^*$, $A$ and $\ell$ are determined only by $m$ and $n$.

Fix $B \in \mathcal{W}_0$ and $x \in 6B$. Consider the keystone ball $B^\# \in \mathcal{W}$ given by $B^\# = \kappa(B)$ which satisfies the conditions in Lemma 99, namely, $\text{diam}(B^\#) \leq \text{diam}(B)$, $\text{dist}(B^\#, B) \leq C_4\text{diam}(B)$, and $AB^\# \subset 2B_0$. By Lemma 101,

$$\sigma_\ell(z_{B\#}) \cap V \subset \mathcal{B}_{z_{B\#},A\text{diam}(B\#)} \subset A^m\mathcal{B}_{z_{B\#},\text{diam}(B\#)} \subset A^m\mathcal{B}_{z_{B\#},\text{diam}(B)}. \quad (4.45)$$

Now, note that $|x - z_{B\#}| \leq 6\text{diam}(B) + \text{dist}(B, B^\#) + \frac{6}{5}\text{diam}(B^\#) \leq C_5\text{diam}(B)$ for $C_5 = C_4 + 8$. Thus, Lemma 56 gives

$$\sigma_{\ell+1}(x) \subset \sigma_\ell(z_{B\#}) + C_T\mathcal{B}_{z_{B\#},C_5\text{diam}(B)} \subset \sigma_\ell(z_{B\#}) + C_TC_5^m\mathcal{B}_{z_{B\#},\text{diam}(B)}.$$

Property (c) in the Main Decomposition Lemma states that $\sigma(z_{B\#})$ is $(z_{B\#}, C_*\delta, R_{\text{huge}})$-transverse to $V$ for all $\delta \in [\text{diam}(B^\#), \text{diam}(B_0)]$. We take $\delta = \text{diam}(B)$ in this statement, and apply Lemma 73 to deduce that $\sigma(z_{B\#})$ is $(z_{B\#}, \text{diam}(B), R_1)$-transverse to $V$, for $R_1 = R_{\text{huge}}\cdot(C_*)^m$. Thus, in particular,

$$\mathcal{B}_{z_{B\#},\text{diam}(B)}/V \subset R_1\cdot(\sigma(z_{B\#}) \cap \mathcal{B}_{z_{B\#},\text{diam}(B)})/V \subset R_1\cdot(\sigma_\ell(z_{B\#}) \cap \mathcal{B}_{z_{B\#},\text{diam}(B)})/V.$$

Combined with (4.45), this shows that $\sigma_\ell(z_{B\#})$ is $(z_{B\#}, \mathrm{diam}(B), R_2)$-transverse to $V$, for $R_2 = \max\{R_1, A^m\}$. Furthermore, by Lemma 69, $\sigma_\ell(z_{B\#}) + C_T C_5^m \mathcal{B}_{z_{B\#},\mathrm{diam}(B)}$ is $(z_{B\#}, \mathrm{diam}(B), R_3)$-transverse to $V$, for $R_3 = R_2 + 3R_2^2 C_T C_5^m$. We conclude that

$$\sigma_{\ell+1}(x) \cap V \subset (\sigma_\ell(z_{B\#}) + C_T C_5^m \mathcal{B}_{z_{B\#},\mathrm{diam}(B)}) \cap V \tag{4.46}$$

$$\subset R_3 \mathcal{B}_{z_{B\#},\mathrm{diam}(B)} \subset R_4 \mathcal{B}_{x,\mathrm{diam}(B)}, \tag{4.47}$$

for $R_4 = R_3 \widetilde{C} C_5^{m-1}$. Here, (4.5) and $|x - z_{B\#}| \leq C_5 \mathrm{diam}(B)$ are used to obtain the last containment.

On the other hand, property (c) of the Main Decomposition Lemma shows that $\sigma(x)$ is $(x, 6C_* \mathrm{diam}(B), R_{\mathrm{huge}})$-transverse to $V$, and hence $\sigma(x)$ is $(x, \mathrm{diam}(B), R_1)$-transverse to $V$, for $R_1 = R_{\mathrm{huge}} \cdot (6C_*)^m$ (by Lemma 73). In particular,

$$\mathcal{B}_{x,\mathrm{diam}(B)}/V \subset R_1 \cdot (\sigma(x) \cap \mathcal{B}_{x,\mathrm{diam}(B)})/V \subset R_1 \cdot (\sigma_{\ell+1}(x) \cap \mathcal{B}_{x,\mathrm{diam}(B)})/V. \tag{4.48}$$

Combining (4.46) and (4.48), we see that $\sigma_{\ell+1}(z_B)$ is $(z_B, \mathrm{diam}(B), \max\{R_1, R_4\})$-transverse to $V$. This finishes the proof of Lemma 95, with $\epsilon^* = \epsilon^*(m,n)$, $\ell^* = \ell(\epsilon^*) + 1$, and $R^* = \max\{R_1, R_4\}$.

## 4.8 The Main Induction Argument III: Putting it all together

Here we finish the proof of the containment (4.37). Namely, for suitable constants $\ell^\# \in \mathbb{Z}_{\geq 0}$ and $C^\# \geq 1$, we will prove

$$\Gamma_{\ell^\#}(x_0, f, 1) \subset \Gamma_{E \cap B_0}(x_0, f, C^\#), \quad \text{for all } x_0 \in B_0,\ f : E \to \mathbb{R}.$$

This will conclude the proof of the Local Finiteness Principle on $B_0$, and complete the Main Induction Argument.

Continuing with the argument outlined in the beginning of section 4.7, we fix $P_0 \in \Gamma_{\ell^{\#}}(x_0, f, 1)$. We apply the Main Decomposition Lemma to the data $(x_0, B_0, E, f, \ell^{\#}, P_0)$ to obtain a Whitney cover $\mathcal{W}$ of $2B_0$, a DTI subspace $V \subset \mathcal{P}$, and families $\{P_B\}_{B\in\mathcal{W}}$ and $\{z_B\}_{B\in\mathcal{W}}$. Recall that $\mathcal{W}_0 \subset \mathcal{W}$ is a finite cover of $B_0$.

We define $\ell^{\#} = \overline{\ell}$, where $\overline{\ell} > \ell_{\text{old}}$ is defined via Lemma 94.

Condition (f) in the Main Decomposition Lemma states that $P_B \in \Gamma_{\ell^{\#}-1}(z_B, f, C)$ for all $B \in \mathcal{W}_0$. By Lemma 90 and the bound $\ell^{\#} - 1 \geq \ell_{\text{old}}$ it follows that $P_B \in \Gamma_{\ell^{\#}-1}(z_B, f, C) \subset \Gamma_{\ell_{\text{old}}}(z_B, f, C) \subset \Gamma_{E\cap\frac{6}{5}B}(z_B, f, C')$. So,

$$P_B \in \Gamma_{E\cap\frac{6}{5}B}(z_B, f, C') \text{ for all } B \in \mathcal{W}.$$

Recall that $z_B \in \frac{6}{5}B$ for all $B \in \mathcal{W}$. By definition of $\Gamma_{E\cap\frac{6}{5}B}(\cdots)$, there exists $F_B \in C^{m-1,1}(\mathbb{R}^n)$ with

$$\begin{cases} F_B = f \text{ on } E \cap (6/5)\cdot B, \; J_{z_B}F_B = P_B, \text{ and} \\ \|F_B\| \leq C'. \end{cases} \tag{4.49}$$

Since $\ell^{\#} = \overline{\ell}$, by Lemma 94 we conclude that

$$|J_{z_B}F_B - J_{z_{B'}}F_{B'}|_{z_B,\text{diam}(B)} \leq \overline{C} \text{ whenever } B, B' \in \mathcal{W}_0, \; (\frac{6}{5})\cdot B \cap (\frac{6}{5})\cdot B' \neq \emptyset. \tag{4.50}$$

Let $\{\theta_B\}_{B\in\mathcal{W}_0}$ be a partition of unity on $B_0$ subordinate to the cover $\mathcal{W}_0$, as in Lemma 65. Define

$$F = \sum_{B\in\mathcal{W}_0} F_B\theta_B \text{ on } B_0.$$

By Lemma 66 (and the conditions (4.49) and (4.50)), $F \in C^{m-1,1}(B_0)$ satisfies $\|F\|_{C^{m-1,1}(B_0)} \leq C$ and $F = f$ on $E \cap B_0$. Recall the points $\{z_B\}_{B\in\mathcal{W}}$ possess the additional property that $z_B = x_0$ if $x_0 \in \frac{6}{5}B$, and the polynomials $\{P_B\}_{B\in\mathcal{W}}$ possess the additional property that $P_B = P_0$ if $x_0 \in \frac{6}{5}B$ (see condition (e) in Lemma 89). Thus, $J_{x_0}F_B = P_0$ whenever $x_0 \in \frac{6}{5}B$. Therefore,

$$\begin{aligned} J_{x_0}F &= \sum_{B\in\mathcal{W}_0 : x_0 \in \frac{6}{5}B} J_{x_0}(F_B\theta_B) = \sum_{B\in\mathcal{W}_0 : x_0 \in \frac{6}{5}B} J_{x_0}F_B \odot_{x_0} J_{x_0}\theta_B \\ &= \sum_{B\in\mathcal{W}_0 : x_0 \in \frac{6}{5}B} P_0 \odot_{x_0} J_{x_0}\theta_B = P_0 \odot_{x_0} 1 = P_0. \end{aligned}$$

We now extend the function $F$ to all of $\mathbb{R}^n$ by a classical extension technique (e.g., Stein's extension theorem). This gives a function $\widehat{F} \in C^{m-1,1}(\mathbb{R}^n)$ with $\|\widehat{F}\| \leq C\|F\|_{C^{m-1,1}(B_0)} \leq C'$ and $\widehat{F} = F$ on $B_0$. In particular, $\widehat{F} = f$ on $E \cap B_0$ and $J_{x_0}\widehat{F} = P_0$ (since $x_0 \in B_0$). Thus, $P_0 \in \Gamma_{E\cap B_0}(x_0, f, C')$.

We define $C^{\#} = C'$ for the constant $C'$ arising above. Since $P_0 \in \Gamma_{\ell^{\#}}(x_0, f, 1)$ is arbitrary, this finishes the proof of the containment (4.37).

## 4.9 Future work and open problems

In this section we will discuss two directions left open by this work. First, we discuss ideas for improving the bounds on the constants in the finiteness principle. Second, we'll discuss a generalized version of this problem for functions on a certain type of nilpotent Lie groups called Carnot groups.

### 4.9.1 Improving the bounds in the finiteness principle

We mentioned in the introduction to this chapter that we believe our bound on $K_0$ in Proposition 82 can be improved. One reason for thinking this is that the current proof does not exploit the special structure of the maps $\tau_{x,\delta}$. The proof works for any dynamical system with semialgebraic dependence on $x$ and $\delta$. This leaves unexploited the fact that the $\tau_{x_0,\delta}$ act diagonally on the monomial basis $\{(x-x_0)^\alpha\}_{|\alpha|\leq m-1}$, a much stronger condition. In this section, we will discuss a promising connection between the problem of bounding $K_0$ and matrix perturbation theory which takes advantage of this structure.

Recall from Remark 68 that a uniform bound on $\mathcal{C}(\mathcal{E})$ for all ellipsoids will imply a uniform bound on $\mathcal{C}(E)$. Also, recall that any ellipsoid $\mathcal{E}$ is defined by the equation $\langle x, Ax\rangle \leq 1$ where $A$ is a positive semidefinite matrix unique to $\mathcal{E}$. We write $\mathcal{E}_A$ to mean the ellipsoid corresponding to the matrix $A$. Without loss of generality, we consider the problem of bounding the complexity at the origin. We write down the action of $\tau_{0,\delta}$ on an ellipsoid. We have,

$$\tau_{0,\delta}(\mathcal{E}_A) = \mathcal{E}_{D_{\delta^{-1}}AD_{\delta^{-1}}}$$

Moreover, the eigenspace of $A$ can be used to describe the transversality properties of $\mathcal{E}_A$; see Remark 68. Therefore, instead of studying the complexity of arbitrary finite sets $E$ we can study how the eigenspaces of positive semidefinite matrices change under a transformation we can write down explicitly.

More specifically, we would like to understand whether the eigenvectors of $D_{\delta^{-1}} A D_{\delta^{-1}}$ are stable for most $\delta$.

### 4.9.2 The finiteness principle for Carnot groups

Carnot groups are nilpotent, connected, simply connected Lie groups. One reason they are studied is that they are close enough to Euclidean space so that many of the same tools are available while still having a very different flavor. The simplest example of a non-Euclidean Carnot group is the Heisenberg group $\mathbb{H}$. Already this case is nontrivial, so we will focus our attention here.

In order to discuss the finiteness principle in this setting we give a brief and informal introduction to the Heisenberg group $\mathbb{H}$. For a complete introduction to the theory of Carnot groups see [6]. We define $\mathbb{H}$ to be $\mathbb{R}^3$ with the group operation

$$(x, y, t) \cdot (x', y', t') = (x + x', y + y', t + t' + 2(yx' - xy'))$$

$\mathbb{H}$ is a Lie group. Its Lie algebra is spanned by the left-invariant vector fields

$$X = \frac{\partial}{\partial x} + 2y\frac{\partial}{\partial t}, \quad Y = \frac{\partial}{\partial y} - 2x\frac{\partial}{\partial t}, \quad T = \frac{\partial}{\partial t}$$

The only nonvanishing commutator between these three elements is $[X, Y] = -4T$. In this sense, we can think of $X$ and $Y$ as first order differential operators and $T$ as a second order differential operator. It turns out (by the Poincaré-Birkhoff-Witt theorem) that a basis for the algebra of differential operators is given by

$$D^I = X^{i_1} Y^{i_2} T^{i_3}$$

where $I = (i_1, i_2, i_3) \in \mathbb{N}^3$ and we define $|I| := i_1 + i_2 + 2i_3$ to be the degree of $D^I$.

Similarly, a basis for the space of polynomials is given by

$$\eta^J := x^{j_1} y^{j_2} t^{j_3}$$

where $J = (j_1, j_2, j_3) \in \mathbb{N}^3$ and we define $|J| := j_1 + j_2 + 2j_3$ to be the degree of $\eta^J$.

We can define a metric on any Carnot group called the Carnot-Carathéodory metric, we will denote this metric by $d_{CC}$. This metric is abstract and is not well suited for computations. On the Heisenberg group, we remark that there exists a (bilipschitz) equivalent metric $d(z, z') := \max\{|x - x'|, |y - y'|, |t - t'|^{1/2}\}$.

We now define the $C^{m-1,1}(\mathbb{H})$ seminorm. For a function $f : \mathbb{H} \to \mathbb{R}$,

$$||F||_{C^{m-1,1}(\mathbb{H})} := \sup_{z,z' \in \mathbb{H}} \left( \sum_{|J|=m-1} \left( \frac{(X^J F)(x) - (X^J F)(y))}{d_{CC}(x, y)} \right)^2 \right)^{1/2}$$

As in the case of $\mathbb{R}^d$, we say that a function $f$ belongs to $C^{m-1,1}(\mathbb{H})$ if this seminorm is finite. Analogues of all of the results of Section 4.1 hold. In particular, we can prove an analog of Taylor's theorem and the classical Whiteny extension theorem (in which we are given jets) in this setting. The classical Whitney extension theorem is known to hold for all Carnot groups [67], and the proof follows from the proof of original mutadis mutandis.

The simplest function space we can attempt to prove this theorem for is $C^{1,1}(\mathbb{H})$. In this case, as in the case of the classical Whitney extension theorem, the proof is a straightforward generalization of the proof on $\mathbb{R}^d$. This relies heavily on the fact that all differential operators commute on polynomials of degree 1.

When we study $C^{2,1}(\mathbb{H})$, we are interested in jets of order 2. In this case, we have the nonvanishing commutator $[X, Y] = -4T$ (this doesn't show up for first degree polynomials because it annihilates them). This fact causes our original argument to fail. In fact, it is no longer true that every Whitney convex set is transverse to a translation invariant subspace. We will now give an example of such a Whitney convex set.

Let $\mathcal{P}$ denote the vector space of polynomials of degree at most 2. We define $\Omega$ to be the ideal in $\mathcal{P}_0$ (i.e. $\mathcal{P}$ with respect to the product $\odot_0$ defined just as in Section 4.1 on $\mathbb{R}^d$) generated by the monomials $x$ and $y$, i.e. $\Omega = \text{span}\{x, y, xy, x^2, y^2\}$. Recall that any ideal is Whitney convex (Section 4.1.2).

Now suppose that $\Omega$ is $R$-transverse to a DTI subspace $V$ for some $0 < R < \infty$. Condition (2) of Definition 67 says that we must have $\Omega \cap V \subset R \cdot \mathcal{B}$. This can only be true if $V \subset \text{span}\{1, t\}$. But the only translation invariant subspace of $\text{span}\{1, t\}$ is $\text{span}\{1\}$. Condition (1) of Definition 67 says that we must have $\mathcal{B}/V \subset R \cdot ((\sigma \cap \mathcal{B})/V)$. This is not possible, however, as $\mathcal{B}/V$ contains a multiple of the monomial $t$, while the right hand side of the inclusion cannot contain a multiple of $t$. Therefore $\Omega$ is not $R$-transverse to any DTI subspace.

This example shows that any proof of a $C^{2,1}(\mathbb{H})$ finiteness principle, and therefore any proof which works for arbitrary Carnot groups, must introduce new ideas.

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