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Abstract

This project is devoted to study the so–called k–generalized chromatic numbers which arise from Low Tree–Depth decompositions of graphs. The k–generalized chromatic number, introduced with this name by Nešetřil and Ossona de Mendez, is the minimum number of colors needed to color a graph such that every i–colors induce a graph of tree–depth at most i. One of the main results in the theory of bounded expansion classes developed by these authors, says that a class of bounded expansion has bounded generalized chromatic numbers, and in fact this provides one characterization of such classes.

We first study the list version of the k–generalized chromatic numbers in connection with the conjecture of Ohba, which says that a large chromatic number ensures the equality with the list chromatic number. We extend to k–generalized chromatic numbers the known results on this conjecture. We next give asymptotic upper bounds on the k–generalized chromatic numbers for classes of graphs with bounded degree by using the Lovász Local Lemma. We finally study the tree–depth of random graphs, closely connected to k–generalized chromatic numbers, and analyse the evolution of this parameter for linear and superlinear numbers of edges.

Aquest projecte està dedicat a estudiar el k-èssim nombre cromàtic generalitzat que sorgeix de les descomposicions Low Tree–Depth en grafs. El k-èssim nombre cromàtic generalitzat, introduït amb aquest nom per Nešetřil i Ossona de Mendez, és el mínim nombre de colors necessaris per colorar un graf de tal manera que qualssevol i classes de colors indueixen un graf amb tree–depth menor o igual a i. Un dels principals resultats en la teoria de grafs d'expansió fitada, desenvolupada per aquests autors, diu que una classe té expansió fitada si el seus nombres cromàtics generalitzats estan fitats. De fet, això proveeix una caracterització d'aquestes classes.

En primer lloc estudiem la versió per llistes del k-èssim nombre cromàtic generalitzat en connexió amb la conjectura de Ohba, que afirma que un nombre cromàtic gran assegura la igualtat entre aquest i el nombre cromàtic per llistes. Estenem als nombres cromàtics generalitzats els resultats coneguts al voltant d'aquesta conjectura. A continuació donem fites superiors asimptòtiques pel k-èssim nombre cromàtic generalitzat per classes de grafs de grau fitat utilitzant el Lema Local de Lovász. Finalment, estudiem el tree–depth dels grafs aleatoris, estretament relacionat amb els nombres cromàtics generalitzats, i analitzem l'evolució d'aquest paràmetre per grafs amb un nombre lineal o supralineal d'arestes.

Introduction

Coloring problems of graphs form a central topic in Graph Theory which has motivated a substantial part of its contemporary form and is still a lively source of challenging problems and theories, as illustrated in the extensive survey of open problems by Jensen and Toft [27]. All of these problems have its origin in the classical question about coloring a graph in such a way that two adjacent vertices do not share a color. Even if coloring problems have often a simple formulation, their solution is usually not simple, a fact that can be illustrated by the algorithmic complexity of the simple question about the value of the chromatic number of a graph being equal to k, only simple (and trivial) when $k = 2$.

The notion of chromatic number, which appears in many basic graph theoretic questions, has been enriched in several directions giving rise to a host of parameters with a wide range of motivations and applications. This work focusses mainly in a recent generalization of the chromatic number with connections with deep structural analysis of sparse graphs developed in the last few years in a series of papers of Nešetřil and Ossona de Mendez [38, 39, 42, 37].

The generalization we are interested in traces back its origins in two variations of the chromatic number: the acyclic and the star chromatic numbers. The acyclic chromatic number $\chi_a(G)$ of a graph G is the minimum number of colors needed to color a graph such that every color class induces a graph with no edges *and* every two color classes induce a graph with no cycles. Similarly, the star chromatic number $\chi_s(G)$ is the minimum number of colors in a proper coloring of G such that every two classes induce a forest of stars. Both parameters have been widely studied in the literature. Perhaps surprisingly, DeVoss et al [15] showed that, for every minor closed class $\mathcal{C}$ of graphs with at least one excluded minor and a given natural number k, there is a constant $k(\mathcal{C})$ such that every graph G in the class admits a coloring with $k(\mathcal{C})$ colors for which a far reaching generalization of the acyclic coloring property holds, namely, every $i \leq k$ colors induce a graph with tree–width at most i. Such a coloring is called a *Low Tree–width Decomposition* of G.

In his structural study of sparse graphs, Nešetřil and Ossona de Mendez introduced the analogous concept of *Low Tree–depth Decomposition* of a graph, related to a new parameter, the tree–depth, which measures the structural complexity of a graph with respect to rooted trees of given depth, in analogy to tree–width, which can be seen as a measure of complexity with respect to trees.

A Low Tree–depth Decomposition is a coloring of a graph in which, for some fixed k, every $i \leq k$ colors induce a subgraph with tree–depth at most i. For $k = 2$, the size of Low Tree–width Decomposition is the acyclic chromatic number, while the size of a Low Tree–depth Decomposition corresponds to the star chromatic number. The fundamental result in the theory of bounded expansion classes of graphs, which provides an effective measure of sparsity, states that for every bounded expansion class $\mathcal{C}$ of graphs exists a parameter $\chi_k(\mathcal{C})$ such that every graph in the class admits a Low Tree–depth Decomposition of level k with $\chi_k(\mathcal{C})$ colors. These authors call $\chi_k(G)$, the minimum number of colors in a Low Tree–depth Decomposition of level k, the *k–generalized chromatic number.* The same notion had already appeared in the literature under different names in many contexts (the ranking number [14], the height of elimination trees [38], etc.).

This work focuses on the study of k–generalized chromatic numbers from the probabilistic perspective. The probabilistic method has a long and fruitful history in combinatorics. The book of Alon and Spencer [4] gives a good account on the probabilistic method and its applications. A good reference on the particularly successful application of the probabilistic method to coloring problems is the monograph by Molloy and Reed [33] on the topic.

The probabilistic method relies on proving that certain statements are true by showing that the probability for a combinatorial element to exist is strictly positive. Whereas many combinatorial proofs imply constructing objects in an explicit way, the probabilistic method gives only existential proofs.

A natural question is whether the restricting variations of graph colorings behave in the same way as ordinary proper colorings. In this direction, it is interesting to see if it is possible to translate certain results that hold for colorings, to these generalized versions of chromatic numbers.

One example of this common behavior is related with the notion of list colorings. The list version of a chromatic number restricts the set of possible colors assigned to a vertex, and we ask for the common length of lists of colors available to each vertex to ensure that a suitable coloring exists. The list version of a coloring has several natural motivations, including the completion of a partial coloring of a graph and the frequency assignment problem in mobile communications. There are several longstanding conjectures on this parameter, which can be much larger than the corresponding ordinary chromatic number, one of them the famous List Coloring Conjecture. We will study another well–known conjecture, due to Ohba, which states that the list chromatic number coincides with the ordinary one if the latter is at least half the order of the graph. Reed and Sudakov [45] proved a weaker form of the conjecture by using a random strategy to color the vertices of the graph. Our first result extends this result for ordinary chromatic numbers to any conceivable generalization of a chromatic number. We also suggest an approach to sharpen the result towards the complete proof of Ohba's conjecture which, unfortunately, we have not been able to complete yet. We also study a related problem posed by Reed and Sudakov, and discuss the relation between fractional chromatic numbers and its list versions.

The second part of this work is based on the application of the Lovász Local Lemma to study the k–generalized chromatic numbers. The Lovász Local Lemma allows for application of the probabilistic method in the context of events with weak dependency relations. As such, the method is suitable for application in coloring problems to graphs with bounded degree, where the conflicts arisen by a random coloring have only local dependencies, see for instance the papers of Alon, Molloy and Reed [3] on the acyclic chromatic index of graphs with bounded degree, or the work of Fertin, Raspaud and Reed on the acyclic chromatic number of various classes of graphs with bounded degree [21]. Since the class of bounded degree graphs is a bounded expansion class, we know that there exist a Low

Tree–depth Decomposition. In other words, the k–generalized chromatic number is well defined in such classes of graphs. We give asymptotic upper bounds on the k–generalized chromatic numbers in classes of graphs with maximum degree d in terms of d by using the Lovász Local Lemma. Our results can be considerably strengthened for classes with large girth. Even if the upper bounds match the best known results in the case of star colorings, we obtain lower bounds based on Hamming graphs which, unfortunately, do not match the upper bounds.

To define Low Tree–depth Decompositions, a tree-like parameter has to be introduced: the tree–depth [38]. Our last chapter is devoted to a different probabilistic approach: random graphs. In the classical Erdős Renyi model, a random graph is obtained by choosing independently every possible edge in a graph with a given probability p. The goal is to identify, if possible, the asymptotically almost sure value of a parameter of a random graph. The usual approach is to let the probability p of an edge to depend on the order n of the graph, and to analyze the evolution of the parameter in terms of the functional relationship of p with n. Equivalently, the value of the parameter under study is analyzed in terms of the edge density of the random graph. In this framework, we study the tree–depth of random graphs and analyze particularly the phase transitions of this parameter. We completely characterize the asymptotic almost surely value of this parameter. As a side result, we give a direct proof of a recently proved conjecture by Kloks on the linear character if tree–width of random graphs with $p = c/n$ and $c > 1$. We also show that this linear character still appears in random regular graphs.

Contents

1

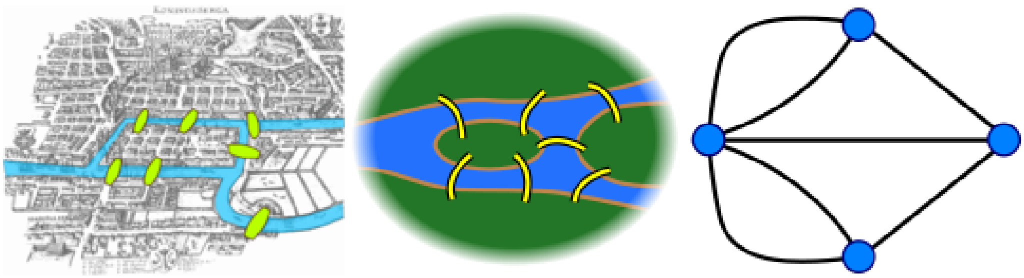
Background

1.1 Why graphs?

Graph theory is a young field of mathematics that has grown exponentially in the last century. Nevertheless, the theory of graphs has started in the XVII-th century.

Roughly speaking, a graph is composed by some elements called vertices and relations among them, called edges.

In 1736 Leonhard Euler, one of the more fruitful mathematics in history, published a paper on the Seven Bridges of Köninsberg. Köninsberg (currently named Kaliningrad) is a city, bathed by the Pregel river, that belonged to the Prussian empire. When the river crosses the city, it forms two little islands. Seven bridges were built to connect these islands with both sides of the river (Fig. 1.1).



(a) (b) (c)

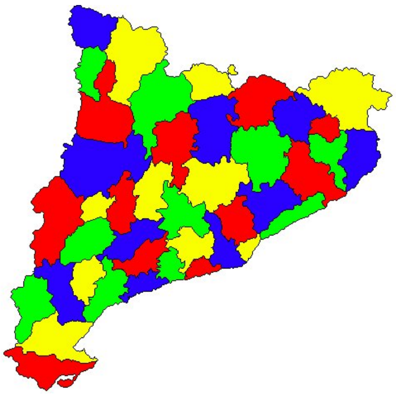
Figure 1.1: Seven Bridges of Köninsberg

Euler, that worked for the prussian court, was asked whether it was possible to do a walk such that each bridge was crossed exactly one time? The answer is no, because there are more than two vertices with odd degree. In his honor, the walks in a graph that use all the edges exactly one time are called *eulerian walks*. Actually, this problem also gave rise to another field of mathematics: the topology.

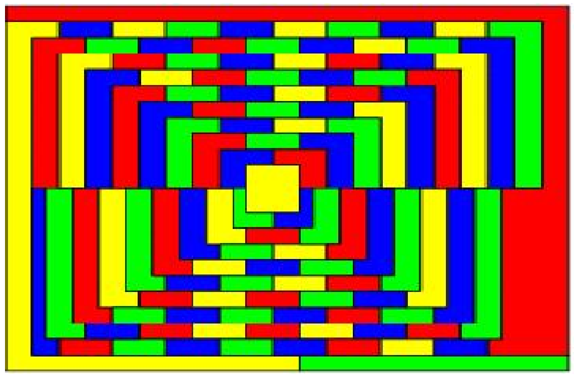
One of the other classic problems in graph theory is graph coloring. It consists in assigning each vertex a color in such a way that two connected vertices do not have the same color. Among problems dealing with colorings, probably the most famous one is the so called four-color problem which asks if it is possible to color any planar graph with at most four colors. A planar graph is a graph that has a representation in $\mathbb{R}^2$ without "crossing" edges.

This problem was set out by Francis Guthrie in 1852 and was solved by Kenneth Appel and Wolfgang Haken. Their proof was based in ruling out a lot of configurations (approx.

2000) using a computer, because it was the only way to do it in a reasonable time. This proof caused a lot of controversy because it was not possible to check its truth. Nowadays, the proof has been simplified but it is still necessary the computer for ruling out the configurations. Despite of this, it is assumed that the proof is correct.



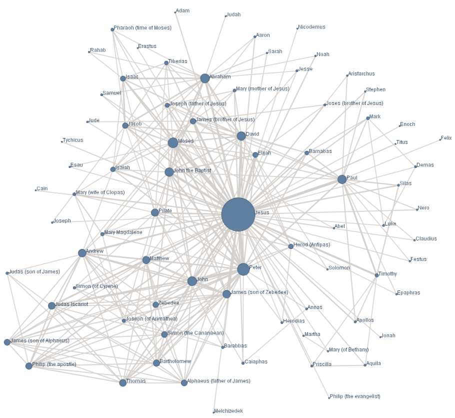
(a) 4-colored map



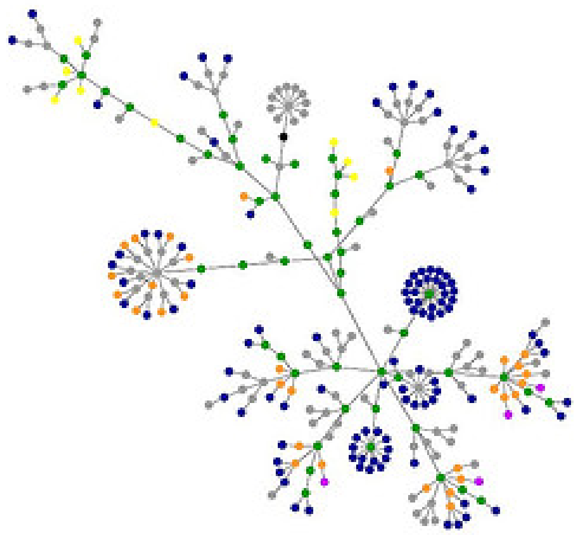
(b) False counterexample from Martin Gardner

Figure 1.2

Graphs also play an important role in other science fields, usually in the analysis of networks. Internet, image processing, social networks, tranport problems or a lot of problems in chemistry, physics and biology, can be represented using graphs.



(a) Social Network



(b) Molecule

Figure 1.3

During this work, we will study list colorings. They arise in problems of frequency assignment. Suppose that we have a set of emitters with a certain scope. We can model this problem by assigning a vertex for each emitter and two vertices are adjacent if they frequency scope overlap. We want to assign each emitter a frequency in such a way that two emitters that are adjacent do not have the same frequency. This problem corresponds to the coloring problem of graphs. If we assign now a subset of fixed frequencies for each emitter, the problem turn out to be a list coloring of a graph. This example motivates the definition of list colorings.

The coloring theory in graphs includes a lot of interesting problems and variations (e.g. see [27]). Moreover, some of the most important open problems and conjectures in graph theory deal with colorings. Among them we stress the Hadwiger conjecture. If a graph has chromatic number k then it is logic to think that there will be a complet subgraph on k vertices. This is not always true, only in the case of perfect graphs. So the conjecture suggests that there must be a subgraph that has the complet graph on k vertices as a minor. We will introduce all this mentioned concepts during this first chapter.

1.2 Basic structures and definitions

We have to state the basic notation that we will use throughout the work. In order to simplify the reading, we will use the standard notation given in the textbook Graph Theory from R. Diestel [16]. A *graph* $G = (V, E)$ is a combinatorial object composed by a pair, where V is the set of *vertices* and $E \subseteq \binom{V}{2}$ is the set of *edges*. We consider a graph to be *simple* if E is not a multiset. Otherwise stated a graph is considered to be *simple*. Usually, the set of vertices is labeled with natural numbers. It is useful to represent graphs with points as vertices and lines linking these points as edges (Fig. 1.4).

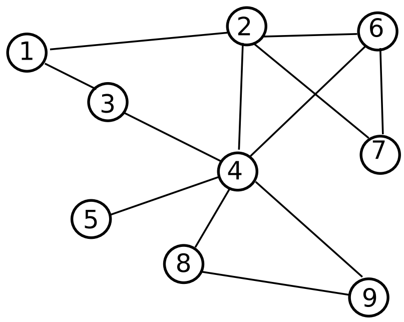


Figure 1.4: Graph with $V = \{1, \ldots, 8\}$ and $E = \{12, 13, 24, 26, 27, 37, 45, 46, 48, 49, 67, 89\}$.

We will use $n = |V|$ and $m = |E|$ to denote the number of vertices and edges respectively. Note that

$$0 \leq m \leq \binom{n}{2}$$

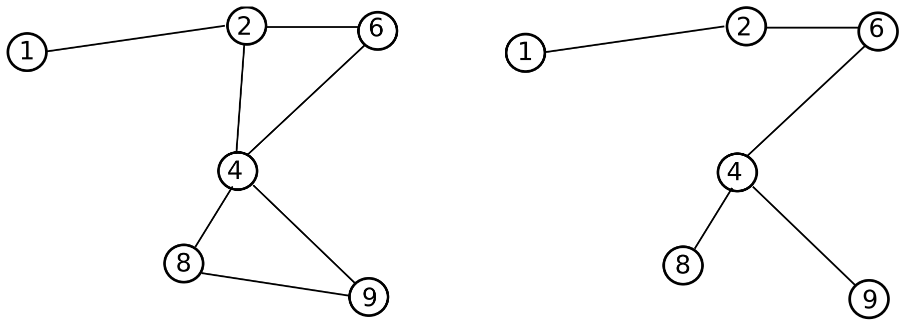
The set of edges that contains v is $E(v)$. The vertices that share an edge with v will be called *neighbors* of e and the set of neighbors of v be denoted by $N(v)$. If $u \in N(v)$, we denote it as $u \sim v$, otherwise $u \nsim v$.

A natural concept is the *degree* of v, $d(v) = |N(v)|$. The *maximum degree* of G is $d(G) = \max_v d(v)$. We call a graph *regular* if $d(v) = d(G)$ for any $v \in V$.

Given a graph G we can construct its *complement* $\overline{G}$ with the same set of vertices, and an edge between to vertices if and only if this edge does not exist in G. The *complete graph*, K_n, is the graph on n vertices with all the possible edges.

Let $X \subseteq V$ a subset of vertices. Then $E(X)$ are the edges totally contained in X and $N(X)$ are the vertices not in X that have a neighbor in X. If Y is another subset of the vertices, $E(X, Y)$ are the edges with one endpoint in each set.

A *subgraph* $H = (X, F) \subset G$ is a graph where $X \subseteq V$ and $F \subseteq E$ with the restriction that the edges F have both ends in X. A subgraph is called *induced* if $F = E(X)$, and is denoted as $G[X]$. Fig. 1.5 shows an induced subgraphs and a subgraph from the graph in Fig. 1.4.



(a) Induced subgraph (b) Subgraph

Figure 1.5

A *path* $P \subseteq G$ is a subgraph such that $V(P) = \{v_0, v_1, \ldots, v_h\}$, where $v_i \neq v_{i+1}$ for any $0 \leq i < h$, and $E(P) = \{\{v_i, v_{i+1}\} : \quad 0 \leq i < h\}$. If this is the case, we say that P has length h. A *cycle* $C \subseteq G$ is a path where $v_0 = v_k$.

The *girth* $g(G)$ of a graph G is the shortest cycle contained in G as a subgraph. Note that $g(G) \geq 3$. If $g(G) > 3$ the graph is called *triangle-free.*

We say that G is *connected* if for every pair of vertices (u, v), there exist a path between them. Otherwise we say that G is disconnected. In this sense, we define the *number of connected components* as the maximum number of vertices that pairwise can not be joined by paths.

A graph G is k*-connected* if removing any set of k vertices leaves the graph connected. In this sense, it extends the notion of connectivity. The connectivity of a graph, $\kappa(G)$ is the maximum k such that G is k-connected.

Let $S \subset V$, then S is a *separator* of G if $V = A \cup S \cup B$ with $A, B \neq \emptyset$, where $|E(A, B)| = 0$. In this case we also say that S disconnects the graph. Note that a graph with $\kappa(G) = k$ has a $k + 1$ separator.

A set of vertices X is called *independent* or *stable* if $|E(X)| = 0$. The independence number $\alpha(G)$ is the maximum cardinal of a stable set. The clique number $\omega(G)$ is the size of the maximum complete subgraph. Note that the independence and the clique number a connected by $\alpha(G) = \omega(\overline{G})$. Stable sets play a crucial role in graph theory.

We highlight a type of graphs: bipartite graphs. A graph is *bipartite* if there exists a partition $V = A \cup B$, where A and B are stable sets. In general we define a k*-partite* graph if $V = A_1 \cup \cdots \cup A_k$ and every A_i is stable.

We can provide a metric in the graph by defining a distance. The distance between u and v, $d(u, v)$ is the minimum length of any path that joins u and v. If there not exist any path joining two vertices we say that their distance is infinity. We can generalize the concept of neighbors to $N_d(v)$, all the vertices that lie at distance d from v. Then, the ball of radius d $B_d(v)$ will be the vertices at distance at most d from v.

The *diameter* $\mathrm{diam}(G)$ of G, is the maximum distance among all pair of vertices.

A *class* of graphs $\mathcal{C}$ is a not necessarily finite set of graphs.

1.2.1 Forests and trees

A *forest* F is a graph without cycles. If such a graph is connected it is called *tree* and it is denoted with T.

Proposition 1.1 *The following statements are equivalent*

- T *is a tree.*
- *for any pair of vertices, there exists a unique path between them.*
- T *is maximally acyclic.*
- T *is minimally connected.*
- T *is connected with* n *vertices and* $n-1$ *edges.*

A *star* is a tree with diameter 2. A *star forest* is a set of disjoint stars.

For some applications it is useful to single out a certain vertex r in the tree. In this case we say that T is a*rooted* tree and the distinguished vertex is the *root* of T. In the same direction, we can define a *rooted forest* as a set of disjoint rooted trees, each one with a root. A good way to see a rooted tree is to draw its vertices in different levels, where v is in the level $d(r, v)$. The *height* of a rooted tree T is the maximum level of its vertices.

A rooted tree also induces a partial ordering on the vertices: $v \leq u$ if $u \in T_v$, where T_v is the subtree that has v as a root.

In this context, we define the *closure* of a tree, $\mathrm{clo}(T)$, as the graph obtained by joining the comparable vertices on a rooted tree (See Figure 1.6).

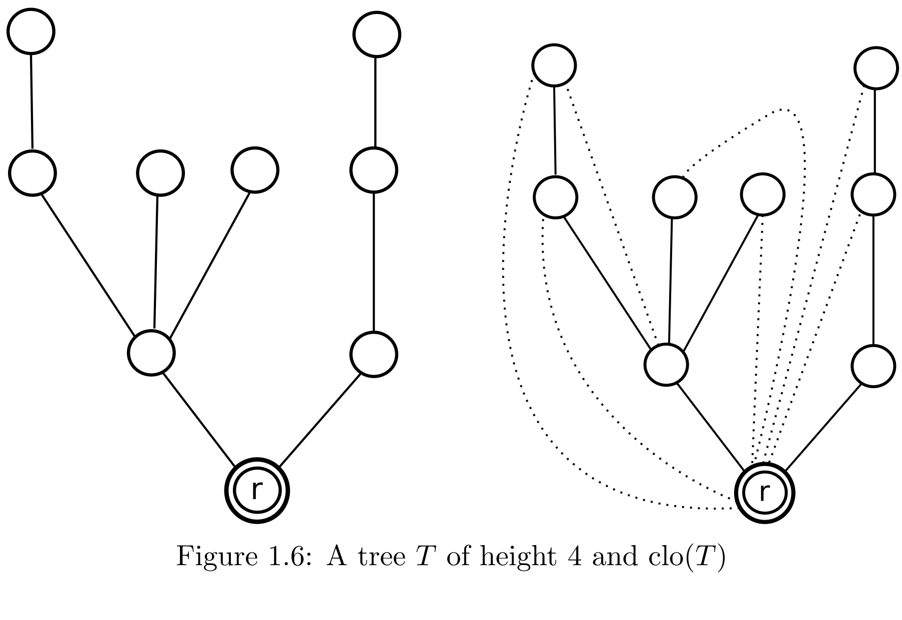


Figure 1.6: A tree T of height 4 and $\mathrm{clo}(T)$

1.2.2 Minor theory

We can define two operations that apply on the edges of the graph. The *contraction* of $e = uv$, is a graph $G \backslash e$ where e is deleted and the vertices u and v are identified. This new vertex is called v_e. If some parallel edges appear (in the case where u and v have common neighbors), we stick them. The *deletion* of e is the graph $G - e$ where the edge e is removed, but the vertices remain untouched.

A *minor* of G is a simple subgraph $H \subset G'$ where G' is obtained from G by applying some deletions and contractions. It is denoted as $H \preceq G$. We call a class $\mathcal{C}$ *minor closed* if for any $G \in \mathcal{C}$ and $H \preceq G$, then $H \in \mathcal{C}$.

A *subdivision* of e is another operation over edges that replaces e with a path of arbitrarily length.

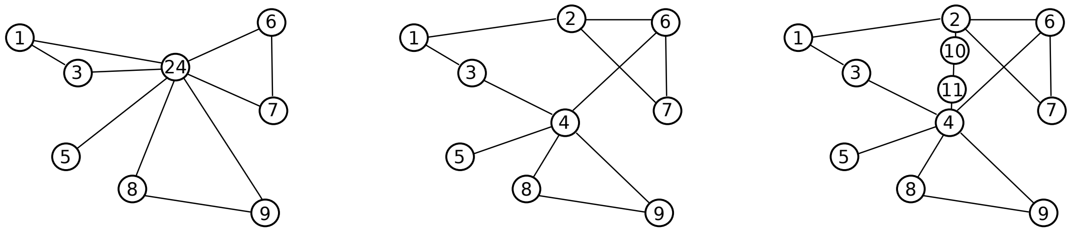


Figure 1.7: Contraction, deletion and subdivision of $e = (2, 4)$

A *topological minor* of G is a subgraph $H \subset G''$ such that G is a subdivision of G'.

The $\preceq$ order induces a partial ordering of graphs, as well as the subgraph order ($\subset$). One of the most relevant theorems in graph theory arises in the context of the minor poset. An *antichain* of a poset is a set of pairwise incomparable elements.

Theorem 1.2 (Robertson and Seymour [47]) *In any minor closed class $\mathcal{C}$ of graphs there are no infinite antichains with respect to the minor ordering.*

An *obstruction set* $\mathcal{O}$ of a graph class $\mathcal{C}$, is a set of graphs such that $G \notin \mathcal{C}$ if and only if $\exists O \in \mathcal{O}$ such that $O \preceq G$. Theorem 1.2 can be stated in terms of obstruction sets: any minor closed class $\mathcal{C}$ has a finite obstruction set. We will use widely this result in chapters 4 and 5.

In the context of graph classes, Nešetřil and Ossona de Mendez defined the bounded expansion class [38].

The graph H is a *shallow minor* of G at depth r if there exists $x_1, \ldots, x_p \in G$ and disjoint $V_1 \subseteq N_r(x_1), \ldots, V_p \subseteq N_r(x_p)$ (each inducing a connected subgraph) such that H is a subgraph of the graph obtained from G by contracting each V_i into x_i, removing loops and sticking multiple edges. The set of the shallow minors of G at depth r is $G \nabla r$.

The *greatest reduced average density* (grad) with rank r of a graph G is defined by the formula,

$$\nabla_r(G) = \max \left\{ \frac{|E(H)|}{|V(H)|} : H \in G \nabla r \right\}$$

Analogously *topological shallow minors* and *topological grads* ($\widetilde{\nabla}_r(G)$) are defined.

A class of graphs $\mathcal{C}$ has *bounded expansion* if,

$$\nabla_r(G) < \infty \quad (\forall r) \quad \iff \quad \widetilde{\nabla}_r(G) < \infty \quad (\forall r)$$

This class is included in the nowhere dense graphs class, which is the class of graphs with at most $O(n)$ edges.

In some sense, bounded expansion classes generalize minor closed and bounded degree classes

1.3 The probabilistic method

The probabilistic method has been developed during the last 60 years. It concerns two huge areas of the mathematics: combinatorics and probability. Probably two of the most relevant examples are due to Erdős: the lower bound for diagonal Ramsey numbers and the proof that there exist graphs with large girth and also large chromatic number. It must be stressed that until now anybody have given an alternative proofs of both problems without using the probabilistic method, and for example, the lower bound for diagonal Ramsey numbers has not been improved yet.

The probabilistic method relies on constructing a probability space over a class of combinatorial objects and then showing that the probability of a randomly chosen element to have a property is greater than 0. Observe that we can show that this element exists but we do not know how is this element, i.e. probabilistic proofs are non constructive.

Although the probabilistic method appears usually in combinatorics and graph theory, there are some applications in other fields like number theory, real analysis, computer science and many others.

Let X be a random variable, we will denote by $\mathbb{E}(X)$ the expected value (or mean) and by $\mathbf{Var}(X)$ or $\sigma^2(X)$ the variance of X.

Next, we present the most relevant concentration inequalities. These inequalities give upper bounds on the probability that a random variable attains values far from its expected value.

The *Markov inequality* guarantees that the tail of a probability distribution has low probability. If X is a positive valued random variable, then

$$\Pr(X \geq k) \leq \frac{\mathbb{E}(X)}{k}$$

where $k \geq \mathbb{E}(X)$.

The *Chebyshev inequality* states that the concentration around the mean depends on the variance. If X is a random variable, then

$$\Pr(|X - \mathbb{E}(X)| \geq k\sigma) \leq \frac{1}{k^2} \tag{1.1}$$

Chernoff bounds give exponential bounds on the tail of certain distributions. There are several variations of them. Here we show one of the more useful. Let $X_1, \ldots, X_n$ mutually independent random variables with $|X_i| \leq 1$ and $X = \sum X_i$, then

$$\Pr(|X - \mathbb{E}(X)| > a) \leq e^{-a^2/2n}$$

If the reader is interested in concentration bound we refer him to survey of Lugosi [32].

1.3.1 Random graphs

The main idea is to turn graphs into a probability space to ask question as, which is the probability that a graph has a certain property or which is the expected value of a parameter.

Let n be the number vertices, for every possible edge $e \in \binom{V}{2}$ we define the probability space $\Omega(e) = \{0_e, 1_e\}$, choosing $\Pr(1_e) = p$ and $\Pr(0_e) = 1 - p$. We will consider p as a constant or as $p = p(n)$ depending on the situation. The probability space $G(n,p)$ is defined as the Cartesian product of all the independent $\Omega(e)$,

$$\Omega = \prod_{e \in \binom{V}{2}} \Omega_e$$

An element of this space is called a random graph and is denoted $G \in G(n,p)$. In fact, we define random graphs by identifying G with the event $\omega \in \Omega$ where $e \in E(G)$ if and only if $\omega(e) = 1_e$.

For any pair of edges e and f, by the definition of the product space, the set of events $A_e = \{\omega : \omega(e) = 1_e\}$, i.e. the graphs that contain the edge e, is independent from the set of events A_f $(f \neq e)$.

This model is known as the Erdős-Rényi model for random graph. There exist several other models such as the regular random graphs or the geometric random graphs.

Let $\mathcal{P}$ be a graph property, e.g. to be connected or to have a hamiltonian path. We will say that this property holds *asymptotically almost sure (a.s.s.)* for random graphs $G \in G(n,p)$, if

$$\lim_{n\to\infty} \Pr(G \text{ has } \mathcal{P}) = 1$$

Throughout the paper, all the results and statements concerning random graphs must be understood in the asymptotically almost sure sense.

Another interesting one is the $G(n,m)$ model. In this case a graph $G \in G(n,m)$ is with equal probability any graph with n vertices and m edges. As it is well–known the two models are closely connected and most of the statements are usually transferred from one model to the other one. Let $\mathcal{P}$ be a convex graph property, i.e. $F \subset G \subset H$ and F and H have $\mathcal{P}$ implies that G has $\mathcal{P}$. The following theorem from Bollobás [10, Theorem 2.2] states precisely this correspondence,

Theorem 1.3 *If $\mathcal{P}$ is convex and $p(1-p)\binom{n}{2} \to \infty$, then $G_p \in G(n,p)$ a.a.s. satisfies $\mathcal{P}$ if and only if, for every fixed x, $G_m \in G(n,m)$ a.a.s. satisfies $\mathcal{P}$, where $m = p\binom{n}{2} + x\sqrt{p(1-p)\binom{n}{2}}$.*

The probability space $G(n,m)$ can be seen as an snapshot of a stochastic process $\{\widetilde{G}_m\}_0^{\binom{n}{2}}$ and at each step we add an edge to an initial empty graph on n vertices. There are two ways to construct $G(n,m)$ but it can be seen that they are equivalent. We present the simpler one: we choose independently m pairs of vertices that will be the edges of G. Then the resulting graph is a.a.s. a simple graph that have m edges. Note that multiedges (repeated pairs in the choice) have probability at most $1/n$ to appear, which tends to 0 when n goes to infinity.

The last model we introduce is the random regular graph model $G(n,d)$ (RRG). It is easy to check that the degree of a certain vertex follows a binomial distribution $\text{Bin}(n-1, p(n))$, therefore the expected degree is $(n-1)p(n)$. In a lot of problems, random graphs come up with the regularity restriction. In this sense the definition of RRGs is necessary. A RRG $G \in G(n,d)$ is constructed in the following way:

1. Let $D_1, \ldots, D_n$ be distinct n sets of size d and $D = \bigcup D_i$.
2. Take a random perfect matching of the complete graph with the elements of D as a vertices.
3. If a vertex in D_i is paired with some vertex in D_j, then add an edge between $v_i, v_j \in G$.
4. This method can provide loops and multiple edges, remove them.

Then the resulting graph is a.a.s. a RRG of degree d.

1.3.2 Lovász Local Lemma

The *Lovász Local Lemma* (LLL) was settled by Lovász and Erdős in 1975 [18] and since then, it has been a really powerful tool for solving a lot of problems. It is well known that if we have some mutually independent events, each with a certain probability less than 1, then the probability that any event occurs is strictly positive. The LLL allows us to slightly relax the independence condition: if we have a set of events that has few dependences among them, and that are sufficiently improbables, then the probability that none of them hold, is strictly positive.

This method will give only existence proofs. Beck in 1991 ([6]) gave a first proof that an algorithmic version was possible. Moser and Tardos in [35] proposed an efficient (polynomial time) randomized algorithm for computing the assignment of the random variables that make no event to hold. Nevertheless, in this project we will not focus on this algorithmic version.

Let $A_1, A_2, \ldots, A_n$ be events in an arbitrary probability space. A directed graph $D = (V, E)$ on the set of vertices $V = \{1, 2, \ldots, n\}$ is called a dependency digraph for the events $A_1, \ldots, A_n$ if for each i, $1 \leq i \leq n$, the event A_i is mutually independent of all the events $D_i = \{A_j : (i, j) \in E\}$. Define $d_i = |D_i|$. Suppose that $D = (V, E)$ is a dependency digraph for the events A_i.

Theorem 1.4 (Lovász Local Lemma) *, General Case, e.g. see [4, pag.68]*

If there are real numbers $x_1 \ldots, x_n$ *such that* $0 < x_i < 1$ *and* $\Pr(A_i) < x_i \prod_{(i,j)\in E}(1 - x_j)$ *for all* $1 \leq i \leq n$. *Then*

$$\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) \geq \prod_{i=1}^{n}(1 - x_i) > 0$$

In particular, with positive probability no event A_i *holds.*

As can be seen, the lemma belong to the theory of probability and the proof of it is based on an induction strategy in the size of the depending events.

There are simpler and weaker forms of the LLL. Here we will announce two more that we will use in Chapter 3.

Corollary 1.5 (Lovász Local Lemma) *, Symmetric Case, e.g. see [4, pag.69]*

Suppose that each event A_i *is mutually independent of a set of all the other events* A_j *but at most* d*, and that* $\Pr(A_i) < p$ *for all* $1 \leq i \leq n$. *If*

$$4pd \leq 1$$

then $\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$. *The above condition can be changed by* $ep(d+1) < 1$*, where* $e = 2.718\ldots$.

Corollary 1.6 (Lovász Local Lemma) *, Weighted Case, e.g. see [33, pag.221]*
If we have integers $t_1, t_2, \ldots, t_n \leq 1$ *and a real* $0 < p < 1/8$ *such that for each* $1 \leq i \leq n$

- $\Pr(A_i) \leq p^{t_i}$
- $\displaystyle\sum_{(i,j)\in E} (2p)^{t_j} \leq \frac{t_j}{4}$

then $\Pr\left(\bigcap_{i=1}^{n} \overline{A_i}\right) > 0$.

The LLL is a strong probabilistic tool to solve problems in different fields, particularly in graph theory and combinatorics. The main versions and some good examples can be found for instance in *The Probabilistic Method* from Noga Alon and Joel Spencer [4]. For applications in coloring theory we strongly recommend the book *Graph Colouring and the Probabilistic Method* from Mike Molloy and Bruce Reed [33].

1.4 Asymptotic notation

During all this work we will use a notation for the asymptotic behaviour of functions. Let $f(n)$ and $g(n)$ be two real valued functions, then

$$g(n) = o(f(n)) \quad \Leftrightarrow \quad \lim_n \frac{g}{f} = 0$$

$$g(n) = O(f(n)) \quad \Leftrightarrow \quad \lim_n \frac{g}{f} \neq \infty$$

$$g(n) = \Theta(f(n)) \quad \Leftrightarrow \quad \lim_n \frac{g}{f} = k,\ k \neq 0$$

$$g(n) = \Omega(f(n)) \quad \Leftrightarrow \quad \lim_n \frac{g}{f} \neq 0$$

$$g(n) = \omega(f(n)) \quad \Leftrightarrow \quad \lim_n \frac{g}{f} = \infty$$

Observe that,

- if $g(n) = o(f(n))$, then $g(n) = O(f(n))$
- if $g(n) = \omega(f(n))$, then $g(n) = \Omega(f(n))$
- if $g(n) = O(f(n))$ and $g(n) = \Omega(f(n))$, then $g(n) = \Theta(f(n))$

2
Coloring graphs

In this chapter we present the problem of graph coloring and we discuss some variations of it.

The first section will be devoted to define the basic coloring concepts. We introduce some other colorings as list, fractional, acyclic or star colorings.

A natural way to generalize the acyclic coloring is through the concept of tree–width. The tree–width is central in the minor theory developed by Robertson and Seymour. It has also important applications in algorithmic analysis thanks to the theorem of Courcelle which says that any problem that can be formulated using Monadic Second-Order logic can be solved in polynomial time for graphs with bounded tree–width. Using the tree-width we can define Low Tree-Width Decompositions, an extension of the acyclic colorings.

On the other hand star coloring can be generalized with the notion of tree–depth. From here on, Low Tree-Depth Decompositions are defined in an analog way.

While the tree–width of a graph measures the similarity of a graph to a forest, the tree-depth has also in account the height of the trees into the forest. In other words, the tree–depth give the similarity of a graph with an star forest.

2.1 Variations on graph coloring

A *k-coloring* of a graph is a map $C : V(G) \longrightarrow \{1 \dots k\}$, such that, if $u \sim v$, then $C(u) \neq C(v)$. The *chromatic number* of G, $\chi(G)$, is the minimum k such that there exists a k-coloring of G.

A greedy procedure gives an upper bound of the chromatic number depending on the maximum degree $d(G)$: $\chi(G) \leq d+1$. This bound is refined by Brooks theorem [12],

$$\chi(G) \leq d$$

An *edge k-coloring* of a graph is a map $C : E(G) \longrightarrow \{1 \dots k\}$, such that, if e and f meet in a vertex, then $C(e) \neq C(f)$.

All the coloring that will be defined from now on, can be also transferred to coloring edges instead of vertices.

Some applications of graph colorings require additional constrains. Hence, it is necessary to define variations of k-colorings.

2.1.1 List colorings

For any given map $f : V(G) \longrightarrow \mathbb{N}$, G is *list f-colorable* if, for any instance of sets $L_v \subset \mathbb{N}$ of size at most $f(v)$ there exists a proper coloring of G such that $C(v) \in L_v$ for any v.

Actually, we are interested in the case where the map is constant ($f(v) = k$), in this case we talk about a *list k-colorable* graph to simplify the notation.

The *list chromatic number* or *choice number* of G, $\chi^L(G)$, is the minimum k, such that G is list k-colorable.

Coloring of graphs with a certain restriction in the list of colors to use for each vertex, appears independently in two papers in the late 70's: Vizing in [49] and Erdős, Rubin and Taylor in [20]. This problem is motivated by partial colorings. Suppose that we have colored some vertices and we want to extend the coloring to the rest of the graph. If a vertex v has a colored neighbor u, then v can not be colored with $c(u)$, so we must remove $c(u)$ from its initial list. Applications arise also in frequency assignment problems.

It is obvious that $\chi(G) \leq \chi^L(G)$, but the equality does not hold in general (Fig.2.1).

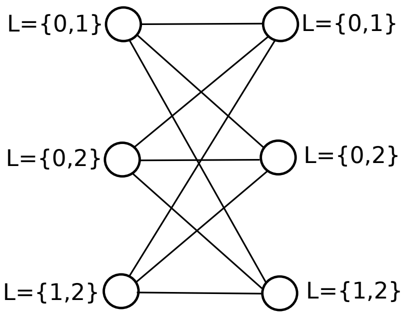


Figure 2.1: List instance ($k = 2$) with no solution: $\chi(K_{3,3}) = 2$, $\chi^L(K_{3,3}) > 3$.

In fact we can have graphs with bounded chromatic number where $\chi^L(G)$ is arbitrarily large, like bipartite graph where $\chi^L(K_{n,n,}) > \log n$.

There is also a theorem for planar graphs concerning list colorings: every planar graph is 5-choosable. So, recalling the 4-color theorem, we need one more color with the list constrain. Thomassen gave a beautiful proof that can be found in [16, Theorem 5.4.2].

One of problems in list coloring is to study when $\chi(G) = \chi^L(G)$. In Chapter 3, we will focus on this problem.

2.1.2 Fractional colorings

A b-fold coloring of a graph G is a coloring that assigns sets of size b to each vertex with the restriction that if two vertices are adjacent, the sets are disjoint. An $(a : b)$-coloring is a b-fold coloring with a colors. We define the b-fold chromatic number, $\chi_b(G)$, as the minimum a such that there exist a $(a : b)$-coloring. The chromatic number of G would be the 1-fold chromatic number. The fractional chromatic number $\chi_f(G)$ is defined as,

$$\chi_f(G) = \lim_{b \to \infty} \frac{\chi_b(G)}{b}$$

The limit exists due to the subadditivity of χ_b with respect to b.

A well known example is the fractional chromatic number of odd cycles: if C_{2k+1} is a cycle with $2k+1$ vertices, then $\chi_f(C_{2k+1}) = \frac{2k+1}{k}$ (Fig.2.2).

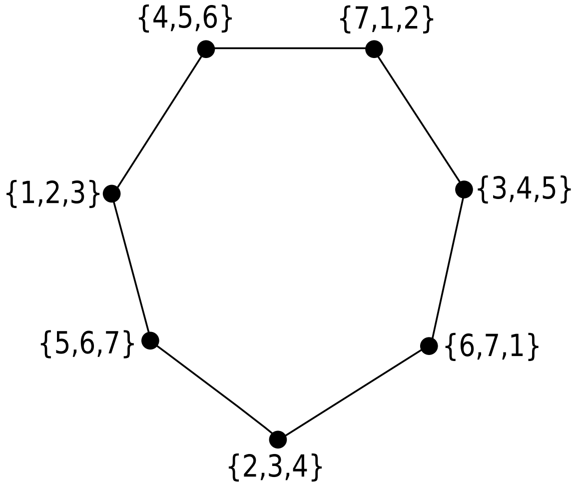


Figure 2.2: Example of a $(7:3)$-fold of C_7, actually $\chi_f(C_7) = 7/3$.

Applications of fractional coloring appear in activity scheduling with conflict graphs or in wireless communication problems in networks.

There is a nice alternative formulation for the fractional chromatic number involving *linear programming* (LP). The coloring problem can be viewed as the solution of an *integer linear programming* (ILP) problem: let x_I be $\{0,1\}$ valued variables associated with the independent set I. Then we want to optimize

$$\min \sum_I x_I$$

where every vertex v gives a restriction

$$\sum_{I \ni v} x_I \geq 1$$

which means that every vertex must be assigned at least one color. In this case $\chi(G) = \sum x_I$.

The dual of this problem solves the problem of finding the clique number $\omega(G)$. Obviously, as we are dealing with ILP the solution is not necessary the same. The graphs where $\omega(G) = \chi(G)$ are called *perfect graphs*. Perfect graphs have an important role in graph theory.

The fractional version has the same formulation, but the variables are not necessarily integers, i.e. we only require that $0 \leq x_I \leq 1$ for each stable I.

With this formulation it is clear that $\chi_f(G) \leq \chi(G)$, every solution of the ILP problem is a solution of the LP problem. It is clear that the feasible region of both problems is not empty (coloring each vertex with a different color, is an assignation of variables that satisfies all the inequalities). The strong duality theorem ensures that in the case of the LP the optimal solution of the dual problem exists and is the same as the primal one. So the fractional clique number is the fractional chromatic number.

In Section 3.5, we discuss the relation between fractional and list colorings.

2.1.3 Acyclic and star colorings

An *acyclic coloring* is a map $C : V(G) \longrightarrow \{1 \ldots, k\}$ such that C is a proper coloring and every two colors form an acyclic graph. The *acyclic chromatic number*, $\chi_a(G)$, is the minimum k such that this coloring is possible.

An *star coloring* is a map $C : V(G) \longrightarrow \{1 \ldots, k\}$ such that C is a proper coloring and every two colors form a star forest. The *star chromatic number*, $\chi_s(G)$, is the minimum k such that this coloring is possible.

Since any star forest is acyclic, note that,

$$\chi(G) \leq \chi_a(G) \leq \chi_s(G) \tag{2.1}$$

Fig. 2.3 shows a case when the inequalities in (2.1) are strict.

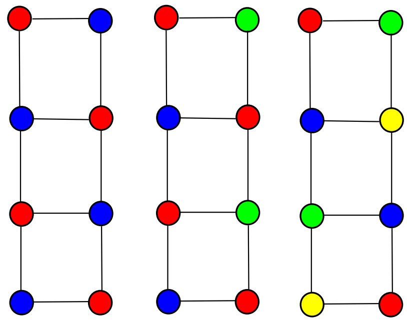


Figure 2.3: A graph with $\chi(G) = 2$, $\chi_a(G) = 3$ and $\chi_s(G) = 4$.

In fact, for any class of graphs $\mathcal{C}$ the star chromatic number is bounded for the graphs $G \in \mathcal{C}$ if and only if the acyclic is bounded. This result arises from the inequalities

$$\chi_a(G) \leq \chi_s(G) \leq \chi_a(G) 2^{\chi_a(G)-1}$$

that are proved in [25]. Acyclic and star colorings were introduced by Grünbaum [25]. They were first studied for planar graphs, see [23, 13, 11, 1]. On this graphs, the most important result, conjectured by Grünbaum and proved by Borodin [11], states that for planar graph G, $\chi_a(G) \leq 5$. In the case of star coloring, the best upper bound known is from Albertson et al. [1], $\chi_s(G) \leq 20$.

Apart from planar graphs, recently an interest has appeared in given upper bounds for these chromatic number in terms of their maximum degree. Let $\mathcal{C}_d$ the class of graphs with maximum degree at most d. We define

$$\chi_*(\mathcal{C}_d) = {}_{G \in \mathcal{C}_d} \max \chi_*(G)$$

where χ_* is any chromatic number.

Then Alon, Mcdiarmid and Reed [3] proved that

$$\frac{d^{4/3}}{log^{1/3} d} \leq \chi_a(\mathcal{C}_d) \leq O(d^{4/3})$$

An analogous result is given by Fertin, Raspaud and Reed [21] for the case of star coloring

$$\frac{d^{3/2}}{log^{1/2} d} \leq \chi_s(\mathcal{C}_d) \leq O(d^{3/2})$$

Regarding to edge coloring, the acyclic case has been studied by Alon, Sudakov and Zaks [5] and we will talk about star edge colorings in more detail on Section 3.2.

The list versions for the acyclic and the star chromatic numbers (denoted as χ_a^L and χ_s^L respectively) are also interesting. Montassier and Serra give a good description of acyclic choosability in [34]. Roughly speaking, there is no significant difference in list colorings if we impose or not the acyclic condition. In Section 3.2 we will support this fact by showing that some results on the Ohba's conjecture can be moved to acyclic colorings.

2.2 Generalizing colorings with graph functions

We have seen in Subsection 2.1.3 that several variation on colorings appear when we impose certain subgraph to have a minimum number of colors.

It is possible to generalize this idea and define different types of colorings via an application $f : \mathcal{P}(V) \longrightarrow \mathbb{N}$, where $\mathcal{P}(V)$ denotes all the subsets of V. If $H \subset G$ is a subgraph, then $f(H)$ gives how many colors have to appear at least in H. This f will be called *graph function* or *coloring maps*. Some example appear in Table 2.1.

$$f_c(H) = \begin{cases} 1 & \text{if } H = \{v\} \\ 2 & \text{if } H = K_2 \end{cases} \qquad f_a(H) = \begin{cases} 1 & \text{if } H = \{v\} \\ 2 & \text{if } H = K_2 \\ 2 & \text{if } H = C_n \end{cases} \qquad f_s(H) = \begin{cases} 1 & \text{if } H = \{v\} \\ 2 & \text{if } H = K_2 \\ 3 & \text{if } H = P_4 \end{cases}$$

Table 2.1: Proper, acyclic and star coloring defined through coloring maps

For the graphs H where the function f is not defined, we will assume that $f(H) = 0$.

Let f be a map, then $\chi(f, G)$ is the minimum number of colors needed to color a graph G in such a way. Then $\chi(f_c, G) = \chi(G)$, $\chi(f_a, G) = \chi_a(G)$ and $\chi(f_s, G) = \chi_s(G)$.

As we will deal with bounded degree graphs, the following concept will be useful,

$$\chi(f, \mathcal{C}_d) := \max_{G \in \mathcal{C}_d} \chi(f, G)$$

The idea behind this definition is to use the minimum number of colors in the *global* context at the same time that we are using at least a fixed number of colors in some subgraphs in a *local* context.

2.3 Tree–like parameters

In this section we define some necessary tree–like parameters of graphs. The most important, which have notorious applications in algorithm complexity, is the tree–width. There are a lot of width-parameters such as path–width, clique–width or rank–width, that have been introduced from the notion of tree–width. For a good introduction on tree–width see [29].

2.3.1 Tree–width

The classic definition of tree-width uses tree–decompositions, but it can also defined through k-trees. For this work, the second definition is more adequate.

A *k-tree* can be defined in a recursive way,

- a $(k+1)$-clique is a k-tree.
- a k-tree can be obtained by adding a new vertex to another k-tree and linking it with all the vertices of a certain k-clique.

A partial k-tree is a subgraph of a k-tree. The *tree-width* of a graph G, $\mathrm{tw}(G)$, is the minimum k such that G is a partial k-tree.

Observe that if T is a tree, then $\mathrm{tw}(T)=1$. In this sense, $tw(G)$ is the similarity of G to some tree structure.

A partition (A,S,B) of the vertices is a *balanced k-partition* if the following three conditions are satisfied:

1. $|S|=k+1$
2. $\frac{1}{3}(n-k-1)\leq |A|,|B|\leq \frac{1}{2}(n-k-1)$
3. S separates A and B.

The following result connecting balanced partitions and tree-width is due to Kloks.

Lemma 2.1 ([29], Lemma 5.3.1.) *Let G be a graph with n vertices and $\mathrm{tw}(G)\leq k$ such that $n\geq k-4$. Then G has a balanced k-partition.*

Therefore, we can identify graphs with large tree–width by the non existence of balanced partitions.

This lemma is inspired in trees ($k=1$). It is easy to see that always exists a vertex/edge such that removing it will provide at least two trees of size stricly less than $n/2$. This vertex or edge will be the set S that have size at most 2.

2.3.2 Tree–depth

Let T be a rooted tree. The *tree-depth* of a graph G is defined to be the minimum height of a rooted forest, whose closure contains G as a subgraph.

This parameter has been introduced under numerous names in the literature. It is equivalent to rank function [40], vertex ranking number (or ordered coloring) [14] and upper chromatic number [38].

There is a recursive way to define the tree–depth of G. Let G be a graph and $C_1,\dots,C_p$ its connected components. Then,

$$\mathrm{td}(G)=\begin{cases} 1 & \text{if } |V(G)|=1\\ 1+\min\limits_{v\in G}\mathrm{td}(G\backslash v) & \text{if } p=1 \text{ and } |V(G)|>1\\ \max\limits_{0<i\leq p}\mathrm{td}(C_i) & \text{if } p>1\end{cases} \tag{2.2}$$

In this case, $\mathrm{td}(G)=1$ if and only if G is a star.

Note that the following inequality holds,

$$\mathrm{td}(G)\leq 1+\mathrm{td}(G\backslash v) \tag{2.3}$$

This implies directly that $\text{td}(G) \leq n$.

By combining the previous inequalities we get that,

$$\text{td}(G) \leq \kappa(G) + \max_i \text{td}(C_i) \tag{2.4}$$

where C_i are the connected components that remain when we remove the minimal separator.

We can still give an alternative definition of the tree–depth. An *elimination tree* of G, is a rooted tree defined recursively:

- If $G = \{v\}$, then $T = \{x\}$.
- Otherwise, we choose a vertex $r \in V(G)$ as the root of T. Let $G_1, \dots, \mathcal{G}_p$ the connected components of $G - r$. For each G_i let T_i be its elimination tree. Then T is constructed by joining r with each r_i, i.e. the root of T_i.

Then the tree–depth of G is the minimum height of an elimination tree of G.

Proposition 2.2 *Let T be a tree and D its diameter, then*

$$\text{td}(T) \leq \min\{\log_2 n + 1, D/2\}$$

Proof. First of all, observe that there exist at least one vertex v such that $T \backslash \{v\} = T_1 \cup \cdots \cup T_p$ where $|T_i| \leq n/2$. Using the recursive definition in (2.2) it is clear that $td(G) \leq \log_2 n + 1$.

Now, take two antipodal vertices u and v. We root the tree in z, the vertex that stays in the middle of the path between them. Note that this vertex is unique, since T is a tree. If there are more than one antipodal pair, the vertex z is in all the cases must be the same, otherwise there would exist an augmentative path and the diameter would be larger. □

In an intuitive way, we can say that the tree–depth not only have in account the similarity of G to a tree, but also the diameter of this tree.

Example 2.3 *Here we have some examples,*

1. $G = K_n \Rightarrow$

   Every vertex must be connected with all the others. Let S be the set of chosen vertices of the tree to represent K_n. Any element of S must be ancestor or predecessor of all the others. Hence $\text{td}\,(K_n) = n$.

2. $G = P_n \Rightarrow$

   Here we consider the path P_n to have n vertices instead of $n+1$. We will prove that $\text{td}(P_n) = \log_2 n + 1$. *Since P is a tree, it is clear that* $\text{td}(P) \leq \log_2 n + 1$.

   We will prove it by induction on n that $\text{td}(P) \geq \log_2 n + 1$. *It is clear that P_1, the path with one vertex has tree-depth 1. Suppose that we have prove it for any $n_0 < n$ and let v be the root of a certain tree of size h. In the subgraph P_n of the closure of T, v must have degree at most 2. So if we remove v we have two paths disjoint as subgraphs of two trees of height $h-1$. In the worst case, each path will have at most $\lfloor n/2 \rfloor$ vertices and then by the induction hypothesis $h - 1 = (\log_2 n + 1) - 1$. Hence, $h = log_2 n + 1$.*

3. $G = K_{n,n} \Rightarrow$

By Equation (2.4) *and noting that* $\kappa(K_{n,n}) = n$, $\mathrm{td}(K_{n,n}) = n+1$.

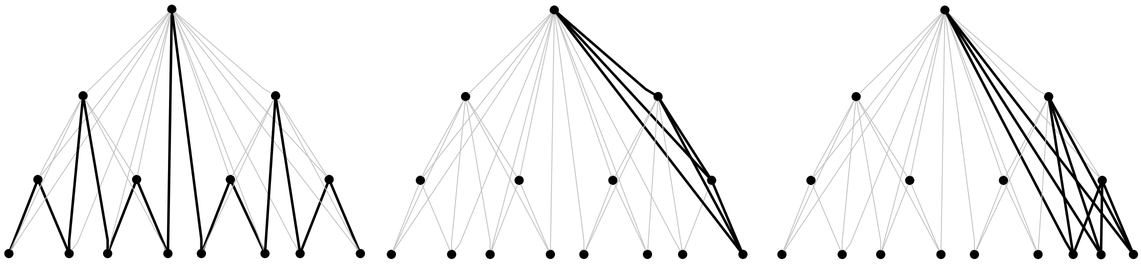


Figure 2.4: The path of length 15, the complete graph K_4 and the complete bipartite $K_{3,3}$ have tree-depth 4

It is well known that the classes of graphs with bounded tree-width are closed under taking minors. By (2.3), the same is true for the class of graphs with tree-depth at most k. Given a class of graphs $\mathcal{C}$, its tree-width is bounded if and only if all the graphs in the class exclude a certain grid as a minor. This role in tree-depth is played by paths. As every minor of a path is a path, it is natural to state the following proposition in terms of subgraphs:

Proposition 2.4 *$\mathcal{C}$ has bounded tree-depth if and only if every $G \in \mathcal{C}$ excludes a certain path as a subgraph*

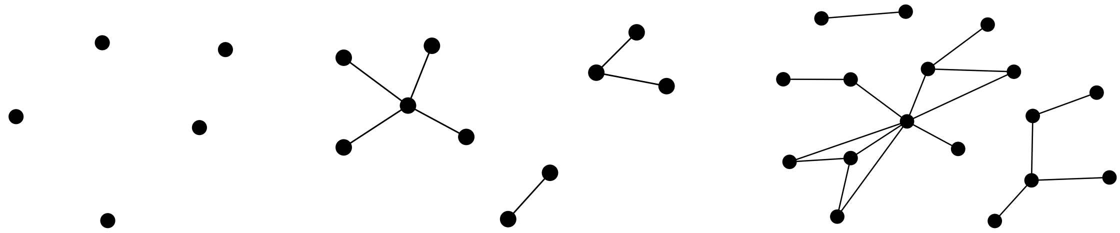
Proof. Suppose that for any $G \in \mathcal{C}$, $\mathrm{td}(G) \leq k$. Then, any graph can not contain a path with more than 2^{k-1} vertices by Example 2.3.(3). So the class excludes the path of length $2^{k-1}+1$.

Let T be an elimination tree of G. If G has no P_{k+1} and, as the edges of the T are edges of the graphs, the height of T must be $\leq k$, and $\mathrm{td}(G) \leq k$. □

2.4 Low tree–depth decompositions

A *low tree-depth decomposition* of order k (k-LTDD) is a map $C : V(G) \longrightarrow \{1 \ldots r\}$, such that, for any subset of i images $I \subset \{1, \ldots r\}$ ($i \leq k$) the induced subgraph H_I formed by the anti-images of I, has $\mathrm{td}\,(H_I) \leq i$ (see Fig.2.5).

In particular, any LTDD is a proper coloring. Thus, we will talk about colors instead of parts.



(a) H_1 (b) H_2 (c) H_3

Figure 2.5: What we see if we look at i colors from a k-LTDD

The *k-th chromatic number* of an LTDD, $\chi_k(G)$, is the minimum number of colors needed such that we have a k-LTDD in G.

This chromatic number has been introduced recently by Nešetřil and Ossona de Mendez [38], and it is inspired in the work of DeVos et al. in [15], who define the same notion in terms of tree–width instead of tree–depth. They talk about low tree-with decompositions of order k (k-LTWD) and they prove that minor closed classes have k-LTWD.

The definition of k-LTDD may seem arbitrarily but it has a crucial role in the theory of bounded expansion classes. Nešetřil and Ossona in [38], proved that for every proper minor closed class $\mathcal{C}$ and every fixed $k \geq 1$, $\chi_k(G)$ is bounded for all $G \in \mathcal{C}$, i.e. there exist a k-LTDD for any k. In [39, Theorem 7.1] they prove it for the bounded expansion class, which include the bounded degree class. Actually, the main theorem on bounded expansion classes states,

Theorem 2.5 *$\mathcal{C}$ has bounded expansion if and only if* $\limsup_{G\in\mathcal{C}} \chi_k(G) < \infty$ *for any k.*

Note that $\chi_1(G) = \chi(G)$ and $\chi_2(G) = \chi_s(G)$. In this sense, $\chi_k(G)$ is a generalization of the star coloring. It must be stressed that star coloring is to k-LTDD what acyclic coloring is to k-LTWD.

By its definition, it is clear that for a fixed graph G, $\{\chi_k(G)\}_{k\in\mathbb{N}}$ is monotonically increasing.

Observe also that, if we define

$$\chi_\infty(G) = \lim_{k\to\infty} \chi_k(G) \tag{2.5}$$

then $\chi_\infty(G) = \mathrm{td}(G)$. So, we recover the notion of tree–depth in the limit. The fact that $\chi_k(G)$ is monotone ensures that $\chi_\infty(G)$ always exists. It must be stressed that if G is infinite, then $\chi_\infty(G)$ can be infinity.

A trivial lower bound if the graph is sufficiently large is,

$$\chi_k(G) \geq k+1 \tag{2.6}$$

This bound can be attained. For example, for a path P_n with n sufficiently large, $\chi_k(P_n) = k+1$. If we enumerate the vertices in the canonical order and we assign to the j-th vertex the color $(j \mod (k+1))$, for all $i \leq k$ any i parts induce path forests as a subgraph with length i, and so that $\mathrm{td}\,(H_I) \leq i$.

As $\mathrm{td}(G)$ is monotone according the subgraph order, if H is a subgraph of G then

$$\chi_k(H) \leq \chi_k(G) \tag{2.7}$$

3

List colorings

The list coloring problem is one of the most studied variations of the standard graph coloring. It is well known that in general the gap between chromatic numbers and their list analogous can be arbitrarily large. For example, the family of complete bipartite graphs, have chromatic number 2 but the list chromatic number depends on the size.

There are many attempts to conjecture which conditions are necessary or sufficient to state that $\chi(G) = \chi^L(G)$. The type of graphs for which this equality holds are called *chromatic–choosable.*

Probably, the most famous conjecture about chromatic–choosable graphs is the *List Coloring Conjecture* (LCC). The line graph of G, $L(G)$, is the graph where $V(L(G)) = E(G)$ and two vertices are connected if and only if the corresponding edges have a vertex in common. We recall that $\chi(L(G)) = d$ or $\chi(L(G)) = d + 1$, by Vizing theorem [16, Theorem 5.3.2], where $d = d(G)$. The LCC says that the list and the standard chromatic numbers of $L(G)$ are the same. The conjecture was stated by Bollobás and Harris [9] in 1985. It has been proved in the case of bipartite graphs (Galvin in [23]).

Recently Molloy and Reed have found the best known bound for the general case,

$$\chi^L(H) < d + 4\sqrt{d}\log^4 d$$

In this Chapter we focus on another well–known conjecture on chromatic–choosable graphs: *the conjecture of Ohba.* In Section 3.1 we introduce this conjecture and explain the known results on it. The version of Ohba's conjecture for general colorings will be presented in Section 3.2. In Section 3.3 we make an attempt to improve the partial result obtained by Reed and Sudakov [45], which is not conclusive but may open a path to the solution of the conjecture. Section 3.4 is devoted to partially answer a question formulated by Reed and Sudakov [46] which can be seen as an extension of Ohba's conjecture. Finally, in Section 3.5 we display a different approach to the problem of chromatic-choosable graphs related with the fractional chromatic number.

3.1 Ohba's conjecture

Erdős, Rubin and Taylor [20] proved that the r-partite graphs with stable of size 2, K_{2*r}, are chromatic–choosable. It is obvious that $\chi(K_{2*r}) = r$. They show that $\chi^L(K_{2*r})$ is also r, by using independent systems of representatives in the set of all vertex lists.

This result motivates the following conjecture in [43],

Conjecture 3.1 (Ohba's conjecture) *If $n \leq 2\chi(G)+1$, then $\chi(G) = \chi^L(G)$, i.e. G is chromatic–choosable.*

The idea behind this conjecture is that when graphs have large chromatic number, then the list chromatic number can not be larger. Actually, K_{2*r} is an almost tight example of the Ohba conjecture. The example which shows that Ohba conjecture can not be sharpened is due to Bohman and Holzman [8]. They realized that the complete multipartite graph with one stable set of size 4 and $k-1$ stable sets of size 2, with k even, has $\chi^L(G) = k+1$, while $n = 2\chi(G)+2$.

Reed and Sudakov have worked in this problem by using the probabilistic method. In a first paper they prove the following weaker version of the conjecture.

Theorem 3.2 ([45]) $\chi(G) = \chi^L(G)$ *provided* $n \leq \frac{5}{3}\chi(G) - \frac{4}{3}$

Later, they give an stronger result, by proving an asymptotic version of the conjecture.

Theorem 3.3 (Asymptotic version in [46]) *For any $0 < \varepsilon < 1$, there exist an $n_0 = n_0(\varepsilon)$ such that G with $n \geq n_0$ is chromatic–choosable, provided*

$$n \leq (2-\varepsilon)\chi(G)$$

Our goal in the next section is to validate the theorem about the Ohba conjecture in [45] for the case of generalized chromatic numbers $\chi_k(G)$. In fact we prove it for any type of partitions.

3.2 General version of the Ohba's conjecture

In this section we generalize the Ohba Conjecture for any type of partitions.

Let $\mathcal{P} = \{V_1, \ldots, V_t\}$ be a partition of a set V.

We say that $\mathcal{P}$ is *choosable* for a set of lists $\{L(v) \in \binom{\mathbb{N}}{t} : \ v \in V\}$ if there is a map $c : V \to \mathbb{N}$, such that $c(v) \in L(v)$ for each $v \in V$, and the partition $\{c^{-1}(i) : \ i \in C\}$ is a refinement of $\mathcal{P}$, where $C = \cup_{v\in V} L(v)$ will denote the list of possible colors. We say that $\mathcal{P}$ is *choosable* if it is choosable for any set of lists, each with cardinality $|\mathcal{P}|$.

Thus, if a graph G is chromatic–choosable, then there is a proper coloring of G whose color classes form a choosable partition. However the definition adapts to any notion of coloring of a graphs. For instance, if G is *acyclic* chromatic–choosable, meaning that the list acyclic chromatic number and acyclic chromatic numbers are the same, then there is an acyclic coloring of G whose color classes form a choosable partition.

For proper colorings the multipartite graph K_{2*r} is an example that Ohba's conjecture can not be sharpened. In this direction, we will provide an analogous example of a nearly-sharp family of graphs for partitions where $|V| = 2\,|\mathcal{P}|$ and $\mathcal{P}$ is choosable.

A partition $\mathcal{P}$ of $|V|$ in sets of size k, is called *equipartition of size k*. We will see that equipartitions of size 2 are choosable.

The proof of the choosability of K_{2*r} by Erdős, Rubin and Taylor [20], is based in an idea which was then reformulated in a Lemma by Reed and Sudakov [45]. We give here a more general version suitable to our generalized colorings.

Lemma 3.4 *Let $\mathcal{P} = \{V_1, \ldots, V_t\}$ be a partition of a set V. If $\mathcal{P}$ is choosable for every set of lists $\{L(v) \in \binom{C}{t} : v \in V\}$ with $|C| < |V|$, then $\mathcal{P}$ is choosable.*

Proof. Suppose the result false and let $\{L(v) \in \binom{C}{t} : v \in V\}$ be a counterexample with minimum $|C|$ of a set of lists for which $\mathcal{P}$ is not choosable. Consider the incidence bipartite graph X with stable sets V and C with an edge (v, c) whenever $c \in L(v)$.

Note that there is no matching of size $|V|$ in X, since otherwise we can find a trivial partition with singletons which is a refinement of $\mathcal{P}$, contradicting that the partition is not choosable for the given set of lists. Hence, by Hall's theorem, there is a set $D \subset C$ such that $|N_X(D)| < |D|$. Choose a minimal set D with this property. By its minimality, there is a matching M in X incident to the points of a subset $D' \subset D$ with cardinality $|D'| = |D| - 1$. Let W be the subset of V that belongs to M. Moreover, by the condition $|W| = |N_X(D)| = |D| - 1$, for every vertex $v \in V \backslash W$, we have $N_X(v) \cap D = \emptyset$.

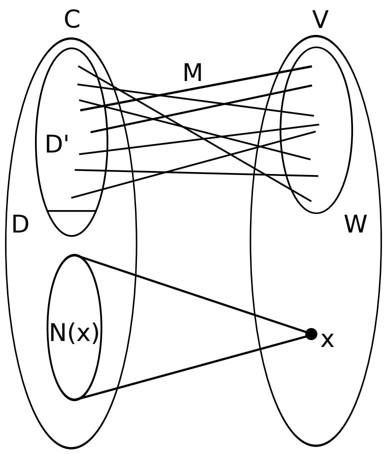


Figure 3.1

Choose a vertex $x \in V \backslash W$ and replace each list $L(v)$ of a vertex $v \in W$ by $L(x)$. The resulting set of lists $\{L'(v) : v \in V\}$ satisfies $| \cup_{v \in V} L'(v)| = |C| - |D| < |C|$. By the minimality of $|C|$, the partition $\mathcal{P}$ is choosable for this new set of lists. Let $\chi : V \to C \setminus D$ be a coloring whose color classes form a refinement of $\mathcal{P}$. Then, by defining $\overline{\chi} : V \to C$ as

$$\overline{\chi}(v) = \begin{cases} \chi(v), & v \in V \backslash W \\ c, & vc \in E(M) \end{cases}$$

we get a coloring whose color classes form a refinement of $\mathcal{P}$. Indeed, the vertices incident to M get a color in their original list and, since they receive pairwise distinct colors, the resulting color classes are still a refinement of $\mathcal{P}$. This contradiction proves the Lemma. □

As we have already mentioned, the above Lemma has the following useful consequence.

Corollary 3.5 *If the list k–generalized chromatic number of a graph G satisfies $\chi_k^L(G) > t$, then there is a set of lists $\{L(v) \in \binom{C}{t} : v \in V\}$ with $|C| < |V(G)|$ such that no k–generalized coloring exists with every vertex getting a color in its list.*

Proposition 3.6 *Any equipartition $\mathcal{P}$ of size 2 of G is choosable.*

Proof. By definition, we have the graph G partitioned in sets of size 2. We will prove by induction that a graph G of size $2r$ is choosable.

$r = 1$ The graph has only one part with two vertices. The length of the lists is one. Therefore, if the color in each vertex is the same, the list partition is the original one. Otherwise we color each vertex with different colors and the resulting partition is a refinement of the first one.

$r > 1$ Now we want to see if $\mathcal{P}$ with $|\mathcal{P}| = r$ is choosable. We will construct a partition $\mathcal{Q}$ that satisfies the list contrains. By Lemma 3.4 we can restrict our number of possible classes to be less strictly than the number of vertices, i.e. $|C| < 2r$.

Given an arbitrary class of the partition $U = \{x, y\}$, since $|C| < 2r = r + r = |L(x)| + |L(y)|$, there exists $c \in L(x) \cap L(y)$.

Suppose that U is deleted, and the color c is removed from all the lists. Then we have a set of lists of size $\geq r-1$ on a graph G' of size $2(r-1)$. By induction hypotesis we can find an appropriate list partition $\mathcal{Q}'$ that refines the previous one with the reduced lists.

Note that we can extend this partial list partitioning of G' to G, by coloring U with the color c, i.e. $\mathcal{Q} = \mathcal{Q}' \cup U$. The list partition $\mathcal{Q}$ refines $\mathcal{P}$. Therefore, $\mathcal{P}$ is choosable.

□

As an example, we define H_r to be an extremal graph with acyclic chromatic number r and color classes of size 2. In this case the partition $\mathcal{P}$ is an acyclic coloring (Fig. 3.2). By extremal we mean that adding an edge will increase $\chi_a(H_r)$.

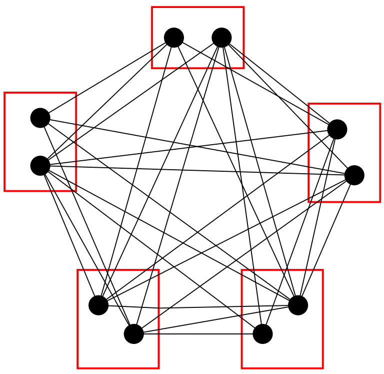


Figure 3.2: H_5

Note that H_r is a subgraph of K_{2*r}, obtained by removing exactly an edge between each pair of stables. Then by Proposition 3.6

$$\chi_a(H_r) = \chi_a^L(H_r)$$

In what follows we prove the weaker form by Reed and Sudakov [45] of Ohba's conjecture for the general case of choosable partitions. We generalize their proof for the partition case.

Proposition 3.7 *A partition $\mathcal{P}$ of a set V with $|V| \leq \frac{5}{3}(|\mathcal{P}| + \frac{4}{3})$ is choosable.*

Proof. As usual in our graph theoretic context we use the terminology of colorings: we regard $\mathcal{P}$ as a coloring and every refinement of $\mathcal{P}$ as a $\mathcal{P}$–coloring. The parts of a partition are called color classes.

Suppose the result false and choose a counterexample with minimum $|V|$.

First of all we suppose that there is no part of size two in $\mathcal{P}$.

Define $k = \lfloor\frac{|\mathcal{P}|}{3}\rfloor$. Then $|\mathcal{P}| = 3k+r$ where $r \in \{0, 1, 2\}$ and $|V| \leq 5k+(5/3)r-4/3 \leq 5k+r$. On the other hand, if x is the number of singletons, since the other components has size at least 3, we have $x + 3(3k + r - x) \leq |V| \leq 5k + r$. Therefore, $x \geq 2k + r$.

We want to see that there exists a $\mathcal{P}$–coloring on an arbitrarily list assignment $\mathcal{L}$ with $|L(v)| = |\mathcal{P}| = 3k+r$. Color r singletons (call them T) and delete r positions of every list.

Now it remains to color $V\backslash T$ with the lists $\mathcal{L}\backslash c(T)$. Making an abuse of notation and for the simplicity of notation, we still denote the resulting set by V and the resulting set of lists $\mathcal{L}$.

Note that now, $L(v) = 3k$ and, by Lemma 3.4, we can assume that we do not have more than $|V| \leq 5k$ colors. Thus, for all $u, v \in V$, we have $|L(u) \cap L(v)| \geq 3k + 3k - 5k = k$.

To clarify we will denote $s_1, \ldots, s_{2k}$ the singleton classes and $U_1, \ldots, U_k$ the largest classes. Construct $C_1, \ldots, C_k$ set of colors, pairwise disjoint, in the following way.

For each pair (s_{2i}, s_{2i-1}) $1 \leq i \leq k$, we select greedily one color c_i that appears in both lists. Indeed, we select also greedily, another disctint color c'_i from each list of s_i. Then $C_i = \{c_i, c'_{2i}, c'_{2i-1}\}$. Note that for each pair of colors in C_i there is an appropriate coloring, using these two colors for (s_{2i}, s_{2i-1}).

G will be colored in two steps. To begin we will color some vertices with the colors of $\mathcal{C} = \bigcup C_i$ and after we will color greedily the rest.

Select a random permutation $\sigma \in \mathcal{S}_k$ and, for each $1 \leq i \leq k$, select randomly $t_i \in C_i$. We color s_{2i} and s_{2i-1} with the colors in $C_i\backslash\{t_i\}$. With $t_{\sigma(i)}$ we color the vertices of U_i that contain this color in their lists.

Note that for every color $c \in \mathcal{C}$, the probability that the vertices in U_i are colored with c is exactly $1/|\mathcal{C}| = 1/3k$.

After this procedure we have some vertices uncolored with some new lists $\mathcal{L}' = \mathcal{L}\backslash\mathcal{C}$, with size $|L'(v)| = t(v)$. Then the probability that $v \in U_i$ remains uncolored after the first coloration is,

$$\Pr(t_{\sigma(i)} \notin L(v)) = 1 - \Pr(t_{\sigma(i)} \in L(v)) = 1 - \frac{3k - t(v)}{3k} = \frac{t(v)}{3k}$$

Now, we define a random variable x_v,

$$x_v = \begin{cases} \frac{1}{t(v)} & \text{if } t(v) > 0 \\ 0 & \text{if } t(v) = 0 \end{cases}$$

We can bound the following expected value.

$$\begin{aligned} \mathbb{E}(\sum_{v \in \bigcup_i U_i} x_v) &= \sum_{v \in \bigcup_i U_i} \mathbb{E}(x_v) = \sum_{t \geq 1} \sum_{v:\, t(v)=t} \mathbb{E}(x_v) = \\ &= \sum_{t \geq 1} \sum_{v:\, t(v)=t} \frac{1}{t} \frac{t}{3k} = \sum_{v \in \bigcup_i U_i} \frac{1}{3k} \leq \frac{5k - 2k}{3k} = 1 \end{aligned}$$

Hence there exist a choice of σ and t_i, such that $\sum_v x_v \leq 1$. Sort all vertices uncolored by $t(v)$ and color greedily each node. This process constructs an acceptable coloring.

Suppose that a vertex v with list size $t(v)$, has no colors available when we try to color it. This means that at least $t(v)$ vertices have been colored previously, and these vertices have list sizes at most $t(v)$. Hence, in the best case, we have at least $t(v)+1$ vertices with list size at most $t(v)$,

$$\sum_{v \in \bigcup_i U_i} x_v \geq \frac{t(v)+1}{t(v)} > 1$$

and we get a contradiction. In fact, behind this reasoning we have the Hall theorem for the existence of a complete matching.

Note that all the coloring is just a refinement of $\mathcal{P}$.

At this moment, we have proved the statement if $\mathcal{P}$ contains no parts of cardinality two. Suppose now that $\mathcal{P}$ has k parts of cardinality two.

By Lemma 3.4 there exists a list assignment that have lists of size $|\mathcal{P}| \geq |V|/2$, and with less colors than vertices, which can be not colored. Let $U = \{x, y\}$ be one of the 2-classes. If we remove it, we get a new pair of a set $V^1 = V \setminus U$ and a partition $\mathcal{P}^1 = \mathcal{P} \setminus \{U\}$ with $(k-1)$ 2-classes, that also satisfies the hypothesis of the statement. By the Lemma, there is a common color in the lists of x and y. We remove this color from all lists and get a new set of lists $\mathcal{L}^1$. We remove successively all the 2-classes, until we reach a set V^k and a partition $\mathcal{P}^k$ that do not has a part with cardinality two. Note by that removing these 2-classes the hypothesis of the statement is fulfilled by V^k and $\mathcal{P}^k$. As we have proved that $\mathcal{P}^k$ is choosable, we can extend the list colouring to the original partition by using all the deleted colors for the parts of cardinality two.

This coloring is a refinement of the former one, and so it is also a refinement of $\mathcal{P}$. This contradiction completes the proof. $\square$

We can formulate Proposition 3.7 in term of colorings as follows.

Corollary 3.8 *Let G be a graph of order n with k–generalized chromatic coloring $n \leq \frac{5}{3}\chi_k(G) - \frac{4}{3}$. Then $\chi_k^L(G) = \chi_k(G)$.*

3.3 On the bounding constant

In this section we go back to the original proper colorings in Ohba's conjecture.

Let T_t be the event that every graph with $|V(G)| \leq t\chi(G)$ is chromatic–choosable. Then the conjecture of Ohba turns to the question: *is T_2 true?*. The best known value for which positive answer is $t = 5/3$ ([45]). We will show some results that lead us to think that may open a path to improve the bound on t and eventually prove the whole conjecture.

Our idea is to take an extremal counterexample with minimum order and, restricted to this, with maximal size. If $t \leq 2$, then by the proof of Proposition 3.7 and Lemma 3.4, we may assume that our coloring of G contains no classes of cardinality two.

We will study the case $t = 7/4$.

Suppose that we are in one of the following situations,

- there is no 3-class in G

- all the non-singleton classes of G are 3-classes.

Then, as we will show, $T_{7/4}$ is true. This suggests that the general case, in which we may have some 3–classes, can be derived from the two extremal cases which are known. Unfortunately, we do not know how to do it yet.

The second statement is proved by Ohba in [44] and refined by Shen et al. in [48]. Actually, the last authors prove the Ohba's conjecture for this case. Both proofs are based in a result of Kierstead ([28]), who showed that the list chromatic number of the multi–partite graph with all the parts of size 3 is chromatic–choosable, i.e. $\chi(K_{3*n}) = \chi^L(K_{3*n}) = n$. We next show the first statement.

Proposition 3.9 *If G has no 3-classes then $T_{7/4}$ is true.*

Proof. We denote by x_i the number of classes of size i and by a_j the classes of size j with $j \geq i$. Let G be an extremal that accomplish $|V| \leq 7/4\chi(G) - c$ where c is a constant. As we have seen, there are no classes of size two, i.e. $x_2 = 0$. By the hypothesis of the proposition, we also have $x_3 = 0$. We can assume that $\chi(G)$ is multiple of 4, otherwise we can fix it with the parameter c. If $k = \lfloor \chi(G)/4 \rfloor$, then

$$x_1 + a_4 = x_1 + x_4 + x_5 + \cdots = \chi(G) = 4k$$

$$x_1 + 4a_4 \leq x_1 + 4x_4 + 5x_5 + \cdots = |V| \leq 7k$$

Recall that, by Lemma 3.4, we may assume that the total number of colors is less than $|V|$.

The above inequalities ensure that $x_1 \geq 3k$ and $a_4 \leq k$. We may assume the worst case in which the equality holds in each inequality. Denote by $s_1, \ldots, s_{3k}$ the x_1 singletons and by $U_1, \ldots, U_k$ the largest classes. Construct $S_i = \{s_{3i-2}, s_{3i-1}, s_{3i}\}$ for all $1 \leq i \leq k$. Given an arbitrarily choice of $\sigma \in \mathcal{S}_k$, $r_i \in \{1, 2, 3\}$ and $t_i \in \{1, 2, 3, 4\}$. We do the following partial coloring procedure:

1. For each S_i, if v and u are the vertices in S_i not indexed by r_i, choose a color c_i in $|L(v) \cap L(u)|$

2. For each S_i, choose a color $c'_{3i-2} \in L(s_{3i-2})$, $c'_{3i-1} \in L(s_{3i-1})$ and $c'_{3i} \in L(s_{3i})$.

3. For each S_i, if $C_i = \{c_i, c'_{3i-2}, c'_{3i-1}, c'_{3i}\}$ let the color indexed by t_i color as much nodes as posible of $U_{\sigma(i)}$.

4. For each S_i, if t_i selects the vertex r_i we color the other two with c_i and then we can choose a new colour for r_i not used yet. If not we can color the nodes in S_i with the remaining colors.

Note that we always can pick colors in steps (1) and (2) because every two lists intersect at least in k colors, and lists have size $4k$. Also note that every color of the set $C = \bigcup C_i$ has the same probability $1/4k$ to color each U_i. In the last step we have made a little modification that alterates the randomness of the coloring of S_i's but it does not modify the probability of the U_i's.

We give the same definition for the variables x_v. If we compute the expectation of $\sum_{v\in\bigcup_i U_i} x_v$,

$$\begin{aligned}\mathbb{E}\left(\sum_{v\in\bigcup_i U_i} x_v\right) &= \sum_{v\in\bigcup_i U_i} \mathbb{E}(x_v) = \sum_{t\geq 1}\sum_{v:\,t(v)=t} \mathbb{E}(x_v) \\ &= \sum_{t\geq 1}\sum_{v:\,t(v)=t} \frac{1}{t(v)}\frac{t(v)}{4k} = \sum_{v\in\bigcup_i U_i} \frac{1}{4k} \\ &\leq \frac{7k-3k}{4k} = 1\end{aligned}$$

Then we use the same argument as in Proposition 3.7 and we obtain a proper coloring of G. □

We can generalize Proposition 3.9 in the following way:

Corollary 3.10 *If G has no T-classes then $T_{(2T+1)/(T+1)}$ is true.*

Proof.

With the former notation we have, $x_1 \geq (T-1)k$ and $a_{T+1} \leq k$. The proof proceed with the same argument and S_i of size $T-1$. We take one color in the intersection of 2 random list and another one for each singleton. An analogous proof gives the desired result. □

3.4 Relaxing the conditions

Reed and Sudakov propose in [46] the following problem,

> *In conclusion we would like to propose a related problem, which was motivated by Ohba's conjecture. Let t be an integer and let G be a graph with at most $t\chi(G)$ vertices. Find the smallest constant c_t such that for any such a graph G its list chromatic is bounded by $c_t\chi(G)$.*

With this formulation the conjecture of Ohba turns to: *is $c_2 = 1$?*

We will deal with the inverse problem, given t find the constant c_t such that for any graph G with at most $c_t\chi(G) + O(1)$ vertices, its list chromatic is upper bounded by $t\chi(G)$.

We use the same type of proof used in [45], to give a bound on c_t

Proposition 3.11 *For any $t \leq 3$, $c_t \geq (1 + 2t/3)$, i.e. if $|V(G)| \leq (1+2t/3)\chi(G) + O(1)$, then $\chi^L(G) \leq t\chi(G)$.*

Proof. Our idea is to obtain the general inequalities that must be satisfied to adapt the proof of [45, Theorem 1.1], and then search the feasible region of some parameters. In fact, we only need to to have a nonempty region. We introduce a modification over the given proof. When we construct the sets $\{s_{2i-1}, s_{2i}, U_{\sigma(i)}\}$, we will have a parameter s that will allow us to assign s of the U_i's for each pair of singletons. In the proof from Proposition 3.7, $s = 1$.

We set p and q such that $c_t = p/q$ and define $k = \chi(G)/q$ to make the proof clearer. We assume that $\chi(G)$ is divisible by 4, otherwise with a constant number of colors (strictly less than q) we can color some singletons. Recall that, by Lemma 3.4, we may assume that we have less than $|V(G)| = pk$ colors and that the length of the lists is tqk. We will also assume that the parameters are chosen in such a way that we can remove classes of cardinality two as in the proof of Proposition 3.7.

Suppose that x_1 denotes the number of singletons of the coloring and x_3 the number of classes of size at least 3. Then,

$$x_1 + x_3 = qk \quad x_1 + 3x_3 \leq pk$$

The following inequalities are derived,

$$x_1 \geq \frac{3q-p}{2}k \quad x_3 \leq \frac{p-q}{2}k$$

We impose the following conditions:

(a). Every two nodes must have a common color in their lists. Otherwise we could not remove all the independent sets of size 2. Easily this condition becomes,

$$2\chi^L > n \quad \Rightarrow \quad 2t > \frac{p}{q}$$

(b). If we want to obtain a color for each intersection of $L(s_{2i-1}) \cap L(s_{2i})$, we must have at least $x_1/2$ repeated colors when we join the two list ($2qk$). We know that the number of colors is at most $n = pk$.

$$2\chi^L - n \geq \frac{x_1}{2} \quad \Rightarrow \quad \frac{8t-3}{3} \geq \frac{p}{q}$$

(c). Our proof will assign at most s number of U_i's (there are x_3 of them) for each of the $x_1/2$ pairs $\{s_{2i-1}, s_{2i}\}$, which leads us to,

$$\frac{x_1}{2} \geq sx_3 \quad \Rightarrow \quad \frac{3s+2}{2+s} \geq \frac{p}{q}$$

(d). We also want to have enough colors for randomly color the U_i's with the singleton colors. Recall that we have to assign $s+2$ colors for every consecutive pair of singletons $\{s_{2i-1}, s_{2i}\}$. Thus,

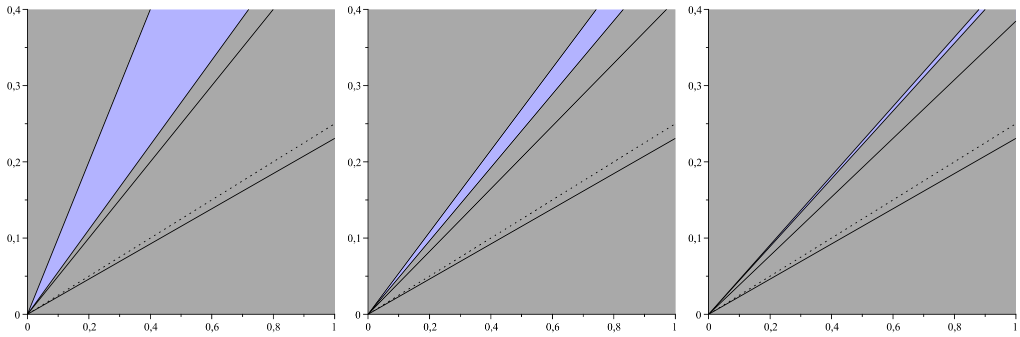
$$\chi^L \geq (2+s)\frac{x_1}{2} \quad \Rightarrow \quad \frac{3s+6-4t}{s+2} \leq \frac{p}{q}$$

(e). Finally, in the computation of the expected value of $\sum x_v$, to ensure that this value is smaller than one,

$$\frac{3x_3}{(2+s)\frac{x_1}{2}} \leq 1 \quad \Rightarrow \quad \frac{3(s+4)}{s+8} \geq \frac{p}{q}$$

Our goal is to maximize the quotient p/q satisfying all the inequalities.

It can be easily checked that (e) implies (a), (b) and (c).



(a) $s = 2$ (b) $s = 5$ (c) $s = 8$

Figure 3.3: Feasible regions with $t = 2$ and different parameters s

Note that the lower bound (d) must be smaller than the upper bound given in (e). Otherwise the feasible region would not exist. Different values of s will give us different feasible regions. As long as we want to maximize p/q, we will also maximize s.

It is easy to see that given a t the optimum is reached when (e)=(d).

Hence,

$$s = \frac{2(4t-3)}{3-t}$$

As $t < 3$ then,

$$\frac{p}{q} = 1 + \frac{2t}{3}$$

For example, in the case of Fig.3.3 the optimum value is for $s = 10$ and $p/q = 7/3$.

□

3.5 Fractional colorings

Fractional colorings provide an interesting generalization of regular colorings. So it could be also interesting to study the relation between fractional and list colorings.

Alon, Tuza and Voigt proved a surprising result,

Theorem 3.12 ([2]) *The fractional chromatic number of a graph equals its fractional list chromatic number, i.e., $\chi_f^L(G) = \chi_f(G)$ for all graphs G.*

In this section we want to analyse if the gap between $\chi_f(G)$ and $\chi(G)$ has a relation with the gap between $\chi(G)$ and $\chi^L(G)$. We will give some examples of graphs G with equal or different $\chi(G)$, $\chi_f(G)$ and $\chi_l(G)$. So the relation between two of these parameters does not affect the other one.

It is clear that

$$\chi_f(G) \leq \chi(G) \leq \chi^L(G)$$

- $\chi_f(G) = \chi(G) = \chi^L(G)$

Our example is the even cycles C_{2k}. As

$$\chi_f(G) \geq \omega(G) = 2$$

and clearly $\chi(G) = 2$ because it is bipartite, we have that $\chi_f(G) = 2$.

To see that $\chi^L(G) = 2$, delete the node v_{2k} and take a path of length $2k-1$. We can color the path greedily starting from v_1 with lists of size 2, but it is not so obvious than we can extend this partial coloring to C_{2k}. Note that we can freely choose the color of the first element of the path among the two colors in its list.

If $L(v_1) \neq L(v_{2k})$, we choose for v_1 the color that only appears in $L(v_1)$, and after the greedy procedure, we can color v_{2k}. If $L(v_1) = L(v_{2k})$ we start coloring with v_{2k-1}. If $L(v_{2k-1}) \neq L(v_{2k})$ we have finished. Otherwise, we also have $L(v_{2k-1}) = L(v_{2k})$ and, as the path has odd length, v_1 and v_{2k-1} must be colored with the same color, allowing us to obtain a proper coloring of C_{2n}.

- $\chi_f(G) = \chi(G) < \chi^L(G)$

  We take $G = K_{3,3}$. Using the argument displayed above and the biepartitness of G, $\chi_f(G) = \chi(G) = 2$. But it is widely known, that $\chi^L(G) = 3$. In general, $K_{n,n}$ is also an example.

- $\chi_f(G) < \chi(G) = \chi^L(G)$

  Taking an odd cycle, $G = C_{2k+1}$, we have $\chi(G) = \chi^L(G) = k+1$. It is also known that $\chi_f(G) = 2 + 1/k < 2$ if $k \geq 2$.

- $\chi_f(G) < \chi(G) < \chi^L(G)$

  This is the most complex example. We take advantage that $\chi^L(K_{n,n}) = (1 + o(1))\log n$, while $\chi(K_{n,n}) = 2$.

  Suppose that n is large enough and we set $G = K_{n,n} \cup \{e_1, e_2, e_3, e_4, e_5\}$. This edges form a cycle with all vertices in the same stable set. Figure 3.4 represent G.

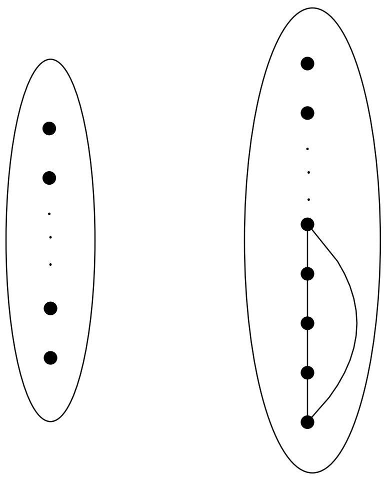


Figure 3.4: G

We claim that $\chi(G) = 4$. It is clear that $\chi^L(G) \geq \chi^L(K_{n,n}) \geq \log n$.

Now we claim that it exists a 2-fold coloring using 7 colors, and so that $\chi_f(G) \leq \frac{7}{2}$. We can use only two colors (a, b) for every vertex in the non modified stable set. We assign two other colors (c, d) for the vertices not in the cycle, and we need 5 colors (c, d, e, f, g) to obtain the 2-color sets for each vertex in the cycle, because $\chi_f(C_5) = 5/2$.

4

Using the Lovász Local Lemma

The Lovász Local Lemma (LLL) is one of the most useful techniques in the probabilistic method. We refer to Subsection 1.3.2 for their basic versions. In our case we want to take advantage of the graph locality. Therefore, we will deal with bounded degree graphs.

In the last years, the LLL has been widely used for giving bounds in the cases of acyclic and star colorings when the degree is bounded. For example, the upper bounds on $\chi_a(G)$ and $\chi_s(G)$ given in 2.1.3 are proved with the general form of LLL. It is also useful for edge colorings.

To warm up with the LLL, in this chapter we will start in Section 4.1 giving a new bound on the edge star chromatic number. In the Section 4.2 we give bounds on some generalized coloring numbers. For the upper bounds we use the LLL in a more involved way. Finally we will apply in Section 4.3 the results from the previous section to state Theorem 4.20 on the k-th chromatic number for low tree–depth decompositions when the maximum degree is bounded.

4.1 Star Edge colorings

Alon, McDiarmid and Reed [3] proved that the acyclic edge chromatic number is linear on the maximum degree of a graph. In particular, if $G \in \mathcal{C}_d$, the class of graphs with maximum degree d, they show that $\chi'_a(G) \leq 64d$. Molloy and Reed [33] reduced the constant to 16. Alon, Sudakov and Zaks [5] conjectured that, in fact

$$\chi'_a(G) \leq d + 2$$

and they proved this inequality for graphs with large girth.

Nešetřil and Wormald [41] proved that a.a.s. for random regular graphs $G \in G(n, d)$,

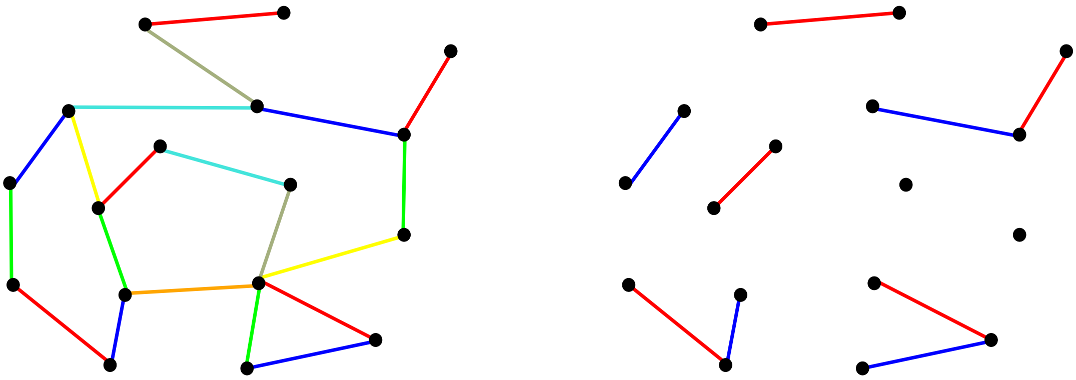
$$\chi'_a(G) = d + 1$$

On the other hand, the star edge chromatic number has not been deeply studied. As far as we know there is only one result about it. Liu and Deng [31] defined their version of $\chi'_s(G)$ where every path with 4 edges must be colored with at least 3 colors. They proved that

$$\chi'_s(G) = 16(d + 1)^{\frac{3}{2}}$$

if $d \geq 6$.

However, it is natural to think that a star coloring of edges must be defined in a way that, when we take any two classes, the induced subgraph is a star forest. In this sense we redefine the *star edge chromatic number* $\chi'_s(G)$ as the minimum number of colors needed to color in a proper edge coloring of a graph such that every two classes induce a star forest. The aim of this section is to prove that, in this case, $\chi'_s(G)$ is linear in terms of the maximum degree.



(a) Star Edge Coloring (b) Every two classes induce a star forest

Figure 4.1

Theorem 4.1 *For each $G \in \mathcal{C}_d$*

$$\chi'_s(G) = O(d)$$

In fact we can assert that $\chi'_s(G) \leq 25d$.

Proof. We will use a proof construction based on [3]. Suppose that $s = 25d$.

Let $f : E \longrightarrow \{1, \ldots, s\}$ a random coloring, such that $f(e)$ are independent random variables, uniformly distributed amongst $\{1, \ldots, s\}$.

We define the following events.

I) For every pair of incident edges e_1 and e_2, let $A_{\{e_1,e_2\}}$ be the event that both have the same color.

II) For every path containing 4 vertices, $P = \{e_1e_2e_3\}$, let $B_{\{e_1,e_2,e_3\}}$ be the event that P is bichromatic.

Note that if none of the former events happen, we have a proper star coloring of the edges.

We will use the Lovász Local Lemma to see that there exist some random coloring that avoid all these events.

Claim 4.2 *For every event of type I, we have that* $\Pr(A) = 1/s$, *and for every event of type II,* $\Pr(B) = 1/s^2$.

We construct the graph H of dependencies, having the previously defined events as vertices $V(H)$. Two events will be adjacent if they are not mutually independent, i.e. if they share at least one edge. Clearly, an stable set is formed by mutually independent events.

Claim 4.3 *Given an edge e, there are at most $2d$ edges incident with e. There are at most $3d^2$ paths with 3 edges containing e.*

Proof. The first assert is trivial as the maximum degree is d and an edge is adjacent to two nodes. Actually there are at most $2(d-1)$ edges incidents at most, but using $2d$ the proof follows easier.

Suppose that e is in the path. Then we have at most $(d-1)^2$ options to construct the path with e in any position (first, middle or last). So we have $3d^2$ incident paths with e.

□

The following table shows us the maximum number of neighbors of type X can have a vertex of type Y

$X \backslash Y$	I	II
I	$4d$	$6d^2$
II	$6d$	$9d^2$

We will use the generalized version of LLL on Lemma 1.4.

The constant x_1 and x_2 for the events of type I and II respectively may have the same order as the probability of each event to happen. In this case, for convenience, we set the constants $x_1 = 2/s$ and $x_2 = 2/s^2$. Then we have to prove the following inequalities that arise from the hypothesis of the generalized version of LLL:

$$\frac{1}{s} \leq \frac{2}{s}\left(1-\frac{2}{s}\right)^{4d}\left(1-\frac{2}{s^2}\right)^{6d^2} \tag{4.1}$$

$$\frac{1}{s^2} \leq \frac{2}{s^2}\left(1-\frac{2}{s}\right)^{6d}\left(1-\frac{2}{s^2}\right)^{9d^2} \tag{4.2}$$

These inequalities will imply that $\Pr(\cap\overline{A_{e_1,e_2}} \cap \overline{B_{e_1,e_2,e_3}}) > 0$, and hence, there is a proper edge acyclic coloring with $O(d)$ colors.

It is easy to check that if (4.2) holds, then (4.1) holds as well. Let us check (4.2).

As $s = 25d$ and using $(1-x)^n \geq 1 - nx$, if x small enough, then

$$\begin{aligned}
\left(1-\frac{2}{s}\right)^{6d}\left(1-\frac{2}{s^2}\right)^{9d^2} &\geq \left(1-\frac{12d}{s}\right)\left(1-\frac{18d^2}{s^2}\right) \\
&\geq \left(1-\frac{12}{25}\right)\left(1-\frac{18}{25^2}\right) \\
&> \frac{1}{2}
\end{aligned}$$

Then by the local lemma, there exists some assignment of $25d$ colors that produces a proper star edge coloring, completing the proof.

□

During the proof, no attempts in improving the constant are made. We stress that it is not difficult to bring this constant down, but we are only interested in its asymptotic order of magnitude.

Remark 4.4 *As we have the following inequalities,*

$$\chi'(G) \leq \chi'_a(G) \leq \chi'_s(G)$$

and using Vizing's theorem (see [16, pag.119]), we can deduce that $\chi'_s(G) = \Theta(d)$

Question 4.5 *Alon, Sudakov and Zaks [5], conjectured that* $\chi'_a(G) = d + 2$.

Exist any c *constant, such that* $\chi'_s(G) \leq d + c$*? We believe that the answer is positive.*

Corollary 4.6 *The version for lists is also true,*

$$\chi'^l_s(G) = O(d)$$

Proof. Note that the proof only uses a random edge coloring with s values but nothing more that its number is said about these s values we choose. Hence, the same proof give the list version. □

All the proofs for upper bounds of chromatic numbers given in this chapter can be adapted to list colorings.

4.2 Generalized colorings

For simplicity, in this section we will denote by P_n the path with $n-1$ edges and n vertices.

The following function will be useful,

$$f_{s,t}(H) = \begin{cases} 1 & \text{if } H = \{v\} \\ 2 & \text{if } H = K_2 \\ s & \text{if } H = P_t \end{cases} \tag{4.3}$$

The Lovász Local Lemma allows us to find an upper bound for $\chi(f_{s,t}, G)$.

Proposition 4.7 ([34]) *Let* d *be the maximum degree of* G*. The chromatic number associated with* $f_{s,t}$ *satisfies*

$$\chi(f_{s,t}, G) \leq h(s,t) d^{\frac{t-1}{t-s+1}}$$

with $h(s,t) = 20st^2\binom{t}{t-s}$.

4.2.1 The function f_k

Based on the former function for the paths, we define,

$$f_k(H) = \begin{cases} 1 & \text{if } H = \{v\} \\ 2 & \text{if } H = K_2 \\ i & \text{if } H = P_i \quad \forall i \leq k+1 \end{cases}$$

Observe that the chromatic number $\chi(f_{k+1,k+1}, G)$, where $f_{k+1,k+1}$ is defined as (4.3), is the same as $\chi(f_k, G)$ with

$$f_k(H) = \begin{cases} 1 & \text{if } H = \{v\} \\ 2 & \text{if } H = K_2 \\ k+1 & \text{if } H = P_{k+1} \end{cases}$$

Proposition 4.8 *Let d be the maximum degree of G, then the chromatic number associated with f_k satisfies*

$$\chi(f_k, G) \leq f(k)d^k$$

with $f(k) = 2k(k+1)^3$.

Proof. Let $x = f(k)d^k$ and $c : V \longrightarrow \{1, \ldots, x\}$ a random coloring of G, where each vertex takes a color with the same probability. We want to prove that there exists some coloring c such that every $k+1$ path has $k+1$ colors. In this direction, we define the following events, for every path P of with $k+1$ vertices,

$$A_P = \{P \text{ is colored with } \leq k \text{ colors}\}$$

We claim that $\Pr(A_P) \leq \frac{k(k+1)}{x}$. It is clear that we have $k+1$ ways to choose k elements, that gives x^k choices to color them. Then there exist at most k colors suitable for the element not colored, so that we have at most k colors in P. The number of choices to color all the vertices without any restriction is x^{k+1}. Thus the probability $\Pr(A_P)$ is at most $\frac{k(k+1)}{x}$.

Any vertex v is adjacent to at most $\lceil \frac{k+1}{2} \rceil d^k$ paths of size $k+1$. Observe that if we construct the path starting from v, in each step we have at most d options to grow the path. But vertex v can take $k+1$ positions in the path, and, by taking into account the symmetry of the paths, we have at most $\lceil \frac{k+1}{2} \rceil$ positions in the d^k posible paths.

Hence, an event of a path with $k+1$ vertices is adjacent to at most $(k+1)\lceil \frac{k+1}{2} \rceil d^k$ events.

We can use the symmetric version of the Lovász Local Lemma 1.5 because we have only one type of event. By this lemma, if $e(d+1)p < 1$ with d the maximum degree in the dependency graph of the events and p the probability of A_P, then $\Pr(\cap \overline{A_P}) > 0$, there is a choice in which no path P_{k+1} is colored with less that $k+1$ colors. Hence, $\chi(f_k, G) \leq f(k)d^k$.

$$e\left((k+1)\left\lceil \frac{k+1}{2} \right\rceil d^k + 1\right) \frac{k(k+1)}{x} < 1 \tag{4.4}$$

Inequality (4.4) is easily checked to hold if $x \geq 2k(k+1)^3 d^k$. □

"Unfortunately" there is a much simpler way to prove even a better result.

Proposition 4.9 *For any graph G, the chromatic number associated with f_k satisfy*

$$\chi(f_k, G) \leq \max_{v \in V} |B^k(v)|$$

where $B^k(v)$ is the ball of radius k centered in v.

Proof.

Let G^k be the graph with vertex set $V(G^k) = V(G)$ and two vertices x and y are adjacent if $d(x, y) \leq k$. The graph G^k has maximum degree $d(G^k) = \max_v |B^k(v)|$ and its chromatic number is at most $d(G^k)$ by Brooks theorem. A greedy coloring of G^k induces an f_k-coloring of G.

Otherwise, if P would be a path with $k+1$ vertices in G colored with $\leq k$ colors, the coloring of G^k would not be proper. □

Corollary 4.10 *Let d be the maximum degree of G, then the chromatic number associated with f_k satisfy*

$$\chi(f_k, G) \leq 2d^k$$

Proof. *If d is the maximum degree it is not difficult to check that the i-th neighborhood $N^i(v)$ of v satisfy, $|N^i(v)| \leq d(d-1)^{i-1}$. So,*

$$|B^k(v)| = \sum_{0 \leq i \leq k} |N^i(v)| \leq 1 + d \sum_{1 \leq i \leq k} (d-1)^{i-1} \leq \sum_{0 \leq i \leq k} d^i \leq 2d^k$$

□

The Proposition 4.9, gives the intuition that the graphs that are not very good expanders will have $\chi(f_k, G)$ far from the bound provided in Corollary 4.10. Grids or forests are good examples of this. But this bound is tight for graphs that have large expanding ratio. There are graphs that reach the bound asymptotically.

Proposition 4.11 $\chi(f_k, d) = \Omega(d^k)$

Proof. *Let $G = Q(h, k)$ the Hamming graph where vertices are the words of length k with h different letters and two of them are adjacent if they differ in only one component. Note that $Q(2, k)$ are k-hypercubes. We will see that $\chi(f_k, d) \geq \chi(f_k, Q(h, k)) = \Omega(d^k)$.*

First of all, note that for every pair of nodes u and v, it exists a path of length at most $k+1$.

As we want to avoid the $(k+1)$-path colored with less than $k+1$ colors, every node must have a different color. It is easy to check that $|V(G)| = h^k$ and therefore, we need h^k colors.

Note that $d = k(h-1)$, and so

$$\chi(f_k, Q(h, k)) = h^k = \left(\frac{d}{k} + 1\right)^k = \Omega(d^k)$$

□

Hence,

$$\chi(f_k, \mathcal{C}_d) = \Theta(d^k)$$

4.2.2 The function g_k

Let define another graph function that will be useful in Section 4.3,

$$g_k(H) = \begin{cases} 1 & \text{if } H = \{v\} \\ 2 & \text{if } H = K_2 \\ k+1 & \text{if } H = P_{k+2} \end{cases}$$

Proposition 4.12 *Let $G \in \mathcal{C}_d$, then the chromatic number associated with g_k satisfy*

$$\chi(g_k, G) \leq g(k) d^{\frac{k+1}{2}}$$

with $g(k) = 10(k+1)^2(k+2)^3$.

Proof. It follows easily from Proposition 4.7 setting $t = k+2$ and $s = k+1$. □

Remark 4.13 *Note that our bounds shows that $\chi(f_1, G) = \chi(\mathcal{C}_d) \leq O(d)$ and $\chi(g_2, \mathcal{C}_d) = \chi_s(\mathcal{C}_d) \leq O(d^{3/2})$.*

The first inequality give the correct order for the chromatic number in terms of the maximum degree by the Brooks Theorem [16, pag.115].

In the second case, we can not ensure that this bound is tight in order. Fertin, Raspaud and Reed in [21] give some asymptotic bounds for the star chromatic number. They adapte a proof from Alon, McDiarmid and Reed in [3] involving random graphs and the LLL, to show that,

$$\Omega\left(\frac{d^{\frac{3}{2}}}{log^3 d}\right) \leq \chi_s(\mathcal{C}_d) \leq O\left(d^{\frac{3}{2}}\right)$$

We strongly believe that $\chi(g_k, G) = O(d^{\frac{k+1}{2}})$. It must be stressed that, if this is the case, our upper bound is sharp. The lower bound will be extracted from the Hamming graphs.

Proposition 4.14 *Let d be the maximum degree of G, then the chromatic number associated with g_k satisfy*

$$\chi(g_k, \mathcal{C}_d) \geq \Omega\left(d^{\frac{k+1}{4}}\right)$$

Proof.

We will also use the Hamming graphs $Q(h, k+1)$. We will fix k and let h variate, i.e. $O(f(k)) = O(1)$ for any function f.

The volume of a $B^s(v)$ is the number of vertices at distance at most s from v. At distance j, we have $\binom{k+1}{j}$ possibilities of positions to change the letter and in every change we have $h-1$ different letters to put. Therefore,

$$|B^s(v)| = \sum_{j=0}^{s} \binom{k+1}{j}(h-1)^j \geq \sum_{j=0}^{s} \binom{s}{j}(h-1)^j = (1+(h-1))^s = h^s = \frac{1}{(k+1)^s} d^s$$

We consider $s = (k+1)/4$, then $|B^{\frac{k+1}{4}}(v)| = \Omega(d^{\frac{k+1}{4}})$. Let X be the maximum number of pairwise disjoint balls of radius s. Then, X is $O(d^{\frac{3(k+1)}{4}})$ since we have $h^{k+1} = \Theta(d^{k+1})$ vertices.

The function g_k must color every path with $k+2$ vertices with $k+1$ colors.

Suppose that there is a g_k-coloring and it uses $C = h^k/(2X+1) = O(d^{\frac{k+1}{4}})$ colors . Then there exist a color c that has colored at least $2X+1$ vertices. By the pigeonhole principle, there exist a ball that intersects at least two other balls, and the center of the three balls have been colored with c. Fig.4.2).

Let x the center of this ball, y and z the other two centers, then $d(x,y), d(x,z) \leq (k+1)/2$ because the balls intersects. Let us check that under this conditions there exist a 2 disjoint paths with at most $(k+1)/2+1$ vertices that contain x,y and x,z respectively.

Suppose that $x = (x_1, \dots, x_{k+1})$, $y = (y_1, \dots, y_{k+1})$ and $z = (z_1, \dots, z_{k+1})$. As x, y and z are different, they must have at least one different component. W.l.o.g, $x_1 \neq y_1$, $x_2 \neq z_2$ and $y_3 \neq z_3$. Let P one of the minimal paths that has endpoints x and y, and contains the vertex $u = (y_1, x_2, \dots, x_{k+1})$. If Q is the minimal path from x to z passing through

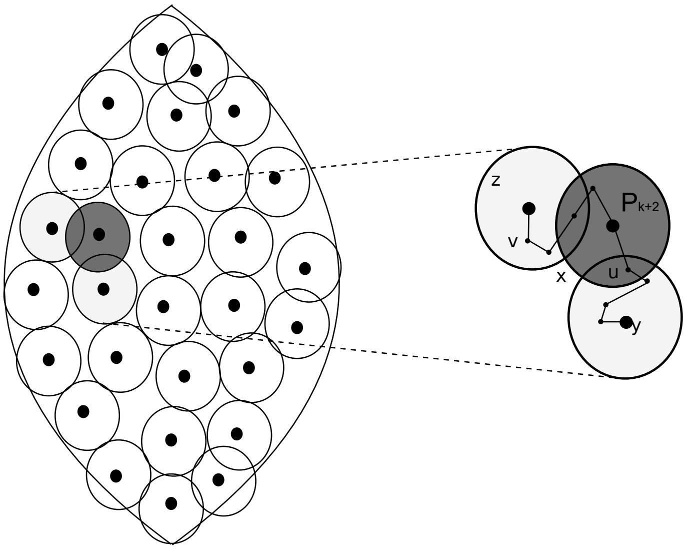


Figure 4.2: Hamming graph

$v = (x_1, z_2, \ldots, z_{k+1})$ (observe that it is possible that $z = v$), then P and Q are the disjoint minimal paths required.

Joining this paths we get a P_{k+2} with endpoints y and z that has 3 vertices with the same color. This is a contradiction with the fact that the coloring satisfies g_k.

So it is not possible to color $Q(h, k+1)$ with $2X + 1 = O(d^{\frac{3(k+1)}{4}})$ colors and $\chi(g_k, \mathcal{C}_d) \geq \Omega(d^{\frac{k+1}{4}})$. □

4.2.3 The functions h_k and $h_{g,k}$

Finally we introduce some weaker colorings,

$$h_k(H) = \begin{cases} 1 & \text{if } H = \{v\} \\ 2 & \text{if } H = K_2 \\ i+1 & \text{if } H = P_{2i} \quad \forall i \leq k \end{cases} \qquad h_{g,k}(H) = \begin{cases} 1 & \text{if } H = \{v\} \\ 2 & \text{if } H = K_2 \\ i+1 & \text{if } H = P_{2i} \quad \forall i,\, 2i \leq g \\ k+1 & \text{if } H = P_g \end{cases}$$

where g is a parameter, which will be later specified as the girth of some graph.

Note that $h_{2k,k} = h_k$.

Proposition 4.15 *Let $G \in \mathcal{C}_d$. The chromatic number associated with h_k satisfies*

$$\chi(h_k, G) \leq h(k) d^{\frac{2k-1}{k}}$$

with $h(k) = (2k+1)(k+1)^2(k+2)\binom{2k+1}{k+1}$.

Proof. Let $x = (2k+1)(k+1)^2(k+2)\binom{2k+1}{k+1} d^{\frac{2k}{k+1}}$ and let $c : V \longrightarrow \{1, \ldots, x\}$ be a random coloring of G, where the color of a vertex is chosen uniformly among the x colors and independently of the other vertices.

For every path P of length $2i$ we define the event of type i as,

$$B_i^P = \{P \text{ is colored with } \leq i \text{ colors}\}$$

We show that

$$\Pr(B_i^P) \leq \left(\frac{i}{x}\right)^i \binom{2i}{i} \tag{4.5}$$

If we want to color the path with i colors, we choose a set of i elements and we assign them a color. Define A as the vertices uncolored.

We have $\binom{2i}{i}$ options for the set A and, for every vertex on A, the probability that it is colored with an already used color is $\frac{i}{x}$. This gives the upper bound (4.5).

We will use the weighted version of the Lovász Local Lemma. Let choose $p = \frac{k}{x}\binom{2k}{k}$, and $t_{B_i^P} = i$. It is easy to check that $\Pr(B_i^P) \leq p^{t_{B_i^P}}$ and, so, the first hypothesis of the weighted version holds.

Focus now in the dependencies graph, with set of vertices the events B_i^P. Observe that a vertex is contained in at most jd^{2j-1} paths with $2j$ vertices, for $1 \leq j \leq k$. Hence, an event of type i, that contains $2i$ vertices, is mutually dependent to at most $2ijd^{2j-1}$ events of type j (paths with $2j$ vertices).

Recalling the weighted version of LLL (Corollary 1.6), we must check some inequalities. For any event X of type i, the following expression must be satisfied,

$$\sum_{B_1^P \sim X} (2p)^{t_{B_1^P}} + \sum_{B_2^P \sim X} (2p)^{t_{B_2^P}} + \cdots + \sum_{B_k^P \sim X} (2p)^{t_{B_k^P}} \leq \frac{t_X}{2}$$

As $x = 2k(k+1)k^2\binom{2k}{k}d^{\frac{2k-1}{k}}$ then $p = \frac{1}{2k(k+1)kd^{\frac{2k-1}{k}}}$.

If the event X is of type i, then

$$\begin{aligned}
S &\leq 2id(2p) + 2i2d^3(2p)^2 + \cdots + 2ikd^{2k-1}(2p)^k \\
&= 2i\left(d(2p) + 2d^3(2p)^2 + \cdots + kd^{2k-1}(2p)^k\right) \\
&\leq 2k\left(\frac{2}{2k(k+1)k} + 2\left(\frac{2}{2k(k+1)k}\right)^2 + \cdots + k\left(\frac{2}{2k(k+1)k}\right)^k\right) \\
&\leq 2k\frac{2}{2k(k+1)k}(1 + 2 + \cdots + k) \\
&\leq 2k\frac{2}{2k(k+1)k}\frac{(k+1)k}{2} \leq \frac{1}{2} \leq \frac{i}{2} = \frac{t_X}{2}
\end{aligned}$$

So the weighted version of the Lovász Local Lemma ensures that there exists an acceptable coloring. □

An analogous proof shows the following bound for $h_{g,k}$-colorings.

Proposition 4.16 *The chromatic number associated with $h_{g,k}$ satisfies*

$$\chi(h_{g,k}, G) \leq h(g,k)d^{\frac{g-1}{g-k}}$$

with $h(g,k) = 2g(g+1)k^2\binom{2k}{k}$.

Proof. Let $x = 2g(g+1)k^2\binom{2k}{k}d^{\frac{g-1}{g-k}}$ and $c : V \longrightarrow \{1, \ldots, x\}$ a random coloring of G, where each vertex take any color with the same probability. Let P be a path with g vertices, then it must be colored with at least $k+1$ colors. So the following event is defined naturally

$$A^P = \{P \text{ is colored with } \leq k \text{ colors}\}$$

As in the former proof, for every path Q of length $2i \leq k$ we define events,

$$B_i^P = \{P \text{ is colored with } \leq i \text{ colors}\}$$

As we have shown, we have $\Pr(B_i^P) \leq \left(\frac{i}{x}\right)^i \binom{2i}{i}$. With an analogous argument we can show that $\Pr(A_P) \leq \left(\frac{k}{x}\right)^{g-k} \binom{g}{k}$.

With $p = \frac{k}{x}\binom{2k}{k}$, $t_{A_P} = g - k$ and $t_{B_i^P} = i$, we show that the hypothesis of the weighted LLL hold.

Let X and Y some events that involve i and j vertices respectively. Every vertex is adjacent to at most jd^{2j-1} paths of size $2j$. Hence, an event X is adjacent in the graph of dependencies to at most $2ijd^{2j-1}$ events like Y.

Then we must check that for any event X of size i, the following expression holds,

$$S = \sum_{B_1^P \sim X} (2p)^{t_{B_1^P}} + \sum_{B_2^P \sim X} (2p)^{t_{B_2^P}} + \cdots + \sum_{B_{g/2}^P \sim X} (2p)^{t_{B_{g/2}^P}} + \sum_{A_P \sim X} (2p)^{t_{A_P}} \leq \frac{t_X}{2}$$

As $x = 2g(g+1)k^2\binom{2k}{k}d^{\frac{g-1}{g-k}}$ then $p = \frac{1}{2g(g+1)kd^{\frac{g-1}{g-k}}}$.

Observe that the following two inequalities hold if $k \leq g$,

$$\frac{g}{2} \leq g - k \qquad \frac{2g-1}{g} \leq \frac{g-1}{g-k}$$

If the event X is of size i, then

$$\begin{aligned}
S &\leq 2id(2p) + 2i2d^3(2p)^2 + \cdots + 2ikd^{g-1}(2p)^{g/2} + 2igd^{g-1}(2p)^{g-k} \\
&= 2i\left(d(2p) + 2d^3(2p)^2 + \cdots + kd^{2k-1}(2p)^k + gd^{g-1}(2p)^{g-k}\right) \\
&\leq 2k\left(\frac{2}{2k(g+1)g} + 2\left(\frac{2}{2k(g+1)g}\right)^2 + \cdots + k\left(\frac{2}{2k(g+1)g}\right)^k + g\left(\frac{2}{2k(g+1)g}\right)^{g-1}\right) \\
&\leq 2k\frac{2}{2k(g+1)g}(1 + 2 + \cdots + g) \\
&\leq 2k\frac{2}{2k(g+1)g}\frac{(g+1)g}{2} \leq \frac{1}{2} \leq \frac{t_X}{2}
\end{aligned}$$

So the weighted version of the Lovász Local Lemma ensures that the probability that a random coloring is an $h_{g,k}$-coloring is strictly positive. So there exist an $h_{g,k}$-coloring of the graph G with at most $O(d^{\frac{g-1}{g-k}})$ □

4.3 LTDD on bounded degree classes

The goal of this section is to take advantage of the generalized coloring defined in Section 4.2 to give bounds on the k-th chromatic number $\chi_k(G)$.

Let $\mathcal{O}_k = O_1 \cup \cdots \cup O_s$ be the subgraph obstruction set of the class of graphs $\mathcal{C}$ with tree-depth at most k. This set is finite but, unfortunately, this set grows exponentially in k, as showed by Nešetřil and Ossona de Mendez.

In the proof of Proposition 2.4 we have seen that if a graph G excludes a path with $k+1$ vertices as a subgraph, then $\mathrm{td}(G) \leq k$. That fact directly implies that P_{k+1} is a subgraph of any graph in $\mathcal{O}_k$.

Fig. 4.3 shows an schema of the subgraphs poset. The cone generated by P_{k+1} contains all the other obstructions.

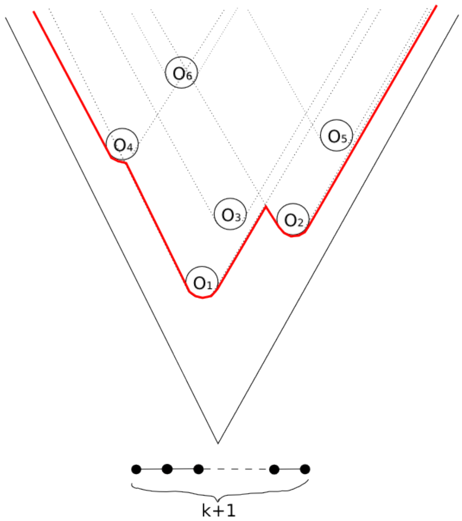


Figure 4.3: P_{k+1}'s cone "covers" $\mathcal{O}_k$

For example $\mathcal{O}_1 = \{K_2\}$ and $\mathcal{O}_2 = \{K_3, P_4\}$. In Fig. 4.4 we can see the set O_3, and paths with 4 vertices which are subgraphs.

Recall that an f_k-coloring, colors in a way that every $k+1$ path with $k+1$ colors. Suppose that we color G with enough colors, such that f_k is satisfied. Then any i ($i \leq k$) classes contain no subgraph which is a path with $k+1$ vertices in H_I, i.e. for any $O \in \mathcal{O}_k$, $O \nsubseteq H_I$.

It is straightforward to see that,

$$\chi_k(G) \leq \chi(f_k, G)$$

And we can ensure that $\chi_k(G) = O(d^k)$.

But we can do better.

Proposition 4.17 *Let $G \in \mathcal{C}_d$. Then,*

$$\chi_k(G) = O(d^{\frac{k+1}{2}})$$

Proof. For any subset I of i elements, the subgraph H_I of G, induced by the color classes $i \in I$, satisfies $\omega(H_I) \leq k$. In other words, this means that H_I does not contain a complete

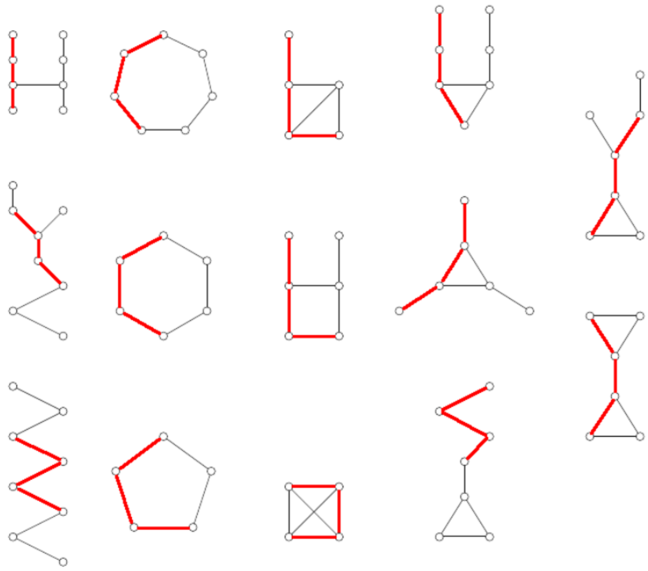


Figure 4.4: Obstruction set for $k = 3$ and their P_4 subgraphs

graph of $k+1$ vertices. This assertion follows from the fact that it is a proper coloring with k colors, and $\chi(K_{k+1}) = k + 1$. Observe that K_{k+1} is always an obstruction for having tree-depth at most k, i.e. $K_{k+1} \in \mathcal{O}_k$, for any k as $\mathrm{td}(K_n) = n$.

We claim that any graph $O \in \mathcal{O}_k \backslash \{K_{k+1}\}$, contains a P_{k+2} path.

This had motivate the definition of the graph function g_k. As we know that no K_{k+1} will appear in H_I it is clear that we can change the condition *every path with* $k+1$ *vertices is colored with* $k+1$ *colors*, for *every path with* $k+2$ *vertices is colored with* $k+1$ *colors*, which is weaker. Then every g_k-coloring is also a k-LTDD and $\chi(g_k, G)$ upper bounds the k-th chromatic number. □

4.3.1 When the girth grows...

In this subsection we will try to sharpen the given upper bounds taking advantage of a large girth. The intuition probably induces us to think that the girth and the chromatic number of graphw are related in some way. In general, if G has large girth we expect a low chromatic number, and vice versa, but this is not always the case. For example, one of the most famous theorems that use the probabilistic method states that there exist graphs with arbitrarily chromatic number and girth. For a reader not familiar with graphs this might seem strange, because large girth implies in some sense sparsity and high chromatic number implies density.

However, in the study of the k-th chromatic number, the girth can have an important effect, or at least allows us to refine the upper bounds.

Take $G \in \mathcal{C}_d$ with girth g.

Proposition 4.18 *If* $g \geq 2k$, *then*

$$\chi_k(G) \leq O(d^{\frac{2k-1}{k}})$$

Proof. As the girth is at least $2k$, no cycles will appear in the induced subgraphs H_I, i.e. H_I is a forest. It is well-known that if H_I if the disjoint union of the trees $T_1, \ldots, T_s$, then $\chi_k(F) = \max \chi_k(T_i)$. So we can assume that H_I is a tree (connected subgraph). If H_I has diameter at most $2i$, by Lemma 2.2, we have $\mathrm{td}\,(H_I) \leq i$. Hence, it will be a tree and a subgraph of the closure for a tree T of height i.

So, we want enough colors to ensure that any subgraph of H_I that is a path, has length at most $2i - 1$, i.e. every $2i$ path of G has at least $i + 1$ colors.

Recalling the Proposition 4.15 and the graph function h_k it is clear

$$\chi_k(G) \leq \chi(h_k, G) \leq O(d^{\frac{2k-1}{k}})$$

□

If the girth is not so large we can also say something.

Proposition 4.19 *If $k < g \leq 2k$, then*

$$\chi_k(G) \leq O(d^{\frac{g-1}{g-k}})$$

Proof.

Doing the same reasoning as the former proof, but adding a little variation we get the result.

We must ensure that the graph H_I is acyclic for any I of size at most k. Therefore, any path of length g must be colored with $k + 1$ colors.

Clearly, this give the function $h_{g,k}$ and

$$\chi_k(G) \leq \chi(h_{g,k}, G) \leq O(d^{\frac{g-1}{g-k}})$$

□

The bounds on k-th generalized chromatic numbers clearly motivate the graph function definitions in Section 4.2.

We can summarize all this section in the following theorem,

Theorem 4.20 *Let G be a graph with maximum degree d and girth g, then*

1. *if $2k \leq g$, $\chi_k(G) = O(d^{\frac{2k-1}{k}})$*
2. *if $k < g < 2k$, $\chi_k(G) = O(d^{\frac{g-1}{g-k}})$*
3. *if $g \leq k$, $\chi_k(G) = O(d^{\frac{k+1}{2}})$*

Providing lower bounds is more difficult. Observe that the upper bounds are given through the graph function. Since we want to have tree–depth at most k, we are avoiding a path with $k + 1$ nodes, but the vice versa is not true. For example $\mathrm{td}(P_{k+1}) = \Theta(\log k)$. Probably, it is possible to give lower bounds by finding certain classes of graphs with bounded degree but large tree–depth.

5

On the tree-depth of Random Graphs

The tree-depth $\mathrm{td}(G)$ of a graph G is a measure introduced by Nešetřil and Ossona de Mendez [38] in the context of bounded expansion classes (see Subsection 2.3.2 for a precise definition). The notion of the tree-depth is closely connected to the tree-width. The tree-width of a graph tells us how similar is G to a tree, while the tree-depth takes also into account the height of the tree.

Bounded expansion classes are defined in terms of shallow minors and its connection to tree–depth is highlighted by the Theorem 2.5 showed in Section 2.4. This is a clear motivation to study tree-depth.

The following inequalities relate the tree–width and tree–depth of a graph:

$$\mathrm{tw}(G) \leq \mathrm{td}(G) \leq \mathrm{tw}(G)(\log_2 n + 1) \tag{5.1}$$

Note that there are graphs that have bounded tree-width but unbounded tree-depth, for example trees. On the other hand, if a class of graphs has bounded tree–depth, then it also has bounded tree–width.

To understand this new parameter, it is useful to know about its behaviour in certain classes of graphs. The main goal of this chapter is to analyze how does it behave on random graphs.

The first result of this chapter states the value of tree-depth for dense random graphs.

Theorem 5.1 *Let $G \in G(n,p)$ be a random graph with $p = \omega(n^{-1})$, then G satisfy a.s.s.*

$$\mathrm{td}(G) = n - o(n)$$

Note that if $p = \omega(n^{-1})$ we will expect a super-linear number of edges. In this context, the theorem says that the tree-depth of G almost attains its maximum possible value. Actually our proof of Theorem 5.1 provides the same result for tree-width. To our knowledge, the tree-width of a dense random graph has neither been studied until now.

But, what happens if the number of edges is linear? This case, the sparse case, is solved by the following theorem,

Theorem 5.2 *Let $G \in G(n,p)$ be a random graph with $p = \frac{c}{n}$, with $c > 0$,*

(1) *if $c < 1$, then a.a.s.* $\mathrm{td}(G) = \Theta(\log \log n)$

(2) *if* $c = 1$, *then a.a.s.* $\mathrm{td}(G) = \Theta(\log n)$

(3) *if* $c > 1$, *then a.a.s.* $\mathrm{td}(G) = \Theta(n)$

This last theorem is closely related with a conjecture of Kloks announced in [29] on the linear behaviour of tree-width for random graphs with $c > 1$. This conjecture has been recently proved by Lee, Lee and Oum [30]. Here we give a proof of Theorem 5.2.(3) which also provides a simpler proof of Kloks conjecture. Our proof uses, as the one in [30], the same essential result of Benjamini, Kozma and Wormald [7] on the existence of an expander of linear size in a sparse random graph for $c > 1$.

The chapter is organized as follows. Section 5.1 contains the proof of Theorem 5.1, which uses the relation connecting tree-width with balanced partitions. Finally Theorem 5.2 will be proved in Section 5.2. For $c < 1$ the result follows from the fact that the random graph is a collection of trees and unicyclic graphs of logarithmic size, which gives the upper bound, and there is one of these components with large diameter with respect to its size, providing the lower bound. For $c = 1$ we show that the giant component in the random graph has just a constant number of additional edges exceeding the size of a tree, which gives the upper bound, and rely on a result of Nachmias and Peres [36] on the concentration of the diameter of the giant component to obtain the lower bound. Finally, as we have already mentioned, for $c > 1$ the result follows readily from the existence of an expander of linear size in a sparse random graph for $c > 1$, a fact proved in Benjamini, Kozma and Wormald [7].

5.1 Tree-depth for dense random graphs

This section is devoted to prove Theorem 5.1. Using the relation between the tree–width of a graph and balanced partitions.

In order to obtain a lower bound of the tree–depth we will get one for the tree-width, and, by inequality (5.1), this will also provide the desired lower bound.

Proof of Theorem 5.1. Using the relation connecting tree-width with balanced partitions (Lemma 2.1), the idea is to prove that G does not contain a balanced separator of size at most $n - o(n)$.

Fix $\beta < 1$. Suppose that there exist a balanced separator S of size $k \leq \beta n$. This set separates the graph into two subsets A and B. As S is balanced, we can assume that $|A| \geq |B| \geq \frac{1-\beta}{3} n$.

The probability that a given set $S \subset V$ separates the graph is,

$$\Pr\left(S \text{ balanced sep. } G\right) = (1 - p(n))^{|A||B|} \leq (1 - p(n))^{\frac{(1-\beta)(2-\beta)}{9} n^2}. \tag{5.2}$$

For the last inequality we consider the worst case where $|A| = ((2 - \beta)/3)n$ and $|B| = ((1 - \beta)/3)n$.

We can bound the number of possible balanced partitions with separators S of size at most βn by the number of labelled partitions in three sets that will represent A,B and S, which is obviously 3^n.

$$\begin{aligned}
\Pr(\exists \text{ a balanced sep. } G) &= \Pr\left(\bigcup_{S \text{ set}} \{S \text{ is a balanced sep. } G\}\right)\\
&\leq \sum_{S \text{ set}} \Pr(S \text{ is a balanced sep. } G)\\
&\leq \sum_{S \text{ set}} (1-p(n))^{\frac{(1-\beta)(2-\beta)}{9}n^2}\\
&\leq 3^n (1-p(n))^{\frac{(1-\beta)(2-\beta)}{9}n^2}\\
&\leq 3^{n+\log(1-p(n))\frac{(1-\beta)(2-\beta)}{9}n^2}
\end{aligned}$$

In order to have a.a.s the non existence of separators, it only remains to prove that the exponent tends to $-\infty$ as $n \to \infty$. As $p(n) < 1$, we can use the inequality of $\log(1-x) \leq -x$.

$$\begin{aligned}
n + \log(1-p(n))\Theta(n^2) &\leq n - p(n)\Theta(n^2)\\
&= n - \frac{\Theta(n^2)}{o(n)} \to -\infty \qquad (n \to \infty).
\end{aligned}$$

Hence $\Pr(\exists S \text{ balanced sep. } G) \to 0$.

Since there is no set of this size separating G, we have $\mathrm{td}(G) \geq \mathrm{tw}(G) > \beta n$. Since the above inequality is valid for all $\beta < 1$, we have $\mathrm{td}(G) \geq n - o(n)$. Observe that, $\mathrm{td}(G) \leq n$.

Remark 5.3 *In this case the same argument can be applied, but now the problem deals with tighter calculations. It is easy to see that the property of non existence of balanced separators depend on the positivity of the following expression.*

$$\beta^2 - 3\beta + 2 - \frac{9\ln 3}{c}$$

As $0 \leq \beta \leq 1$, it is clear that in this way we can not prove the same result for sufficiently large c, but for some cases, we obtain a lower bound on the constant of the tree-depth linearity for the case $c > 1$.

In fact, if $c > c_0 \approx 4.94$, then

$$\mathrm{td}(G(n, p = c/n)) \geq f(c)n$$

where $f(c) = \frac{3-\sqrt{1+\frac{36\ln 3}{c}}}{2}$.

In the next section we will prove that, actually, $\mathrm{td}(G(n, c/n))$ is linear for any c but we will not give a value for the linear constant.

5.2 Tree-depth for sparse random graphs

In this section Theorem 5.2 will be proved.

5.2.1 $c < 1$

Let $G \in G(n, p = c/n)$ with $0 < c < 1$. Our objective is to show that $\mathrm{td}(G) = \Theta(\log \log n)$.

First we will prove the upper bound.

A *unicyclic* graph is a connected graph that has the same number of vertices than edges. Note that such a graph consists of a cycle C, with some attached trees to its vertices. A *pseudotree* is a graph that is either a tree or a unicyclic graph. A *pseudoforest* is a graph composed by different connected components that are pseudotrees.

Assume that $\mathcal{C}$ is the set of connected components.

Lemma 5.4 *If G is a pseudoforest, then* $\mathrm{td}(G) \leq \log n_c + 2$, *where* $n_c = \max_{C \in \mathcal{C}} |C|$.

Proof. Take the largest connected component of G. For any tree T there is an unicycle U such that $T \subset U$. As tree-depth is an increasing parameter with the subgraph partial ordering, we can assume that G is a unicyclic graph. Let C the cycle of U and $x \in V(C)$, then $T = G \backslash \{x\}$ is a tree.

By using (2.3) and Proposition 2.2 it follows that

$$\mathrm{td}(G) \leq \log n_c + 2.$$

□

A famous result of Erdős and Rényi [19] states that, if $0 < c < 1$, then G is a pseudoforest. Now we are interested in bound the size of each connected component of G by a logarithmic expression. Bóllobas [10, Corollary 5.11] showed that, if $0 < c < 1$, then the size of the largest tree component in the random graph has order $\Theta(\log n)$. Moreover, for any function $\vartheta(n) \to \infty$, there are at most $\vartheta(n)$ vertices belonging to unicyclic graphs [10, Corollary 5.8]. Taking $\vartheta(n) < \log n$ we can ensure that the largest connected component of G is of size $\Theta(\log n)$. Therefore, $n_c = \Theta(\log n)$ and by Lemma 5.4,

$$\mathrm{td}(G) = O(\log \log n).$$

The lower bound is more involved.

As we have seen, G is composed of pseudotrees of logarithmic size.

Let T be a tree and denote by d its diameter. Every graph G contains a path of length at least the diameter of the graph, and in particular, if G is a tree, then the maximal path is attained with the minimal path between two antipodal vertices. Since the tree–depth of a path is $\Theta(\log n)$, the fact that the tree-depth is monotonically increasing with respect to the subgraph partial ordering and that the diameter is a path subgraph of G,

$$\mathrm{td}(T) \geq \log d.$$

Recall that every labeled tree on k vertices has the same probability to appear in G as a connected component.

The question now is: which is the diameter of these random trees? Rényi and Szekeres [45] proved that, if H_k is the height of a random labeled rooted tree on k vertices, then

$$\mathbb{E}(H_k) \sim \sqrt{2\pi k}$$

and

$$\mathbf{Var}(H_k) \sim \frac{\pi(\pi-3)}{3}k$$

Let D_k be the diameter of a labelled rooted tree. This random variable is also the one that computes the diameter of unrooted labelled trees since any tree on k vertices has exactly k ways to be rooted.

Since $H_k \le D_k \le 2H_k$, asymptotically the two parameter have the same value. Therefore D_k satisfies $\mathbb{E}(D_k) = \Theta(\sqrt{k})$ and $\mathbf{Var}(D_k) = \Theta(k)$. Hence our proof for this case will be completed if we show that the random graph contains a tree T with size $\Theta(\log n)$ and diameter $\Theta(|T|) = \Theta(\log n)$.

Since the expectation of D_k has the same order of magnitude as the standard deviation, we can not ensure that its value is highly concentrated. The diameter of each individual tree can be arbitrarily small as n tends to infinity, but in general this will not be true. Thus, we must show that the mean of the tree diameters is a.a.s large enough to ensure the existence of a tree with relatively large diameter.

Changing the model an taking now $H \in G(n, m(n))$ Erdős and Rényi [19] showed that X_k, the number of trees of order k in H with $m(n)/n^{\frac{k-2}{k-1}} \to \infty$, has a normal distribution with expectation and variance M_n, where

$$M_n(k) = n\frac{k^{k-2}}{k!}\left(\frac{2m}{n}\right)^{k-1}\exp\left(-\frac{2km}{n}\right)$$

Moving back to the random graph model $G(n,p)$ with $p = c/n$, and noting that $\mathbb{E}(m) = \frac{cn}{2}$ we get the analogous result, where

$$M_n(k) = n\frac{k^{k-2}}{k!}c^{k-1}\exp(-kc) \tag{5.3}$$

We are interested in the trees of logarithmic size. Hence, if $k = \log n$ then,

$$M_n = \frac{n^{\log\log n - \alpha}}{c(\log^2 n)(\log n)!}$$

where $\alpha = c - 1 - \log c$.

As the logarithm is a monotonically increasing function and (5.3) tends to infinity as $k \to \infty$, $M_n \to \infty$, i.e. $M_n = \omega(1)$.

By using Chebyshev's inequality with $\mu = \sigma^2 = M_n$, we can see that, for some $\vartheta(n) = o(\sqrt{M_n})$ s.t. $\vartheta(n) \to \infty$,

$$\Pr(|X_{\log n} - M_n| \ge \vartheta(n)\sqrt{M_n}) \le \frac{1}{\vartheta(n)^2} \longrightarrow 0 \qquad (n \to \infty)$$

This ensures that at least $K(n) = M_n - o(M_n) \to \infty$ tree components have size $\log n$.

Let $\overline{D}$ be the mean of the diameter among all the components of size $\log n$. Clearly $\mathbb{E}(\overline{D}) = \Theta(\sqrt{\log n})$, since $\mathbb{E}(D) = \Theta(\sqrt{k})$, but now we have that $\mathbf{Var}(\overline{D}) = O(\log n/K))$, where $K = o(1)$, and so, $\sigma(\overline{D}) = o(\sqrt{\log n})$. Hence, by using again Chebyshev inequality on $\overline{D}$, we can ensure that $\overline{D} = \Theta(\sqrt{\log n})$ a.a.s. and by Markov inequality there exists some tree T with diameter $\Omega(\sqrt{\log n})$.

Finally,

$$\mathrm{td}(G) = \Omega(\log\sqrt{\log n}) = \Omega\left(\frac{1}{2}\log\log n\right) = \Omega(\log\log n).$$

5.2.2 $c = 1$

Now we look at the critical point where $c = 1$.

Another famous result of Erdős and Rényi [19] states that, in this case, the random graph has some giant components (GC) of order $\Theta(n^{2/3})$. Nevertheless, the number of giant components is constant. In this situation we can only assert that the graph has at most $n/2 + O(\sqrt{n})$ edges.

The idea is to prove, that the GC is almost a tree. In [26, pag.112] a useful concept is defined. For $l \geq -1$ an *l-component* is a connected component with k vertices and $k + l$ edges. For example, (-1)-components are trees and 0-components are unicyclic graphs. A *complex component* is an l-component with $l > 0$. Let $\{\tilde{G}_t\}_0^N$ denote a graph process where edges are added at random on n points and $N = \binom{n}{2}$.

Janson, Łuczak and Rucinski [26, Theorem 5.19] proved that the excess $l(C)$ of the complex components when $m = n/2 + O(n^{2/3})$ is constant. In other words, $l(C) = O(1)$.

The GC of G contain k vertices and $k + l$ edges, and if we delete $l = O(1)$ vertices, we will get a tree on $(k - l) = O(n^{2/3})$ vertices. Since the remaining components have similar or negligible order, the tree-depth satisfies

$$\mathrm{td}(G) \leq l + O\left(\log\left(n^{2/3}\right)\right) = O(1) + O\left(\frac{2}{3}\log n\right) = O(\log n)$$

To prove the lower bound, we use the following result which follows from a more general statement due to Nachmias and Peres [36].

Theorem 5.5 ([36]) *Let C be the largest component of a random graph in $G(n, p)$ with $p = 1/n$. Then, for any $\varepsilon > 0$, there exists $A = A(\varepsilon)$ such that*

$$\Pr(\mathrm{diam}(C) \notin (A^{-1}n^{1/3}, An^{1/3})) < \varepsilon.$$

This diameter agree with the expected one. We have proof that in this case the GC are almost trees, but for a constant number of edges, and we also know, that the expected diamter of a tree of size k is $\sqrt{k}$. As the size of this component is $O(n^{2/3})$, the diameter that make sense is its square root, $O(n^{1/3})$.

It follows from the monotonicity of tree–depth (2.3) and from $\mathrm{td}(P_n) = \Theta(\log n)$, that a graph with diameter d satisfies $td(G) = \Omega(\log d)$. Hence, it follows from Theorem 5.5 that

$$\mathrm{td}(G) = \Omega(\log n^{1/3}) = \Omega(\log n).$$

This concludes the proof of the case $c = 1$.

The same reasoning can be applied in the case of width parameters. A parameter $\mathrm{w}(G)$ of the graph is called *width parameter* if is bounded by a certain function of $\mathrm{tw}(G)$, i.e. exists f such that $\mathrm{w}(G) \leq f(\mathrm{tw}(G))$. Some examples of that could be the branch–width, the path–width, the rank–width or the clique–width.

Proposition 5.6 *Let $G \in G(n, p = 1/n)$. For any width parameter $\mathrm{w}(G)$,*

$$\mathrm{w}(G) = O(1)$$

Proof. The idea is to prove that $\mathrm{tw}(G)$ is constant.

We know that the r largest components C of G have size $O(n^{2/3})$ and constant excess by the former proposition. Hence, C are almost trees.

We claim that if $\mathrm{tw}(G) = k$, then for any edge $e \notin E(G)$, $\mathrm{tw}(G \cup e) \leq k+1$.

It follows from the definition of k-trees. If G is a partial k-tree and we add an edge $e = uv$ it results a $(k+1)$-tree. In fact it will be a $(k+1)$-tree if and only if we put the edge between two vertices that are completely connected to the same k-clique. □

This remark improves, for example, the upper bound given in [30] about the rank–width of G. If $c = 1$ we can replace an upper bound of order $O(n^{2/3})$ by $O(1)$, which is a large improve.

5.2.3 $c > 1$

With $c > 1$ we want to proof that the tree–depth is linear.

The remark given in Section 5.1 ensures that for $c > c_0 \approx 4,94$ the tree-depth of G is linear. Moreover, we gave an explicit linear constant.

Kloks in [29] conjectured that c_0 must be one. He provides a proof for $c_0 = 1,18$. In [24], Gao lowered the constant to $c_0 = 1,081$. Lee, Lee and Oum proved in [30] the conjecture using the Theorem 5.8.

During this subsection we will give another proof to the Kloks conjecture which in our opinion is more direct. This will allow us to proof that the tree-depth is also linear.

The *Cheeger constant* or *isoperimetric number* of a graph G can be defined as:

$$\Phi(G) = \min_{0<|X|\leq n/2} \frac{E(X, V\backslash X)}{|X|}$$

This coefficient measures the expansion of the graph. A graph G is said to be an α–expander for $\alpha > 0$, if $|N(X) \setminus X| \geq \alpha|X|$ for every set X of vertices with $|X| \leq |V(G)|/2$, where $N(X)$ denotes the vertex neighborhood of X. Note that, for an α–expander graph, $\Phi(G) \geq \alpha$.

Proposition 5.7 *Let $\alpha > 0$. Let G be a graph that contains H an α-expander, of size k. Then* $\mathrm{tw}(G) = \Omega(k)$.

Proof. Denote by $\mathrm{tw}(G) = t_G$ and $\mathrm{tw}(H) = t_H$. As tree-width is closed under subgraph relation and $H \subset G$, we know that $t_H \leq t_G$.

By Theorem 2.1, we know that there is a balanced partition $V(H) = (A, S, B)$, where S is a vertex separator of cardinality $t_H + 1$ and we can assume that $k/2 \geq |A| \geq (k - t_H - 1)/3$. Hence,

$$t_G \geq t_H \geq |S| - 1 \geq \alpha|A| - 1 \geq \alpha\frac{k - t_H - 1}{3} - 1 \geq \alpha\frac{k - t_G - 1}{3} - 1,$$

where we have used the fact that H is an α–expander. Thus, $t_G \geq \frac{\alpha(k-1)-3}{\alpha+3}$, and $\mathrm{tw}(G) = \Omega(k)$.

□

The recent proof of Benjamini, Kozma and Wormald about the value of mixing time of the random walk on the giant component of a random graph $p = c/n$, $c > 1$, relies on the existence of an α–expander subgraph of linear size in the giant component. The following result follows from [7, Theorem 4.2] where the fact that the expander has linear size is stated within the proof of that Theorem. They show that there exist a certain subgraph $R_N(G)$ such that its kernel is of linear size. Then they prove that this subgraph $R_N(G)$ is an α-strong core, and as its kernel is a subgraph, $R_N(G)$ has also linear size. By the definition of an α-strong core, it is included in a "decorated" expander, and $\Phi(R_N(G)) \geq \alpha$. Therefore, there exists an α-expander subgraph.

Theorem 5.8 ([7]) *Let G be a random graph in $G(n,p)$ with $p = c/n$, $c > 1$. There are $\alpha, \delta > 0$ and a subgraph H of G such that H is an α–expander and $|V(H)| = \delta n$.*

The linearity of $\mathrm{td}(G)$ in this case follows from Proposition 5.7 and Theorem 5.8.

Corollary 5.9 *Random regular graphs have linear tree-width*

Proof. For d–regular graphs $G \in G(n,d)$ there is a two–sided equivalence between expanders and a nonzero value of the Cheeger constant. The latter can be lower bounded by $(d - \lambda_2(G))/2$, where $\lambda_2(G)$ is the second largest eigenvalue of the adjacency matrix of the graph. Friedman, Kahn and Szemerédi [22] prove that this eigenvalue in d–regular random graphs is $O(\sqrt{d})$. Therefore it follows from Proposition 5.7 that $td(G) \geq tw(G) = \theta(n)$.

We note that Proposition 5.7 also shows that random d-regular graphs (d-RRG) have linear tree–width. $\square$

A

Evolution of random graphs

In Chapter 4 we make a strong use of the structure of the connected components from the random graph. There, we distinct the sparse and the dense case. In this appendix our propose is to sum up all this properties to give to the reader an idea of the random graphs evolution. We refer the interested ones, to [10] and [26].

When a certain paramater is studied in random graphs, is really useful the idea of critical points or phase transitions. This is close to the notion of threshold function. Given a graph property $\mathcal{P}$ we call a real valued function $f = f(n)$ a *threshold function* if the following holds

$$\lim_{n\to\infty} \Pr(G \in \mathcal{P}) = \begin{cases} 1 & \text{if } p/f \to \infty \text{ as } n \to \infty \\ 0 & \text{if } p/f \to 0 \text{ as } n \to \infty \end{cases}$$

The phase transition is the point where p/f tends to a constant. In general these are the most difficult point to analyze because $\lim\limits_{n\to\infty} \Pr(G \in \mathcal{P})$ can be any value in $[0, 1]$.

A.1 The double jump

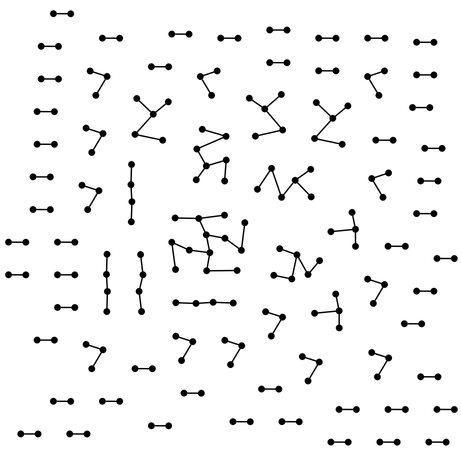
Recall that we talk about sparse random graphs, when the probability of an edge is $p(n) = O(n^{-1})$, and otherwise, the graphs is dense.

Observe that if $p(n) = c/n$, then the expected degree of a vertex is approximately c. This simple fact gives an idea on why is so important this point. Obviously, $f(n) = c/n$ will be a threshold function for many properties, but we can refine even more and study the behaviour in dependence of c.

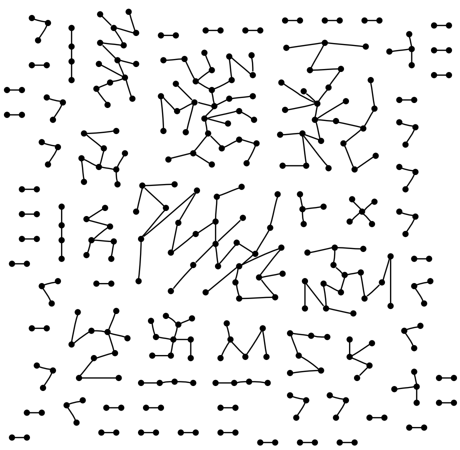
Just before and after $c = 1$ the random graph behaves in a completely different way. The behaviour change between $c < 1$ and $c = 1$ and between $c = 1$ and $c > 1$, is known as the double jump.

It is a well known result [19] that the structure of a random graph with $p < c/n$, $c < 1$, is composed by trees and unicycles size at most $\frac{3}{(1-c)^2} \log n$. This can be justified because the expected degree of the vertices is < 1. This reason cause the small size and the low density of the connected components. In fact, $n - o(n)$ of the elements belong to trees, therefore there are not many vertices in unicycles, moreover for any $\omega(n) \to \infty$. In Fig. A.1a we can appreciate this union of small sparse components.

When $p = 1/n$, a huge change happen. The different components "join" and there are some components that start to grow. This phenomena is known as the birth of the giant component (GC). However, their size is not linear but $O(n^{2/3})$ and are denser than trees. Nevertheless the vast majority of vertices still belong to logarithmic size trees, that will be attached slowly to this GCs. Fig. A.1b show an example of one of this graphs. Later, we will zoom on this phase to understand this transition.



(a) Random graph: $n = 500$ and $c = 1/2$



(b) Random graph: $n = 500$ and $c = 1$

Figure A.1

When $c > 1$ the GC is unique and of linear size. All the other components are pseudotrees of logarithmic size. This GC is considerably denser than in the critical point. However, the graph is still not connected. We have also logarithmic size trees that will be attached to the GC as can be appreciated in Fig. A.2a. In fact, the GC is so dense that its diameter if of order $O(\log n)$.

It is not strange, that when the expected degree is constant, asymptotically the graph is not connected. The following question seems natural: which is the threshold function for the graph property of being connected? It can be shown that if $p(n) = (\log n + c)/n$ with $c > 0$, then X, the number of isolated vertices, is asymptotically a Poisson with $\lambda = e^{-e^{-c}}$. Also, the probability that there exist a connected component of size at least 2 appart from the GC tends quickly to zero. This implies that

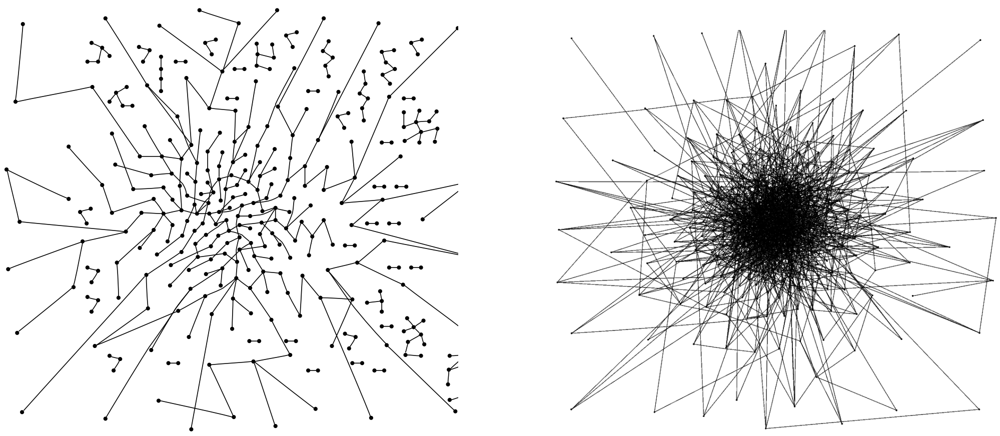
$$\Pr(G \text{ is connected}) = e^{-e^{-c}}$$

If $np(n) - logn \to \infty$, then G is a.a.s. connected. In Fig. A.2b we have an example of such a random graph.

Table A.1 summarize the behaviour exposed above.

$p(n)$	sz lgst cmp	tree V	unicycle V	denser V
c/n, $0 < c < 1$	$\log n$	$n - O(1)$	$\leq \omega(n)$	0
c/n, $c > 1$	$\Theta(n)$	few	few	$O(n)$
c/n, $c = 1$	$\Theta(n^{2/3})$	$n - o(n)$	$O(n^{2/3})$	$O(n^{2/3})$
$\log n/n$	n	0	0	n

Table A.1



(a) Random graph: $n = 500$ and $c = 3/2$ (b) Random graph: $n = 500$ and $c = \log n$

Figure A.2

A.2 The growth window of the giant component

It is really interesting to study what happen around the point where $p(n) = 1/n$. During this section we will deal with the edge model $G(n, m)$ and use $m(n) = n/2 \pm s$ where $s = o(n)$, $s \geq 0$. In this case, we divide in three regions according to the behaviour of the connected component in the random graph.

The first region is the *subcritical region* and is characterised by $p(n) = 1/n - s$, $\Omega(n^{2/3}) \leq s \leq o(n)$. Recall that when $c < 1$ no GC exists. During this stage, the GC begin its growth. In fact, in [26] it is proved that G is composed vastly by trees and few unicycles, as in the case when $c < 1$. The size of the r-th largest component, $L_r(G)$, satisfy,

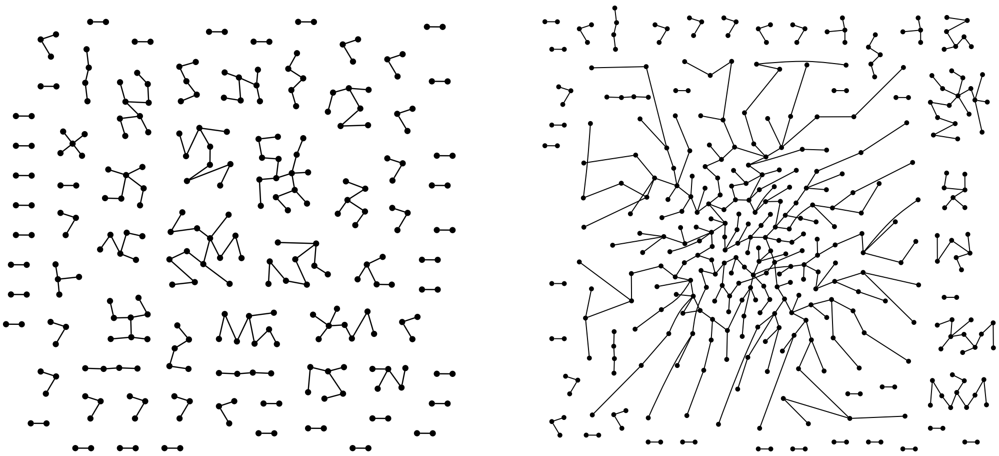
$$L_r(G) = \Theta\left(\frac{n^2}{s^2} \log \frac{s^3}{n^2}\right)$$

Observe that if $s = \Theta(n)$, then $L_r(G) = \Theta(\log n)$ and when $s = \Theta(n^{2/3})$, $L_r(G) = \Theta(n^{2/3})$, so the result is coherent with Table A.1.

The second region is the *critical phase* and occurs when $p(n) = 1/n \pm s$ with $s = O(n^{2/3})$. It is really curious that the behaviour in this region depend on the parameter s, but not on the sign of it, i.e. it has a symmetric behaviour. For this reason, $L_r(G) = \Theta(n^{2/3})$ for any s in this region, and there are several GCs. They are almost trees, in fact, the excess of edges respect the number of vertices is constant.

The third region, appear when $p(n) = n/2 + s$ with $\omega(n^{2/3}) \leq s \leq o(n)$. This region, called *supercritical phase*, is the most difficult one to study because the components become denser and greater. Some GC and trees quickly merge giving a component of size $\Theta(s)$. This is the unique GC, i.e. all the other will be of size $O(n^{2/3})$. The excess of the GC is of order $\Theta(16s^3/3n^2)$, so it gets denser when s grows. Ding, Kim, Lubetzky and Peres in [17] show that there exist an expander subgraph of size $O(s)$ in this stage.

Finally, Fig.A.4 gives us an idea of the number of connected components and their size, while $p(n)$ increases.



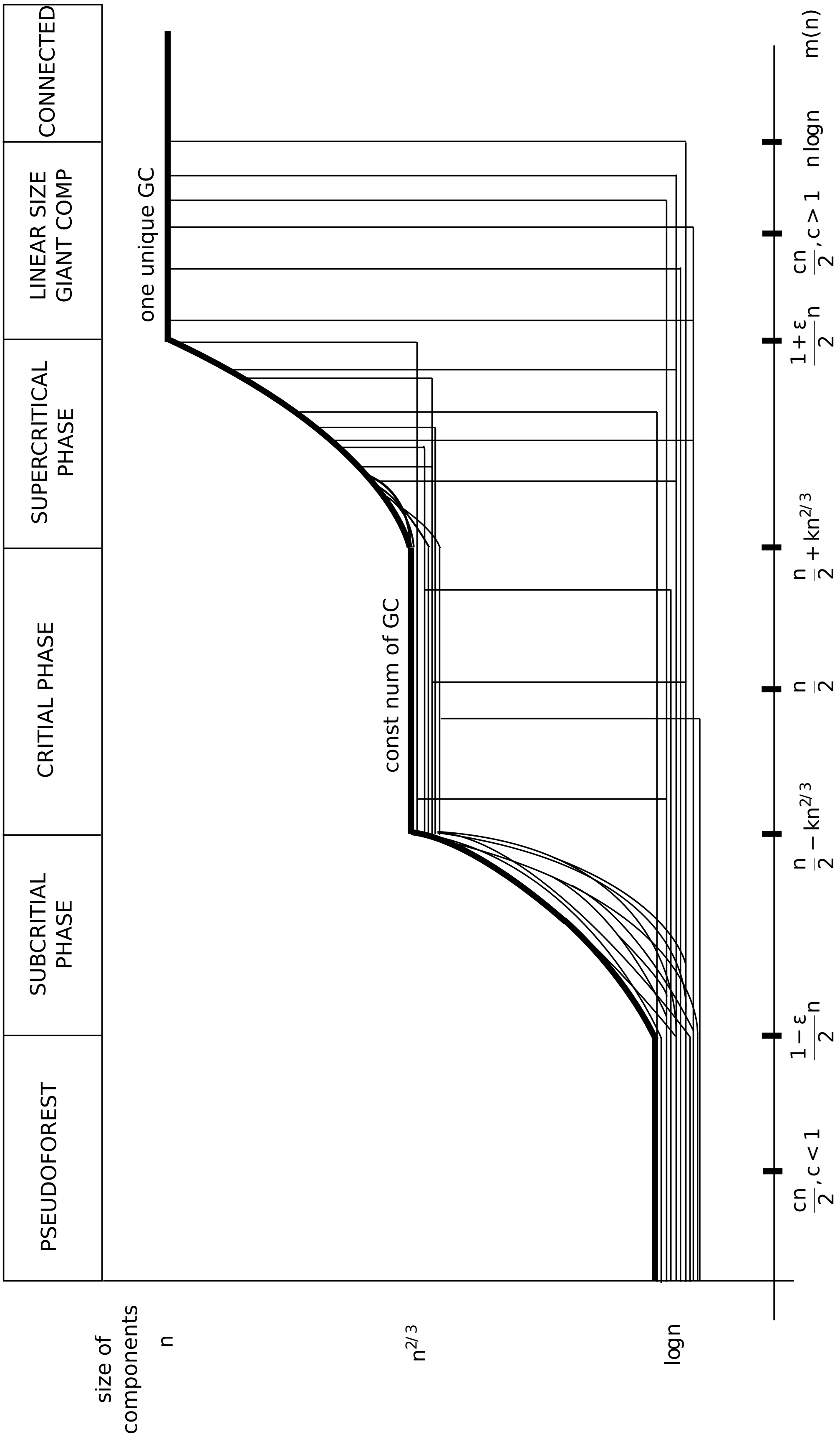
(a) Random graph: $-n \ll s \ll -n^{2/3}$

(b) Random graph: $n^{2/3} \ll s \ll n$

Figure A.3

p	sz lgst cmp	tree V	unicycle V	denser V
$-n \ll s \ll -n^{2/3}$	$O\left(\frac{n^2}{s^2}\log\frac{s^3}{n^2}\right)$	$n - O(1)$	$\leq \omega(n)$	0
$s = O(n^{2/3})$	$O(n^{2/3})$	$n - o(n)$	$\Theta(n^{2/3})$	$\Theta(n^{2/3})$
$n^{2/3} \ll s \ll n$	$O(s)$	$n - O(s)$	$\Theta(n^{2/3})$	$O(s)$

Table A.2


PSEUDOFOREST
SUBCRITIAL PHASE
CRITIAL PHASE
SUPERCRITICAL PHASE
LINEAR SIZE GIANT COMP
CONNECTED
one unique GC
const num of GC
size of components
n
$n^{2/3}$
log n
$\frac{cn}{2}, c < 1$
$\frac{1-\varepsilon}{2}n$
$\frac{n}{2} - kn^{2/3}$
$\frac{n}{2}$
$\frac{n}{2} + kn^{2/3}$
$\frac{1+\varepsilon}{2}n$
$\frac{cn}{2}, c > 1$
n log n
m(n)

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