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The Hiring Problem: An Analytic and Experimental Study

STUDENT: AHMED MOHAMED HELMI MOHAMED
DIRECTOR: CONRADO MARTÍNEZ

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To my father and my mother.
They are praying for me all the time.
They give me the advice to succeed not only in my work but also in my life.
To my wife and my beautiful Salma.
They support me a lot and give me the hope for tomorrow.

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Finally, the most important thing which I learned while working in this thesis is that there is no place for desperation in the scientific research, because even “no solution” may be a result.

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Chapter 1

Introduction

1.1 Hiring in general

When a small, start-up company intends to grow, it has to hire employees. Because the company requires high quality staff, the employer has to interview a lot of candidates and thus, she may take long time to collect the required staff. Of course, there is another important demand which is the time taken by the hiring process which is required to be as short as possible or the company's rate of growth which is required to go as quickly as possible. So, the company has two main demands and needs to achieve balance between them. This is an intuitive idea about hiring from which this case study problem bears its name.

The *hiring problem* is just an abstract model of the real hiring process. It is clear that the hiring problem will not cover every aspect of real hiring processes but it investigates some important parameters under a simplified mathematical model. On the other hand, the statement of the problem - as we will see- will give a general mathematical question with many possible applications; it is relevant in many instances where one must make decisions under uncertainty.

1.2 History of the problem

The problem of searching the best candidate of a sequence represents a simple model for decision making under uncertainty because the decision maker -at some moment- has to select one out from this sequence without interviewing the candidates who may come after the selected one. So, the decision maker needs to maximize the probability of choosing the best one. Also the decision of hiring or discarding a candidate is taken on-line and it is irrevocable. This problem is first modeled as the well-known problem called *the secretary problem*[3]. In the secretary problem, the employer is looking for only one candidate to fill in one secretarial position. The secretary problem is well studied and have many extensions. One important extension is to consider the case when the employer is looking for many employees to grow her company. Broder et al. [1] introduced this extension as *the hiring problem*. The hiring problem has the same spirit as the secretary problem but with some changes as shown later. Broder et al. presented a continuous probabilistic model of the problem and analysed some hiring strategies in this model.

Archibald and Martínez [2] have reformulated the problem in a discrete combinatorial model. They used techniques from the analytic combinatorics field, e.g. generating functions. They introduced their model and some results in Archibald and Martínez [2]. Now, we (Archibald, Martínez and me) have a current work [4] to complete the study of the problem including analysing new parameters and considering some extensions of the problem.

1.3 Modeling the problem

In the hiring process, candidates are interviewed sequentially and each one has a score or quality. Depending on the used hiring strategy an applicant will be hired or discarded. The decisions made are irrevocable and there is no knowledge on the future. The decisions must be taken, hence, on the scores of the candidates seen so far.

1.3.1 Broder et al.'s model

The first step in modeling the problem is how to determine the scores of the candidates. Broder et al. consider candidate scores as uniformly distributed on the interval $(0, 1)$ where each candidate has a quality score Q_i . Thus these Q_i 's are i.i.d. random variables with common distribution $\text{Unif}(0, 1)$. Broder et al. consider the absolute quality scores of candidates to be the input for the hiring strategy. In this light they introduced hiring strategies like

- Above a threshold t .
- *Lake Wobegon strategies*:
 - Above the current mean.
 - Above the current median.

1.3.2 Archibald and Martínez's model

Here, the interviewed candidate is given a *rank* which is relative to the ranks of the previous candidates. Thus, at step i , $\sigma(i)$ denotes the rank of the i th candidate among all interviewed ones, where the best candidate seen so far among the n gets a rank n , while the worst one gets rank 1. Then the n ranks or the relative scores of the n candidates form a permutation $\sigma = \{1, \dots, n\}$. In this context, the *hiring set* of a permutation σ is the set of indices that would be hired by applying a specific strategy to the permutation σ . Considering relative ranks is one similarity between this model and the secretary problem, which doesn't hold for the model of Broder et al.. In this model any permutation σ representing the scores is equally likely. Due to dealing with relative ranks, the considered strategies here are *rank-based strategies* like

- Above the best hired candidate.
- Above the m best hired candidates.
- Above the $P\%$ quantile of hiring set (with $P = 50$: Above the median).

In this model, we can state the problem as follows

1. Input: a sequence of relative scores $S = s_1, s_2, \dots, s_i, \dots$ of the candidates.
2. The score of the i th candidate, $1 \leq s_i \leq i$, is uniformly distributed.
3. Each finite sequence s_1, \dots, s_n represents a random permutation of length n .
4. A decision must be taken whether to hire the i th candidate or not at step i .
5. Decisions are irrevocable.
6. No information of the future.
7. Goals: hire candidates in some reasonable rate and improve the mean quality of the hired staff.

1.3.3 Similarities and Differences with the secretary problem

First, it is better to state the features of the secretary problem in its simplest form [3]

1. There is one secretarial position available.
2. The number n of applicants is known.
3. The applicants are interviewed sequentially in random order, each order being equally likely.
4. It is assumed that you can rank all the applicants from best to worst without ties. The decision to accept or reject an applicant must be based only on the relative ranks of those applicants interviewed so far.
5. An applicant once rejected cannot later be recalled.
6. You are very particular and will be satisfied with nothing but the very best. (That is, your payoff is 1 if you choose the best of the n applicants and 0 otherwise.)

The most important similarity with Archibald and Martínez's model is that the decision maker can rank all the applicants without ties. But there is a clear difference which is the number of applicants n . This n is known in advance in the secretary problem but it is unknown (may be infinite) in the hiring problem. So, in the secretary problem if the employer reach this n , then she must accept the n th applicant. Another difference is the measure of quality; this measure is clear in the secretary problem where the optimal strategy should hire the best applicant: the optimal strategy has the maximum probability of choosing the best one.

1.4 Personal Contributions

We have done theoretical work (Chapter 3) where we introduce some parameters of the hiring problem under the combinatorial model of Archibald and Martínez. These parameters are the distance between the last two hirings, the size of the firing set and the score of best not hired candidate. We introduce these parameters to know more about the dynamics and the quality of hiring strategies. Also, hiring above the $P\%$ quantile strategy is studied in depth and we have general results for the following parameters

- The expected size of hiring set is $\Theta(n^{1-P})$.
- The expected gap of the last hired candidate is $\Theta(1/n^P)$.
- The expected size of firing set is $\Theta(n^{1-P} - \ln n)$.

The expectations for these three parameters with hiring above the median strategy are obtained in equations (3.15), (3.18) and (3.21). We analyse another strategy called hiring above the worst and obtain the results for the above parameters in equations (3.22) and (3.23) (as explained later, the study of firings does not have any interest, as they are trivially zero). We also obtain the average size of the firing set for hiring above the best strategy in equation (3.15).

We have developed a software simulator for rank-based strategies. The main features of this simulator are mentioned in Chapter 4. Using the software simulator, we carried out an experimental study for the strategies introduced in [2] and this thesis.

The introduced rank-based strategies in [4] and this thesis are deterministic (the decision to hire or discarded the following candidate is based on the whole hiring set). In section (3.4),

we introduce randomized hiring, in which a sample (committee) is randomly chosen from the current hiring set. The decision to hire or discard a candidate is based on a deterministic hiring strategy w.r.t. the committee rather than the whole hiring set. From the experimental results developed in Chapter 3, we also pose several conjectures about the expected size of hiring set for two randomized hiring strategies, above the median and above the best as reported in Chapter 4. As a summary of the experimental results for randomized hiring strategies, we have the following results:

1. Randomized hiring above the median:

The expected size of hiring set is $\Theta(\sqrt{n})$. The recommended sample size is $\sqrt{h_n^{(RM)}}$, where $h_n^{(RM)}$ denotes the current hiring set size.

2. Randomized hiring above the best:

Let s denote the sample size and $h_n^{(RB)}$ denote the current hiring set size. Then,

- For $s = c \cdot h_n^{(RB)}$, $0 < c \leq 1$, the expected size of hiring set is $\Theta(\ln n)$.
- For $s = \left(h_n^{(RB)}\right)^\alpha$, $0 < \alpha \leq 1$, the expected size of hiring set is $\Theta\left((\ln n)^\beta\right)$ for $0 < \beta < 1$.
- For $s = \Theta(1)$, the expected size of hiring set is $\Theta(n^\gamma)$ for $0 < \gamma < 1$.

Chapter 2

Previous Work

2.1 Lake Wobegon and other strategies [1]

Before introducing the Lake Wobegon strategies which are studied in the paper of Broder et al., there are two other natural hiring strategies that were considered also in that paper.

2.1.1 Hiring above a threshold t

This strategy hires only the candidates with quality scores above a pre-specified threshold t . So, with a large value of t , this strategy guarantees high quality from the beginning. But there are two clear drawbacks: the first, the strategy does not lead to continual improvement because the threshold is fixed; the second one, the hiring rate is $1 - t$ and since t is fixed, it is difficult to choose t in order to achieve a reasonable balance between the overall quality of employees and the rate of company's growth.

2.1.2 *Max* strategy

In this strategy, the company will start with a single employee with a quality score (I will use just *score* to mean quality score) g , and only hires applicants that have scores higher than the scores of all the hired employees. To analyze this strategy, they considered the *gap* $G_i = 1 - Q_i$ between the score Q_i and 1.

Let $G_0 = g = 1 - q$ denotes the gap of the first candidate, then after n hirings: $\mathbb{E}\{G_n\} = g/2^n$. So, the expected gap shrinks exponentially as the number of hirings grows and this reflects the high quality gained in this strategy. On the other hand, since all scores between $(0, 1)$ are equally likely to happen then $G_i = G_{i-1}U_i$ where U_i are independent uniform $(0, 1)$ random variables and the expected number of interviews required between any two hirings is actually infinite. So the strategy has large lags between hirings. Also Broder et al. showed that the gap has approximately *lognormal* distribution where

$$\ln G_n \approx N(\ln g - n, n).$$

2.1.3 Hiring above the Mean

This is the first Lake Wobegon strategy. Let A_i denote the average quality after i hirings, with $A_0 = g$ being the quality of the initial employee, so that A_i refers to the average quality of $i + 1$ hired candidates. Then, at any step, this strategy will hire only scores that are above the *mean* quality of the hiring set. The results obtained here are

- With probability 1, infinitely many candidates will be hired and $\lim_{i \rightarrow \infty} A_i = 1$. From this proposition, it is clear that the quality of the hiring set is improved all the time and also the hiring rate will be reasonable as shown later.
- The expected gap after n hirings is

$$\mathbb{E}\{G_n\} = \Theta(1/\sqrt{n}).$$

The gap expectation gives us an indicator of the quality. So, for large values of n , the quality will be close to 1 (the maximum quality).

- Let T_n be the number of candidates that are interviewed after the size of the hiring set is n , then

$$\mathbb{E}\{T_n\} = \Theta(n^{3/2}).$$

Following the derivation in this paper, we find that the initial starting gap has a multiplicative effect on the expected gap and the expected size of the hiring set. And this is true also when starting with more than one employee.

- The distribution of the gap (under suitable initial conditions) weakly converges to lognormal distribution, that means that there may be larger error at the tails.

Finally, Broder et al. claim that hiring above the mean -with these expectations of gap and number of interviews- is within a constant factor of optimal. However, they do not explicitly state what do they mean by “optimal” in this context.

2.1.4 Hiring above the Median

This is the other Lake Wobegon strategy. The hiring set here starts with one employee with quality $q \in (0, 1)$ and when ever we have $2k + 1$ hired candidates, the next two hired candidates must have at least the median score M_k of the $2k + 1$ candidates. The results of this strategy are stated as follows

- With probability 1, infinitely many candidates will be hired and $\lim_{k \rightarrow \infty} M_k = 1$. Again, since $G_k \geq 1 - M_k$, then the gap converges to 0 as $k \rightarrow \infty$.
- The gap expectation is

$$\mathbf{E}\{G_k\} = \Theta(1/k).$$

- Let T_k be the number of interviews until there are $2k + 1$ hired candidates. Then

$$\mathbf{E}\{T_k\} = k(k + 1)/g.$$

where $g = 1 - q$, the initial starting gap.

And we need the following quantity to compare hiring above the median with hiring above the mean,

- Let A_n denote the mean quality score of the first n hired candidates for hiring above the median. Then

$$\mathbf{E}\{A_n\} = 1 - \Theta(\log n/n).$$

Thus, $\mathbb{E}\{G_n\} = \Theta(\log n/n)$, which is asymptotically smaller than $\Theta(1/\sqrt{n})$ of hiring above the mean strategy.

- Hiring above the median also yields a weak convergence to a lognormal distribution as hiring above the mean.

From these results, we can note that hiring above the median leads to *smaller* gap (*higher* quality) than hiring above the mean, but with *fewer* hirings (*slower* rate of growth) because the number of interviews between hirings is much larger. Also this strategy is seen as within a constant factor of optimal.

2.2 Rank-based strategies [2]

2.2.1 A general framework

Before introducing the main tool used in this model, it is better to be familiar with the meaning of some terms. Given a permutation σ of length $n - 1$ and a value (relative rank) j , $1 \leq j \leq n$, $\sigma \circ j$ denotes the resulting permutation of size n after relabelling $j, j + 1, \dots, n$ and appending j to the end. For example, if we have this sequence of relative ranks: 1, 1, 3, 2, 2 then the corresponding permutations are $\sigma_1 = 1$, $\sigma_2 = \sigma_1 \circ 1 = 21$, $\sigma_3 = \sigma_2 \circ 3 = 213$, $\sigma_4 = \sigma_3 \circ 2 = 3142$ and $\sigma_5 = \sigma_4 \circ 2 = 41532$. The notation $\mathcal{H}(\sigma)$ will denote the set of the indices of the hired candidates or the *hiring set* of the permutation σ . This hiring set has some parameters to be studied w.r.t. a given hiring strategy such as its size $h(\sigma)$, the index of last hired candidate $L(\sigma)$, the distance between the last two hirings $\Delta(\sigma)$ and other useful parameters as we will see later. The letters h_n, L_n, r_n, \dots will denote the corresponding random variables of these parameters. A rank-based strategies is *pragmatic* if the following two conditions are met:

1. For all σ and all j , $X_j(\sigma) = 1$ implies $X_{j'}(\sigma) = 1$ for all $j' \geq j$.
2. For all σ and all j , $X(\sigma \circ j) \leq X(\sigma) + X_j(\sigma)$

The first condition states that whenever a pragmatic strategy hires a candidate with score j , it would hire a candidate with a higher score. The second condition bounds the hiring rate and guarantees that the potential of hiring $X(\cdot)$ does not change if no new candidate is hired.

The idea of having permutation representing the relative scores of the candidates allows the use of bivariate generating functions (*BGF*) [5] to analyse several parameters.

The size of the hiring set

The *BGF* (more precisely, the exponential *BGF* because we are dealing with permutations which are labelled objects) of the size of the hiring set $h(\sigma)$ is

$$H(z, u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)},$$

where: z marks the size of the permutation and u marks the value of the parameter being analysed which is the size of the hiring set $h(\sigma)$ in this case. \mathcal{P} denotes the set of all permutations. To obtain the generating functions of the moments of h_n , we have to take successive derivatives of $H(z, u)$ w.r.t. u and set $u = 1$, then

$$h_r(z) = \left. \frac{\partial^r}{\partial u^r} H(z, u) \right|_{u=1} = \sum_{\sigma \in \mathcal{P}} \mathbb{E}\{h_n^r\} z^n,$$

Where $\mathbb{E}\{X^r\} = \{X(X - 1) \dots (X - r + 1)\}$ denotes the r th falling factorial of the random variable X [6].

Now, the first moment gives us the expected value of the size of the hiring set as follows

$$h(z) = \frac{\partial}{\partial u} H(z, u) \Big|_{u=1} = \sum_{\sigma \in \mathcal{P}} h(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} ,$$

Hence $\mathbb{E}\{h_n\} = [z^n]h(z)$.

There is an important variable indicator called $X_j(\sigma)$ which is defined as

$$X_j(\sigma) = \begin{cases} 1, & \text{if a candidate with score } j \text{ is hired after } \sigma , \\ 0, & \text{otherwise.} \end{cases}$$

Thus,

$$h(\sigma \circ j) = h(\sigma) + X_j(\sigma),$$

and the quantity

$$X(\sigma) = \sum_{1 \leq j \leq |\sigma|+1} X_j(\sigma),$$

tells us how many candidates from the $\sigma + 1$ possible ones, would be hired after processing the permutation σ under the applied strategy.

Then the recurrence of $h(\sigma)$ is translated to the following *PDE*

$$(1 - z) \frac{\partial}{\partial z} H(z, u) - H(z, u) = (u - 1) \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} \quad (2.1)$$

Since each hiring strategy is characterized by its corresponding definition of $X(\sigma)$, then substituting $X(\sigma)$ in the last equation and solving for $H(z, u)$, will give us a closed form of $H(z, u)$.

The index of the last hired candidate

This parameter helps us to study the dynamics of the hiring problem. We have the random variable $L(\sigma)$ that represents the maximum index of the hiring set. So, $L(\sigma)$ denotes the last “time” where a candidate was hired. Then, the *BGF* of $L(\sigma)$ is defined as

$$L(z, u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)},$$

with the following recurrence

$$L(\sigma \circ j) = \begin{cases} L(\sigma), & \text{if the } j\text{th candidate is not hired,} \\ |\sigma| + 1, & \text{otherwise.} \end{cases}$$

And $L(\emptyset) = 0$. Then, we have the following *PDE* of $L(z, u)$

$$(1 - z) \frac{\partial}{\partial z} L(z, u) - L(z, u) = u \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{(zu)^{|\sigma|}}{|\sigma|!} - \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)} , \quad (2.2)$$

The gap of last hired candidate

Let $r(\sigma)$ denote the absolute score (the final score after processing the permutation) of the last hired candidate in a permutation σ , then the *gap* is defined as follows

$$g(\sigma) = 1 - \frac{r(\sigma)}{|\sigma|}.$$

This parameter is an indicator of the strategy quality. It gives the relative distance of the score of the last hired candidate to the maximum possible score. Thus $g(\sigma) = 0$ if the last hired candidate has maximum score and $g(\sigma) = 1 - 1/|\sigma|$ if the last hired candidate has the worst score. Then, the expected value of the gap is

$$\mathbb{E}\{g_n\} = \frac{1}{2n}(\mathbb{E}\{X_n\} - 1), \quad (2.3)$$

where $\mathbb{E}\{X_n\} = [z^n] \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!}$.

2.2.2 The Results

Hiring above the m th best

In this strategy, a candidate j (with relative rank j) is hired if her score is better than the score of the m th best currently hired employee.

So, if the j th candidate is hired, then she will be member of the “elite” of the m th best candidates and the candidate with relative score $|\sigma| + 1 - m$ will exit the elite (but stills in the hiring set). We can understand that this strategy has two phases: the first, before collecting the elite of size m where every candidate will be hired; the second one, when there are only m possible relative ranks to be hired. As a short example let $m = 3$; then the first three candidates will be hired whatever their scores. The 4th candidate will be hired if she has one of the scores $\{2, 3, 4\}$ and discarded if her score is 1.

For analysing the parameters of this strategy, we have first to define the quantity $X(\sigma)$ as mentioned before:

$$X(\sigma) = \begin{cases} |\sigma| + 1, & \text{if } \sigma < m, \\ m, & \text{if } \sigma \geq m. \end{cases} \quad (2.4)$$

In this strategy, there are two cases of m : first, m is fixed value and the general case where m is a function of the number of candidates.

1. The average size of hiring set h_n

(a) For fixed m

Replacing $X(\sigma)$ in equation (2.1) of $H(z, u)$ by its corresponding value in equation (2.4), will give us the following closed form

$$H^{(m)}(z, u) = \frac{1}{(mu - m + 1) \cdot (mu - m) \dots (mu - 1)} \left(\left(\frac{1}{1 - z} \right)^{mu - m + 1} P_m(u, z) + \frac{1}{(1 - z)^m} Q_m(z, u) \right).$$

where $P_m(u, z)$ and $Q_m(u, z)$ are polynomials in z and u .

And the superscript is used to make the dependence on m explicit. Then,

$$h^{(m)}(z) = \sum_{n \leq 0} \mathbb{E}\{h_n^{(m)}\} z^n = \frac{\partial}{\partial u} H^{(m)}(z, u) \Big|_{u=1} = m \frac{\ln\left(\frac{1}{1-z}\right)}{1-z} - \frac{P_m(z)}{1-z},$$

with $P_m(z)$ a polynomial of degree $m - 1$.

Finally,

$$\mathbb{E}\{h_n^{(m)}\} = m \ln n + O(1). \quad (2.5)$$

Hiring above the Best

This strategy is a special case when $m = 1$. So, by setting $m = 1$ in equation (2.5), we obtain the expected size of hiring set

$$\mathbb{E}\{h_n\} = \ln n + O(1). \quad (2.6)$$

We have also the closed form of $H(z, u)$ as

$$H^{(1)}(z, u) = \left(\frac{1}{1-z} \right)^u,$$

This can be obtained simply by setting $m = 1$ in $X(\sigma)$ equation (2.4), and substituting $X(\sigma)$ in equation (2.1).

Thus, the coefficient $[z^n u^k]H^{(1)}(z, u)$ are the Stirling number of the first kind $\left[\begin{matrix} n \\ k \end{matrix} \right]$ which coincides with the number of permutations of size n that have exactly k left-to-right maxima [7]. This is what we are doing in this strategy where we only hire the best candidate seen so far and for any processed permutation, at any step, we will find that only the left-to-right maxima ranks are hired.

(b) For variable m

Here, the size of the elite m is function of n . Then, the expected size of the hiring set is variable with m . To reflect this dependency, the *BGF* will take this form

$$\mathbf{H}(z, u, v) = \sum_{m \geq 1} v^m H^{(m)}(z, u),$$

Now, the *PDE* of the average size of the hiring set is

$$(1-z) \frac{\partial}{\partial z} \mathbf{H}(z, u, v) - \mathbf{H}(z, u, v) - (u-1)v \frac{\partial}{\partial z} \mathbf{H}(z, u, v) = (1-u) \frac{v^2}{(1-v)^2} \ln \left(\frac{1}{1-zuv} \right).$$

Similarly, differentiating w.r.t. u and setting $u = 1$, we have an *ODE* for $\mathbf{h}(z, v)$

$$(1-z) \frac{\partial}{\partial z} \mathbf{h}(z, v) - \mathbf{h}(z, v) = \left(\frac{v}{1-v} \right)^2 \left(\frac{1}{1-zv} \right),$$

Since $\mathbf{H}(z, 1, v) = \frac{v}{(1-z)(1-v)}$, then the solution is

$$\mathbf{h}(z, v) = \frac{v \ln \frac{1}{1-z}}{(1-z)(1-v)^2} - \frac{v \ln \frac{1}{1-zv}}{(1-z)(1-v)^2}$$

Using the initial condition $\mathbf{h}(0, v) = 0$. The quantity we are looking for is

$$[z^n v^m] \mathbf{h}(z, v) = \mathbb{E}\left\{ h_n^{(m)} \right\} = m(H_n - H_m + 1), \text{ for } m \leq n.$$

So, for any m ,

$$\mathbb{E}\left\{ h_n^{(m)} \right\} \sim m \ln \left(\frac{n}{m} \right) + m + O(1), \text{ for } n \rightarrow \infty. \quad (2.7)$$

2. The average index of last hired candidate L_n

(a) For fixed m

In equation (2.2) of this parameter, substituting $X(\sigma)$ by its value in equation (2.4) resulting in the following *PDE*

$$(1-z)\frac{\partial}{\partial z}L^{(m)}(z,u) = \frac{m - muz - (u-1)(u^m z^m - 1)}{(1-uz)^2} + (1-m)L^{(m)}(z,u)$$

Following the solution of this equation in [4], Then

$$\mathbb{E}\{L_n^{(m)}\} = \begin{cases} \frac{m(n+1)}{m+1}, & \text{if } m \leq n \\ n, & \text{if } m > n \end{cases} \quad (2.8)$$

For $m = 1$, Hiring above the best strategy is expected to lastly hire the candidate with index $(n+1)/2$ which makes no surprise because the random variable $L_n^{(m)}$ is uniformly distributed in $\{1, \dots, n\}$.

(b) For arbitrary m

Again, since m varies, then the *BGF* of $L_n^{(m)}$ is defined as

$$\mathbb{L}(z, u, v) := \sum_{m \leq 1} v^m L^{(m)}(z, u),$$

Then, we are looking for this quantity

$$l(z, v) := \frac{\partial}{\partial u} \mathbb{L}(z, u, v) \Big|_{u=1}.$$

Following the derivations in [4], the explicit form of $l(z, v)$ is

$$l(z, v) = \frac{1-v+ vz}{(1-v)(1-z)^2} - \frac{1}{v(1-z)^2} \log \left(\frac{1}{1-vz} \right).$$

The $[z^n v^m]l(z, v)$ coefficients give the solution

$$\mathbb{E}\{L_n^{(m)}\} = \begin{cases} \frac{m(n+1)}{m+1}, & \text{if } m \leq n \\ n, & \text{if } m > n \end{cases} \quad (2.9)$$

3. The average gap g_n

Since $\mathbb{E}\{X_n\} = m$ if $n \geq m$, then

$$\mathbb{E}\{g_n^{(m)}\} = \frac{m-1}{2n}, \text{ if } n \geq m. \quad (2.10)$$

For $m = 1$: Hiring above the best strategy hires -all the time- the best candidate seen so far. So, one can say that this strategy is *optimal* in quality which is translated by having the expected gap is 0. On the other hand the hiring set of this strategy is the smallest between all the strategies.

4. **The probability distribution of h_n for fixed m**

This is one useful result which can be obtained simply from the explicit form of $H^{(m)}(z, u)$. The probability generating function of the random variable h_n is obtained as follows

$$\mathbb{E}\{u^{h_n^{(m)}}\} = [z^n]H^{(m)}(z, u) \sim A_m(u) \cdot n^{m(u-1)} \cdot \left(1 + O\left(\frac{1}{n}\right)\right),$$

with mean $\mathbb{E}\{h_n^{(m)}\} \sim m \ln n$, and the variance is of the same order. Then, by applying Hwang's quasi-power theorem [5], we have

$$\frac{h_n^{(m)} - m \ln n}{\sqrt{m \ln n}} \xrightarrow{d} \mathcal{N}(0, 1). \quad (2.11)$$

Thus $h_n^{(m)}$ converges to a normal distribution.

Chapter 3

Theoretical Work

3.1 New parameters

In this section we introduce new parameters to obtain more information about the quality and the dynamics of hiring strategies. The analysis of the new parameters follows the general framework of section (2.2).

The distance between last two hirings

This parameter helps (with L_n) to study the dynamics of the hiring problem. The distance between the last two consecutive hirings will be denoted by $\Delta(\sigma)$. The *BGF* of $\Delta(\sigma)$ is defined as

$$\Delta(z, u) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{\Delta(\sigma)}.$$

Now if the j th candidate is hired, then the index of last hired candidate will be $|\sigma| + 1$. Since $L(\sigma)$ is the index of last hired candidate in the permutation σ , then $\Delta(\sigma \circ j) = |\sigma| + 1 - L(\sigma)$. But if the j th candidate is discarded, then $\Delta(\sigma \circ j)$ is the same as $\Delta(\sigma)$. Thus we have the following recurrence

$$\Delta(\sigma \circ j) = \begin{cases} \Delta(\sigma), & \text{if the } j\text{th candidate is not hired,} \\ |\sigma| + 1 - L(\sigma), & \text{otherwise.} \end{cases}$$

with $\Delta(\emptyset) = 0$, and $\Delta(\sigma) = i$ if $\mathcal{H}(\sigma) = \{i\}$. Then, we obtain the *PDE* of $\Delta(z, u)$ [4]

$$(1 - z) \frac{\partial}{\partial z} \Delta(z, u) - \Delta(z, u) = u \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{(zu)^{|\sigma|}}{|\sigma|!} u^{-L(\sigma)} - \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{\Delta(\sigma)} \quad (3.1)$$

The size of the firing set

Here we handle a very important extension of the hiring problem which is *hiring with replacement* or *hiring and firing*. This extension violates the 5th condition of the problem statement which states that “Decisions are irrevocable”. Now the hiring process is working as follows: the new candidate j has three possibilities: 1) She will be hired directly by applying the used strategy. 2) She will be hired because her rank is better than the rank of the worst hired candidate; she replaces that worst one. 3) She will be discarded because her rank is worse than the rank of the worst. The goal of hiring with replacement is to have the ultimate best quality at all. Notice that the size of the hiring set does not change when firing takes place, one candidate replaces the worst one, then the size of the hiring set is the same as if hiring without firing. Thus, we

have a new parameter f_n for the average number of firings made if using a given strategy. The random variable f_n gives another indicator of the quality of a specific strategy where a good strategy should not make a lot of firings.

For analyzing the parameter f_n , we introduce the following trivariate generating function

$$F(z, u, v) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{f(\sigma)} v^{h(\sigma)},$$

And the quantity which we are looking for is

$$\mathbb{E}\{f_n\} = [z^n]f(z) = [z^n] \sum_{\sigma \in \mathcal{P}} f(\sigma) \frac{z^{|\sigma|}}{|\sigma|!}.$$

From the previous discussion, $f(\sigma)$ has the recurrence

$$f(\sigma \circ j) = \begin{cases} f(\sigma), & \text{if } 1 \leq j \leq |\sigma| + 1 - h(\sigma), \text{ “}j \text{ is discarded”} \\ f(\sigma) + 1, & \text{if } |\sigma| + 2 - h(\sigma) \leq j \leq |\sigma| + 1 - X(\sigma), \text{ “}j \text{ replaces the worst”} \\ f(\sigma), & \text{if } |\sigma| + 2 - X(\sigma) \leq j \leq |\sigma| + 1, \text{ “}j \text{ is hired”} \end{cases}$$

Then the PDE of $F(z, u, v)$ takes the following form

$$(1 - z) \frac{\partial}{\partial z} F(z, u, v) - F(z, u, v) = (u - v) \left(\frac{\partial}{\partial v} F(z, u, v) - \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{f(\sigma)} v^{h(\sigma)} \right).$$

Following the derivations in [4], we have

$$(1 - z) \frac{d}{dz} f(z) - f(z) = h(z) - \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!}, \quad (3.2)$$

Where

$$h(z) = \frac{\partial}{\partial v} H(z, v) \Big|_{v=1}$$

is the generating function for the expected values h_n , and the initial condition is $f(0) = 0$.

The score of the best not hired candidate

We introduce this parameter to know more information about some particular region of the ranks. For any *pragmatic* hiring strategy and any permutation σ , $\mathcal{H}(\sigma)$ contains at least the $X(\sigma)$ best candidates of σ [2], that is, the candidates with scores $|\sigma|, |\sigma| - 1, \dots, |\sigma| + 1 - X(\sigma)$. And the region of the ranks that are less than the worst rank will be discarded in any strategy. Thus, the region of ranks that starts from 1 to $|\sigma| - X(\sigma)$ will contain some hired ranks and others discarded. The random variable $M(\sigma)$ will denote the score of the best not hired candidate. So that, $1 \leq M(\sigma) \leq |\sigma| - X(\sigma)$ in hiring without replacement; and $M(\sigma) = |\sigma| - h(\sigma)$ in hiring with replacement. We introduce the following trivariate generating function

$$M(z, u, v) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{h(\sigma)} v^{M(\sigma)},$$

Where we are looking for

$$\mathbb{E}\{M_n\} = [z^n]m(z) = [z^n] \sum_{\sigma \in \mathcal{P}} m(\sigma) \frac{z^{|\sigma|}}{|\sigma|!}.$$

The recurrence of $M(\sigma)$ is

$$M(\sigma \circ j) = \begin{cases} M(\sigma) + 1, & \text{if } j \leq M(\sigma), \text{ “}j \text{ is discarded”} \\ M(\sigma), & \text{if } j > |\sigma| + 1 - X(\sigma), \text{ “}j \text{ is hired”} \\ j, & \text{if } M(\sigma) < j \leq |\sigma| + 1 - X(\sigma), \text{ “}j \text{ is discarded”} \end{cases}$$

We use this recurrence to obtain a *PDE* for $M(z, u, v)$ as follows

$$\frac{\partial M}{\partial z} - u^2 \frac{\partial M}{\partial u} + \frac{u}{u-1} M(z, u, v) = v \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{M(\sigma)} v^{h(\sigma)} + \frac{u^2}{u-1} \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{(zu)^{|\sigma|}}{|\sigma|!} u^{-X(\sigma)} v^{h(\sigma)}. \quad (3.3)$$

Appendix A contains the complete derivation of this equation.

3.2 Hiring above the P% quantile

In this strategy, we have a special case strategy which is **Hiring Above The Median**. It is more reasonable to start with stating this special case before analysing the general strategy. Since the strategy will hire only the candidates that have scores above the median hired score, then there are two cases of the size of hiring set $h(\sigma)$: the first, $h(\sigma) = 2k$ then there are two median ranks and selecting the lower one, there are $k + 1$ possible ranks to be hired (the best k hired and $|\sigma| + 1$). The other case: $h(\sigma) = 2k + 1$ then there is just one median value and there will be also $k + 1$ possible ranks to be hired. So that, for hiring above the median: $X(\sigma) = \lceil (h(\sigma) + 1)/2 \rceil$. This ceiling has two possible outcomes as follows

$$X(\sigma) = \left\lceil \frac{h(\sigma) + 1}{2} \right\rceil = \begin{cases} \frac{h(\sigma)+1}{2}, & \text{if } \frac{h(\sigma)+1}{2} \text{ is integer,} \\ \frac{h(\sigma)}{2} + 1, & \text{otherwise.} \end{cases} \quad (3.4)$$

Let $P = 1 - a$. We can say that hiring above the $(1 - a)h(\sigma)$, with $0 < a < 1$, can be analyzed in the same way, with $X(\sigma) = \lceil a \cdot (h(\sigma) + 1) \rceil$. In general, we will study lower and upper bounds of the form $X(\sigma) = a \cdot h(\sigma) + b$ and $0 < a, b \leq 1$ for $\sigma \neq \emptyset$, $X(\emptyset) = 1$. It is worth mentioning that one can not set both a and b to 1 because this will give a non-natural value of $X(\sigma)$ which is $h(\sigma) + 1$, and this means that the resulting strategy will hire every candidate interviewed (of course it will be the optimal in the number of hirings and the minimal in the quality). But we can say that the *lower bound* strategy in this type is when setting a to 1 and b to 0, So that the strategy will hire $h(\sigma)$ possible ranks which are $|\sigma| + 1$ and all the hired ranks except the worst one, thus this strategy is called **Hiring Above The Worst**.

Before introducing the obtained results, it is worth to mention that *MAPLE 12* is used to obtain the solutions for the differential equations and any required intermediate mathematical derivations.

1. The average size of hiring set h_n

Now, returning to equation (2.1) of the expected size of hiring set and replacing $X(\sigma)$ by

$ah(\sigma) + b$, then we have the following *PDE*

$$(1-z)\frac{\partial}{\partial z}H(z,u) - au(u-1)\frac{\partial}{\partial u}H(z,u) - (1+b(u-1))H(z,u) = (1-b)(u-1) \quad (3.5)$$

It is difficult to obtain a closed form of $H(z,u)$ from the last equation, so we will go directly to the next step by differentiating w.r.t. u and setting $u = 1$, then

$$(1-z)\frac{\partial}{\partial z}h(z) - (1+a)h(z) - \frac{b}{1-z} = 1-b, \quad (3.6)$$

Remember that $h(z) = \frac{\partial}{\partial u}H(z,u)\Big|_{u=1} = \sum_{\sigma \in \mathcal{P}} h(\sigma) \frac{z^{|\sigma|}}{|\sigma|!}$.

And the solution for the *ODE* (3.6) will be

$$h(z) = \frac{-1}{(1-z)^{a+1}} \left(\frac{(1-z)^a (az(b-1) + a+b)}{a(a+1)} + C \right), \quad (3.7)$$

Where C is a constant and $\mathbb{E}\{h_n\} = [z^n]h(z)$.

2. The average gap g_n

Returning to equation (2.3) of $\mathbb{E}\{g_n\}$, we have first to compute $\mathbb{E}\{X_n\}$ as follows

$$\begin{aligned} \mathbb{E}\{X_n\} &= [z^n] \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} \\ &= [z^n] \sum_{\sigma \in \mathcal{P}} (ah(\sigma) + b) \frac{z^{|\sigma|}}{|\sigma|!} \\ &= a \cdot [z^n] \sum_{\sigma \in \mathcal{P}} h(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} + b \cdot [z^n] \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} \\ &= a \cdot \mathbb{E}\{h_n\} + b. \end{aligned}$$

Then equation (2.3) turns to

$$\mathbb{E}\{g_n\} = \frac{1}{2n} (a \cdot \mathbb{E}\{h_n\} + b - 1). \quad (3.8)$$

3. The average number of firings f_n

We have $X(\sigma) = ah(\sigma) + b$, then substituting in equation (3.2)

$$(1-z)\frac{d}{dz}f(z) - f(z) = (1-a)h(z) - \frac{bz}{1-z}. \quad (3.9)$$

The solution for this equation is valid for $n \geq 1$ and for the empty set, we have $f(0) = 0$. Then, by defining a and b , we have $X(\sigma)$ and the corresponding $h(z)$, plugging them in the last equation resulting in an *ODE* of $f(z)$.

4. The average index of last hired candidate L_n

Starting from equation (2.2), replacing $X(\sigma)$ with $a \cdot h(\sigma) + b$ resulting in

$$(1-z)\frac{\partial L}{\partial z} - L(z,u) = u \sum_{\sigma \in \mathcal{P}} (ah(\sigma) + b) \frac{(zu)^{|\sigma|}}{|\sigma|!} - \sum_{\sigma \in \mathcal{P}} (ah(\sigma) + b) \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)},$$

Then in the right hand side we will find this term $\sum_{\sigma \in \mathcal{P}} h(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)}$ which can not be reduced to a known quantity (like $\frac{\partial L}{\partial u}$, $\frac{\partial H}{\partial u}$, ...) in order to have a *PDE* in $L(z, u)$ only. To overcome this situation, we introduce a trivariate generating function as

$$L(z, u, v) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)} v^{h(\sigma)},$$

Then we have the following *PDE*

$$(1-z) \frac{\partial l}{\partial z} - (1-b)l(z, v) + av \frac{\partial l}{\partial v} = av^2 \left(\frac{\partial}{\partial v} H(z, v) + \frac{\partial^2}{\partial z \partial v} H(z, v) \right) + bv \left(H(z, v) + z \frac{\partial}{\partial z} H(z, v) \right) \quad (3.10)$$

where $l(z, v) = \frac{\partial L}{\partial u} \Big|_{u=1} = \sum_{\sigma \in \mathcal{P}} L(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} v^{h(\sigma)}$ and $H(z, v) = \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} v^{h(\sigma)}$

Appendix A contains the complete derivation of this equation.

3.2.1 Hiring above the median

1. The average size of hiring set h_n

The lower bound: $X(\sigma) = (h(\sigma) + 1)/2$:

Then, set $a = 1/2$ and $b = 1/2$ in the solution equation (3.7), we have

$$h(z) = \frac{-4}{3(1-z)} + \frac{z}{3(1-z)} + \frac{C}{(1-z)^{3/2}},$$

Since the hiring set for the empty set is also empty, then the initial condition is $h(0) = 0$, so that the constant $C = 4/3$. Then

$$h(z) = \frac{-4}{3(1-z)} + \frac{z}{3(1-z)} + \frac{4}{3(1-z)^{3/2}}, \quad (3.11)$$

There is a correspondance between the asymptotic expansion of a function near its dominant singularities and the asymptotic expansion of the function's coefficient [5]. In equation (3.11), $h(z)$ has a singularity at $z = 1$ and the dominant function is $\frac{4}{3(1-z)^{3/2}}$ so that we have the coefficients of $h(z)$ as follows

$$[z^n]h(z) = [z^n](4/3(1-z)^{-3/2}) \sim \frac{4}{3} \frac{n^{3/2-1}}{\Gamma(3/2)} \left(1 + O\left(\frac{1}{n}\right) \right)$$

Finally,

$$\mathbb{E}\{h_n^{(l.b.)}\} = [z^n]h^{(l.b.)}(z) = \frac{8\sqrt{n}}{3\sqrt{\pi}} \left(1 + O\left(\frac{1}{n}\right) \right) \quad (3.12)$$

Where $(l.b.)$ refers to the lower bound case to be distinguished from the other case.

The upper bound: $X(\sigma) = (1/2)h(\sigma) + 1$:

Then, set $a = 1/2$ and $b = 1$ in the solution equation (3.7), we have:

$$h(z) = \frac{C}{(1-z)^{3/2}} - \frac{2}{(1-z)},$$

Where C is a constant. Similar to the previous analysis, we have

$$h(z) = \frac{2}{(1-z)^{3/2}} - \frac{2}{(1-z)}, \quad (3.13)$$

Then

$$\mathbb{E}\{h_n^{(u.b.)}\} = [z^n]h^{(u.p.)}(z) = \frac{4\sqrt{n}}{\sqrt{\pi}} \left(1 + O\left(\frac{1}{n}\right)\right) \quad (3.14)$$

Also $(u.p)$ refers to the upper bound case.

In this particular case, we were lucky to obtain a closed form of $H(z, u)$ by setting $a = 1/2$ and $b = 1$ in equation (3.5). Then solving that *PDE* gives

$$H(z, u) = \frac{F\left(\frac{1-u}{u\sqrt{1-z}}\right)}{(1-z)u^2},$$

where F is an arbitrary function. To find the absolute form of this function, we have to use the initial conditions $H(0, u) = 1$ and $H(z, 1) = 1/(1-z)$. Then

$$F\left(\frac{1-u}{u\sqrt{1-z}}\right) = \left(\frac{1}{1 + \frac{1-u}{u\sqrt{1-z}}}\right)$$

And doing small check, this function satisfies the initial conditions. Hence,

$$H(z, u) = \frac{1}{(1-u + u\sqrt{1-z})^2}.$$

If we follow the same way of derivation to obtain $[z^n]h^{(u.b.)}(z)$, then we have a typical result to equation (3.14). On the other hand, the closed form of $H(z, u)$ will be useful to obtain the probability distribution of the hiring set.

Finally, in general

$$\mathbb{E}\{h_n\} = \Theta(\sqrt{n}). \quad (3.15)$$

2. The average gap g_n

The lower bound: $X(\sigma) = (h(\sigma) + 1)/2$:

From equation (3.12) we can say that $\mathbb{E}\{h_n\} \sim (8/(3\sqrt{\pi})) \cdot \sqrt{n}$, Then substituting in equation (3.8)

$$\mathbb{E}\{g_n^{(l.b.)}\} \sim \frac{1}{2n} \left(\frac{4}{3\sqrt{\pi}} \cdot \sqrt{n} - 1\right),$$

Then

$$\mathbb{E}\{g_n^{(l.b.)}\} \sim \frac{2}{3\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}}. \quad (3.16)$$

The upper bound: $X(\sigma) = (1/2)h(\sigma) + 1$:

Again from equation (3.14), we have $\mathbb{E}\{h_n\} \sim (4/\sqrt{\pi}) \cdot \sqrt{n}$, Then substituting in equation (3.8)

$$\mathbb{E}\{g_n^{(u.b.)}\} \sim \frac{1}{2n} \left(\frac{2}{\sqrt{\pi}} \cdot \sqrt{n} - 1\right),$$

Then

$$\mathbb{E}\{g_n^{(u.b.)}\} \sim \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}}. \quad (3.17)$$

Thus, it is clear that in general :

$$\mathbb{E}\{g_n\} = \Theta(1/\sqrt{n}). \quad (3.18)$$

3. The average size of firing set f_n

The Lower bound: we have $a = 1/2$, $b = 1/2$ and $h(z)$ as in equation (3.12). Using these values in equation (3.9), we get

$$(1-z) \frac{d}{dz} f(z) = \frac{1}{2} \left[\frac{z-4}{3(1-z)} + \frac{4}{3(1-z)^{3/2}} \right] - \frac{z}{2(1-z)},$$

The solution is

$$f(z) = \frac{4}{3(1-z)^{3/2}} - \frac{1}{1-z} \ln \frac{1}{1-z} + \frac{C}{1-z},$$

Since $f(0) = 0$, then $C = -4/3$. The coefficients of $[z^n]f(z)$ give the solution

$$\mathbb{E}\{f_n^{(l.b.)}\} = \frac{8\sqrt{n}}{3\sqrt{\pi}} - H_n - \frac{4}{3}$$

Where $H_n = \sum_{1 \leq k \leq n} (1/k)$ denotes the n th harmonic number, $H_n \sim \ln n$ [7] Finally

$$\mathbb{E}\{f_n^{(l.b.)}\} \sim \frac{8\sqrt{n}}{3\sqrt{\pi}} - \ln n. \tag{3.19}$$

The upper bound: we have $a = 1/2$, $b = 1$ and $h(z)$ as in equation (3.14). Using these values in equation (3.9), solving for $f(z)$, we get

$$f(z) = \frac{2}{(1-z)^{3/2}} - \frac{2}{1-z} \ln \frac{1}{1-z} + \frac{2-z}{1-z}.$$

Then

$$\mathbb{E}\{f_n^{(u.p.)}\} \sim \frac{4\sqrt{n}}{\sqrt{\pi}} - 2 \ln n. \tag{3.20}$$

And in general

$$\mathbb{E}\{f_n\} = \Theta(\sqrt{n} - \ln n). \tag{3.21}$$

3.2.2 Hiring above the worst

This strategy may be not natural (practical) but it represents a special case of the general strategy so that it helps to check that the used tools are working well, also to compare it with other strategies in order to see the difference in quality and hiring time. On the other side, this strategy was mentioned in the paper of Broder et. al [1] as *hiring above the mimimum* and they comapred it with hiring above the mean.

1. The average size of hiring set h_n

In this strategy we have $X(\sigma) = h(\sigma)$ -as explained above- so that we set $a = 1$ and $b = 0$ in equation (3.7) to have the following closed form

$$h(z) = \frac{-1}{2(1-z)^2} + \frac{z}{(1-z)^2} - \frac{z^2}{(1-z)^2} + \frac{C}{(1-z)^2},$$

Applying the initial condition $h(0) = 0$, the constant $C = 1/2$, Then

$$\mathbb{E}\{h_n\} = \frac{n+1}{2}. \tag{3.22}$$

This result also should not make surprise because for a permutation of length n , the size of hiring set is uniformly distributed on the set $\{1, \dots, n\}$.

2. The average gap g_n

From equation (3.8) of $\mathbb{E}\{g_n\}$, replacing $\mathbb{E}\{h_n\}$ by its value in equation (3.22), we have directly

$$\begin{aligned}\mathbb{E}\{g_n\} &= \frac{1}{2n} \left(\frac{n+1}{2} - 1 \right), \\ \mathbb{E}\{g_n\} &= \frac{1}{4} - \frac{1}{4n} = \frac{1}{4} - o\left(\frac{1}{n}\right).\end{aligned}\tag{3.23}$$

Thus, we can say that $\mathbb{E}\{g_n\} \sim 1/4$ which is fixed for any n and reflects the low quality of this strategy (in deed this quality is the worst!) but it hires more people and is expected to hire people rapidly which can be tested by analyzing the Δ_n parameter.

3. The average size of firing set f_n

No firing in this strategy, because every candidate better than the worst will be hired, otherwise she is discarded. From the equations point of view; setting $a = 1$ and $b = 0$ in equation (3.9) gives the solution $f(z) = C/(1-z)$. Since $f(0) = 0$ then $C = 0$ and $f(z) = 0$. Hence, this strategy is optimal according to the number of firings measure.

3.2.3 The general case

1. The average size of hiring set $\mathbb{E}\{h_n\}$

In the first discussion of this chapter, we said that, in general $X(\sigma) = [a(h(\sigma) + 1)]$. Then equation (3.4) becomes

$$X(\sigma) = \left[a \cdot (h(\sigma) + 1) \right] = \begin{cases} ah(\sigma) + a, & \text{if } a(h(\sigma) + 1) \text{ is integer,} \\ ah(\sigma) + 1, & \text{otherwise.} \end{cases}$$

Let $X(\sigma) = ah(\sigma) + b$ with $b = a$ for lower bound and $b = 1$ for upper bound. Now, we drive a general case for $h(z)$, from equation (3.7)

$$h(z) = \frac{-1}{(1-z)^{a+1}} \cdot \frac{(1-z)^a (az(b-1) + b+a)}{a(a+1)} + \frac{C}{(1-z)^{a+1}}$$

with $h(0) = 0$, $C = \frac{b+a}{a(a+1)}$, then

$$h(z) = \frac{b+a}{a(a+1)} \cdot \frac{1}{(1-z)^{a+1}} - \frac{b+a+az(b-1)}{a(a+1)} \cdot \frac{1}{1-z},\tag{3.24}$$

Then form the singularity analysis -as in section (3.2.1)- we have

$$\begin{aligned}\mathbb{E}\{h_n\} &= [z^n]h(z) \sim \frac{n^a}{\Gamma(a+1)} \cdot \frac{b+a}{a(a+1)} \cdot \left(1 + O\left(\frac{1}{n}\right) \right) \\ &\sim \frac{n^a}{\Gamma(a+1)} \cdot \frac{b+a}{a(a+1)} + O\left(\frac{1}{n^{1-a}}\right),\end{aligned}$$

Then asymptotically,

$$\mathbb{E}\{h_n\} \sim \frac{n^a}{\Gamma(a+1)} \cdot \frac{b+a}{a(a+1)}.\tag{3.25}$$

The lower bound: in the last equation replacing b by a , we have

$$\mathbb{E}\{h_n^{(l.b.)}\} \sim \frac{n^a}{\Gamma(a+1)} \cdot \frac{2a}{a(a+1)},$$

Replacing a by $1 - P$

$$\mathbb{E}\{h_n^{(l.b.)}\} \sim \frac{2}{2 - P} \cdot \frac{n^{(1-P)}}{\Gamma(2 - P)}. \quad (3.26)$$

Doing the same, setting $b = 1$ in equation (3.25), we have the upper bound

$$\mathbb{E}\{h_n^{(u.b.)}\} \sim \frac{1}{1 - P} \cdot \frac{n^{(1-P)}}{\Gamma(2 - P)}. \quad (3.27)$$

So that for hiring above the $P\%$ quantile: $\mathbb{E}\{h_n\} = \Theta(n^{1-P})$.

Finally, the relation between upper and lower bounds is

$$\mathbb{E}\{h_n^{(u.b.)}\} \sim \mathbb{E}\{h_n^{(l.b.)}\} \cdot \frac{2 - P}{2(1 - P)}.$$

For hiring above the median ($P = 0.5$), the ration between upper and lower bounds of the average size of hiring set is

$$\frac{\mathbb{E}\{h_n^{(u.b.)}\}}{\mathbb{E}\{h_n^{(l.b.)}\}} \sim \frac{3}{2}.$$

2. The average gap $\mathbb{E}\{g_n\}$

Recalling equation (3.8) of $\mathbb{E}\{g_n\}$, replacing $\mathbb{E}\{h_n\}$ by its value in equation (3.25), then

$$\mathbb{E}\{g_n\} \sim \frac{1}{2n} \left(a \cdot \frac{n^a}{\Gamma(a + 1)} \cdot \frac{b + a}{a(a + 1)} + b - 1 \right),$$

Lower bound: $a = 1/2$ and $b = 1/2$:

$$\mathbb{E}\{g_n^{(l.b.)}\} \sim \frac{1}{2n} \left(\frac{2}{a + 1} \cdot \frac{n^a}{\Gamma(a + 1)} \cdot \frac{b + a}{a(a + 1)} + a - 1 \right),$$

Replacing a by $1 - P$

$$\mathbb{E}\{g_n^{(l.b.)}\} \sim \frac{1}{2 - P} \cdot \frac{n^{-P}}{\Gamma(2 - P)}. \quad (3.28)$$

Upper bound: $a = 1/2$ and $b = 1$:

$$\mathbb{E}\{g_n^{(u.b.)}\} \sim \frac{1}{2n} \left(\frac{1 + a}{a(a + 1)} \cdot \frac{n^a}{\Gamma(a + 1)} \right),$$

Replacing a by $1 - P$

$$\mathbb{E}\{g_n^{(u.b.)}\} \sim \frac{1}{2(1 - P)} \cdot \frac{n^{-P}}{\Gamma(2 - P)}. \quad (3.29)$$

The ratio between upper and lower bounds is

$$\mathbb{E}\{g_n^{(u.b.)}\} \sim \mathbb{E}\{g_n^{(l.b.)}\} \cdot \frac{2 - P}{2(1 - P)}.$$

Which is the same ratio -after approximation- of upper and lower bounds for the average size of hiring set; because approximately $\mathbb{E}\{g_n\}$ is inversely proportional to $\mathbb{E}\{h_n\}$ (equation (3.8)). Also, for hiring above the median ($P = 0.5$)

$$\frac{\mathbb{E}\{g_n^{(u.b.)}\}}{\mathbb{E}\{g_n^{(l.b.)}\}} \sim \frac{3}{2}.$$

In general, for hiring above the $P\%$ quantile: $\mathbb{E}\{g_n\} = \Theta(1/n^P)$.

3. The average size of firing set $\mathbb{E}\{f_n\}$

Doing the same as above, $X(\sigma) = ah(\sigma) + b$ and we have the general case of $h(z)$ in equation (3.24), then equation (3.2) becomes

$$(1-z)\frac{d}{dz}f(z) - f(z) = (1-a)\left(\frac{b+a}{a(a+1)(1-z)^{a+1}} - \frac{b+a+az(b-1)}{a(a+1)(1-z)}\right) - \frac{bz}{1-z},$$

Solving this equation gives us the following bounds,

Lower bound: set $b=a$:

$$\mathbb{E}\{f_n^{(l.b.)}\} \sim \frac{2(1-a)}{a(a+1)} \cdot \frac{n^a}{\Gamma(a+1)} - \ln n,$$

Replacing a by $1-P$

$$\mathbb{E}\{f_n^{(l.b.)}\} \sim \frac{2P}{(1-P)(2-P)} \cdot \frac{n^{1-P}}{\Gamma(2-P)} - \ln n. \quad (3.30)$$

Upper bound: set $b=1$:

$$\mathbb{E}\{f_n^{(u.b.)}\} \sim \frac{1-a^2}{a^2(a+1)} \cdot \frac{n^a}{\Gamma(a+1)} - \frac{1}{a} \cdot \ln n,$$

Replacing a by $1-P$

$$\mathbb{E}\{f_n^{(u.b.)}\} \sim \frac{P}{(1-P)^2} \cdot \frac{n^{1-P}}{\Gamma(2-P)} - \frac{1}{1-p} \cdot \ln n. \quad (3.31)$$

So that, for hiring above the $P\%$ quantile: $\mathbb{E}\{f_n\} = \Theta(n^{1-P} - \ln n)$.

3.3 The average size of firing set for hiring above the best

We have equation (3.2) of $f(z)$ as follows

$$(1-z)\frac{d}{dz}f(z) - f(z) = h(z) - \sum_{\sigma \in \mathcal{P}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!},$$

Since $H(z, u) = \left(\frac{1}{1-z}\right)^u$ for hiring above the best, then

$$h(z) = \frac{\partial}{\partial u} H(z, u) \Big|_{u=1} = \frac{1}{1-z} \ln \frac{1}{1-z},$$

And $X(\sigma) = 1$, then $f(z)$ equation takes this form

$$(1-z)\frac{d}{dz}f(z) - f(z) = \frac{1}{1-z} \ln\left(\frac{1}{1-z}\right) - \frac{z}{1-z},$$

The solution is

$$f(z) = \frac{1}{2(1-z)} \ln\left(\frac{1}{1-z}\right)^2 - \frac{1}{1-z} \ln \frac{1}{1-z} - \frac{z+C}{1-z},$$

Where C is a constant. From singularity analysis [5], we have the n th coefficient as follows

$$\mathbb{E}\{f_n\} = [z^n]f(z) = \frac{1}{2}(\ln^2 n - \ln n) + O(1). \quad (3.32)$$

3.4 Randomized hiring strategies

All the strategies discussed so far are *deterministic*. That means that there is a fixed rule to decide whether a newcomer would be hired or not. In this section, we try to investigate what happens when we consider some randomness within the hiring process. The randomization idea is useful and practical because for a large hiring set, it may be difficult to collect information about the whole set, but if the decision maker picks a sample of hired candidates at random to collect representative information about the full staff, then it is more easier and faster to take the next decision. In a *randomized hiring strategy*, the scenario is: after processing some sequence of candidates, a sample (committee) of the currently hired staff is randomly chosen and the decision to hire or discard is based upon an ordinary (deterministic) rank-based strategy with respect to the committee rather than with respect to the whole current staff. Thus, we have some parameters to be analysed like the committee size, the cost -in terms of quality- and the gain -in terms of time- of applying a randomized strategy instead of a deterministic one. The *BGF* of $H(z, u)$ takes this form

$$H(z, u) = \sum_{\sigma \in \mathcal{P}} \sum_{k \geq 0} \text{Prob}\{h(\sigma) = k\} \frac{z^{|\sigma|}}{|\sigma|!} u^k = \sum_{n \geq 0, k \geq 0} z^n u^k \text{Prob}\{h_n = k\} ,$$

So that $[z^n u^k]H(z, u) = \text{Prob}\{h_n = k\}$; in other words, $[z^n]H(z, u)$ is the probability generating function of h_n .

By similar derivation to the deterministic case, we have this *PDE*

$$(1 - z) \frac{\partial H}{\partial z} - H(z, u) = (u - 1) \sum_{n \geq 0, k \geq 0} z^n u^k \mathbb{E}\{X_n | h_n = k\} \text{Prob}\{h_n = k\}.$$

We have the expectation of the gap where for any pragmatic randomized hiring strategy,

$$\mathbb{E}\{g_n\} = \frac{1}{2n} (\mathbb{E}\{X_n\} - 1) ,$$

where $\mathbb{E}\{X_n\}$ is the expected number of relative ranks that would be hired right after processing a random permutation of size n .

Chapter 4

Experimental Work

In this chapter we develop a lot of experiments to test the *rank-based* strategies. We consider both deterministic and randomized hiring. For each parameter, we analyse its behaviour in each hiring strategy. For the parameters that are mathematically analyzed, we compare the obtained solutions to the experimental results to verify the theory and the used tools. For other parameters that are difficult to be analyzed mathematically -until now-, we try to investigate approximate solutions from the experimental results.

Experiment setup

The experiment has two main parts: the first part is a code which simulates the hiring process under the mentioned assumptions in subsection (1.3.2) and the rules for each hiring strategy. This code is written in C++ language of Microsoft Visual Studio environment. This C++ code is object-oriented and the project runs as a console application. The mission of this code ends up in writing the numerical values of each parameter for each hiring strategy in an individual text file. The second part is a code on Matlab environment. This Matlab code reads the text files generated by the C++ code, computes the values of the existing mathematical solutions and plots both experimental values and mathematical solutions together.

It is worth mentioning to clarify the method used to generate a random permutation. The standard C++ has the method `rand()` to generate a uniformly distributed random number. `rand()` works well only for small spans; but for large spans, lower numbers are slightly more likely. So that we use another uniform random number generator (URNG) from a C++ class library developed by Agner Fog. [8]. For each permutation, at each step i , we use the URNG to generate uniformly a value between 1 and i . Thus, the formed permutation of length n is totally random with probability $1/n!$ to happen.

4.1 Deterministic hiring

Here, we consider a permutation length from 500 to 50000 with step 500, thus we have 100 permutations of different sizes. For each permutation, we run 100 trials. And as a road map for the following experiments, we introduce the experimental results; followed by comments and conclusions, for the following parameters

1. The average size of the hiring set h_n .
2. The average size of the firing set f_n .
3. The probability distribution of the hiring set.

4. The average gap of the last hired candidate g_n .
5. The average index of the last hired candidate L_n .
6. The average distance between the last two hirings Δ_n .
7. The average score of the best not hired candidates M_n .

With the following hiring strategies

1. Hiring above the median.
2. Hiring above the best.
3. Hiring above the m th best with these values of m :
 - $m = 30$.
 - $m = 0.05n$.
 - $m = \ln n$.
 - $m = \sqrt{n}$.
4. Hiring above the worst.

We consider hiring with replacement only in the case of analysing f_n . For hiring with replacement, the results will be different for all other parameters except for h_n and the probability distribution of hiring set because the hiring set size does not change. Also for M_n , if replacement is considered, its value will always be $n - h_n$ for all strategies.

4.1.1 The average size of hiring set

This is a legend for figure (4.1)

- The **blue points** represent the experimental results for h_n .
- The **blue solid lines** represent the mean value \pm the standard deviation.
- The **red solid lines** represent the mathematical expectation of h_n .

Comments and Conclusion

In figure (4.1a): the **red solid lines** represent equations (3.12) and (3.14) of the following expected lower and upper bounds

$$\mathbb{E}\{h_n^{(l.b.)}\} \sim \frac{8}{3\sqrt{\pi}} \cdot \sqrt{n} \quad , \mathbb{E}\{h_n^{(u.b.)}\} \sim \frac{4}{\sqrt{\pi}} \cdot \sqrt{n} .$$

In figure (4.1b): the **red solid line** represents equation (2.6) of the expectation: $\mathbb{E}\{h_n\} \sim \ln n$.

In figure (4.1c): the **red solid line** represents the expectation: $\mathbb{E}\{h_n\} \sim 30 \ln\left(\frac{n}{30}\right) + 30$ which obtained by substituting m by 30 in equation (2.7).

In figure (4.1d): the **red solid line** represents the expectation: $\mathbb{E}\{h_n\} \sim 0.2n$ which obtained by substituting m by $0.05n$ in equation (2.7).

In figure (4.1e): the **red solid line** represents the expectation: $\mathbb{E}\{h_n\} \sim \sqrt{n}(\ln \sqrt{n} + 1)$ which obtained by substituting m by \sqrt{n} in equation (2.7).

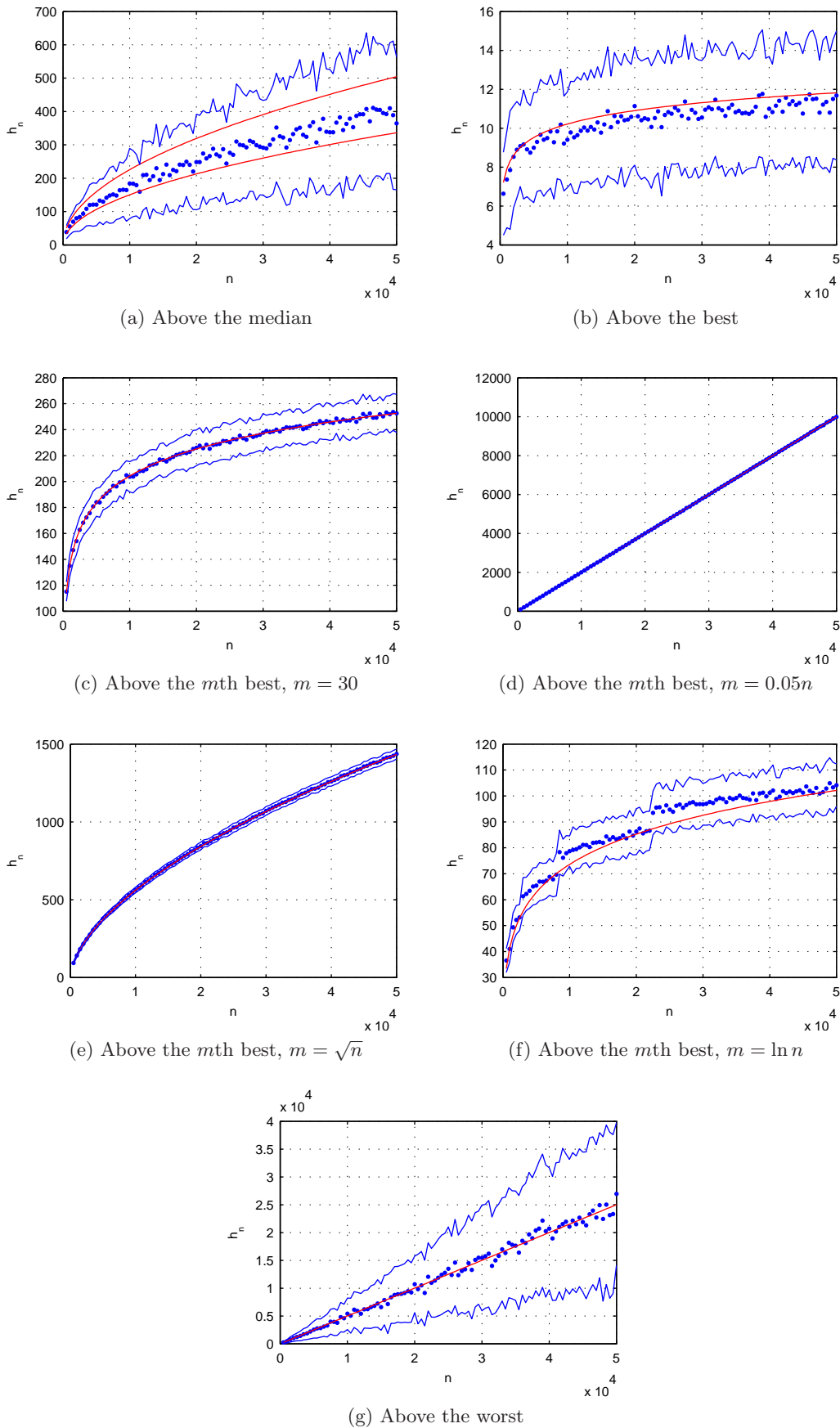


Figure 4.1: The average size of the hiring set.

In figure (4.1f): the **red solid line** represents the expectation: $\mathbb{E}\{h_n\} \sim \ln n(\ln n - \ln \ln n + 1)$ which obtained by substituting m by $\ln n$ in equation (2.7).

In figure (4.1g): the **red solid line** represents equation (3.22) of the following expectation: $\mathbb{E}\{h_n\} = (n + 1)/2$.

Thus, the experimental results fully validate the results of the theoretical derivations.

4.1.2 The average size of firing set

This is a legend for figure (4.2):

- The **blue points** represent the experimental results for f_n .
- The **blue solid lines** represent the mean value \pm the standard deviation.
- The **red solid lines** represent the mathematical expectation of f_n .

Comments and Conclusion

In figure (4.2a): the **red solid lines** represent equations (3.19) and (3.20) of the following expected lower and upper bounds:

$$\mathbb{E}\{f_n^{(l.b.)}\} \sim \frac{8}{3\sqrt{\pi}} \cdot \sqrt{n} - \ln n \quad , \mathbb{E}\{f_n^{(u.b.)}\} \sim \frac{4}{\sqrt{\pi}} \cdot \sqrt{n} - 2 \ln n .$$

In figure (4.2b): the **red solid line** represents equation (3.32) of the expectation: $\mathbb{E}\{f_n\} \sim \frac{1}{2}(\ln^2 n - \ln n)$.

Note that for hiring above the worst strategy, no firings. For this parameter also, the experimental results support the theoretical analysis.

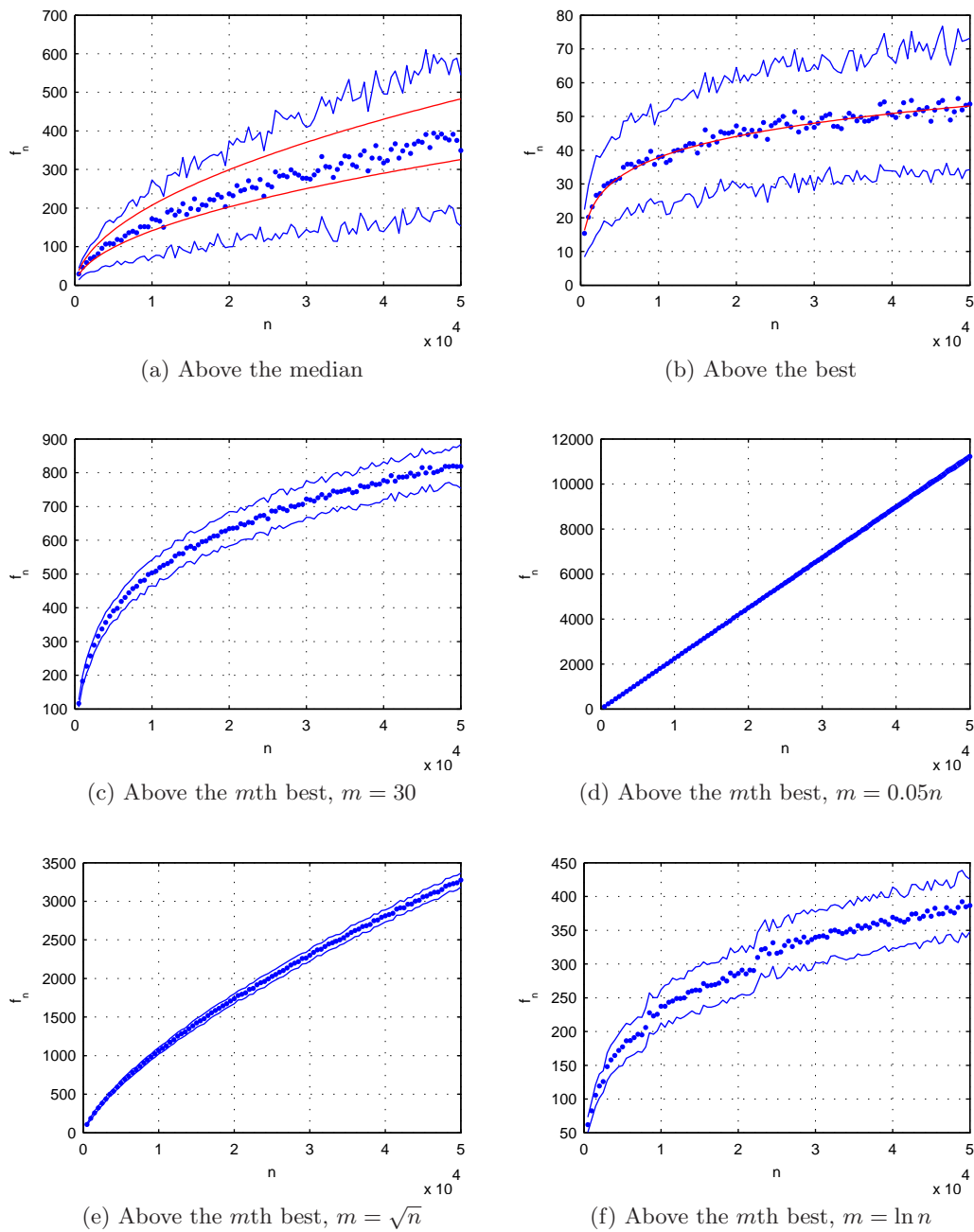


Figure 4.2: The average size of the firing set.

4.1.3 The probability distribution of hiring set

For hiring above the m th best, with fixed m strategy; it is expected that the probability distribution of hiring set tends to a Gaussian distribution with mean $\mathbb{E}\{h_n\} \sim m \ln n + O(1)$ and variance of the same order. For the other strategies, except hiring above the worst, the hypothesis is that the probability distribution of hiring set tends also to $\mathcal{N}(0,1)$. To check the expectation and test the hypothesis we make this experiment: for each n , we compute a normalized random variable \hat{h}_n^* for the size of hiring set as follows

$$\hat{h}_n^* = \frac{\hat{h}_n - \mathbb{E}\{\hat{h}_n\}}{\sqrt{\text{Var}\{\hat{h}_n\}}},$$

where \hat{h}_n represents the observed size of hiring set. By computing the frequencies of each unique value of \hat{h}_n , we obtain an approximation of the probability distribution.

For hiring above the worst, the size of hiring set for a permutation of size n is uniformly distributed in the set $\{1, \dots, n\}$. Since we run 100 trials for each n and n strat from 500 to 50000, then with high probability; each possible hiring set size for any n should appear just once. If we use the above formula to plot \hat{h}_n^* for hiring above the worst strategy, the corresponding probability for each value of \hat{h}_n^* should be 1/100.

In figure (4.3): x -axes represent \hat{h}_n^* for all n and y -axes represent its probability.

Comments and Conclusion

From figure (4.3), for hiring above the m th best strategy with fixed m ; the probability distribution is close to the Gaussian distribution, as expected. The hypothesis for arbitrary m seems also to hold. For hiring above the median, we can say that the probability distribution of hiring set tends to a Gaussian distribution but not as quickly as in hiring above the m th best strategy. For hiring above the worst, it is clear that the probability of h_n is fixed at 0.01; there are only a few cases in which the probability exceeds 0.01. Due to this uniform distribution, we can understand figure (4.1g) where the variance of h_n is very large (the largest among all analysed strategies).

Then, we conclude that all the rank-based strategies considered in the study, except hiring above the worst, collect a hiring set with approximately Gaussian distribution

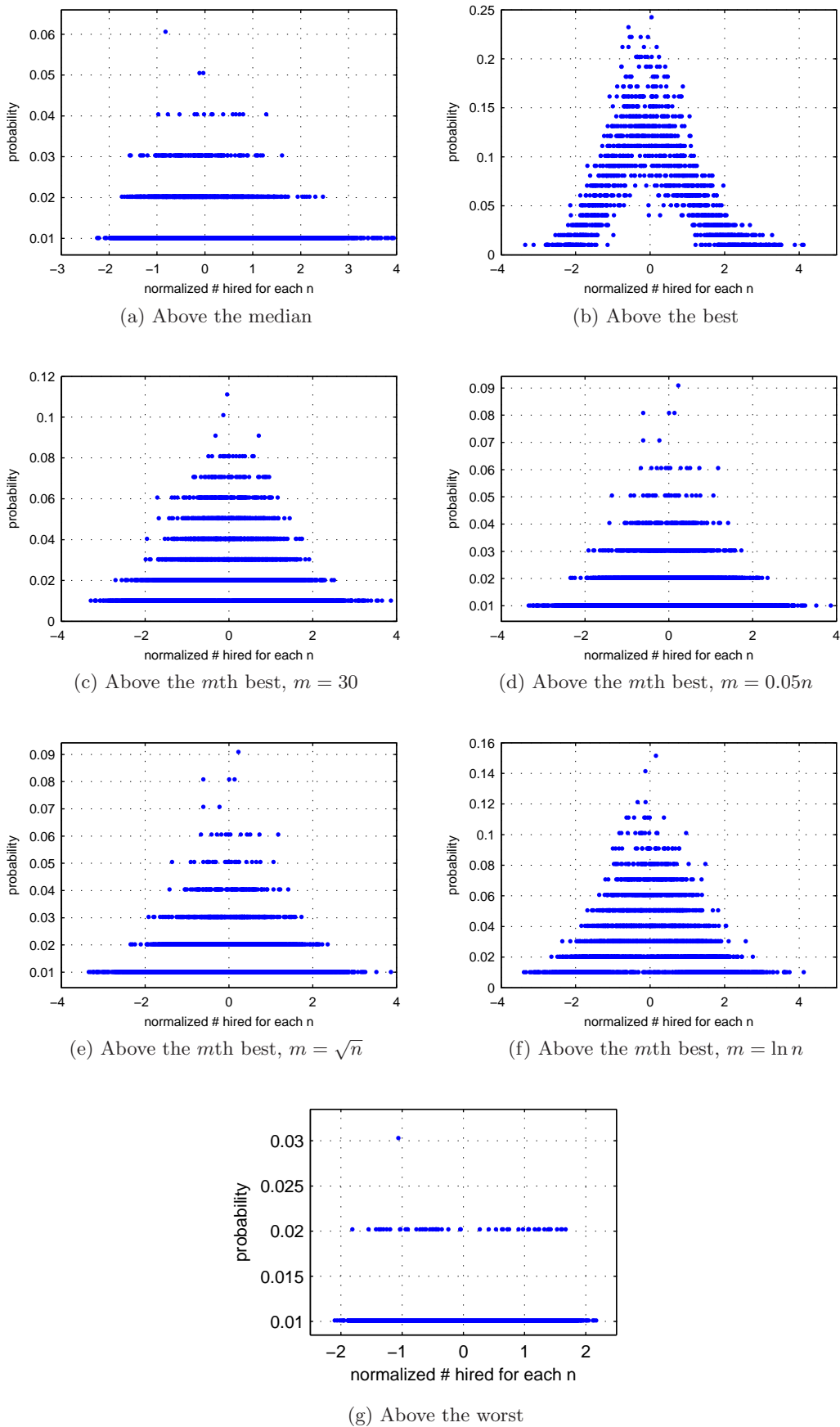


Figure 4.3: Probability distribution of the hiring set.

4.1.4 The average gap of the last hired candidate

This is a legend for figure (4.4):

- The **blue points** represent the experimental results for g_n .
- The **red solid lines** represent the mathematical expectation of g_n .

Comments and Conclusion

In figure (4.4a): the **red solid lines** represent equations (3.16) and (3.17) of the following expected lower and upper bounds

$$\mathbb{E}\{g_n^{(l.b.)}\} \sim \frac{2}{\sqrt{3}\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}} \quad , \mathbb{E}\{g_n^{(u.b.)}\} \sim \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}} .$$

In figure (4.4b): the gap is zero as discussed in subsection (2.2.2).

In figure (4.4c): the **red solid line** represent the following expectation: $\mathbb{E}\{g_n\} \sim \frac{29}{2n}$, which yields by substituting m by 30 in equation (2.10).

In figure (4.4d): the **red solid line** represent the following expectation: $\mathbb{E}\{g_n\} \sim \frac{0.05n-1}{2n}$, which yields by substituting m by $0.05n$ in equation (2.10).

In figure (4.4e): the **red solid line** represent the following expectation: $\mathbb{E}\{g_n\} \sim \frac{\sqrt{n}-1}{2n}$, which yields by substituting m by \sqrt{n} in equation (2.10).

In figure (4.4f): the **red solid line** represent the following expectation: $\mathbb{E}\{g_n\} \sim \frac{\ln n-1}{2n}$, which yields by substituting m by $\ln n$ in equation (2.10)

In figure (4.4g): the **red solid line** represent the following expectation: $\mathbb{E}\{g_n\} \sim \frac{1}{4}$, which is equation (3.23).

From figure (4.4), We can say that the theoretical expectations of g_n are supported by the experimental results.

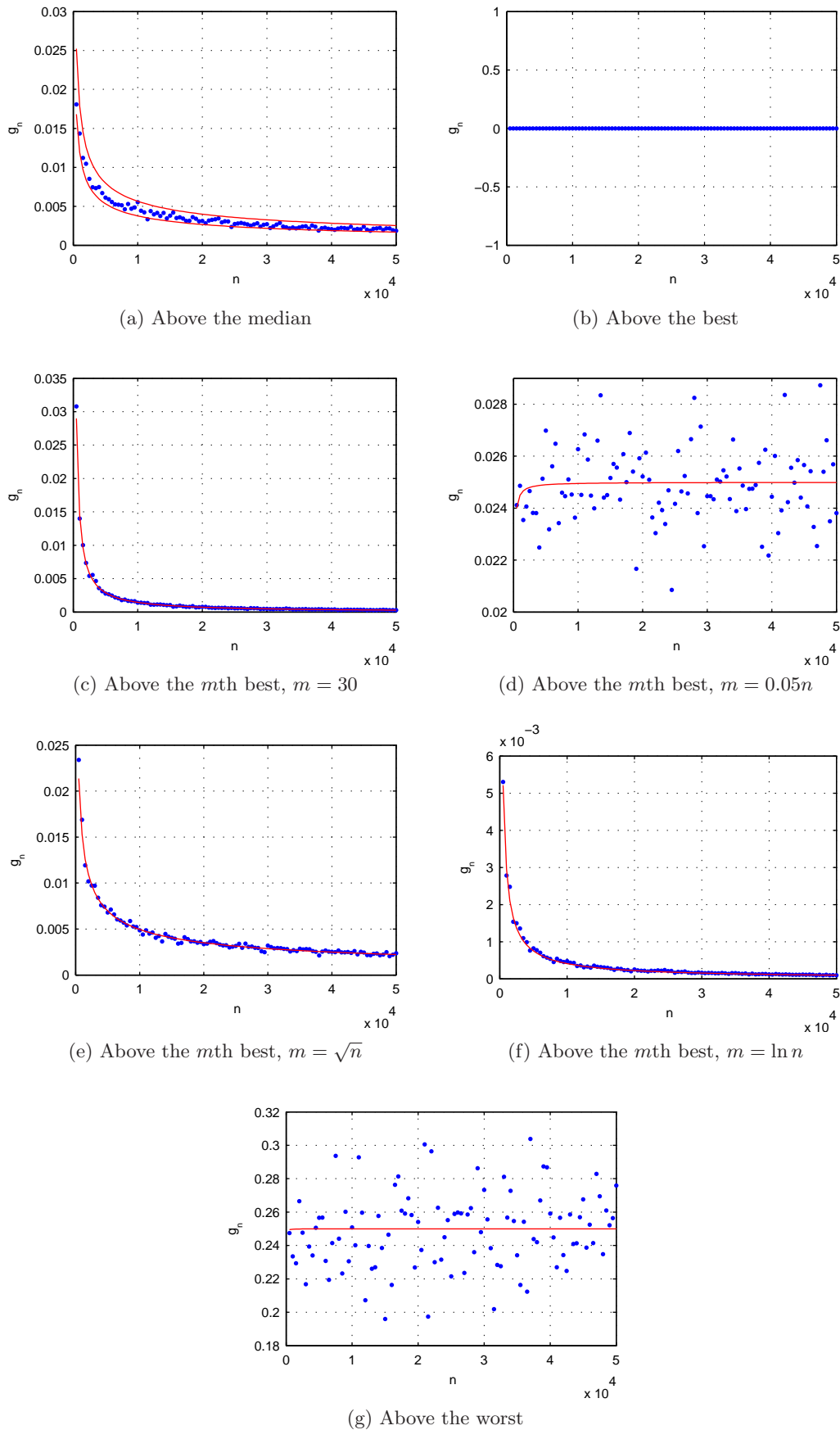


Figure 4.4: The average gap.

4.1.5 The average index of last hired candidate

This is a legend for figure (4.5):

- The **blue points** represent the experimental results for L_n .
- The **red solid lines** represent the mathematical expectation of L_n .

Comments and Conclusion

In figure (4.5b): the **red solid line** represent the following expectation: $\mathbb{E}\{L_n\} \sim (n + 1)/2$, which is equation (2.8) with $m = 1$.

In figure (4.5c): the **red solid line** represent the following expectation: $\mathbb{E}\{L_n\} \sim \frac{30(n+1)}{31}$, obtained from equation (2.8) with $m = 30$.

In figure (4.5d): the **red solid line** represent the following expectation: $\mathbb{E}\{L_n\} \sim \frac{n(n+1)}{n+20}$, obtained from equation (2.8) with $m = 0.05n$.

In figure (4.5e): the **red solid line** represent the following expectation: $\mathbb{E}\{L_n\} \sim \frac{\sqrt{n}(n+1)}{\sqrt{n+1}}$, obtained from equation (2.8) with $m = \sqrt{n}$.

In figure (4.5f): the **red solid line** represent the following expectation: $\mathbb{E}\{L_n\} \sim \frac{\ln n(n+1)}{\ln n+1}$, obtained from equation (2.8) with $m = \ln n$.

Again, the theoretical expectations of L_n for hiring above the m th best strategy are supported by the experimental results.

4.1.6 The average distance between last two hirings

In figure (4.6): the **blue points** represent the experimental results for Δ_n .

Comments and Conclusion

From figure (4.6), we note that hiring above the best strategy, figure (4.6b), has the maximum Δ_n which is expected because this strategy has the smallest hiring set; the same case for hiring above the m th best with $m = \ln n$, figure (4.6f). For hiring above the m th best with $m = 0.05n$, figure (4.6d), Δ_n is small -compared to the other strategies- because it hires a lot of candidates; also it is noticed that the mean value of Δ_n is approximately constant to 20 which is $1/0.05$, then we guess that $\mathbb{E}\{\Delta_n\} = 1/c$ for $m = c \cdot n$. From figure (4.6g), it is clear that $\mathbb{E}\{\Delta_n\}$ is at most 10 for hiring above the worst; which is the smallest distance because hiring above the worst strategy has the largest hiring set, $(n + 1)/2$; which insures that there is dependence between some parameters. This dependence makes comparing strategies w.r.t. the introduced parameters more difficult.

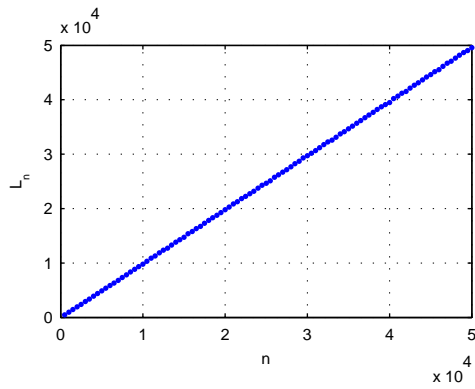
4.1.7 The average score of best not hired candidate

In figure (4.7): the **blue points** represent the experimental results for M_n .

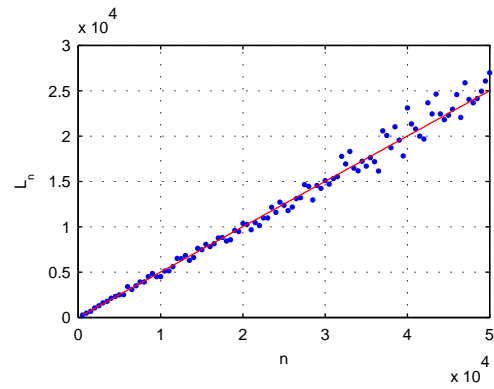
Comments and Conclusion

From figure (4.7), for most of strategies, we note that M_n is close to n . But for figure (4.7d), M_n is far from n because for $m = 0.05n$, the strategy will hire a lot of people, then the probability of missing a good score will be smaller than other strategies that have smaller hiring set. From figure (4.7g), it is clear that $\mathbb{E}\{M_n\} \sim (n - 1)/2$ which is expected; because in hiring above the worst, every candidate better than the worst one will be hired. Then $M(\sigma) = |\sigma| - h(\sigma)$ and

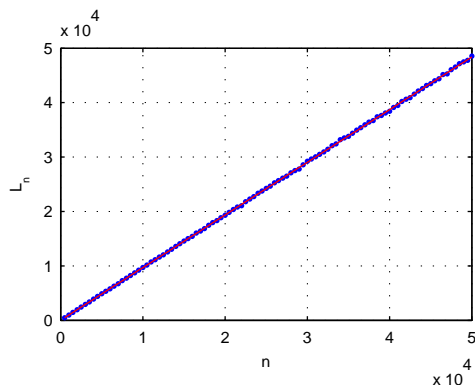
$$\mathbb{E}\{M_n\} = n - \mathbb{E}\{h_n\} = n - \frac{n+1}{2} = \frac{n-1}{2}.$$



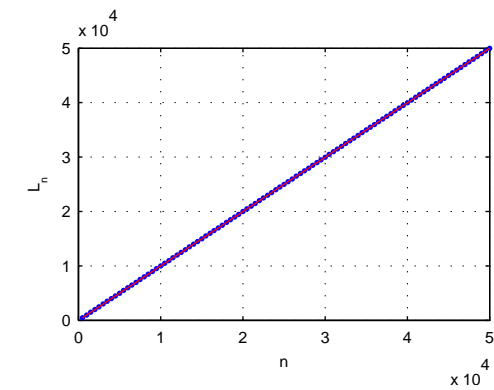
(a) Above the median



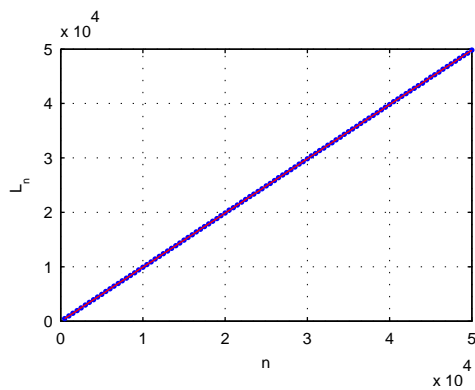
(b) Above the best



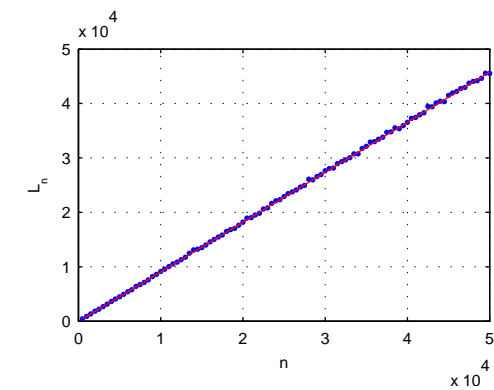
(c) Above the m th best, $m = 30$



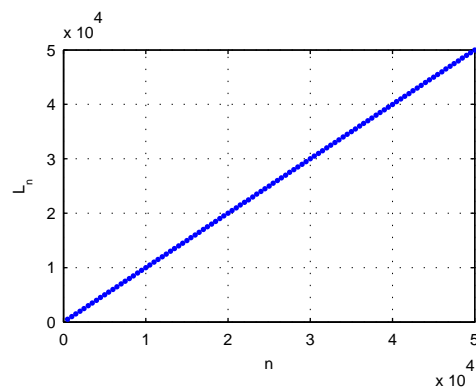
(d) Above the m th best, $m = 0.05n$



(e) Above the m th best, $m = \sqrt{n}$



(f) Above the m th best, $m = \ln n$



(g) Above the worst

Figure 4.5: The average index of the last hired candidate.

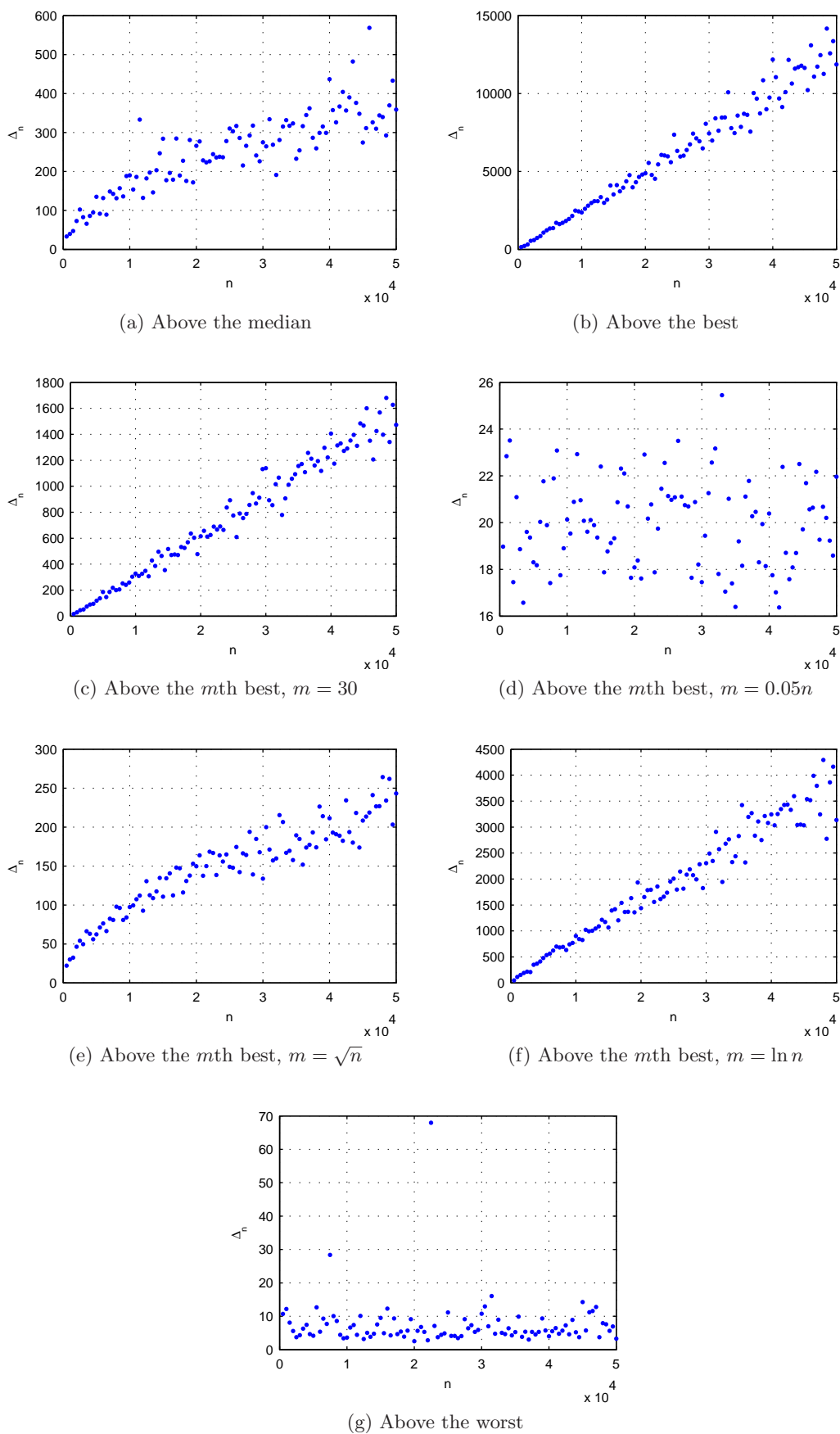


Figure 4.6: The average distance between the last two hirings.

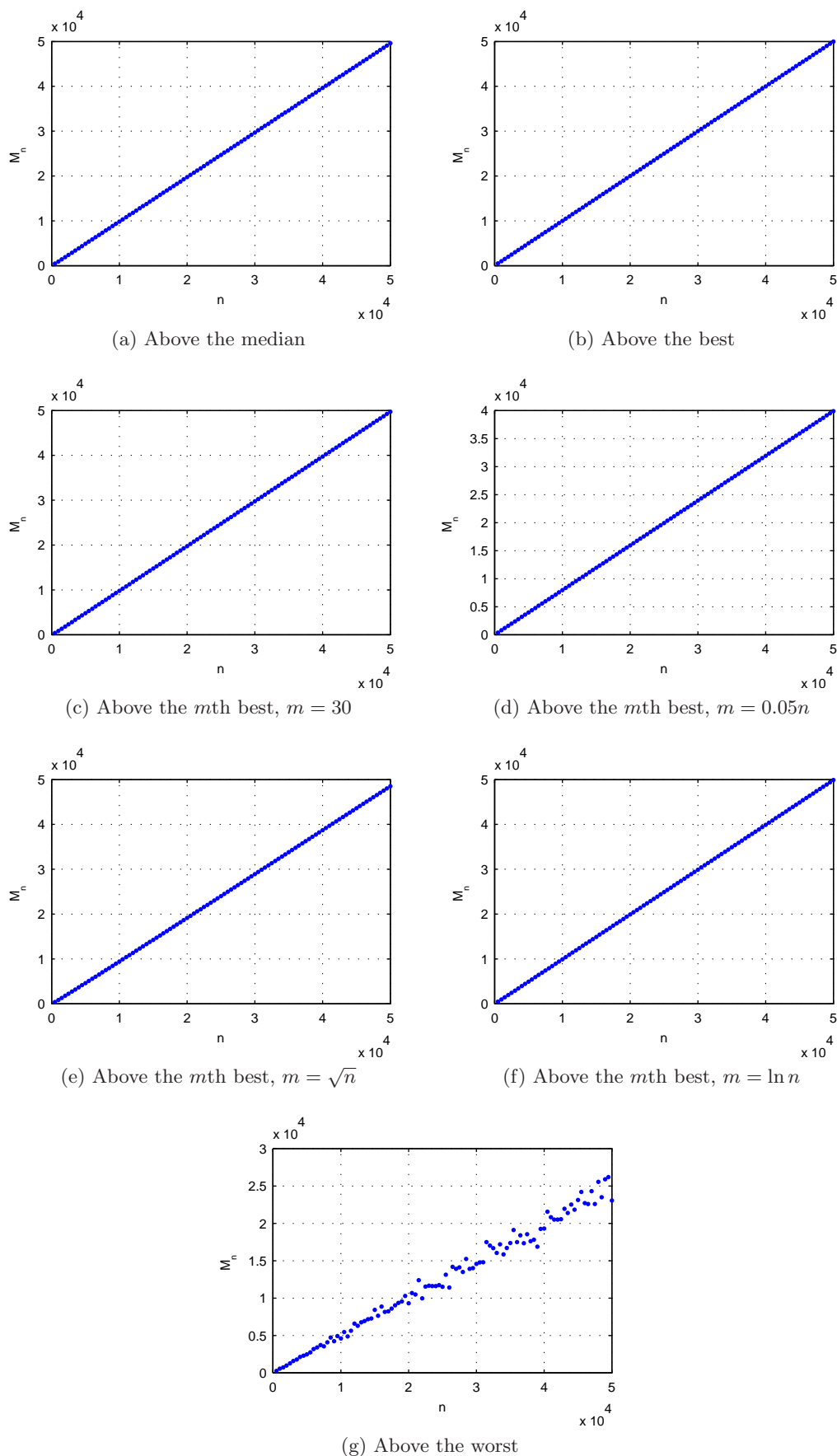


Figure 4.7: The score of best not hired candidate.

4.2 Randomized hiring

Here we introduce the experimental results for the average size of hiring set for two randomized hiring strategies, above the median and above the best. For both strategies, the committee must have distinct values (ranks) and for collecting it we use simply the rejection method. Notice that when the committee size is fixed and the size of the hiring set is smaller, then the committee contains all the hired people. We will always use s to denote the committee size, recall that h_n represents the size of the hiring set and n is the permutation size.

4.2.1 Randomized hiring above the median

The scenario of this strategy is: after processing some sequence of the coming ranks, a committee of the hiring set is chosen randomly for each interview and the new candidate will be hired if her score is better than the median rank of the committee, and discarded otherwise. We develop an experiment to make two tests:

1. For the same strategy: computing the ratio between the average sizes of the hiring sets for deterministic and randomized hiring.
2. Our hypothesis is that the expectation of h_n for randomized hiring above the median will have the same order as deterministic hiring. Since $\mathbb{E}\{h_n^{(DM)}\} = \Theta(\sqrt{n})$, equations (3.12) and (3.14), where DM is a notation for deterministic hiring above the median strategy, then the exponent $c^{(DM)} = \log h_n^{(DM)} / \log n$ should be approximately constant and converge to $1/2$. To check our hypothesis, we plot $c^{(DM)}$ against $c^{(RM)} = \log h_n^{(RM)} / \log n$ where RM is a notation for randomized hiring above the median strategy.

In the C++ code, we test both deterministic and randomized hiring for the same strategy under the same sequence of relative ranks. In this experiment, we consider different s against n from 100 to 10000 with step 100, thus we have 100 different n , and for each n we run 100 trials.

This is a legend for figure (4.8):

- The **blue points** represent the results of deterministic h_n .
- The **blue lines** represent the results of deterministic $h_n \pm$ the standard deviation.
- The **red points** represent the results of randomized h_n .
- The **red lines** represent the results of randomized $h_n \pm$ the standard deviation.

For figure (4.10):

- The **blue line** represents $c^{(DM)}$.
- The **red line** represents $c^{(RM)}$.

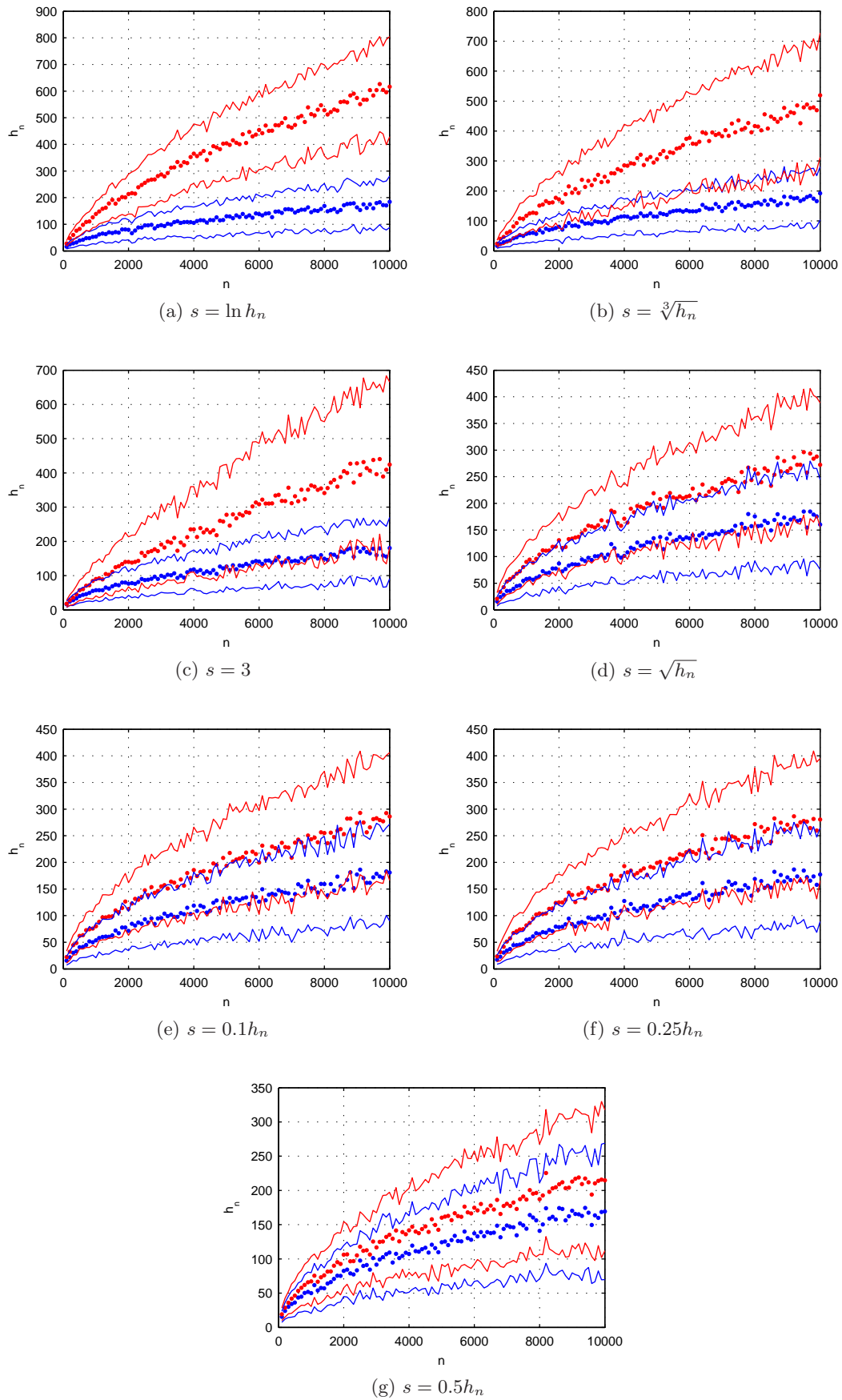


Figure 4.8: The average size of the hiring set for randomized hiring above the median.

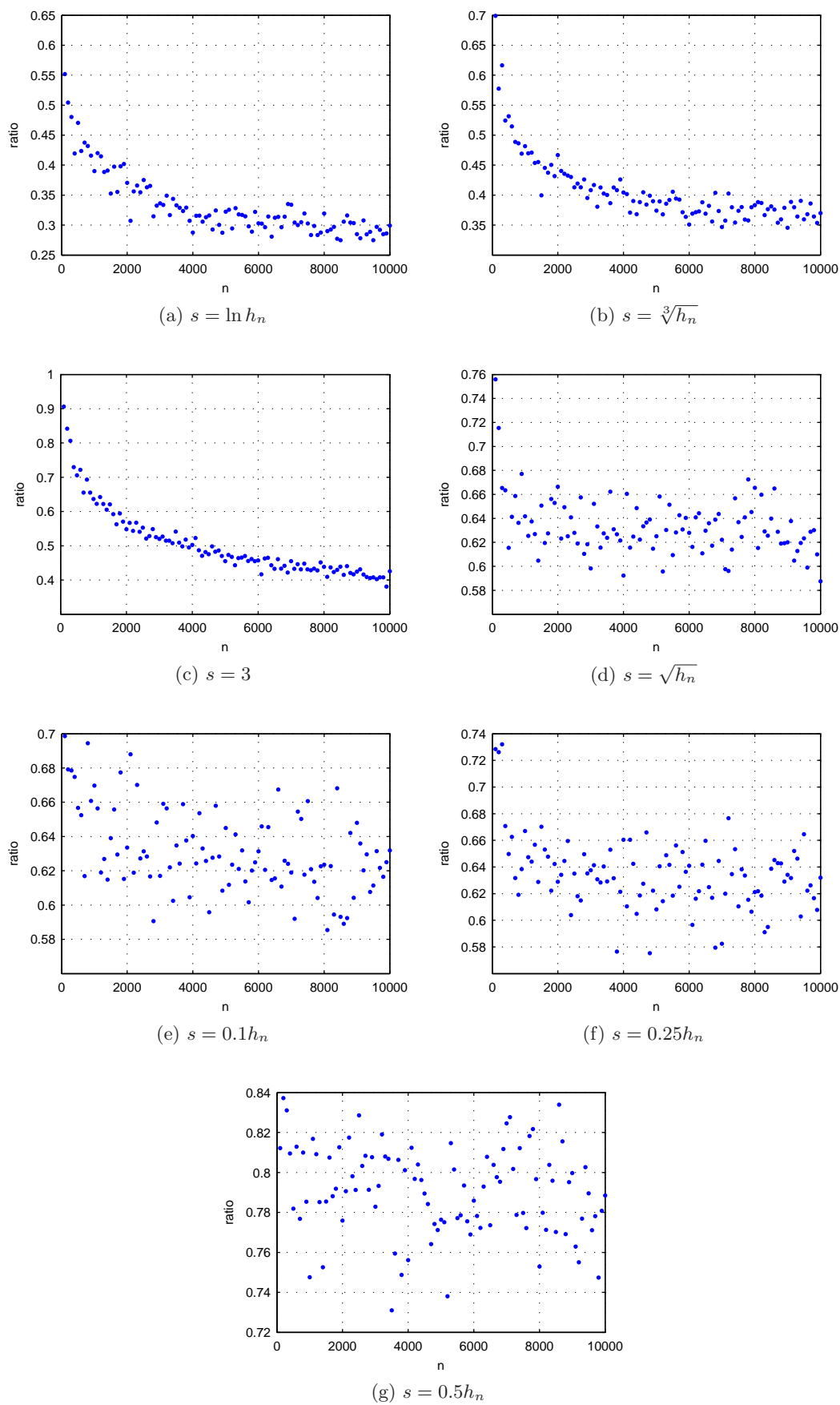


Figure 4.9: The ratio $h_n^{(DM)}/h_n^{(RM)}$.

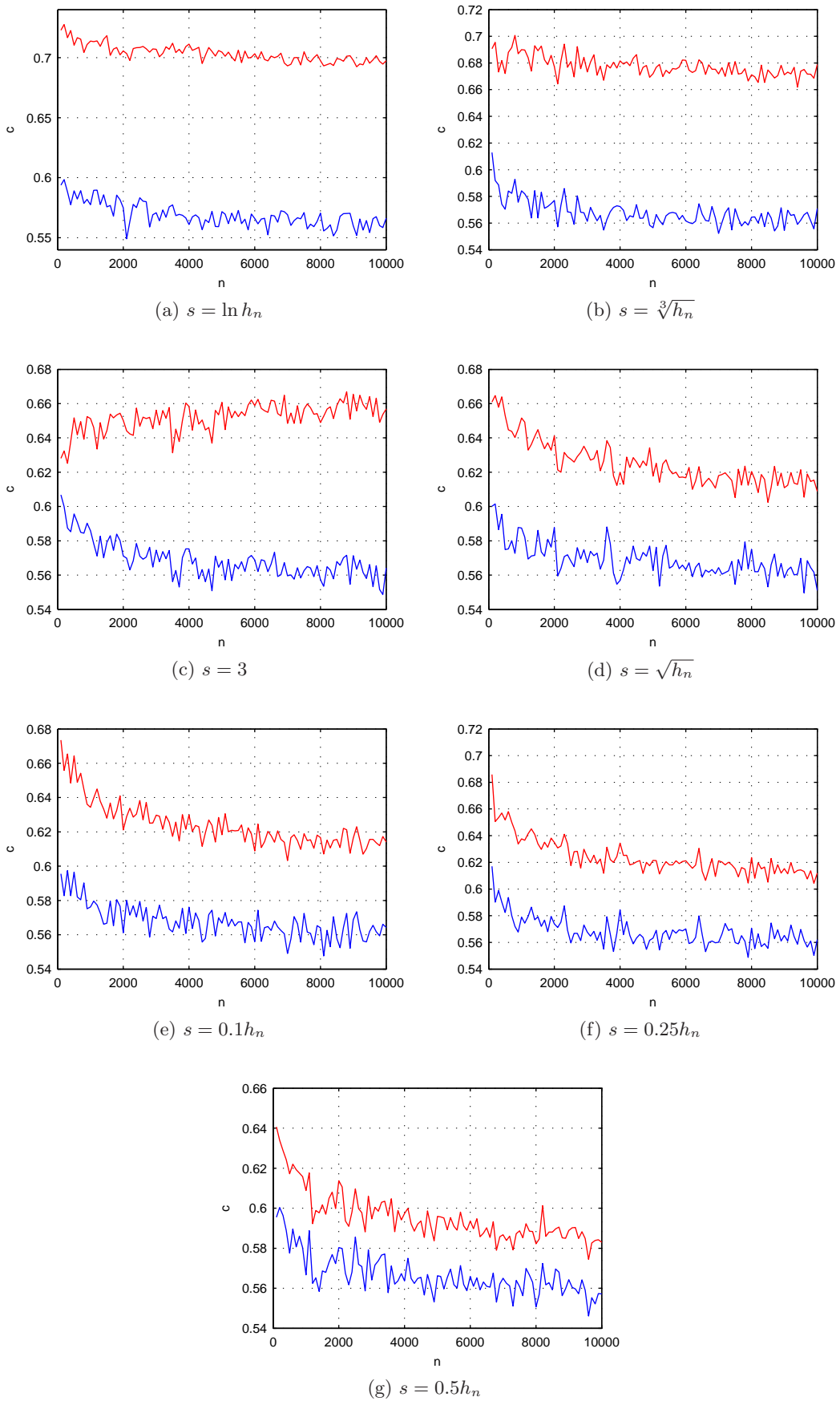


Figure 4.10: The exponents $C^{(DM)}$ and $C^{(RM)}$.

Comments and Conclusion

From figure (4.8), we note that $s = 3$ is a small sample size, then the ratio $h_n^{(DM)}/h_n^{(RM)}$ goes slowly to zero. Also for $s = \ln h_n^{(RM)}$ and $s = \sqrt[3]{h_n^{(DM)}}$, they are small and grow very slowly and with larger n , we expect that $h_n^{(RM)}$ will be far from $h_n^{(DM)}$. From figure (4.10), the difference between $C^{(RM)}$ and $C^{(DM)}$ increases for $s = 3$ because it is fixed while n grows. For $s = \ln h_n^{(RM)}$ and $s = \sqrt[3]{h_n^{(DM)}}$, that difference decreases very slowly but it is still large. Thus, the hypothesis $\mathbb{E}\{h_n^{(RM)}\} \sim n^c$ does not hold for $s = \ln h_n^{(RM)}$, $s = \sqrt[3]{h_n^{(DM)}}$ and $s = 3$. For the other sample sizes in figure (4.8), we note that $h_n^{(RM)}$ is close to $h_n^{(DM)}$, the ratio $h_n^{(DM)}/h_n^{(RM)}$ seems to converge to a constant value rather than zero and the difference between $C^{(RM)}$ and $C^{(DM)}$ decreases slowly as n increases. So that the hypothesis holds for these sample sizes. We conjecture that $\mathbb{E}\{h_n^{(RM)}\} = \Theta(\sqrt{n})$, for reasonable sample sizes, and the recommended sample size is $\sqrt{h_n^{(RM)}}$ because the experimental results for $h_n^{(RM)}$ are within the standard deviation of $h_n^{(DM)}$. Also $\sqrt{h_n^{(RM)}}$ is not as large as $0.25h_n^{(RM)}$ or $0.5h_n^{(RM)}$, then we make use of the idea of randomness in collecting this committee rapidly and speed up the hiring process.

4.2.2 Randomized hiring above the best

The scenario of hiring is: after processing some sequence of the coming ranks, then a committee of the hiring set is chosen randomly for each interview and the new candidate will be hired if her score is better than the best rank of the committee, and discarded otherwise. We are interested in comparing the sizes of hiring sets for deterministic and randomized hiring. In the next experiment, n goes from 100 to 10000 with step 100 and for each n , we run 100 trials. For figure (4.11), the legend is

- The **blue points** represent the results of deterministic h_n .
- The **blue lines** represent the results of deterministic $h_n \pm$ the standard deviation.
- The **red points** represent the results of randomized h_n .
- The **red lines** represent the results of randomized $h_n \pm$ the standard deviation.

Comments and Conclusion for figures (4.11) and (4.12)

It is noticed that as the committee size increases, $h_n^{(RB)}$ becomes closer to $h_n^{(DB)}$, where RB and DB denote randomized and deterministic hiring above the best respectively. From figures (4.12b), (4.12d), (4.12f) and (4.12g) where $s = c \cdot h_n^{(RB)}$, the ratio $h_n^{(DB)}/h_n^{(RB)}$ converges slowly to this constant c . So that we make this hypothesis: $\mathbb{E}\{h_n^{(RB)}\} \approx c^{-1}\mathbb{E}\{h_n^{(DB)}\}$. To test this hypothesis, we continue with the last experiment and plot $C^{(DB)} = \ln \mathbb{E}\{h_n^{(DB)}\}/\ln \ln n$ which converges to 1 and $C^{(RB)} = \ln \mathbb{E}\{h_n^{(RB)}\}/\ln \ln n$. For figure (4.13), the legend is

- The **blue line** represents $C^{(DB)}$.
- The **red line** represents $C^{(RB)}$.

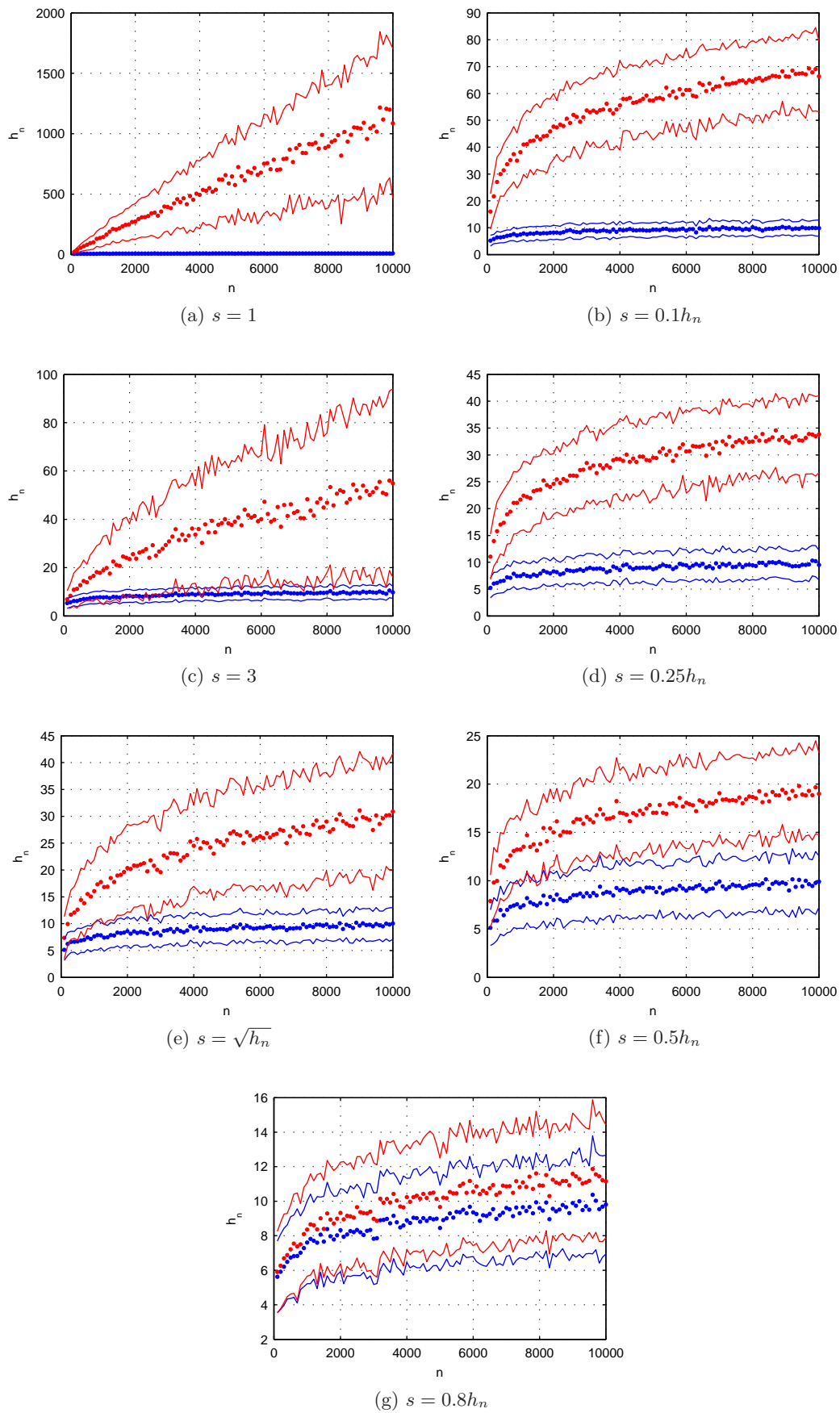


Figure 4.11: The average size of the hiring set for randomized hiring above the best.

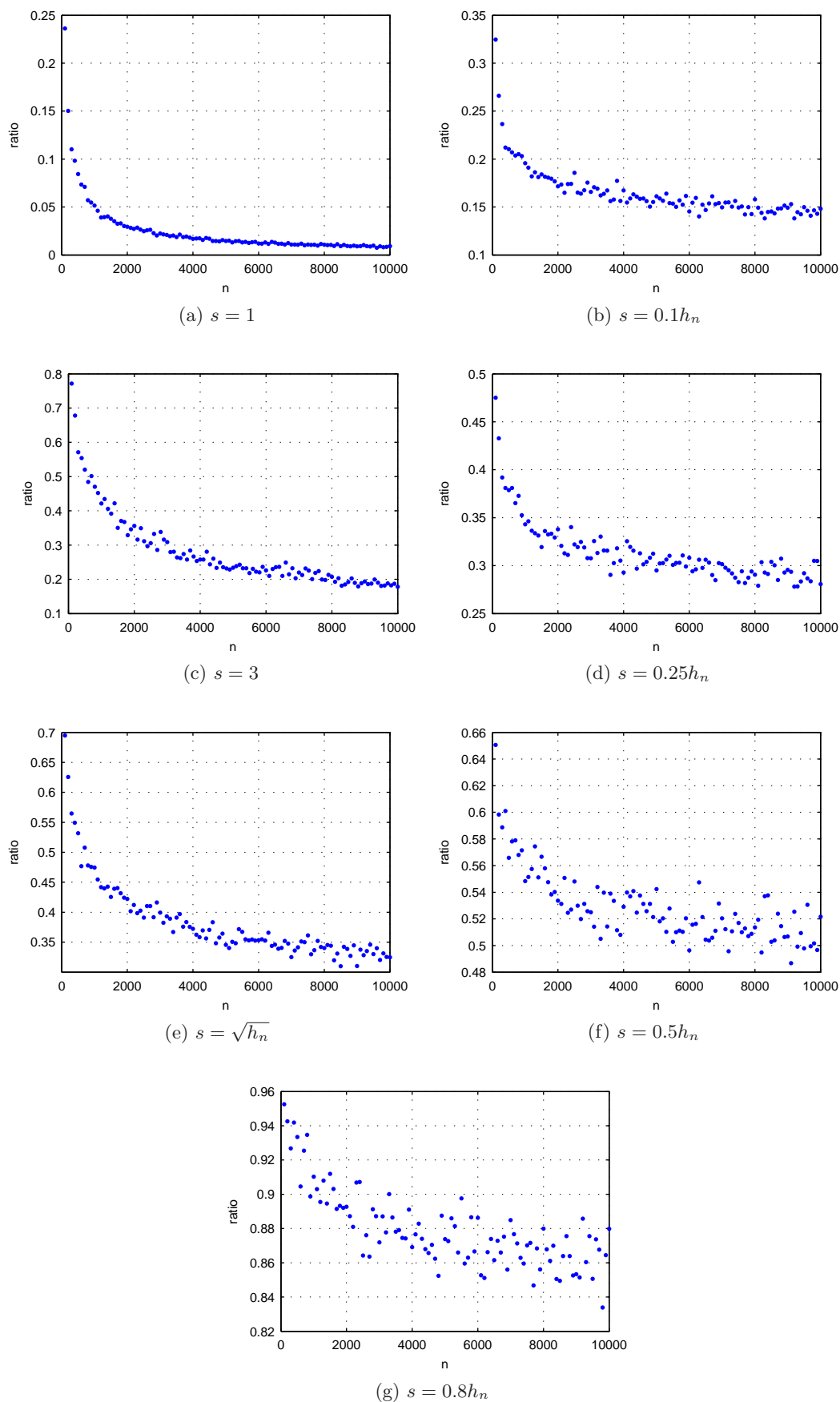


Figure 4.12: The ratio $h_n^{(DB)}/h_n^{(RB)}$.

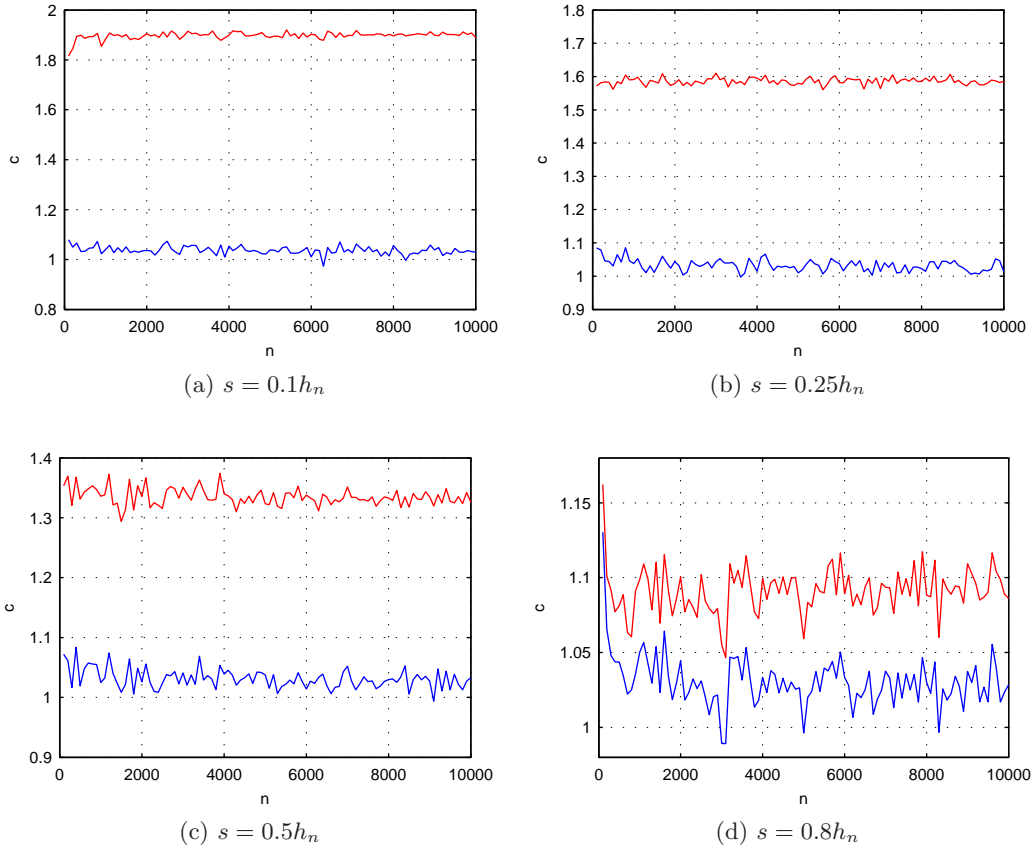


Figure 4.13: The exponents $C^{(DB)}$ and $C^{(RB)}$ for $s = c \cdot h_n^{(RB)}$.

Comments and Conclusion for figure (4.13)

We can say that our hypothesis holds. For instance, if $s = 0.25h_n^{(RB)}$, then $C^{(RB)} = \frac{\ln \mathbb{E}\{h_n^{(RB)}\}}{\ln \ln n} = \frac{\ln 0.25 \mathbb{E}\{h_n^{(DB)}\}}{\ln \ln n} = \frac{\ln 0.25 \ln n}{\ln \ln n} = 1 + \frac{\ln 0.25}{\ln \ln n} \sim 1.6$; the same check can be done for other sample sizes in figure (4.13).

Thus, we conjecture that for $s = c \cdot h_n^{(RB)}$, $\mathbb{E}\{h_n^{(RB)}\} = \Theta(\ln n)$.

For $s = \left(h_n^{(RB)}\right)^\alpha$, e.g. $s = \sqrt{h_n^{(RB)}}$, our hypothesis is that $\mathbb{E}\{h_n^{(RB)}\} = \Theta\left((\ln n)^\beta\right)$. To test this hypothesis, we plot the exponent $\ln \mathbb{E}\{h_n^{(RB)}\} / \ln \mathbb{E}\{h_n^{(DB)}\} = \ln \mathbb{E}\{h_n^{(RB)}\} / \ln \ln n$ which should converge to β . The next experiment plots the exponents $C^{(DB)} = \ln \mathbb{E}\{h_n^{(DB)}\} / \ln \ln n$ which converges to 1 and $C^{(RB)} = \ln \mathbb{E}\{h_n^{(RB)}\} / \ln \ln n$. We consider two sample sizes, $s = \sqrt{h_n^{(RB)}}$ and $s = \left(h_n^{(RB)}\right)^{2/3}$ for n from 5000 to 50000 with step 5000, with 100 trials for each n . Figure (4.14) shows the results of this experiment. The legend of figure (4.14) is

- In figures (4.14a) and (4.14b), the **blue color** for $h_n^{(DB)}$ and the **red color** for $h_n^{(RB)}$.
- In figures (4.14c) and (4.14d), we plot the ratio $h_n^{(DB)} / h_n^{(RB)}$.
- In figures (4.14e) and (4.14f), the **blue color** for the $C^{(DB)}$ and the **red color** for $C^{(RB)}$.

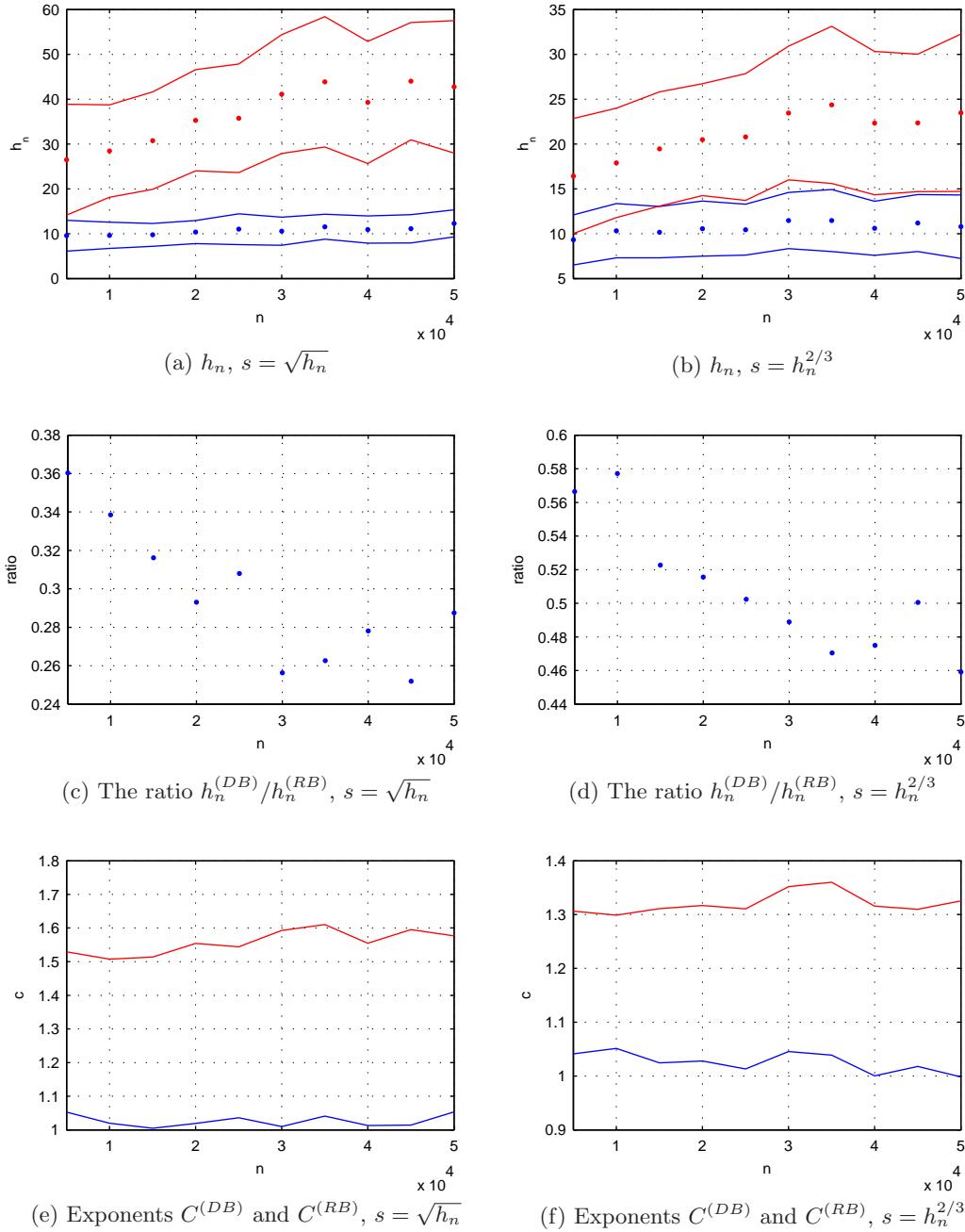


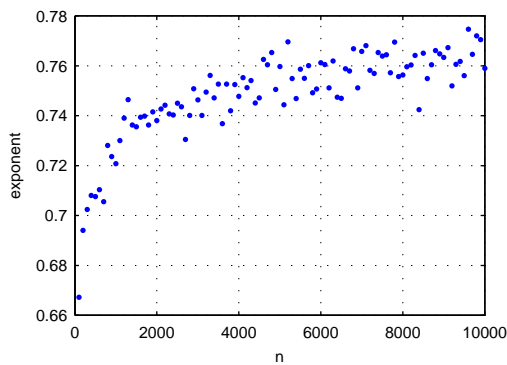
Figure 4.14: The average size of the hiring set for sublinear sample size for randomized hiring above the best.

From figures (4.14e) and (4.14f), we can say that $\mathbb{E}\{h_n^{(RB)}\} \sim (\ln n)^{3/2}$ for $s = \sqrt{h_n^{(RB)}}$ and $\mathbb{E}\{h_n^{(RB)}\} \sim (\ln n)^{4/3}$ for $s = \left(h_n^{(RB)}\right)^{2/3}$. Our hypothesis seems to hold. Formally, we conjecture that for $s = \left(h_n^{(RB)}\right)^\alpha$ for $0 < \alpha \leq 1$, $\mathbb{E}\{h_n^{(RB)}\} = \Theta\left((\ln n)^\beta\right)$ and $\beta \sim 2 - \alpha$ although a theoretical explanation, for all these conjectives, is still missing.

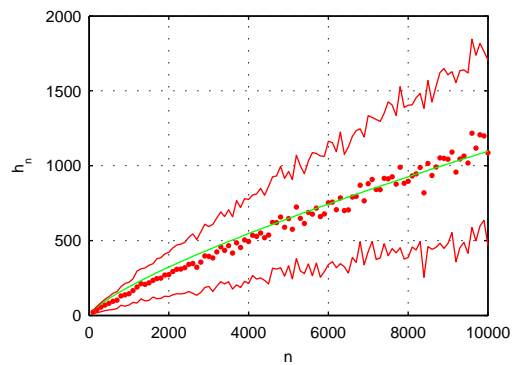
For $s = \Theta(1)$, e.g. figures (4.11a) and (4.11c), our hypothesis is that $\mathbb{E}\{h_n^{(RB)}\} = \Theta(n^\gamma)$.

To test this hypothesis, we make the following experiment; n from 100 to 10000 with step 100 and 100 trials for each n . We plot the exponent $\gamma = \ln \mathbb{E}\{h_n^{(RB)}\} / \ln n$. Figure (4.15) holds the results of this experiment for different s . The legend is

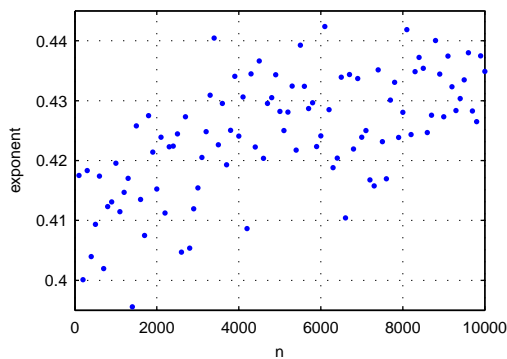
- In figures (4.15a), (4.15c) and (4.15e), we plot the exponent γ .
- In figures (4.15b), (4.15d) and (4.15f), the red color for $h_n^{(RB)}$ and the green line represents n^γ .



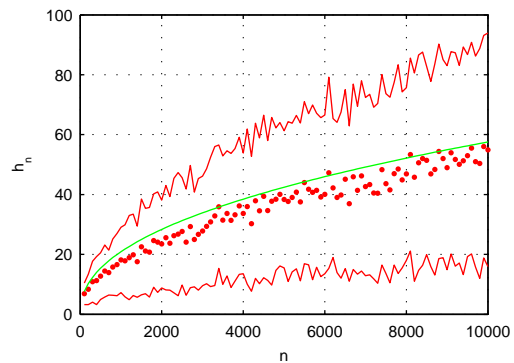
(a) The exponent γ , $s = 1$



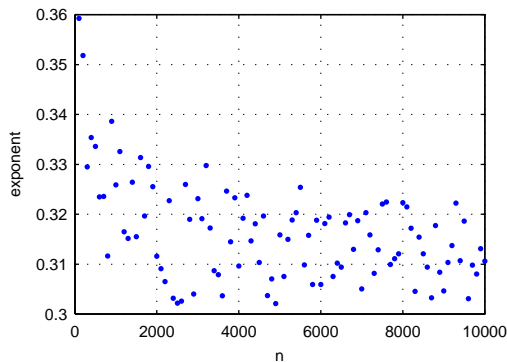
(b) h_n , $s = 1$



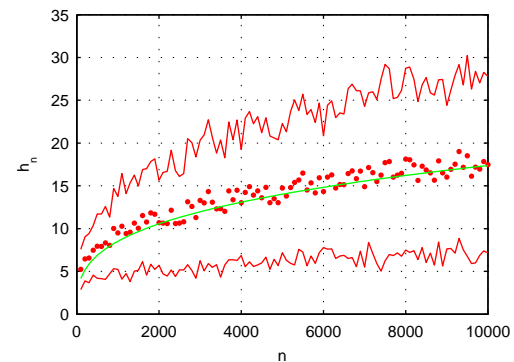
(c) The exponent γ , $s = 3$



(d) h_n , $s = 3$



(e) The exponent γ , $s = 5$



(f) h_n , $s = 5$

Figure 4.15: The average size of the hiring set with constant sample size for randomized hiring above the best.

Comments and conclusion for figure (4.15)

For $s = 1$, from figure (4.15a), the exponent γ approximately converges to 0.76; so that we plot $n^{0.76}$ which is the **green line** in figure (4.15a) and it is clear that it matches the experimental results of $h_n^{(RB)}$. Also for figure (4.15d), the **green line** represents $n^{0.44}$. The exponent γ seems to converge to 0.44 for $s = 3$. For figure (4.15f), the **green line** represents $n^{0.31}$. The exponent γ seems to converge to 0.31 for $s = 5$. so that, as s increases, the exponent γ decreases. We develop additional experiments of the same type but not reported here support the hypothesis that for $s = \Theta(1)$, $\mathbb{E}\{h_n^{(RB)}\} = \Theta(n^\gamma)$ for $0 < \gamma < 1$.

Chapter 5

Conclusions and future work

This thesis in principle represents an extension for the work of Archibald and Martínez's [2]. We introduced new parameters of the hiring problem under the combinatorial model; e.g. the distance between last two hirings, the size of firing set for the strategies that consider hiring with firing and the score of best not hired candidate. Also, we analysed hiring above the $P\%$ quantile strategy which has hiring above the median strategy as a special case. We obtained the expectations of the average size of hiring set, average gap of the last hired candidate and average size of firing set. The results we obtained for hiring above the median strategy match the results of Broder et al. [1] for the average size of hiring set and the average gap. In the experimental part, we have a code that simulates the rank-based hiring strategies; using this code simplifies testing new strategies or new parameters. All the mathematical expectations obtained in [2] and [4] are verified by the experimental results. We also introduced the idea of injecting some randomness to the hiring process. Using the simulator code, we test randomized hiring above the best and above the median strategies, and obtained approximate expectations for the average size of hiring set in both strategies under different sample sizes.

On the other hand; we still have some questions and extensions that represent the future work of this problem, like

- Completing the analysis of some parameters which have not analysed yet, where there are some difficulties to obtain a result. The difficulty comes sometimes from reasons like obtaining arbitrary functions in the solution of a *PDE*; these arbitrary functions are significant to obtain a closed form to the analysed parameter.
- How could we evaluate a hiring strategy, then comparing different strategies?. However, the goal of a hiring strategy is clear (to hire candidates in reasonable rate and improve the mean quality of the hired staff) but there are some dependence between the two demands of the hiring process. The experiment clarifies the dependence between some parameters like the size of the hiring set and the distance between last two hirings; where as the size of hiring set increases, the distance between last two hirings decreases and vis versa.
- Doing the mathematical analysis for the discussed randomized hiring strategies and developing a mathematical theory that explain the observed data in experiments.
- Firing strategies: one useful option may be to fire the worst candidate every k hirings, or your median reaches a certain value (dependant on the size of hiring set or not). Thus, your company at least continues to grow.
- Sampling: discard all the initial K candidates; then apply some standard strategy using the statistics gathered in the initial interviews.

The following extensions are well studied for the secretary problem and we would like to apply them for the hiring problem

- “Batch hiring”: candidates come in blocks or batches of b , and decisions for all the candidates in the block are simultaneously taken.
- “Sliding window”: we can change our mind and hire some candidate who we already interviewed and provisionally discarded if we have not interviewed more than $w - 1$ candidates afterwards.
- Multicriteria hiring: the candidates have more than one quality score.

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Appendix A

Mathematical Derivations

A.1 Proof/Derivation of equation (3.3)

$$\begin{aligned}
M(z, u, v) &= \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{M(\sigma)} v^{h(\sigma)} \\
&= 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{M(\sigma)} v^{h(\sigma)} = 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \sum_{1 \leq j \leq |\sigma|+1} \frac{z^{|\sigma \circ j|}}{|\sigma \circ j|!} u^{M(\sigma \circ j)} v^{h(\sigma \circ j)} \\
&= 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} \sum_{1 \leq j \leq |\sigma|+1} u^{M(\sigma \circ j)} v^{h(\sigma \circ j)}
\end{aligned}$$

Since the recurrence of $M(\sigma)$ is

$$M(\sigma \circ j) = \begin{cases} M(\sigma) + 1, & \text{if } j \leq M(\sigma), \text{ “}j \text{ is discarded”} \\ M(\sigma), & \text{if } j > |\sigma| + 1 - X(\sigma), \text{ “}j \text{ is hired”} \\ j, & \text{if } M(\sigma) < j \leq |\sigma| + 1 - X(\sigma), \text{ “}j \text{ is discarded”} \end{cases}$$

Then,

$$\begin{aligned}
\sum_{1 \leq j \leq |\sigma|+1} u^{M(\sigma \circ j)} v^{h(\sigma \circ j)} &= u^{M(\sigma)} v^{h(\sigma)} \left[\sum_{j=1}^{M(\sigma)} v^0 u^1 + \sum_{j=M(\sigma)+1}^{|\sigma|+1-X(\sigma)} v^0 u^{-M(\sigma)+j} + \sum_{j=|\sigma|+2-X(\sigma)}^{|\sigma|+1} v^1 u^0 \right] \\
&= u^{M(\sigma)} v^{h(\sigma)} \left[uM(\sigma) + \frac{u^{-M(\sigma)+|\sigma|+2-X(\sigma)} - u}{u-1} + vX(\sigma) \right]
\end{aligned}$$

Substituting in the right hand side of the equation of $M(z, u, v)$, we have

$$M(z, u, v) = 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{M(\sigma)} v^{h(\sigma)} \left[uM(\sigma) + \frac{u^{-M(\sigma)+|\sigma|+2-X(\sigma)} - u}{u-1} + vX(\sigma) \right]$$

Taking derivatives w.r.t. z ,

$$\begin{aligned}
\frac{\partial M}{\partial z} &= \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{M(\sigma)} v^{h(\sigma)} \cdot u M(\sigma) + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{M(\sigma)} v^{h(\sigma)} \cdot v X(\sigma) \\
&+ \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{M(\sigma)} v^{h(\sigma)} \cdot \frac{u^{-M(\sigma)+|\sigma|+2-X(\sigma)}}{u-1} - \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{M(\sigma)} v^{h(\sigma)} \cdot \frac{u}{u-1} \\
&= u^2 \frac{\partial}{\partial u} M(z, u, v) + v \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{M(\sigma)} v^{h(\sigma)} + \frac{u^2}{u-1} \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{(zu)^{|\sigma|}}{|\sigma|!} u^{-X(\sigma)} v^{h(\sigma)} \\
&- \frac{u}{u-1} M(z, u, v)
\end{aligned}$$

Finally, we have

$$\frac{\partial M}{\partial z} - u^2 \frac{\partial M}{\partial u} + \frac{u}{u-1} M(z, u, v) = v \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{M(\sigma)} v^{h(\sigma)} + \frac{u^2}{u-1} \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{(zu)^{|\sigma|}}{|\sigma|!} u^{-X(\sigma)} v^{h(\sigma)}.$$

A.2 Proof/Derivation of equation (3.10)

$$\begin{aligned}
L(z, u, v) &= \sum_{\sigma \in \mathcal{P}} \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)} v^{h(\sigma)} \\
&= 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)} v^{h(\sigma)} = 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \sum_{1 \leq j \leq |\sigma|+1} \frac{z^{|\sigma \circ j|}}{|\sigma \circ j|!} u^{L(\sigma \circ j)} v^{h(\sigma \circ j)} \\
&= 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} \sum_{1 \leq j \leq |\sigma|+1} u^{L(\sigma \circ j)} v^{h(\sigma \circ j)}
\end{aligned}$$

We have $L(\sigma \circ j) = L(\sigma) + X_j(\sigma)(|\sigma|+1 - L(\sigma))$ and $h(\sigma \circ j) = h(\sigma) + X_j(\sigma)$ where

$$X_j(\sigma) = \begin{cases} 1, & \text{if the last candidate of } \sigma \circ j \text{ is hired,} \\ 0, & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned}
\sum_{1 \leq j \leq |\sigma|+1} u^{L(\sigma)+X_j(\sigma)(|\sigma|+1-L(\sigma))} v^{h(\sigma)+X_j(\sigma)} &= u^{L(\sigma)} v^{h(\sigma)} \sum_{1 \leq j \leq n+1} u^{X_j(\sigma)(|\sigma|+1-L(\sigma))} v^{X_j(\sigma)} \\
&= u^{L(\sigma)} v^{h(\sigma)} \left[(|\sigma|+1 - X(\sigma)) + X(\sigma) u^{|\sigma|+1-L(\sigma)} v \right]
\end{aligned}$$

Hence,

$$L(z, u, v) = 1 + \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|+1}}{(|\sigma|+1)!} u^{L(\sigma)} v^{h(\sigma)} \left[(|\sigma|+1 - X(\sigma)) + X(\sigma) u^{|\sigma|+1-L(\sigma)} v \right]$$

Taking derivatives w.r.t. z ,

$$\begin{aligned}\frac{\partial}{\partial z}L(z, u, v) &= \sum_{n>0} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)} v^{h(\sigma)} \left[(|\sigma| + 1 - X(\sigma)) + X(\sigma) u^{|\sigma|+1-L(\sigma)} v \right] \\ &= z \frac{\partial L}{\partial z} + L(z, u, v) - \sum_{\sigma \in \mathcal{P}_{n-1}} X(\sigma) \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)} v^{h(\sigma)} + uv \sum_{\sigma \in \mathcal{P}_{n-1}} X(\sigma) \frac{(zu)^{|\sigma|}}{|\sigma|!} v^{h(\sigma)}\end{aligned}$$

For hiring above the $P\%$ quantile strategy, $X(\sigma) = ah(\sigma) + b$, then

$$\begin{aligned}(1-z) \frac{\partial L}{\partial z} - L(z, u, v) &= uv \sum_{\sigma \in \mathcal{P}_{n-1}} (ah(\sigma) + b) \frac{(zu)^{|\sigma|}}{|\sigma|!} v^{h(\sigma)} - \sum_{\sigma \in \mathcal{P}_{n-1}} (ah(\sigma) + b) \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)} v^{h(\sigma)} \\ &= auv^2 \frac{\partial}{\partial v} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{(zu)^{|\sigma|}}{|\sigma|!} v^{h(\sigma)} + buv \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{(zu)^{|\sigma|}}{|\sigma|!} v^{h(\sigma)} - av \frac{\partial}{\partial v} \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)} v^{h(\sigma)} \\ &\quad - b \sum_{\sigma \in \mathcal{P}_{n-1}} \frac{z^{|\sigma|}}{|\sigma|!} u^{L(\sigma)} v^{h(\sigma)} \\ &= auv^2 \frac{\partial}{\partial v} H(zu, v) + buv H(zu, v) - av \frac{\partial L}{\partial v} - bL(z, u, v)\end{aligned}$$

After organizing the terms of this equation, we obtain

$$(1-z) \frac{\partial L}{\partial z} - (1-b)L(z, u, v) + av \frac{\partial L}{\partial v} = auv^2 \frac{\partial}{\partial v} H(zu, v) + buv H(zu, v),$$

Taking derivatives w.r.t. u and set $u = 1$ yields in equation (3.10).