

## MSc in Photonics

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## *Master in Photonics*

### MASTER THESIS WORK

# Local discrimination of rotationally invariant states

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# Local discrimination of rotationally invariant states

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**Abstract.** We provide lower bounds for the minimum error discrimination probability of multipartite rotationally invariant states using separable measurements. The separability of the measurement operators has been investigated, and we have found PPT-based conditions which can be directly tested in the total angular momentum basis.

## 1. Introduction

Discrimination of states plays a key role in quantum communication and quantum computation, but it also offers great insight in fundamental aspects of quantum mechanics, since the notion of indistinguishability lies at the very heart of the theory. Moreover, if the states to discriminate consist of several subsystems distributed among some parties, entanglement becomes possible and can affect the distinguishability of the states. We will study the discrimination of orthogonal rotationally invariant states, which are completely distinguishable when global measurements can be performed, but when they can not, some information becomes lost and thus the discrimination becomes non-trivial. This contrasting performance of different strategies can be used to design communication protocols, for example.

### *1.1. Entanglement and separability*

Entanglement is a pure quantum property which has several and very astonishing consequences, many of them with direct applications in quantum communication, cryptography and computation. Formally, a bipartite state  $\rho$  acting on the composite space  $\mathcal{H}_A \otimes \mathcal{H}_B$  is entangled iff it can not be expressed as a convex sum of product states:

$$\rho = \sum_i p_i \rho_i^A \otimes \rho_i^B \quad (1)$$

where  $p_i$  are probabilities. A state which can be expressed in this way is called separable. The case of multipartite states is more complex, since they can be entangled in many different ways, corresponding to the different partitions of the parties. For

instance, a 3-party state  $\rho_{ABC}$  can be entangled respect to the following partitions:  $\{\{A\}, \{B\}, \{C\}\}, \{\{A\}, \{B, C\}\}, \{\{A, B\}, \{C\}\}$ , etc. The first corresponds to a fully entangled state whereas the rest correspond to states entangled respect to a particular partition.

A multipartite state  $\rho$  of  $N$  parties, acting on a space  $\mathcal{H} = \bigotimes_{i=1}^N \mathcal{H}_i$ , is *fully* separable, or  $N$ -separable, iff it can be written in the form:

$$\rho = \sum_{i=1}^N p_i \bigotimes_{j=1}^N \rho_i^j \quad (2)$$

otherwise it must contain some form of entanglement.

Now, the question is how can we detect if a state is separable or not. Although extensive research has been carried out, until now there is no definite answer to that question<sup>†</sup>, yet some criteria have been stated [4, 5, 6]. One of the most useful is the *Positive Partial Transposition (PPT)* or *Peres-Horodecki* criterion [4]:

**Proposition 1.** *The state  $\rho$  acting in  $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$  is separable only if its partial transposition  $\rho^{TA}$  is positive. The condition is also sufficient for the cases  $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^2$  and  $\mathcal{H} = \mathbb{C}^2 \otimes \mathbb{C}^3$*

The partial transposition is nothing but the standard transposition applied only to some of the parties, which is an operation that will clearly affect in a different way entangled and separable states. It is a particular case of the non-completely positive maps as in the reference [5].

Although the formal separability conditions can be straightforwardly generalized to the multipartite case, the PPT criterion would only detect entanglement between bipartite splits of a  $N$ -partite state, since it is naturally designed for bipartite states. Nevertheless, the positivity of the partial transposition corresponding to each of the  $2^{N-1} - 1$  possible bipartite splits is still a necessary condition for full-separability.

### 1.2. Collective versus local measurements

When the states to discriminate are multipartite, entanglement can be essential for achieving the lowest error probability. If possible, the best strategy is to use global measurements, acting over the total quantum state (all parties simultaneously) without any restriction upon it.

On the other hand, one can consider local measurements, that act on each party independently. In this case the discrimination reduces to a  $N$  different measurement processes and a post-measurement statistical analysis of the results. This kind of strategies range from the most simple ones, where the same POVM is used for each

<sup>†</sup> Formally, necessary and sufficient separability conditions based on positive maps exist [5], although they are not operational. Entanglement witnesses, for example, must be constructed for particular states, thus not detecting entanglement for arbitrary states.

party (local fixed), to a very sophisticated ones, where the POVM used on each party depends on the results obtained in the previous measurements (local measurements and classical communication, *LOCC*).

If the states to be discriminated contain entanglement, the reduced states (the state of some parties when the information from the others is ignored) will be noisy, hence any local strategy would be worse than a global one (provided that both are optimal). However, local strategies are very important, since they are feasible with current technology. One can design discrimination scenarios involving several parties where one achieves low error rates with collective measurements but high error with local strategies. An example can be found in reference [8].

Since local strategies are difficult to characterize, due to the many ways in which each measurement can depend on the previous ones, a more convenient idea is to analyze collective strategies with *separable* POVM's, which include *LOCC*. Any POVM element can be turned into a state by normalizing it. Using that fact, a POVM is separable iff the corresponding state for each element is separable<sup>††</sup>.

Since separable strategies are more general than the local ones, the minimal error achievable with the former represents a lower bound to the latter.

## 2. Rotationally invariant states

Any problem can be significantly simplified if a convenient symmetry can be found. Many composite states used in quantum communication are symmetric under the permutations of its parties (for instance, multi-copy states  $\rho^{\otimes N}$ ), which already simplifies many things. Here, we will focus on rotational symmetry.

A multipartite state is invariant under the action of the  $SO(3)$  group iff  $[\mathcal{D}(R), \rho] = 0$  for all  $R$ , where  $\mathcal{D}(R)$  is the representation of an element of  $SO(3)$ . By Schur's lemma,  $\rho$  must be proportional to the identity on each irreducible subspace spanned by the eigenvectors of the total angular momentum. It must have the form

$$\rho = \sum_{j=j_{min}}^{N/2} \frac{A(j)}{n_j(2j+1)} \sum_{m=-j}^{+j} \sum_{\alpha=1}^{n_j} |j m \alpha\rangle \langle j m \alpha| = \sum_{j=j_{min}}^{N/2} \frac{A(j)}{n_j(2j+1)} \mathbb{I}_j \quad (3)$$

where  $n_j = \binom{N}{N/2-j} - \binom{N}{N/2-j-1}$  is the multiplicity of the representation with angular momentum  $j$ , and  $\mathbb{I}_j$  is the identity in this particular invariant subspace.

The former expression is completely general, and can be simplified when dealing with bipartite systems. In our case, we will consider multipartite states with  $1/2$ -spin subsystems (q-bits) shared by  $N$  parties. This kind of states offer some advantages besides their simple mathematical expression, like robustness against noise, not to mention the importance of the projectors  $\mathbb{I}_j$  in quantum mechanics.

<sup>††</sup> Notice that both PPT and entanglement witnesses can be directly applied to an operator, since a normalization constant will not affect the test.

### 3. Minimum error discrimination

Our goal is to discriminate between a given pair of states  $\rho_{j_1}$  and  $\rho_{j_2}$  each corresponding to a different invariant subspace, namely, states of the simple form:

$$\rho_j = \frac{1}{n_j(2j+1)} \mathbb{I}_j \quad (4)$$

For simplicity we assume an equal *a priori* probability:  $\pi_{j_1} = \pi_{j_2} = 1/2$ .

An expression for the minimum error discrimination with a general POVM can be given for these kind of states. First, we try to determine the form of the measurement operators.

Any optimal POVM for rotationally invariant states can be fully characterized by a set of  $\lfloor N/2 \rfloor + 1$  parameters  $\{\gamma_j\}$  in the following way:

$$\begin{aligned} E_1 &= \sum_j \gamma_j \mathbb{I}_j \\ E_2 &= \sum_j (1 - \gamma_j) \mathbb{I}_j \\ 0 &\leq \gamma_j \leq 1 \end{aligned} \quad (5)$$

The reason is that given an optimal POVM element, the rotational invariance of the states allows us to choose it also invariant:

$$\begin{aligned} \text{tr}(E_i \rho_i) &= \int dR \text{tr}[E_i \mathcal{D}(R)^\dagger \rho_i \mathcal{D}(R)] = \int dR \text{tr}[\mathcal{D}(R) E_i \mathcal{D}(R)^\dagger \rho_i] = \\ &= \text{tr} \left\{ \left[ \int dR \mathcal{D}(R) E_i \mathcal{D}(R)^\dagger \right] \rho_i \right\} = \text{tr} \{ E'_i \rho_i \} \end{aligned} \quad (6)$$

Hence, by Schur's lemma,

$$E'_i = \int dR \mathcal{D}(R) E_i \mathcal{D}(R)^\dagger = \sum_j \gamma_j \mathbb{I}_j \quad (7)$$

Therefore, with the POVM defined in (5), the error probability in discriminating the states  $\rho_{j_1}$  and  $\rho_{j_2}$  is:

$$p_{err} = \pi_1 \rho(2 | 1) + \pi_2 \rho(1 | 2) = \frac{1 - (\gamma_{j_1} - \gamma_{j_2})}{2} \quad (8)$$

Only two out of the  $\lfloor N/2 \rfloor + 1$  coefficients show up in the error probability, the rest will be necessary to ensure that the POVM is separable. Obviously, if no further constraint is imposed on the POVM, perfect discrimination will be achieved with  $p_{err} = 0$  for  $\gamma_{j_1} = 1$ ,  $\gamma_{j_2} = 0$ , with the rest of the coefficients being completely arbitrary. This corresponds to projecting onto the subspaces corresponding to the given states, and since they are orthogonal, they are also completely distinguishable. However, that does not hold when global measurements are not possible. Note, for instance, that even

though  $\mathbb{I}_{Jmax}$  is a separable operator<sup>§</sup>,  $\mathbb{I}_{Jmax}$  is not.

Our main goal now is to find a suitable separability criterium, that takes full advantage of the symmetry of the problem.

#### 4. Separability of rotationally invariant states

A huge effort has been devoted to determine under which conditions a given state is separable or not, and in the first section some results have been commented. Most of the results are for bipartite systems, and this is specially the case concerning rotationally invariant states. The main difficulty when dealing with a multipartite rotationally invariant state is that all criteria can be applied easily in the computational basis (product basis), and rotationally invariant states are easily characterizable only in the total angular momentum basis. Only for bipartite systems the Clebsch-Gordan decomposition becomes easier and some interesting results can be obtained [9].

Most of the criteria found in the literature are based directly or indirectly on PPT, and since partial transposition requires to change basis, they are not suitable for rotationally invariant states. Entanglement witnesses may seem appropriate since they are operators and can be constructed in any basis; however, finding the right witness for a mixed state is quite a difficult task, and may involve partial transpositions or related operations. Other criteria based on expected values inequalities may seem suitable for our states [10, 11], but they proved to be useless<sup>||</sup>.

Finding no suitable conditions for our states in the literature, we have tried to analyze how an operator of the form (5) behaves under partial transpositions, by performing the change of basis and computing the resulting eigenvalues with *Mathematica*. Since our states are permutational invariant the only relevant parameter is the number of transposed parties, but not which ones are. Therefore we define a  $k$ -PT as the partial transposition performed on the first  $k$  parties. Obviously,  $k$  only ranges from 1 to  $\lfloor N/2 \rfloor$ , for the rest can be obtained by a further total transposition.

We find that 1-PPT is equivalent to the following simple conditions:

$$(2j + 1)\gamma_j \geq \gamma_{j-1} \quad \forall j \tag{9}$$

2-PPT and above yield more complex relations.

<sup>§</sup>  $\mathbb{I}_{Jmax} = \int dR \mathcal{D}(R) |JJ\rangle\langle JJ| \mathcal{D}(R)^\dagger = \int dR \mathcal{D}(R) (|\uparrow\rangle^{\otimes N} \langle \uparrow|^{\otimes N}) \mathcal{D}(R)^\dagger$

<sup>||</sup> The depolarization process used in the first destroys all entanglement in rotationally invariant states, and the second is not useful since it is affected by the normalization constant, which in turn depends on the variables  $\{\gamma_j\}$ , hence making more difficult the optimization.

## 5. Minimum-error lower bounds

### 5.1. Optimal POVM

Now we can use the separability condition to determine the optimal values of  $\{\gamma_j\}$  in order to make the operators positive under 1-PT. Since we need both elements of the POVM, we require condition (9) and the complementary one:

$$(2j+1)(1-\gamma_j) \geq 1-\gamma_{j-1} \quad \forall j \quad (10)$$

both to hold. Assuming  $j_1 > j_2$ , it is clearly desirable to minimize  $\gamma_{j_2}$  in order to minimize the error probability. In fact,  $\gamma_{j_2} = 0$  is compatible with the conditions, provided all the coefficients with  $j < j_2$  are also zero. All coefficients with  $j > j_1$  are not a problem, since for them the choice  $\gamma_j = \gamma_{j_1}$  trivially fulfills the conditions. Thus, we just have to find the maximum value of  $\gamma_{j_1}$ , which is bounded by the complementary condition  $(2j_1+1)(1-\gamma_{j_1}) \geq 1-\gamma_{j_1-1}$ . If  $\gamma_{j_2} = \gamma_{j_1-1} = 0$  we immediately obtain:

$$\gamma_{j_1} = \frac{2j_1}{2j_1+1} \quad (11)$$

For arbitrary  $j_1$  and  $j_2$  we can write the set of inequalities:

$$\gamma_j \geq \frac{\gamma_{j-1}}{2j+1} \geq \frac{\gamma_{j-2}}{(2j+1)[2(j-1)+1]} \geq \frac{\gamma_{j-3}}{(2j+1)[2(j-1)+1][2(j-2)+1]} \geq \dots \quad (12)$$

And then the relation between both parameters is:

$$\gamma_{j_1} \geq \frac{\gamma_{j_2}}{\prod_{j=j_2+1}^{j_1} (2j+1)} \quad \text{and} \quad 1-\gamma_{j_1} \geq \frac{1-\gamma_{j_2}}{\prod_{j=j_2+1}^{j_1} (2j+1)} \quad (13)$$

Again, setting  $\gamma_{j_2} = 0$  yields the optimal choice:

$$\gamma_{j_1} = \frac{\prod_{j=j_2+1}^{j_1} (2j+1) - 1}{\prod_{j=j_2+1}^{j_1} (2j+1)} \quad (14)$$

which conveniently reduces to (11) when  $j_2 = j_1 - 1$ .

### 5.2. Bounds

Substituting (14) in (8) we obtain the following bound for arbitrary states (assuming  $j_1 > j_2$ ):

$$p_{err}(j_1, j_2) \geq \frac{1}{2 \prod_{j=j_2+1}^{j_1} (2j+1)} \quad (15)$$

An interesting particular case is when the states to discriminate are  $\rho_{j_{max}}$  and  $\rho_{j_{max}-1}$ . Using (11) we obtain the following simple lower bound for the error probability:

$$p_{err}(j_{max}, j_{max}-1) \geq \frac{1}{2(2j_{max}+1)} = \frac{1}{2(N+1)} \quad (16)$$

It is interesting to note that the bounds do not depend on the number of parties  $N$ , despite what (16) may suggest. This is reasonable, since the difficulty of the discrimination rest on the difference of singlets contained in each state, hence on the values  $j_1, j_2$ . A different behaviour is expected for truly local measurements, for in that case a larger number of parties imply the possibility of a more complex dependence between the  $N$  different measurements, allowing to extract more information from the reduced states. In any case, the final error probability will be larger than the lower bounds obtained for separable measurements.

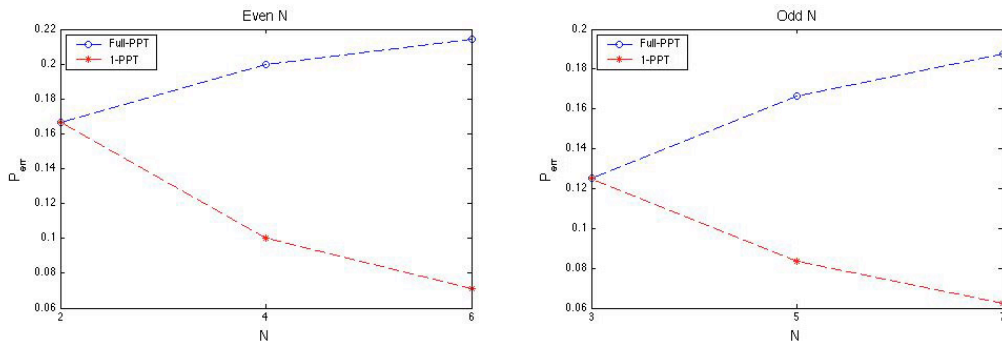
### 5.3. Full-PPT bound and discussion

Now we can compute numerically the error probability imposing  $k$ -PPT for all  $k$  and compare this strong bound with the weaker one resulting from 1-PPT. The problem of discrimination subjected to full-PPT conditions can be formulated as a *semidefinite program*[3] and efficiently solved with the Yalmip toolbox [12] for matlab, and the solver SeDuMi [13]. We show the results from  $N = 2$  to  $N = 7$ . For  $N > 7$  the computation time becomes excessively large.

- $J_{max}$  vs.  $J_{max} - 1$ .

First we analyse the bounds for the states corresponding to the maximum total angular momentum subspace and the immediately subsequent. The values for the bounds on the minimum error probabilities can be seen in fig.1. The cases with even and odd values of  $N$  are plotted apart in order to appreciate the monotonicity.

The 1-PPT-based bound goes to zero very fast, while the much stronger full-PPT-based one keeps growing with  $N$ . Hence, the bound, although strictly valid, is too weak to be useful for large values of  $N$ , when the error is expected to grow, because the states to discriminate differ only by a singlet, which is harder to detect as  $N \rightarrow \infty$ .

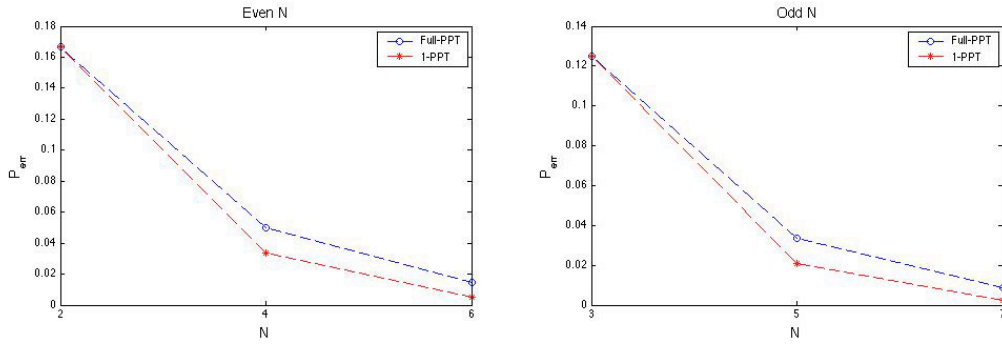


**Figure 1.** Minimum error probabilities for the discrimination of  $\rho_{J_{max}}$  and  $J_{max} - 1$ . On the left are plotted the probabilities for even parties  $N$ , while on the right are plotted the ones for odd  $N$ . The probabilities found with the full-PPT condition are pictured in blue, while the red ones correspond to the 1-PPT condition.



- $J_{max}$  vs.  $J_{min}$ .

We analyze a second particular case, which is also of interest. States  $\rho_{J_{max}}$  and  $\rho_{J_{min}}$  are clearly the most distinguishable ones, for a given number of parties, so in that case we expect the bound to go to zero as  $N$  grows. We see in fig.2 that now, although 1-PPT still yields a weaker bound than full-PPT, both bounds are very close and converge to zero quite fast.



**Figure 2.** Minimum error probabilities for the discrimination of states corresponding to  $J_{max}$  and  $J_{min}$ . On the left are plotted the probabilities for even parties  $N$ , while on the right are plotted the ones for odd  $N$ . The probabilities found with the full-PPT condition are pictured in blue, while the red ones correspond to the 1-PPT condition.

## 6. Summary and conclusions

We have studied the separability of multipartite rotationally invariant states and have found a simple condition for the positivity of the partial transposition respect to a single party. It provides a way to test states directly in the basis of total angular momentum eigenstates with simple inequalities.

As a way to obtain a bound to the local discrimination error, we have used the former criterion to bound the separable error. The bound can be obtained analytically and has a very simple form.

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## Appendix A: Quantum states and measurement

Here we provide a brief summary of concepts and formulae used throughout the work. For more detailed explanations see references [1, 2].

Any physical system is associated to a Hilbert space. The state of the system is fully described by a state vector  $|\psi\rangle$  of this space. Given two possible states, a system can be also in any superposition of them, due to the linearity of the Hilbert space.

A more general picture is the density matrix formalism: a system is described by a collectivity  $\{p_i, |\psi_i\rangle\}$ , where  $p_i$  represents the probability of the system being in the quantum state  $|\psi_i\rangle$ . The state is now represented by a matrix operator  $\rho$ :

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \quad (17)$$

Such a state is called *mixed*, in contrast to the pure states  $|\psi\rangle$ , and contain classical randomness besides the intrinsically quantum one.

*Quantum measurement.* The information of a quantum state can be retrieved via measurements. Formally, a measurement is represented by a POVM, which is a set of *positive* operators  $\{E_i\}$  such that  $\sum_i E_i = \mathbb{I}$ . This two conditions allow us to define a probability rule:

$$P(i | \rho) = \text{tr}(E_i \rho) \quad (18)$$

Then, if we associate an operator  $E_i$  to each possible outcome of a measurement, equation (18) represents the probability for the outcome  $i$ , provided we measure the state  $\rho$ .

*Quantum state discrimination with two hypothesis.* Given a state  $\rho$  which is equal to one of the two states  $\{\rho_1, \rho_2\}$  with *a priori* probabilities  $\{\pi_1, \pi_2\}$ , the problem of discrimination consist on determining with the minimum error which one of the states is  $\rho$ . Assigning each hypothesis to an operator of a two-element POVM and using (18), the error probability is:

$$p_{err} = \pi_1 \times P(\rho_2 | \rho_1) + \pi_2 \times P(\rho_1 | \rho_2) = \pi_1 \times \text{tr}(E_2, \rho_1) + \pi_2 \times \text{tr}(E_1, \rho_2) \quad (19)$$

The success probability is clearly:

$$p_{succ} = \pi_1 \times P(\rho_1 | \rho_1) + \pi_2 \times P(\rho_2 | \rho_2) = \pi_1 \times \text{tr}(E_1, \rho_1) + \pi_2 \times \text{tr}(E_2, \rho_2) \quad (20)$$

## Appendix B: Semidefinite programming

Semidefinite programming is a subfield of convex optimization. A semidefinite program can be cast in the following form

$$\begin{aligned} \min. & \langle c | x \rangle \\ \text{s.t.} & F(x) \succeq 0 \end{aligned} \quad (21)$$

with

$$F(x) = F_0 + \sum_{i=1}^m x_i F_i \quad (22)$$

where  $c \in \mathbb{R}^m$  and  $F_i \in \mathbb{R}^{n \times n}$  are given and  $x \in \mathbb{R}^m$  is the vector of variables. This is called the *primal* SDP. Alternatively, one can formulate the dual problem:

$$\begin{aligned} \max. \quad & -\mathbf{tr}F_0Z \\ \text{s.t.} \quad & \mathbf{tr}F_iZ = c_i, \quad i = 1, \dots, m \\ & Z \succeq 0 \end{aligned} \tag{23}$$

where  $F_0$ ,  $F_i$  and  $c_i$  are the same as in the primal problem, and  $Z \in \mathbb{R}^{n \times n}$  is the new variable.

Primal and dual forms of the problem are equivalent and provide bounds to the solution. That is the basis of many efficient algorithms for solving the problems.

*Minimum-error discrimination as a SDP*

$$\min. \quad p_{err} = \eta_1 \text{Tr}(E_2 \rho_1) + \eta_2 \text{Tr}(E_1 \rho_2) \tag{24}$$

$$\text{s.t.} \quad E_1 \succeq 0 \tag{25}$$

$$E_2 \succeq 0$$

$$E_1 + E_2 = \mathbb{I}$$

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