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MASTER THESIS WORK

BROADCASTING IN SPIN NETWORKS

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Broadcasting in spin networks

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Abstract. We present an implementation of $N \to M$ quantum broadcasting of mixed qubits in the equatorial plane in an unmodulated spin network. The broadcasting transformation is performed by the time evolution of the network. We focus in the XX Hamiltonian because its appropriate symmetry in the XY plane and its easy physical implementation. We study two scenarios with different initial state of the network: one in a pure state $|M/2, m\rangle$ and the other in the tensor product of maximally mixed states. We also focus in the $3 \to 4$ scenario, where in principle perfect phase covariant broadcasting is possible.

1. Introduction

In the same way as classical and quantum systems have fundamental differences in their description and behaviour, quantum information appears to be fundamentally different to its classical counterpart. New quantum features, such as entanglement, open new possibilities in information processing and communication. However, quantum laws also impose new important restrictions. One of the most far-reaching of these is the impossibility to perfectly copy an arbitrary unkown quantum state due to the linearity of quantum evolution. This important feature of quantum information, known as the *no-cloning theorem* [1, 2], is one of the cornerstones of quantum information theory, since classical information can always be perfectly copied. However, it is still possible to produce approximate copies [3], and perfect copies of orthogonal states [4]. There exist also probabilistic protocols for cloning nonorthogonal states [5], where perfect cloning is achieved with probability strictly less than one.

Since the quantum information encoded on pure states can only be cloned aproximately it is of outmost importance to find the optimal cloning protocol. The quality of the clones is usually quantified by the so called *fidelity*. Given two states ρ_1 and ρ_2, their fidelity is defined as [6]

$$\mathcal{F}(\rho_1, \rho_2) = \left(\mathrm{Tr} \left[\sqrt{\rho_1^{1/2} \rho_2 \rho_1^{1/2}} \right] \right)^2 . \tag{1}$$

Its value ranges from 0 to 1, with $\mathcal{F} = 1$ if and only if $\rho_1 = \rho_2$ and $\mathcal{F} = 0$ if and only if they are (pure) orthogonal states. Thus, the fidelity gives a quantitative measure of how similar two quantum states are. When considering qubits, as in our case, this expression can be simplified to

$$\mathcal{F}(\rho_1, \rho_2) = \mathrm{Tr} \left[\rho_1 \rho_2 \right] + 2 \sqrt{\det(\rho_1) \det(\rho_2)} . \tag{2}$$
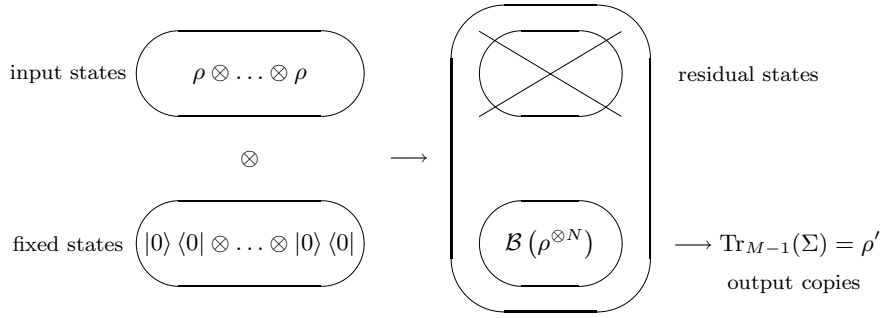
Figure 1. Scheme of the broadcasting transformation, with outer spins initialized to the $|0\rangle$ state. After a unitary evolution, the residual states are traced out and the local copy is obtained by tracing out all spins except one.

There is also another expression for the fidelity in terms of the Bloch vectors of the two density matrices. The Bloch form of a density matrix ρ is $\rho = \frac{1}{2}\left(\mathbb{1} + \mathbf{r}\cdot\sigma\right)$, where $\mathbf{r}$ is its Bloch vector with length $|\mathbf{r}| \leq 1$ and $\sigma = (\sigma_x, \sigma_y, \sigma_z)$ is a vector of the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3)$$

Then Equation (2) gives [6]

$$\mathcal{F}(\rho_1, \rho_2) = \frac{1}{2}\left(1 + \mathbf{r}_1\cdot\mathbf{r}_2 + \sqrt{(1-\mathbf{r}_1^2)(1-\mathbf{r}_2^2)}\right). \quad (4)$$

The process of imperfectly cloning a quantum state is generally performed by the unitary time evolution of an open system that contains a subsystem in the input state, some other subsystems whose state will evolve to become the aproximate copy, and an auxiliary subsystem acting as the environment. For pure qubits, the optimal $1 \rightarrow 2$ universal quantum cloning reaches a fidelity $\mathcal{F} = 5/6 \simeq 0.83$ [3, 7]. The quality of the output copies can be increased if some prior information about the input state is known. One class of states extensively studied is that corresponding to the *phase covariant quantum cloning*, which for pure qubits correspond to the cloning of states of the form $\frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\varphi}|1\rangle\right)$. These states are called *equatorial* because they lie in the equator (XY plane) of the Bloch sphere. They are characterized for having an arbitrary phase but a constant modulus, and are called phase covariant because their cloning fidelity is independent of φ. Its optimal value for the $1 \rightarrow 2$ cloning transformation in this case is $\mathcal{F} = \left(1 + 1/\sqrt{2}\right)/2 \simeq 0.854$ [8].

The no-cloning theorem only tells us about copying pure states. However, the situation changes when the states are mixed, because from the local point of view of a single user the local mixed state is indistinguishable from the partial trace of an entangled state. *Quantum broadcasting*, as first named in [9], is distributing the information encoded in N input systems equally prepared to $M > N$ users, who will have the same local states. Technically, it is a map $\mathcal{B}$ from the input Hilbert space $\mathcal{H}_{\text{in}} = \mathcal{H}^{\otimes N}$ to the output $\mathcal{H}_{\text{out}} = \mathcal{H}^{\otimes M}$. The single-site output or local copy is the trace over all the outer spins but one, $\rho' = \text{Tr}_{M-1}\left[\mathcal{B}(\rho^{\otimes N})\right]$. Figure 1 shows a scheme of the broadcasting tranformation.

In the case when the broadcast states are pure, ideal broadcasting coincides with quantum cloning, where the output is the tensor product of identical states, and is
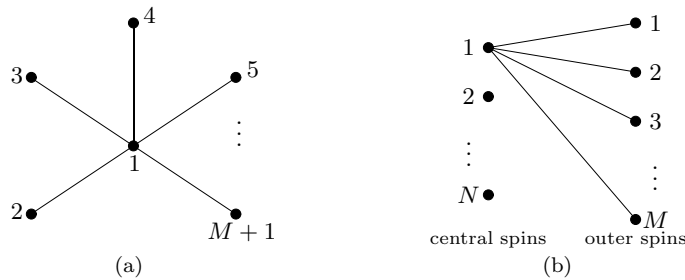
Figure 2. (a) Topology for the spin star network where each dot represents a site and lines represent a coupling J between the two sites. The spin in the first site interacts with the spins 2 to $M+1$. (b) Topology for the generalization of the spin star network for $N \to M$ cloning, where only interactions for the first central spin have been drawn. Each central spin (1 to N, on the left) interacts with every outer spin (1 to M, on the right) with a coupling J, but neither centers nor tips interact between them.

thus forbidden. However, in the case of mixed states only the local state of each final user has to be equal to the input state, while the global output can be correlated. This is still impossible in the case of one input copy and two output copies [9], but can be achieved for four input copies in a universal broadcasting (where an arbitrary state is broadcast) and for three in a phase covariant broadcasting [10, 11]. In this case the local copy has the same direction and length of the original state. However, it is improtant to notice that this does not imply an increase of the available information on the state of the input copies, as there exist correlations among the output copies.

The usual approach to the implementation of cloning transformations, as in most quantum computation tasks, is by means of quantum gates. One of the alternatives is the use of spin networks, which are systems of many spins with a given interaction pattern between them. The coupling constants are chosen in such a way that the task to be done is performed by the time evolution of the network. Cloning transformations of pure states in this framework have been studied recently [12, 13] in a spin star network like the one in Figure 2a. Since the couplings do not need to be time modulated the only control of the system resides on the preparation of its initial state and on the read out. This different approach to quantum protocols, although less flexible than the one with quantum gates, has the main advantatge of preserving the system better isolated and is therefore more robust against noise. The idea comes from *spin chains*, where the qubits are coupled only with the previous and following qubits and are specially suited for acting as quantum channels in the field of quantum communication [14, 15, 16, 17, 18, 19, 20, 21]. Spin networks have also been proposed for some uses in quantum computation apart from cloning [22, 23].

Motivated by the possibility of implementing quantum cloning transformations of pure states by means of spin networks, in this work we have studied the possibility of implementing quantum broadcasting of mixed states in the same framework. We have focused in the phase covariant situation in which the states to be copied are equatorial qubits of the form $\left(\mathbb{1} + r\cos\varphi\sigma_x + r\sin\varphi\sigma_y\right)/2$.

The report is organized as follows. In Section 2 we present the spin network and coupling model considered, and its initializations. In Section 3 we introduce the Wedderburn decomposition as a tool to simplify the interaction between the input

and the output spins, and in Section 4 we study the possibility of phase covariant broadcasting in the considered framework. Finally, in Section 5 we summarize the results and present our conclusions.

2. The spin network

We use an extension of the star network [24, 12] consisting of N centers and M tips, as shown in Figure 2b. In this architecture the centers encode the input qubits, which will be broadcast into the tips. Each center is connected to each tip. This model is fairly general and can be realized experimentally. However, we have not ruled out that other configuration might perform better. Our choice seems to have the necessary ingredients and has some symmetries that simplify considerably the calculations and the interpretation of the results.

The phase covariance in the input copies suggests a coupling Hamiltonian symmetric in the XY plane. We start with the XXZ family

$$H = \frac{1}{4} \sum_{ij} J_{ij} \left(\sigma_x^i \sigma_x^j + \sigma_y^i \sigma_y^j + \lambda \sigma_z^i \sigma_z^j \right) + \frac{B}{2} \sum_i \sigma_z^i \, , \tag{5}$$

which is phase covariant. Here $\sigma_{x,y,z}^i$ are the Pauli matrices acting on the ith site, J_{ij} is the coupling between sites i and j, and B an external magnetic field. The parameter λ gives the anisotropy of the model. There are two simple, limiting Hamiltonians in this family for $\lambda = 0$ and $\lambda = 1$, named the XX model and the Heisenberg model respectively. These interaction models can be easily implemented in solid state devices, which allow to realize the spin networks shown in Figure 2. In the present work we concentrate in one of them, the XX model, because it is the one among all the models studied in [12] that give higher fidelity for the cloning of pure states, and because there are realistic physical implementations with superconducting qubits which provide an effective Hamiltonian very close to the ideal one.

We also consider the couplings J_{ij} between central and outer spins to be unmodulated and equal to J if one of the sites, but not both, is a center, and $J_{ij} = 0$ otherwise. This makes the configuration invariant under permutations of the central and the outer spins, which is a requirement for our broadcasting transformation. Taking a spin network with N centers and M tips with the described couplings we can map the problem to the interaction between a spin-$N/2$ and a spin-$M/2$. The Hamiltonian is then

$$H = J \left(S_+^{\text{in}} S_-^{\text{out}} + S_-^{\text{in}} S_+^{\text{out}} \right) + B F_z \, , \tag{6}$$

where we have defined the input and output spins

$$S_{x,y,z}^{\text{in}} = \frac{1}{2} \sum_{i=1}^N \sigma_{x,y,z}^i \quad \text{and} \quad S_{x,y,z}^{\text{out}} = \frac{1}{2} \sum_{j=1}^M \sigma_{x,y,z}^j \tag{7}$$

with i running only for centers and j for tips, $F_z = S_z^{\text{in}} + S_z^{\text{out}}$ and $S_\pm = S_x \pm i S_y$ the raising and lowering operators. This Hamiltonian commutes with the total input and output spins, $(S^{\text{in}})^2$ and $(S^{\text{out}})^2$ and z component of the total angular momentum F_z, but not with the total angular momentum.

The fidelity depends not only on the Hamiltonian and the network topology but also on the initialization of the target states, which must not have any preferred direction in the XY plane. We have thus studied two extreme scenarios. In the first

one, the outer spins are initialized in the state $|M/2, m\rangle$. Here we focus on the state with $m = M/2$, which is a tensor product of spinup states and is therefore very simple to prepare. We also consider the other limiting case where the spins are initialized in the maximally mixed state $(\mathbb{1}/2)^{\otimes M}$.

Let us now briefly discuss what is the action of this Hamiltonian in a system of two pure spins l and j, one pointing in the x direction and the other one in z. The initial state is $|l, n_x\rangle_x |j, m\rangle_z$, where n_x is the eigenvalue of S_x^{in} and m the eigenvalue of S_z^{out}. This Hamiltonian will produce a rotation of both spins l and j without modifing their length, while holding constant the total projection in the z direction, that is $n_z + m_z$. We are interested in the output subsystem $|j, m\rangle_z$, which will eventually hold the clones. When the global state is let evolve under the action of (6), the expected value of S_x^{out} oscillates around 0 while S_z^{out} also oscillates, getting the maximum (and minimum) values of S_x^{out} when S_z^{out} is closer to 0. For a starting value of $m = 0$, however, no oscillation is observed.

3. Decomposition of the transformation

In our situation, where we have mixed states, the evolution is more involved than for pure states since we need to consider that the input spin can take different values for l. However, the Hamiltonian can be simplified if we consider a decomposition of the initial state. The Wedderburn decomposition is used to deal with unitary group representations in a Hilbert space, based on a decomposition into irreducible components. A tensor product of spaces $\mathcal{H}^{\otimes L}$ decomposes in

$$\mathcal{H}^{\otimes L} = \bigoplus_{j=\langle\langle L/2 \rangle\rangle}^{L/2} \mathcal{H}_l \otimes \mathbb{C}^{d_l} , \tag{8}$$

with $\langle\langle x \rangle\rangle$ denoting the fractional part of x and

$$d_l = \frac{2l+1}{L/2+j+1} \begin{pmatrix} L \\ L/2+l \end{pmatrix} \tag{9}$$

the degeneracy of the representation l. The spaces $\mathcal{H}_l$ are called *representation* spaces and support the irreducible representations of $\mathcal{H}^{\otimes L}$, while $\mathbb{C}^{d_l}$ are the *multiplicity* spaces with dimension d_l. In our case, where we are using qubits, $\mathcal{H} = \mathbb{C}^2$ and $\mathcal{H}_l = \mathbb{C}^{2l+1}$. In this decomposition, an operator X invariant under the permutation group $\mathbb{P}_L$ of the L copies has the form

$$X = \bigoplus_{l=\langle\langle L/2 \rangle\rangle}^{L/2} X_l \otimes \mathbb{1}_{d_l} . \tag{10}$$

This property can be used to decompose the initial state that we want to copy. This decomposition of a multiqubit of the form $\rho^{\otimes N}$ was first given in [25] and used for the construction of the optimal single qubit purification procedure. The form we use is that of [26, 11], which follows more closely our notation. We consider the single input mixed state ρ in the x direction with Bloch vector length r, namely $\rho = \frac{1}{2}(\mathbb{1} + r\sigma_x)$. We can do that without loss of generality because the output states will be initialized without any preferred direction in the XY plane in order to make our device phase covariant. Since $\rho^{\otimes N}$ is obviously invariant under permutations of the N single copies

it can be written, following Equation (10), as $\rho^{\otimes N} = \bigoplus_{l=\langle\langle N/2\rangle\rangle}^{N/2} \rho_l \otimes \mathbb{1}_{d_l}$, with the states in the representation space given by

$$\rho_l = (r_+ r_-)^{N/2} \sum_{n_x=-l}^{l} \left(\frac{r_+}{r_-}\right)^{n_x} |l, n_x\rangle \langle l, n_x| \otimes \mathbb{1}_{d_l} . \tag{11}$$

The states $|l, n_x\rangle$ are eigenstates of $J_x^{(l)}$ with eigenvalue n_x, and $r_\pm = (1 \pm r)/2$.

In the first two cases we initialize the outer spins always as a pure state, so it can be considered as a spin-$M/2$. Since the decomposition of $\rho^{\otimes N}$ is in representations of spin-l, the Hamiltonian can also be decomposed as a direct sum of Hamiltonians representing the interaction of a spin-l with a spin-$M/2$, with l ranging from $\langle\langle N/2\rangle\rangle$ to $N/2$, namely $H = \bigoplus_l H^{l,\frac{M}{2}}$. In the last case where we initialize the outer spins in the random state, the output spin can take values from $j = \langle\langle M/2\rangle\rangle$ to $j = M/2$, so we have to consider a more general interaction of a spin-l with a spin-j.

The Hamiltonian corresponding to the $1 \to M$ broadcasting is similar to the Jaynes-Cummings model [27] and the one studied in [24] but with an extra magnetic field. Its eigenstates are of the form

$$\frac{1}{\sqrt{2}} \left(\left|\frac{1}{2}, -\frac{1}{2}\right\rangle_{\text{in}} \left|\frac{M}{2}, m\right\rangle_{\text{out}} \pm \left|\frac{1}{2}, \frac{1}{2}\right\rangle_{\text{in}} \left|\frac{M}{2}, m-1\right\rangle_{\text{out}} \right) , \tag{12}$$

where states labeled "in" and "out" belong to input and output spaces respectively, and m ranges from $M/2$ to $-M/2 + 1$. It is important to note that states $\left|\frac{1}{2}, \pm\frac{1}{2}\right\rangle$ are not single spins but systems of N spin-1/2 particles with total angular momentum 1/2. There are two more eigenstates where only one of the two terms exist:

$$\left|\frac{1}{2}, \frac{1}{2}\right\rangle_{\text{in}} \left|\frac{M}{2}, \frac{M}{2}\right\rangle_{\text{out}} \quad \text{and} \quad \left|\frac{1}{2}, -\frac{1}{2}\right\rangle_{\text{in}} \left|\frac{M}{2}, -\frac{M}{2}\right\rangle_{\text{out}} . \tag{13}$$

The energy eigenvalues associated with all these eigenstates have the form

$$E = \pm J \sqrt{\left(\frac{M}{2} + m\right)\left(\frac{M}{2} - m + 1\right)} + B\left(m - \frac{1}{2}\right) . \tag{14}$$

The diagonalization of the general Hamiltonian for the $l \to M$ broadcasting is more involved, but can be simplified if it is expressed in blocks with same total magnetization f_z, that is $H^{l,\frac{M}{2}} = \bigoplus_{f_z} H_{f_z}^{l,\frac{M}{2}}$. Each block is tridiagonal, with elements given by

$$\left\langle l, n' \right| \left\langle \frac{M}{2}, f_z - n' \right| H_{f_z}^{l,\frac{M}{2}} \left| l, n \right\rangle \left| \frac{M}{2}, f_z - n \right\rangle = B f_z \delta_{n',n}$$

$$+ J \left[(l-n)(l+n+1)\left(\frac{M}{2} + f_z - n\right)\left(\frac{M}{2} - f_z + n + 1\right) \right]^{1/2} \delta_{n',n+1}$$

$$+ J \left[(l+n)(l-n+1)\left(\frac{M}{2} - f_z + n\right)\left(\frac{M}{2} + f_z - n + 1\right) \right]^{1/2} \delta_{n',n-1} . \tag{15}$$

4. $N \to M$ broadcasting

We have considered two limiting initializations for the outer spins: the pure states $|M/2, m\rangle$ and the tensor product of M maximally mixed states $(\mathbb{1}/2)^{\otimes M}$.

4.1. Initialization in $|M/2, m\rangle$

First we consider broadcasting with outer spins initialized in the $|M/2, m\rangle$ state. In this case, and recalling the initial state for the central spins in Equation (11), the global initial state for the network is

$$\Lambda(0) = (r_+ r_-)^{N/2} \bigoplus_{l=\langle\langle N/2 \rangle\rangle}^{N/2} \sum_{n,n'} \sum_{n''} \left(\frac{r_+}{r_-}\right)^{n''} (W_l)_{nn''} (W_l^\dagger)_{n''n'}$$

$$\times \left| l, n \right\rangle \left| \frac{M}{2}, m \right\rangle \left\langle l, n' \right| \left\langle \frac{M}{2}, m \right| \otimes \mathbb{1}_{d_l} . \tag{16}$$

Here we used the small Wigner d-matrix to write the state in eigenvectors of $J_z^{(l)}$, defining $(W_l)_{ab} \equiv \langle l, a | l, b_x \rangle$, which is usually found in the literature as $d^l_{n,n'}(\pi/2)$.

The time evolution of this state, $\Lambda(t) = e^{-iHt} \Lambda(0) e^{iHt}$, can be easily computed if we first write the state $\Lambda(0)$ in the basis of eigenstates of H. These are of the form $|l, j, f_z, k\rangle = \sum_i a_{i,k}^{l,j,f_z} |l, i\rangle |j, f_z - i\rangle$, where we have defined the coefficient $a_{i,k}^{l,j,f_z} = \langle l, i | \langle j, f_z - i | l, j, f_z, k\rangle$. We write the corresponding eigenvalues as E_k^{l,j,f_z}. The output copy can be found by tracing out the central spins and all but one outer spins. This can be done by means of the Clebsch-Gordan coefficients,

$$|jm\rangle = \sqrt{\frac{j+m}{2j}} \left| j - \frac{1}{2}, m - \frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle + \sqrt{\frac{j-m}{2j}} \left| j - \frac{1}{2}, m + \frac{1}{2} \right\rangle \left| \frac{1}{2}, \frac{1}{2} \right\rangle . \tag{17}$$

The resulting state is

$$\rho' = \begin{pmatrix} \rho'_{00} & \rho'_{01} \\ \rho'_{10} & \rho'_{11} \end{pmatrix} , \tag{18}$$

with elements given by

$$\rho'_{00} = \frac{(r_+ r_-)^{N/2}}{M} \sum_{l=\langle\langle N/2 \rangle\rangle}^{N/2} d_l \sum_{n=-l}^{+l} \sum_{n''} \left(\frac{r_+}{r_-}\right)^{n''} (W_l)_{n,n''} (W_l^\dagger)_{n'',n}$$

$$\times \sum_{k,k'} a_{n,k}^{l,\frac{M}{2},m+n} a_{n,k'}^{l,\frac{M}{2},m+n} e^{-iE_k^{l,\frac{M}{2},m+n} t} e^{iE_{k'}^{l,\frac{M}{2},m+n} t}$$

$$\times \sum_p a_p^{l,\frac{M}{2},m+n,k} a_p^{l,\frac{M}{2},m+n,k'} (\frac{M}{2} + m + n - p)$$

$$\times \Theta\left(\frac{M}{2} - m - n + p\right) \Theta\left(\frac{M}{2} + m + n - p\right) , \tag{19}$$

$$\rho'_{01} = \frac{(r_+ r_-)^{N/2}}{M} \sum_{l=\langle\langle N/2 \rangle\rangle}^{N/2} d_l \sum_{n=-l+1}^{+l} \sum_{n''} \left(\frac{r_+}{r_-}\right)^{n''} (W_l)_{n,n''} (W_l^\dagger)_{n'',n-1}$$

$$\times \sum_{k,k'} a_{n,k}^{l,\frac{M}{2},m+n} a_{n-1,k'}^{l,\frac{M}{2},m+n-1} e^{-iE_k^{l,\frac{M}{2},m+n} t} e^{iE_{k'}^{l,\frac{M}{2},m+n-1} t}$$

$$\times \sum_p a_p^{l,\frac{M}{2},m+n,k} a_{p+1}^{l,\frac{M}{2},m+n-1,k'} \sqrt{(\frac{M}{2} + m + n - p)(\frac{M}{2} - m - n + p + 1)}$$

$$\times \Theta\left(\frac{M}{2} - m - n + p\right) \Theta\left(\frac{M}{2} + m + n - p\right) , \tag{20}$$

$$\rho'_{10} = (\rho'_{01})^* \ , \tag{21}$$

$$\rho'_{11} = 1 - \rho'_{00} \ . \tag{22}$$

Here $\Theta(x)$ is the Heaviside step function with $\Theta(x) = 0$ for $x < 0$ and $\Theta(x) = 1$ for $x \geq 0$.

The matrix element ρ'_{01} has the same input phase, so the projection of the Bloch vector in the XY plane has the same direction of the input state. However, it can be seen that in general the state ρ' is not equatorial since it has a component in σ_z, i.e., the diagonal entries are not equal. This was expected because the output states are initialized in a non equatorial state, and the z component of the total spin is preserved during the evolution of the system. However, the length of the output Bloch vector projected in the equatorial plane, $r'_x = 2\rho_{01}$, still gives some information about the output copy, indicating by which factor its components in the initial x direction have been modified. The fidelity can be obtained following Equations (2) and (4) and comparing the input and the output copy, giving

$$\mathcal{F} = \frac{1}{2} + r\mathrm{Re}\left[\rho'_{01}\right] + \sqrt{1 - r^2}\sqrt{\rho'_{00}\rho'_{11} - |\rho'_{01}|^2} \ . \tag{23}$$

Let us comment the two limiting situations, $m = M/2$ and $m = 0$. The first one is interesting because it is very easy to prepare since all the outer spins are initialized in the spinup state, $|1/2, 1/2\rangle$. In this case, the output copy will be far from the equatorial state compared to thoses situations with an m closer to 0, because the z component of the total angular momentum must be preserved and the input spin can have at most $n_z = N/2$. Therefore, the fidelity will not be one but can be close to it. The second situation, with $m = 0$, would seem better because this problem is not present. However, in this case the local state of the copies remains maximally mixed during the whole evolution.

We succeeded in diagonalizing the Hamiltonian in Equation (6) for the $3 \rightarrow 4$ broadcasting, which is the simplest scenario in which perfect phase covariant broadcasting is possible [11]. We numerically maximized both the fidelity $\mathcal{F}$ and the scaling factor of the projected Bloch vector $p_x = r'_x/r$ over B, J and t in the interval $0 \leq t \leq 2000$. Table 1 shows the obtained results for an input state with $r = 0.5$. The obtained copy is closer to an equatorial qubit when the initial state has m value close to, but not equal to, zero. In this situation the fidelity gets its maximum value. However, this comes at the expense of having a much more mixed state than in the case with initial $m = M/2$. As an example, in the same scenario with $r = 0.5$, the Bloch vector length of the copy is $r' \approx 0.320$ for $m = 2$ and $r' \approx 0.19$. Finally, in the case with $m = 0$ the obtained copy is allways a maximally mixed state, which still gives a fidelity of 0.933. Figure 3 shows the evolution of the fidelity for a state initialized in $|2, 1\rangle$.

Table 1. Maximum fidelity $\mathcal{F}$ and scaling factor p_x for the $3 \rightarrow 4$ broadcasting for the network of Figure 2b and the output spin initialized in $|M/2, m\rangle$, input state with $r = 0.5$. Columns 2, 3 and 4 report the values of B, J and t at which the value $\mathcal{F}$ and p_x is obtained.

m	$\mathcal{F}$	B	J	t	m	p_x	B	J	t
2	0.940	-0.0834	1.2967	847.304	2	0.674	0.3048	1.2661	1365.5
1	0.978	0.9385	0.0910	105.44	1	0.652	0.4537	1.4243	1111.38
0	0.933	–	–	0	0	0	–	–	0

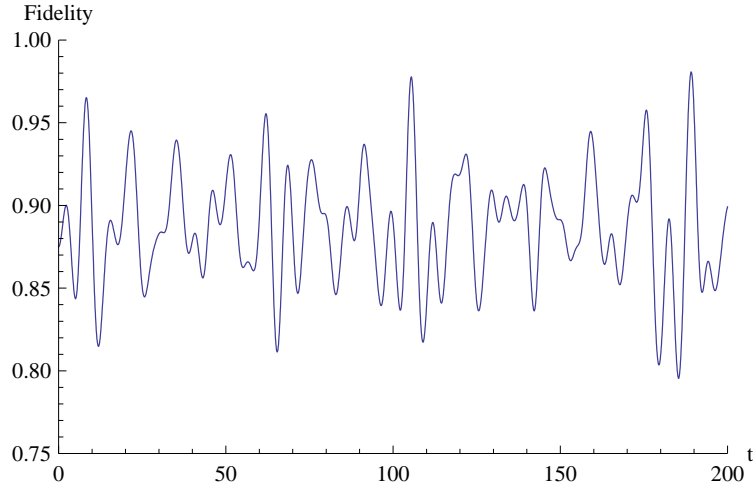**Figure 3.** Exact evolution of the fidelity in the $3 \to 4$ broadcasting for an output state initialized to $|2,1\rangle$ for $0 \leq t \leq 200$, and with $r = 0.5$. The values of B and J are set to those giving the maximum fidelity, as shown in Table 1. Time t is in arbitrary units.

4.2. Initialization in a random state

Finally we consider the outer states initialized in the product state of M maximally mixed states, $(\mathbb{1}/2)^{\otimes M}$. Now we have to use the decomposition in Equation (11) for both the central and the outer spins. However, in the latter case some simplifications can be done, since the length of the Bloch vector is zero and the state $(\mathbb{1}/2)^{\otimes M}$ can be taken diagonal in any direction, for example in z:

$$\frac{1}{2^M} \bigoplus_{j=\langle\langle M/2 \rangle\rangle}^{M/2} \sum_{m=-j}^{j} |j,m\rangle \langle j,m| \otimes \mathbb{1}_{d_j} . \tag{24}$$

The global initial state for the network is thus

$$\Lambda(0) = \frac{(r_+ r_-)^{N/2}}{2^M} \bigoplus_{l=\langle\langle N/2 \rangle\rangle}^{N/2} \sum_{n,n'} \sum_{n''} \left(\frac{r_+}{r_-}\right)^{n''} (W_l)_{nn''} (W_l^\dagger)_{n''n'}$$

$$\times \bigoplus_{j=\langle\langle M/2 \rangle\rangle}^{M/2} \sum_{m=-j}^{j} |l,n\rangle |j,m\rangle \langle l,n'| \langle j,m| \otimes \mathbb{1}_{d_l} \otimes \mathbb{1}_{d_j} . \tag{25}$$

This results in an output copy that, as in the previous case with initial state $|M/2, 0\rangle$, remains maximally mixed during the whole evolution.

5. Conclusions

In this work we have studied the $N \to M$ broadcasting of mixed qubits in the equatorial plane. We have shown that it is possible to implement the broadcasting transformation in an unmodulated spin network, which requires no external control. In particular, we considered XX couplings with the spins where the qubits are to

be copied initialized in two limiting states: $|M/2, m\rangle$ and $(\mathbb{1}/2)^{\otimes M}$. This model is a simple case in a bigger family of Hamiltonians with XY symmetry, and is a first step into a more general scenario concerning also a broader range of states. We calculated the local output copy after the evolution. The first case is the one that gives better results for the broadcasting transformation, although it does not produce equatorial copies. We also focused in the $3 \rightarrow 4$ scenario, the simplest in which perfect phase covariant broadcasting is possible. We succeded in diagonalizing the Hamiltonian and obtained the exact evolution of the system. The maximum fidelity was reached by initializing the output state in $|2, 1\rangle$, although in this case the copies are more mixed than those obtained by the initial state $|2, 2\rangle$. A physical implementation of this model is possible in solid state devices by means of superconducting qubits, which are available with the current technology.

Acknowledgments

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