Classification of plane germs: metric and valorative properties

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Contents

Introduction

The singularities of germs of plane curves constitute an old and nowadays very attractive field of research which combines techniques and viewpoints from different mathematical fields such as Geometry, Algebra, Analysis or Topology.

There is a well-established theory for analysis and classification of curve singularities since the classical time. We emphasize the algebraic approach of Zariski, and the geometric approach of Enriques, with the development of the theory of infinitely near points.

In this memory we follow the geometric approach of Casas' boof [1] for studying of the singularities of plane germs of curves, which updates Enriques' works to modern standards and reviews the modern development of the theorey from the point of view of infinitely near points.

Recently, Favre and Jonsson have considered the valuation theory from a new point of view. They give a Real-Tree structure to the some set of real valuations of the ring of curve germs. This allows them to obtain some important results in dynamical systems (see [3] or [2]).

This memory has a two sided goal: on one hand, we want to acquire skills with the tools and concepts of the singularity theory and the valuative theory, both the classical ones and the more recent ones. On the other hand, we want to study in depth the different implicit concepts and notions involved in the Favre and Jonsson's new approach, such as the ultrametric space structure of the set of irreducible germs of plane curves and the tree structure of the valuations.

The last goal allowed us to obtain new results, such as Theorem 1.3.1 or Theorem 1.5.22, which gives a new characteritzation of the equisingularity class of an irreducible curve.

This memory is structured in the following way: the first chapter is devoted to the study of the ultrametric space of irreducible plane germs of curves.

In Section 1.1 we introduce Casas' theory of infinitely near points, which includes the definition of proximity, the Noether formula and the Enriques Diagrams, which are very strong tools used in all the memory.

In Section 1.2 we give some previous definitions for our study. This section is divided into the classical definitions, mainly coming from Casas, Favre and Jonsson's works, and some other new definitions introduced in this work.

Section 1.3 contains some results on the distance between curves. Theorem 1.3.1 is a remarkable result in the study of the ultrametric space, whereas Proposition 1.3.3 will be very usefull in all the forthcoming sections.

Section 1.4 is devoted to give some methods for computing the distance between curves and comparing them. In particular, the triangles of the ultrametric space of plane germs of curves are studied.

Section 1.5 describes the set of inverse distances of one fixed irreducible curve. This set is a topological invariant, and it is related to most other invariants. In this section a method for computing this invariant is given, and we also prove that this invariant determines the equisingularity class of the curve.

Chapter 2 is devoted to the study of the valuations of the space of the plane germs of curves.

In Section 2.1 we give some basic properties about general classical valuative theory.

Section 2.2 describe $\mathbb{R}$-Trees, which is a central object in this work because of Favre and Jonsson approach of the valuative theory.

Section 2.3 relates the valuative theory in the ring of plane germs of curves with Casas' point of view. Concepts like blowing-up a valuation are introduced, and finally we give a classification of the valuations in that ring.

Section 2.4 is devoted to the Valuative Tree described by Favre and Jonsson in [4]. General properties are given, and we give some other properties, using the concepts introduced in the previous sections.

I gratefully acknowledge the assistance that I have received from many people during the development of this work. In particular I would like to thank Jesús Fernández and Charles Favre from their patience when I asked them questions. Víctor González helped me on the understanding of Casas, Favre and Jonsson's works, and he cooperated in the development of our results. Finally, this text never would have been finished without the constant advice, help and guide of my advisor Maria Alberich-Carramiñana.

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Chapter 1

The ultrametric space of germs of irreducible plane curves

This section is devoted to the study of the space of irreducible germs of plane curves, $\mathcal{C}$, seen as an ultrametric space. We will study properties of $\mathcal{C}$ as a metric space, such as comparison of distances between curves, measures of how far are the curves, etc. In order to achieve this goal, we need to understand the singularities of the germs of plane curves. We have chosen the geometrical approach that uses infinitely near points to describe singularities, following the book of Casas [1], which updates the classical approach of Enriques' theory.

1.1 Infinitely near points

The concept of infinitely near points was first introduced by M. Noether (1884) and their theory was developed by Enriques (1915). We will follow the modern approach that presents infinitely near points as points lying in different surfaces by means of a birational morphism, named blowing-up.

Hence, this section is devoted to the study of the blowing-ups of points in a complex surface and its properties. This concept is very important in the classical complex birational algebraic geometry, and in this section only the main results will be given. All the results given in this section are proved in [1], Chapter 3. An interested reader can found there an extensive study of the concepts introduced here.

1.1.1 Blowing-up

Let S be a complex surface, and fix a point $O \in S$. Let $U \subset S$ be an open neighbourhood of O, and let x, y local coordinates at U. Let us consider the projective (complex) line, $\mathbb{P}_1$, and $[z_0, z_1]$ some projective coordinates. Write

$$\overline{U} = \{(x, y, [z_0, z_1]) \in U \times \mathbb{P}_1 \mid xz_1 - yz_0 = 0\}.$$

Lemma 1.1.1. *$\overline{U}$ is a complex connected surface. The projection*

$$\pi : U \times \mathbb{P}_1 \to U$$

induces an analytic morphism which will be also called π*,*

$$\pi : \overline{U} \to U.$$

Furthermore, the restriction

$$\pi|_{\overline{U}\setminus\pi^{-1}(O)} : \overline{U} \setminus \pi^{-1}(O) \to U \setminus \{O\}$$

is an isomorphism.

Let $\overline{S}$ be the surface obtained by patching together $\overline{U}$ and $S\setminus\{O\}$ by the isomorphism $\pi|_{\overline{U}\setminus\pi^{-1}(O)}$. We can extend π from $\overline{S}$ to S, such that $\pi|_{\overline{S}\setminus\pi^{-1}(O)}$ is an isomorphism from $\overline{S} \setminus \pi^{-1}(O)$ to $S\setminus\{O\}$. Then $\pi : \overline{S} \to S$ is called the *blowing-up* of O on S. The projective line $E = \pi^{-1}(O)$ is called the *exceptional divisor* of π.

Proposition 1.1.2. *Let* $\pi' : \overline{S'} \to S$ *be another blowing-up from* S *to* O*, obtained from an open set* U' *and coordinates* $\{x', y'\}$*. Then there exists an unique* S*-isomorphism* $\varphi : \overline{S} \to \overline{S'}$*. Furthermore,* φ *induces a projectivity (lineal) between* E *and* $E' = \pi'^{-1}(O)$*.*

This proposition justifies that the blowing-up is well defined.

1.1.2 Transforming germs of curves

Lemma 1.1.3. *Let* C *be a curve on* S*, and let* $\overline{C} = \pi^*(C)$ *be its pullback. Then it holds that* $\overline{C} = \tilde{C} + m_O(C)E$*, where* $\tilde{C}$ *is a curve of* $\overline{S}$ *with finite intersection with* E*, that is,* $[E, \tilde{C}] < \infty$*.*

The curve $\tilde{C}$ is called *strict transform* of C, and the curve $\overline{C}$ is called *total transform* of C.

The intersection points between the strict transform of a curve C and the exceptional divisor E depends only on the tangent cone of C.

Theorem 1.1.4. *There is a linear projectivity* τ *between the pencil of tangent lines to* S *at* O *and the exceptional divisor* E*, such that for any curve* C *on* S*,* C *is tangent to the line* l *at* O *if and only if* $\tilde{C}$ *passes through* $\tau(l)$*. Moreover, the multiplicity of* l *as a component of the tangent cone of* C *is equal to* $[E, \tilde{C}]_{\tau(l)}$*.*

Corollary 1.1.5. *It holds that* $m_O(C) = \sum_{p\in E}[\tilde{C}, E]_p$*.*

Corollary 1.1.6. *If* C *is smooth at* O*, then* $\tilde{C}$ *is smooth at* p*, the (unique) point lying on* $\tilde{C}$ *and on* E*. If* C *is reduced, then* $\tilde{C}$ *is also reduced.*

Let C be a non-irreducible curve. Let C_l be the curve formed from all the branches of C with principal tangent l. Then

Proposition 1.1.7. *Let* $p = \tau(l)$*. Then the germ* $\tilde{C}_p = (\tilde{C}, p)$ *depends only on* C_l*. Furthermore, the correspondence* $C \to \tilde{C}_p$ *induces a bijection between the germs at origin with only principal tangent* l *and the germs at* p *with no component equal to* E_p*.*

Corollary 1.1.8. *Let* p *be a point of the exceptional divisor* E*. Let* (D, p) *be a germ of a curve in* $(\overline{S}, p)$ *which does not contain* E_p*. Then there is a unique germ* (C, O) *in* (S, O) *such that* $\tilde{C} = D$*.*

1.1.3 Infinitely near points

The exceptional divisor E is called *first infinitesimal neighbourhood* of O, and its points are called *points in the first infinitesimal neighbourhood* of O or just *points in the first neighbourhood* of O.

By induction, take $i > 0$ and we can define the set of *points in the i-th neighbourhood* of O as all the of the points in the first neighbourhood of p for any p in the $i-1$-th neighbourhood of O. By convention, O is in the 0-th neighbourhood of O. The points in any i-th neighbourhood of O are called *points infinitely near* to O, and the set of that points are denoted by $\mathcal{N}_O$. On the other hand, the points of S are called *ordinary points*.

There is a partial order at $\mathcal{N}_O$: we say $p \leqslant q$ if and only if q is infinitely near to q. Notice that an infinitely near point p is an ordinary point in some surface S_p, obtained after doing some blowing-ups.

$$\pi_p : S_p = S^i \to S^{i-1} \to \cdots \to S^1 = \overline{S} \to S$$

If $p \neq O$ is an ordinary point of S, it will be identified with $\pi^{-1}(p)$.

Let C be a germ on (S, O), and take $p \in \mathcal{N}_O$. Let us consider the successive total and strict transformations of C, until obtaining C_p and $\overline{C_p}$ on S_p. There are called *strict and total transformation of C with origin at p*. We will say that C *passes through* or *contains* p if C_p is not the empty germ. Notice that if C passes through p, then it passes also through any $q < p$. The set of all infinitely near points contained in C will be denoted by $K(C)$.

The *multiplicity of C at p* is defined as the multiplicity at p of C_p, and it will be denoted by $m_p(C)$. In particular, $m_p(C) > 0$ if and only if C passes through p. If $m_p(C) = 1$, p is called *simple point* of C, and if $m_p(C) > 1$, p is called *multiple point* of C.

Remark 1.1.9. *Notice that if C is an irreducible germ, then there is only one point in any infinitessimal neighbourhood. Therefore, $K(C)$ is a totally ordered set.*

Theorem 1.1.10 (Noether Formula)**.** *Let C, D be two germs of curves defined in (S, O). The intersection multiplicity $[C, D]_O$ is finite if and only if C and D share finitely many infinitely near points. In this case it holds*

$$[C, D]_O = \sum_{p \in K(C) \cap K(D)} m_p(C) m_p(D).$$

Remark 1.1.11. *$[C, D]_O < \infty$ if and only if C and D do not have any common branch. In particular, two different branches of a germ share finitely many infinitely near points.*

Lemma 1.1.12. *Let C, D be two germs of curves defined in (S, O), and let $\tilde{C}$ and $\tilde{D}$ be their strict transforms after blowing-up O. Then*

$$[C, D]_O = m_O(C) m_O(D) + \sum_{p \in E} [C, D]_p.$$

Corollary 1.1.13. *Let C and D be reduced germs at O. Then $C = D$ if and only if $K(C) = K(D)$.*

1.1.4 Proximity

Let $p, q \in \mathcal{N}_O$. We say that q is *proximate* to p if it belongs, as an ordinary point or as an infinitely near point, to the exceptional divisor E_p origined after blowing-up p. It is denoted by $q \to p$.

In other words, q is proximate to p if either $q \in E_p$ or $q \in \tilde{E}_p$.

Remark 1.1.14.

1. *Let q be a proximate point to p. Then q is in the first neighbourhood of p or in the first neighbourhood of a point proximate to p.*

2. *E_p is a smooth curve, and therefore it is irreducible at any point. Then, in the first neighbourhood of a point proximate to p there is exactly one point proximate to p.*

3. *$q \to p$ implies that $p \leqslant q$, but the converse does not hold.*

Theorem 1.1.15 (Proximity equalities)**.** *Let C be a germ of a curve on (S, O), and let p be a point of C. Then it holds*

$$m_p(C) = \sum_{q \to p} m_q(C)$$

Corollary 1.1.16.

$$m_p(C) \geqslant \sum m_q(C),$$

where the sum runs on the points q in the first neighbourhood of p.

In particular, if C is irreducible, then the sequence of multiplicities is a non-increasing one.

Lemma 1.1.17. *If q is an infinitely near to O, then q is proximate either exactly to one point, or it is proximate exactly to two points.*

If q is proximate to exactly one point, it is called *free*. Otherwise, it is called *satellite*.

Theorem 1.1.18. *Let C be an irreducible germ, and let p, q be two points on C, q in the first neighbourhood of p. Let $n = m_p(C), n' = m_q(C)$.*

Write

$$\begin{aligned}
&n = a_0 n' + r_1 \\
&n' = a_1 r_1 + r_2 \\
&r_1 = a_2 r_2 + r_3 \\
&\cdots \\
&r_{n-1} = a_n r_n
\end{aligned}$$

the Euclidean divisions. Then the points q_j in the j-th neighbourhood of q are proximate to p for all $1 \leqslant j < a_0$, and have multiplicity $m_{q_j}(C) = n'$. If $r_1 = 0$ (if and only if $n = 1$), then the point in the a_0-th neighbourhood of q is free. Otherwise, it is proximate to p and has multiplicity $m_{q_j}(C) = r_1$.

Furthermore, for any $1 \leqslant k \leqslant n$ and $1 \leqslant j < r_k$, write $i = a_0 + a_1 + \ldots + a_{k-1} + j$. The point q_i in the i-th neighbourhood of q is proximate q_{i-j-1} and has multiplicity $m_{q_i}(C) = a_k$. Write $i' = a_0 + a_1 + \ldots + a_k$. If $k = n$, then the point in the i'-th neighbourhood of q is free. Otherwise, it is proximate to $q_{j'}$, where $j' = a_0 + \ldots + a_{k-1} - 1$, and has multiplicity r_k.

Proposition 1.1.19. *The points in the first neighbourhood of O are free. There is exactly one satellite point in the first neighbourhood of a free point, and there are exactly two satellite points in the first neighbourhood of a satellite point.*

Proposition 1.1.20. *If q is a satellite point of C, then q is proximate to a multiple point of C. In particular, there is no satellite point on a smooth curve.*

1.1.5 Resolution of singularities, equisingularity class and Enriques Diagrams

Theorem 1.1.21. *A reduced germ contains at most finitely many multiple infinitely near points.*

Corollary 1.1.22 (Resolution of Singularities)**.** *Let C be a reduced germ of curve. There exists a finite sequence of blowing-ups such that the strict transform of C is smooth.*

Theorem 1.1.23 (Embedded Resolution of Singularities)**.** *Let C be a reduced germ of curve. There exists a finite sequence of blowing-ups*

$$\pi : S^i \to S^{i-1} \to \cdots \to S^1 \to S$$

such that $\tilde{C}$, the strict transform of C, is smooth and has only normal crossings (that is, transverse intersections) with $\pi^{-1}(O)$.

Let C be a germ of curve, and let $p \in K(C)$. The point p is called *singular* if either p is a multiple point of C, or p is a satellite point, or p precedes a satellite point of C. By Theorem 1.1.23, the number of singular points in a reduced germ is finite. Take a reduced germ C, and let $C_1, \ldots, C_r$ be the branches of C. Let p_i be the first non-singular point of C_i. Then q is a singular point of C_i if and only if $q < p_i$. Let us define the set $S(C)$ as follows:

$$S(C) = \{q \in \mathcal{N}_O \mid q \leqslant p_i \text{ for some } i\}.$$

Let C, D be two reduced germs. We say that C and D are *equisingular* if there exists a bijection $\varphi : S(C) \to S(D)$ such that for any $p, q \in S(C)$, $p > q$ if and only if $\varphi(p) > \varphi(q)$, and $p \to q$ if and only if $\varphi(p) \to \varphi(q)$.

Theorem 1.1.24. *Let C, D be two germs of curves. Then C and D are topological equivalents if and only if they are equisingulars.*

Let us introduce a graph, in fact a tree, which will be called *Enriques Diagram*, and which will be used to describe the singularity of a curve, encoding the information of the nature of the infinitely near points. The Enriques diagram of a curve C is a tree, the root corresponds to the point O, and the other nodes correspond to the other points of $K(C)$. There is an edge between the node of p and the node of q if and only if q is in the first neighbourhood of p. The edge is curved and tangent if q is free, and it is straight otherwise. If p and q have been represented, q is in the first neighbourhood of p, and there are more points on C proximate to p, these points are drawn in a straight halfline which starts at q and is orthogonal to the edge pq.

In this memory the Enriques diagrams will be represented satisfying these supplementary conventions:

- The origin is the bottom left point.
- If there is a straight halfline after a curve arc, this line is oriented to the bottom, that is going down.

1.1.6 Clusters and weighted clusters

A subset $K \subset \mathcal{N}_O$ is called a *cluster* if for all $q \in K$ and $p < q$ it holds that $p \in K$. A pair $\mathcal{K} = (K, \nu)$, where $\nu : K \to \mathbb{Z}$ is a map and K is a cluster, is called *weighted cluster*. It is usual to denote $\nu(p)$ by ν_p. We say that $\mathcal{K}$ is *consistent* if for any $p \in K$ it is satisfied

$$\nu_p \geqslant \sum_{q \to p,\ q \in K} \nu_q.$$

1.2 Preliminary definitions

1.2.1 Classical definitions

In this section we will give some basic definitions about algebraic geometry of plane germs of curves in $\mathbb{C}^2$. The reader can found more details about these definitions in [1] and [4].

Let $\mathcal{C}$ be the set of analytic and formal germs of irreducible curves in $(\mathbb{C}^2, O)$. For any $C \in \mathcal{C}$, we define the *curve valuation* $\nu_C : R \to \mathbb{R} \cup \{\infty\}$ as follows: $\nu_C(\psi) = \frac{C \cdot (\psi = 0)}{m_O(C)}$, where $R = \mathcal{O}_{\mathbb{C}^2, O}$ is the ring of holomorphic germs at the origin in $\mathbb{C}^2$, $\cdot$ means intersection multiplicity between the two curves C and $\psi = 0$, and $m_O(C)$ is the multiplicity of the curve C at the point O. If $C : \psi = 0$, we also write $\nu_\psi = \nu_C$.

The set $\mathcal{C}$ is equipped with an ultrametric distance: $d_{\mathcal{C}}(C, D) = \frac{m_O(C) m_O(D)}{C \cdot D}$. There are some well known properties of ultrametric spaces with the topology defined by the ultrametric distance:

- Every open ball is an open and a closed subset.
- Every point of a ball is its center, that is, if $q \in B_p(r)$, then $B_p(r) = B_q(r)$.
- The intersection of two balls B_1, B_2 is either empty, or B_1, or B_2.
- Ultrametric inequality: If $d(a, b) \neq d(a, c)$, then $d(b, c) = \max\{d(a, c), d(b, c)\}$.

Let $\mathcal{N}_O$ be the set of points infinitely near to O. This set is equipped with a natural order: $p < q$ if and only if $q \in \mathcal{N}_p$. Given a curve $C \in \mathcal{C}$, let $K(C)$ be the set of points lying on C infinitely near to O. We call it *cluster of the curve* C. Let $p \in K(C)$ be a point infinitely near to O. The set of points on C infinitely near to p is denoted by $K_p(C)$.

1.2.2 More definitions

In this section we will give some definitions, which will be used throughout this chapter.

Let $F(C) = \{O = p_0(C), p_1(C), p_2(C), \ldots\} \subset K(C)$ be the (totally ordered) set of free points on C (with $p_0(C) < p_1(C) < p_2(C) < \ldots$). Let $1 = n_0(C) \geqslant n_1(C) \geqslant \ldots$ be the *normalized multiplicity*[1] of the curve C at the points $p_0(C), p_1(C), \ldots$. Let $b_k(C)$ be the normalized multiplicity of C at the immediate predecessor (free or satellite) point of $p_k(C)$. For convention, $b_0(C) = 1$. We define $t_i(C) := \frac{C \cdot D_i}{m_O(C) m_O(D_i)}$, where D_i is any curve which passes through $p_j(C)$ with normalized multiplicity $n_j(C)$ for all j with $0 \leqslant j < i$, through $p_i(C)$ with normalized multiplicity $b_i(C)$, but does not pass through $p_{i+1}(C)$ and it is an smooth curve after $p_i(C)$. We have, for example, that $t_0(C) = 1$, $t_1(C) = 1 + n_1(C)$. It is clear that $p_i(C)$, $n_i(C)$, $b_i(C)$ and $t_i(C)$ depend only on the curve C. The magnitude $t_i(C)$ will be used for calculating the distance between two curves in a quick way. We will show that the set of inverse distances $\{t_i(C)\}$ determines the equisingularity class of the irreducible curve C (see forthcoming Section 1.5).

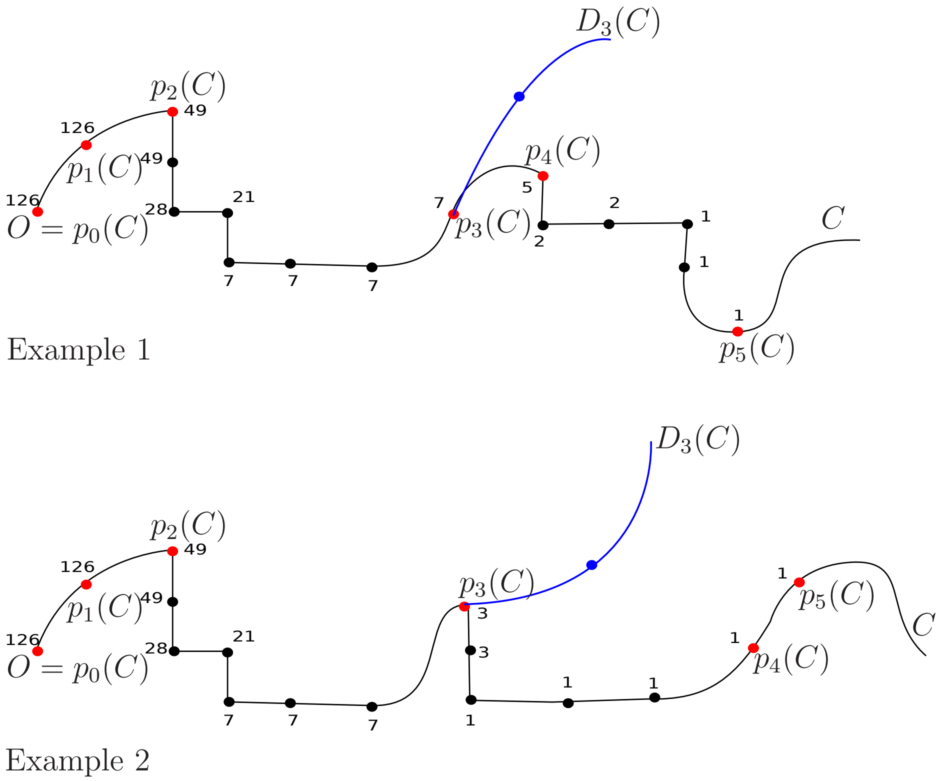


Figure 1.1: Two examples of the Enriques diagram of some curve C. In blue, the curve $D_3(C)$. In red, the points of $F(C)$.

Examples:

1. Let C be a curve with Enriques diagram as in Example 1 of Figure 1.1. The set $F(C)$ contains the points $p_0(C)$ (the origin), $p_1(C), p_2(C), p_3(C), \ldots$

   The normalized multiplicities are $n_0(C) = 126/126 = 1$, $n_1(C) = 1$, $n_2(C) = 7/18$, $n_3(C) = 1/18$, $n_4(C) = 5/126$, $n_i(C) = 1/126$, for all $i > 4$.

   The normalized multiplicities at the immediate predecessor are $b_0(C) = 1$, $b_1(C) = 1$, $b_2(C) = 1$, $b_3(C) = 1/18$, $b_4(C) = 1/18$, $b_i(C) = 1/126$, for all $i > 4$.

[1] The normalized multiplicity of a curve C at a point p is the multiplicity of C in p divided by the multiplicity of C at the origin O.

The inverse distances are $t_0(C) = \frac{126}{126\cdot 1} = 1$, $t_1(C) = \frac{126+126}{126\cdot 1} = 2$, $t_2(C) = \frac{126+126+49}{126\cdot 1} = 43/18$, $t_3(C) = \frac{126\cdot 18+126\cdot 18+49\cdot 7+\cdots}{126\cdot 18} = \frac{5425}{2268} = \frac{775}{324}$.

2. If C is the curve of Example 2, the normalized multiplicities are $n_0(C) = 1$, $n_1(C) = 1$, $n_2(C) = 7/18$, $n_3(C) = 1/42$, $n_i(C) = 1/126$, for all $i > 3$, and $b_0(C) = 1$, $b_1(C) = 1$, $b_2(C) = 1$, $b_3(C) = 1/18$, $b_i(C) = 1/126$, for all $i > 3$.

In the same way, if $\mathcal{K} = (K, m)$ is an unibranched cluster, we define $F(\mathcal{K}) = \{p_1(\mathcal{K}), \ldots\}$, as the set (finite or not) of the free points of K. Similarly, we can define $n_i(\mathcal{K})$, $b_i(\mathcal{K})$ and $t_i(\mathcal{K})$.

These following properties can be easily proved:

Lemma 1.2.1. *1. $p_k(C)$ is the immediate predecessor of $p_{k+1}(C)$ if and only if $b_k(C) = n_k(C)$.*

2. If $p_{k-1}(C)$ is the immediate predecessor of $p_k(C)$ then $b_k(C) = n_{k-1}(C)$.

3.

$$b_k(C) = \frac{\gcd(n_{k-1}(C)m_O(C), b_{k-1}(C)m_O(C))}{m_O(C)}.$$

1.3 On the distance between curves

In this section it will be proved that there are curves at any (rational) distance of any curve. This result will be proved in forthcoming Theorem 1.3.1. Next, let us discuss a case where this result is easily checked to be true, which will help to point out where the difficulty of the proof is hidden.

Keep the notation introduced in the previous section. Assume that C is an irreducible curve and $t \in \mathbb{Q}$, $k \in \mathbb{N}$ satisfy that $t_{k-1}(C) < t < t_k(C)$ and $n_k(C) = b_k(C)$. In this case the successor of $p_k(C)$ in $K(C)$ is free, that is, in the Enriques diagram of $K(C)$ there is not a *stair* beginning at the point $p_k(C)$. In order to attain the desired distance t^{-1}, we will take a curve D that passes through $p_1(C), \ldots, p_{k-1}(C)$ with normalized multiplicity $n_i(C)$ and through $p_k(C)$ with a suitable multiplicity which is fixed by considering only the multiplicities at the points preceding $p_k(C)$. Since, by hypothesis, C does not share any point with D after $p_k(C)$, the result easily follows (see Figure 1.2).

However, in the case $n_k(C) > b_k(C)$, C and any chosen D may share a number of satellite points after $p_k(C)$, and hence the result will not be as easy as before. We need to study carefully the distances in these cases.

Theorem 1.3.1. *Let C be an irreducible curve and take $t \in \mathbb{Q}$, $t \geqslant 1$. There exists $D \in \mathcal{C}$ such that $d_{\mathcal{C}}(C, D) = \frac{1}{t}$.*

Before proceeding to the proof, we need some preliminary results.

Lemma 1.3.2. *Let $0 < n_1 < n_0$ be two natural numbers. Let $n_0 = q_1 n_1 + n_2$, $n_1 = q_2 n_2 + n_3, \ldots,\ n_{r-1} = q_r n_r$ be the Euclidean divisions. Then $n_1 n_0 = q_1 n_1^2 + q_2 n_2^2 + \cdots + q_r n_r^2$.*

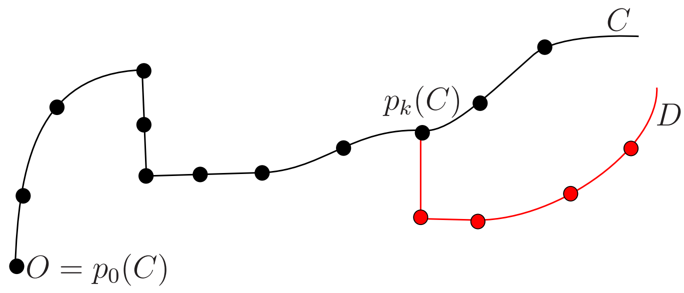


Figure 1.2: If $p_k(C)$ is a free point, for every multiplicity (lower than $n_k(C)$) there are curves that pass through this point with that multiplicity.

Proof. We will argue by induction on r. If $r = 1$, then $n_0 = q_1 n_1$, so $n_0 n_1 = q_1 n_1^2$.

In the general case, we apply the induction hypothesis on n_1 and n_2: $n_1 n_2 = q_2 n_2^2 + \cdots + q_r n_r^2$. So $n_0 n_1 = (q_1 n_1 + n_2) n_1 = q_1 n_1^2 + q_2 n_2^2 + \cdots + q_r n_r^2$. □

Proposition 1.3.3. *Let $C, D \in \mathcal{C}$ be two curves for which $p_i(C) = p_i(D)$ for any $0 \leqslant i \leqslant N$, but $p_{N+1}(C) \neq p_{N+1}(D)$. Suppose that $n_N(C) \geqslant n_N(D)$. Then*

$$\frac{C \cdot D}{m_O(C) m_O(D)} = t_{N-1}(C) + b_N(D) n_N(D).$$

Proof. Since $p_i(C) = p_i(D)$ at any $0 \leqslant i \leqslant N$, we have that $n_i(C) = n_i(D)$ for all $0 \leqslant i < N$. Applying the Noether formula (theorem 3.3.1 of [1]) we have that

$$\frac{C \cdot D}{m_O(C) m_O(D)} = \sum_{q \in K(C) \cap K(D)} \frac{m_q(C)}{m_O(C)} \frac{m_q(D)}{m_O(D)} = \sum_{\substack{q \in K(C) \cap K(D) \\ q < p_N(C)}} \frac{m_q(C)}{m_O(C)} \frac{m_q(D)}{m_O(D)} +$$

$$+ \sum_{\substack{q \in K(C) \cap K(D) \\ p_N(C) \leqslant q < p_{N+1}(C)}} \frac{m_q(C)}{m_O(C)} \frac{m_q(D)}{m_O(D)} = t_{N-1}(C) + \sum_{\substack{q \in K(C) \cap K(D) \\ p_N(C) \leqslant q < p_{N+1}(C)}} \frac{m_q(C)}{m_O(C)} \frac{m_q(D)}{m_O(D)}.$$

If the point $p_{N+1}(C)$ is in the first neighbourhood of the point $p_N(C)$, the result is clearly true, since in the last sum there is only one point, $p_N(C)$, and $n_N(C) = b_N(C) = b_N(D)$, so $\frac{m_q(C)}{m_O(C)} = n_N(C) = b_N(D)$ and $\frac{m_q(D)}{m_O(D)} = n_N(D)$.

Now assume that $p_{N+1}(C)$ is not in the first neighbourhood of $p_N(C)$. We distinguish two cases:

1. $b_N(C) < n_N(C) < n_N(D)$. Let C' be a curve which passes through $p_i(C)$ for any $0 \leqslant i < N$ with normalized multiplicity $n_i(C') = n_i(C)$, and through $p_N(C)$ with normalized multiplicity $n_N(C') = b_N(C)$. Applying the Noether formula (theorem 3.3.1 of [1]) it is obtained

$$d_{\mathcal{C}}(C', C) = \frac{1}{t_{N-1}(C) + n_N(C) b_N(C)} < d_{\mathcal{C}}(C', D) = \frac{1}{t_{N-1}(C) + n_N(D) b_N(C)}.$$

Now, by the ultrametric inequality this implies that $d_{\mathcal{C}}(C, D) = \frac{1}{t_{N-1}(C) + n_N(D) b_N(D)}$, as wanted.

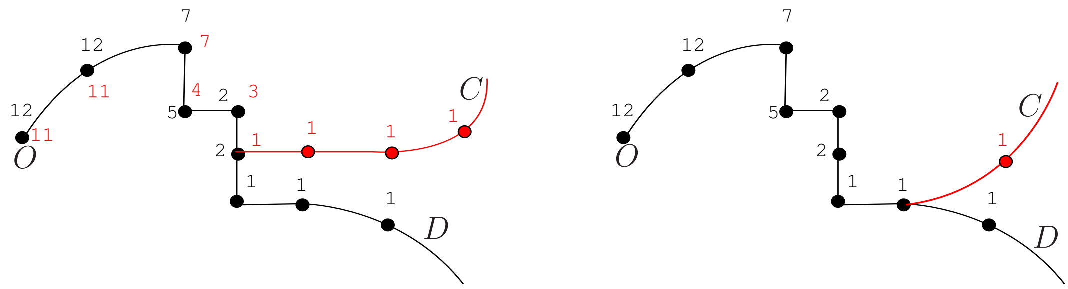
2. $b_N(C) < n_N(C) = n_N(D)$. In this case Lemma 1.3.2 is used for computing the last sum:

$$\sum_{\substack{q \in K(C) \cap K(D) \\ p_N(C) \leqslant q < p_{N+1}(C)}} \left(\frac{m_q(D)}{m_O(D)} \right)^2 = \frac{1}{m_O(D)^2} m_N(D) m_p(D) = n_N(D) b_N(D),$$

where p is the point immediate predecessor of $p_N(C)$.

□

Remark 1.3.4. *Suppose that we have two curves C and D like in the case 2 of the proof of Proposition 1.3.3. Let C' be a curve that passes through $p_i(C)$ for any $0 \leqslant i < N$ with normalized multiplicity $n_i(C') = n_i(C)$, and through $p_N(C)$ with normalized multiplicity $n_N(C') = b_N(C)$. In that case C', C, D form an equilateral triangle.*



Case 1: $n_{N-1}(C) < n_N(C) < n_N(D)$ Case 2: $n_{N-1}(C) < n_N(C) = n_N(D)$

Figure 1.3: Two examples that illustrate the two cases occurring in the proof of Proposition 1.3.3.

Proposition 1.3.3 enable us to calculate the distance between two curves. In particular, it provides a very useful method when $b_N(C) > n_N(C) > n_N(D)$, that is, the last point in $K(C) \cap K(D)$ is a satellite point. In this case, it will be said that *C and D split up at a stair.*

Examples:

1. Let C, D be curves that have Enriques diagram as in case 1 of Figure 1.3. We have that $N = 2$, $n_0(C) = 1$, $n_1(C) = 1$, $n_2(C) = 7/11$, and $n_0(D) = 1$, $n_1(D) = 1$, $n_2(D) = 7/12$. So $t_1(C) = 2$. By the Noether formula:

$$\frac{C \cdot D}{m_O(C) m_O(D)} = \frac{11 \cdot 12 + 11 \cdot 12 + 7 \cdot 7 + 4 \cdot 5 + 3 \cdot 2 + 1 \cdot 2}{12 \cdot 11} = \frac{341}{132} = \frac{31}{12} = \\ = t_1(C) + n_1(C) n_2(D).$$

2. Let C, D be curves that have Enriques diagram as in case 2 of Figure 1.3. We have that $N = 2$, $n_0(C) = n_0(D) = 1$, $n_1(C) = n_2(D) = 1$, $n_2(C) = n_2(D) = 7/12$.

So $t_1(C) = 2$. By the Noether formula:

$$\frac{C \cdot D}{m_O(C)m_O(D)} = \frac{12 \cdot 12 + 12 \cdot 12 + 7 \cdot 7 + 5 \cdot 5 + 2 \cdot 2 + 2 \cdot 2 + 1 \cdot 1 + 1 \cdot 1}{12 \cdot 12} =$$
$$= \frac{372}{144} = \frac{31}{12} = t_1(C) + n_1(C)n_2(D).$$

As a consequence of Proposition 1.3.3 we obtain a recursive formula for $t_i(C)$:

Corollary 1.3.5. *Let C be an irreducible curve. Then the following formula holds:*

$$t_i(C) = t_{i-1}(C) + n_i(C)b_i(C).$$

In particular, if $p_{i+1}(C)$ is in the first neighbourhood of $p_i(C)$, then $t_i(C) = t_{i-1}(C) + n_i(C)^2$.

Now, Theorem 1.3.1 can be proved:

Proof of Theorem 1.3.1. The succession $\{t_i(C)\}_{i \in \mathbb{N}}$ tends to infinity because $d_{\mathcal{C}}(C, C) = 0$. So there exists $N \in \mathbb{N}$ such that $t_N(C) \leqslant t < t_{N+1}(C)$. If $t = t_N(C)$, then the proof is trivial: we take a curve D such that passes through $p_i(C)$ with relative multiplicity $n_i(D) = n_i(C)$ $(0 \leqslant i \leqslant N)$ but that does not pass through $p_{N+1}(C)$. From the definition of $t_N(C)$, the distance between C and D is $1/t_N(C)$.

Suppose now that $t_N(C) < t < t_{N+1}(C)$. We define $k \in \mathbb{Q}$ as follows:

$$k = \frac{t - t_N(C)}{b_{N+1}(C)}.$$

Then $k < n_{N+1}(C)$ because $t_{N+1}(C) = t_N(C) + b_N(C)n_N(C) < t = t_N(C) + b_N(C)k$.

Let D be a curve that passes through $p_i(C)$ with relative multiplicity $n_i(D) = n_i(C)$ $(0 \leqslant i < N)$ and through $p_N(C)$ with relative multiplicity $n_N(D) = k < n_N(C)$. By Proposition 1.3.3, D satisfies what we want. ☐

1.4 Computation of the distance between curves

In this section the previous results will be applied to compare the distance between any two curves. We will show an intuitive method for studying the triangles in the ultrametric space $\mathcal{C}$ and we will also give an easy method for computing the distance between two curves, using the Noether formula and Proposition 1.3.3.

1.4.1 Distance between two curves

Let $C, D \in \mathcal{C}$ be two curves. Let us consider $K(C)$ and $K(D)$ their clusters of infinitely near points. Let us draw the Enriques diagram of $K(C)$ and $K(D)$, and let us write their normalized multiplicities ($n_k(C)$ and $n_k(D)$ respectively) at their free points, and their normalized multiplicities at the immediate predecessors of the free points ($b_k(C)$ and $b_k(D)$ respectively). Then the distance between C and D can be computed by using the following formula:

Proposition 1.4.1. *Let C, D be two irreducible curves. Then*

$$\frac{1}{d_{\mathcal{C}}(C,D)} = \sum b_k(C) \min\{n_k(C), n_k(D)\},$$

where the sum runs over all points $p_k \in F(C) \cap F(D)$.

Proof. This formula is derived directly from Proposition 1.3.3 and Corollary 1.3.5. □

Remark 1.4.2. *Notice that in all points of the set $F(C) \cap F(D)$ holds $b_k(C) = b_k(D)$, and all points but perhaps the last satisfy $n_k(C) = n_k(D)$.*

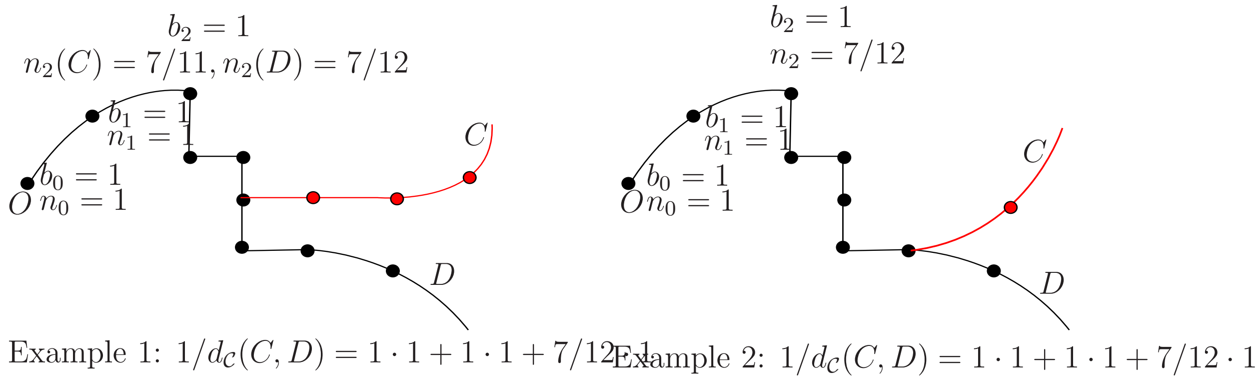


Figure 1.4: Two examples of computing the distance between two curves.

1.4.2 Triangles in $\mathcal{C}$

In this section the ultrametric inequality will be used to compare the relative position of three curves. In an ultrametric space all the triangles are isosceles or equilateral. Therefore, in our case, given three curves, either they form an equilateral triangle, or there are two nearer curves that are equidistant from the other curve.

Given two irreducible curves, if the last point that they share is free, we say that the curves *split up at a free point*; otherwise we say that the curves *split up at a stair* (cf. the Enriques Diagrams of the curves).

First, the case where a pair of curves splits up at a free point is considered.

Proposition 1.4.3. *Let C, D, E be three curves such that any pair of them splits up at a free point. Then the nearer curves are those that share more free points. If the three curves share the same points, then they form an equilateral triangle.*

Proof. The result is obtained by applying directly the Noether formula (see Figure 1.5). □

Now we will show that this fact also applies to the general case. If two curves share more free points than the third, then these two curves are closer than the third. This result will be proved in forthcoming Theorem 1.4.9. Let us check first an easy case:

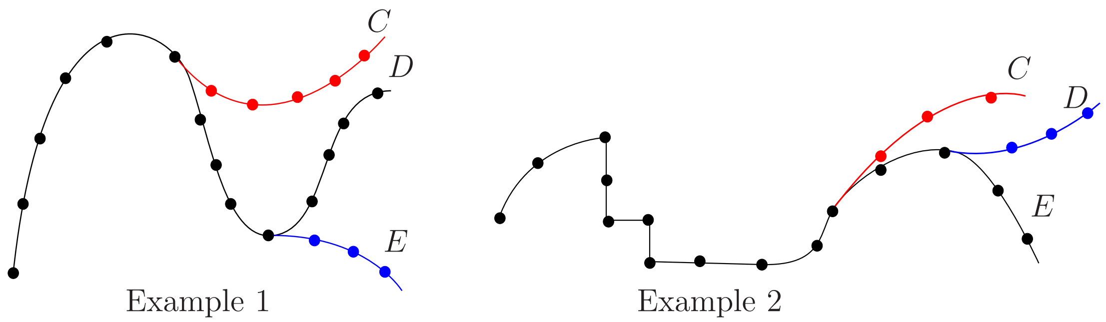


Figure 1.5: Two examples of isosceles triangles. The curves D and E are closer than C.

Proposition 1.4.4. *Let $C, D, E \in \mathcal{C}$ be curves. Suppose that $F(D) \cap F(E) \supsetneq F(C) \cap F(D) \cap F(E)$ and $n_p(C) \leqslant n_p(D) = n_p(E)$ at the last point p in $F(C) \cap F(D) \cap F(E)$. Then $d_{\mathcal{C}}(C, D) = d_{\mathcal{C}}(C, E) > d_{\mathcal{C}}(D, E)$.*

Proof. According to Remark 1.4.2, $n_q(C) \leqslant n_q(D) = n_q(E)$ for all point q in $F(C) \cap F(D) \cap F(E)$. Then the result follows applying the Noether formula (see Examples 1 and 2 in Figure 1.6). □

Remark 1.4.5. *It is worth to notice that if $p_k \in F(D) \cap F(E)$, then $\{p \in F(D) \mid p < p_k\} \subset F(E)$, from the definition of cluster.*

Therefore, it cannot occur that $F(D) \cap F(E) \supsetneq F(C) \cap F(D) \cap F(E)$ and $F(C) \cap F(D) \supsetneq F(C) \cap F(D) \cap F(E)$ at the same time, i.e., Proposition 1.4.4 cannot be applied two times at the same curves for concluding that $d_{\mathcal{C}}(C, D) > d_{\mathcal{C}}(D, E) > d_{\mathcal{C}}(C, D)$.

Now the case of three curves sharing the same common free points is considered. Let $C_1, C_2, C_3 \in \mathcal{C}$ be curves such that $F(C_1) \cap F(C_2) = F(C_1) \cap F(C_3) = F(C_2) \cap F(C_3) = \{p_1, \ldots, p_N\}$. Let C be a curve which passes through p_N with multiplicity $n_N(C) = b_N(C)$ (i.e., such that the point $q \in C$ in the first neighbourhood of p_N is also a free point), and such that $F(C) \cap F(C_i) = \{p_1, \ldots, p_N\}$ for all $i = 1, 2, 3$. (see figure 1.7).

Let d_i be the distances between the curves C_i and C for all $i = 1, 2, 3$. These distances $d_i = d_{\mathcal{C}}(C_i, C)$ can be easily computed by virtue of Proposition 1.3.3:

$$\frac{1}{d_i} = t_N(C) + b_N(C) n_N(C_i).$$

Lemma 1.4.6.

$$d_{\mathcal{C}}(C_i, C_j) = \max\{d_i, d_j\}.$$

Proof. Suppose that $d_i \neq d_j$. Then

$$d_{\mathcal{C}}(C_i, C_j) = \max\{d_{\mathcal{C}}(C, C_i), d_{\mathcal{C}}(C, C_j)\} = \max\{d_i, d_j\}.$$

Suppose now that $d_i = d_j$. Then C, C_i, C_j form an equilateral triangle (see Remark 1.3.4). Therefore, $d_{\mathcal{C}}(C_i, C_j) = d_i = d_j$. □

After ordering the three curves if needed, we can assume that $d_1 \leqslant d_2 \leqslant d_3$.

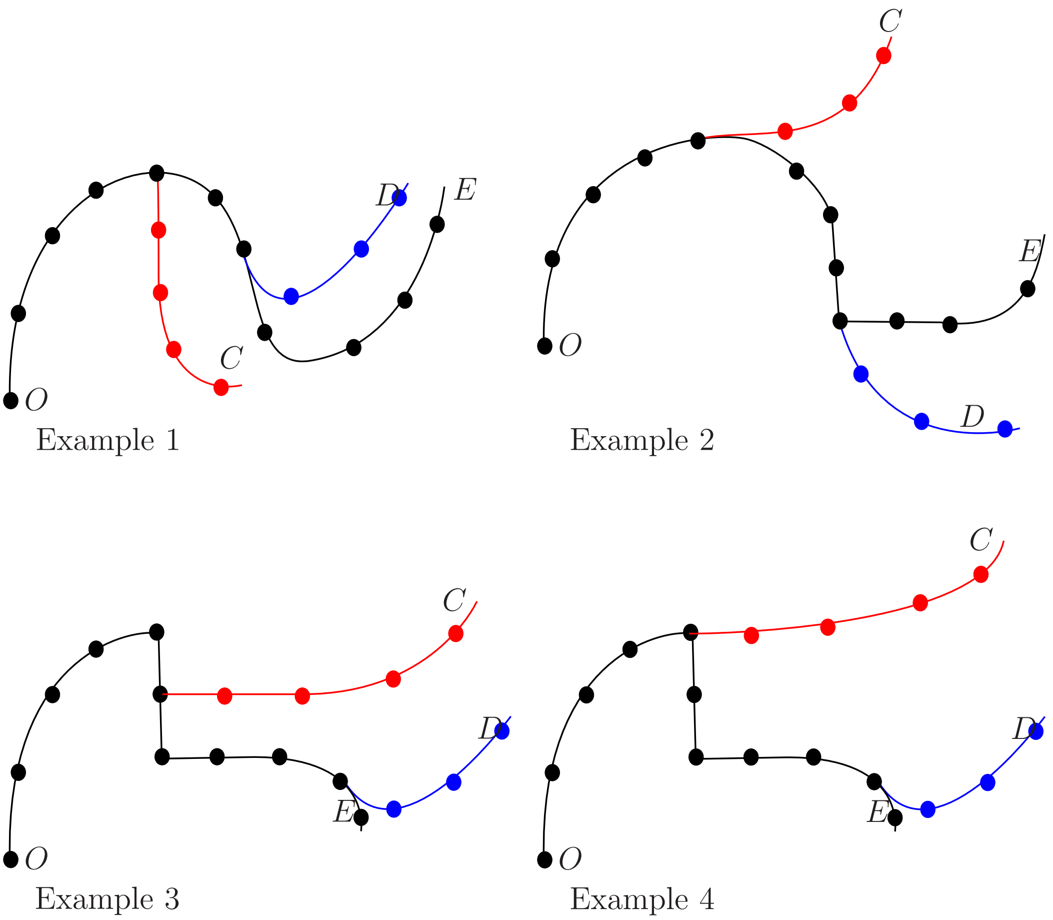


Figure 1.6: More examples of isosceles triangles. The curves D and E are closer than C.

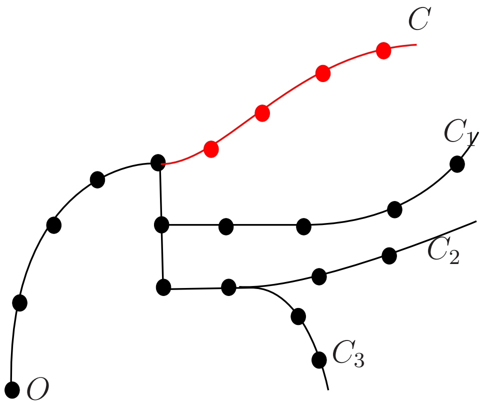


Figure 1.7: Three curves C_1, C_2 and C_3 with the same common free points and the curve C.

Proposition 1.4.7. *The curves C_1, C_2 and C_3 form an equilateral triangle if and only if $d_2 = d_3$. Furthermore, if $d_2 < d_3$, then C_1 and C_2 are closer than C_3.*

Proof. By Lemma 1.4.6, the following formulas hold:

$$d_{\mathcal{C}}(C_1, C_2) = d_2 \quad d_{\mathcal{C}}(C_1, C_3) = d_{\mathcal{C}}(C_2, C_3) = d_3$$

We distinguish two cases:

$d_1 \leqslant d_2 < d_3$. In this case, $d_{\mathcal{C}}(C_1, C_2) < d_{\mathcal{C}}(C_1, C_3) = d_{\mathcal{C}}(C_2, C_3) = d_3$, namely the three curves form an isosceles triangle.

$d_1 \leqslant d_2 = d_3$. In this case the curves form an equilateral triangle.

□

Figure 1.8 illustrates all the cases listed in the proof of Proposition 1.4.7.

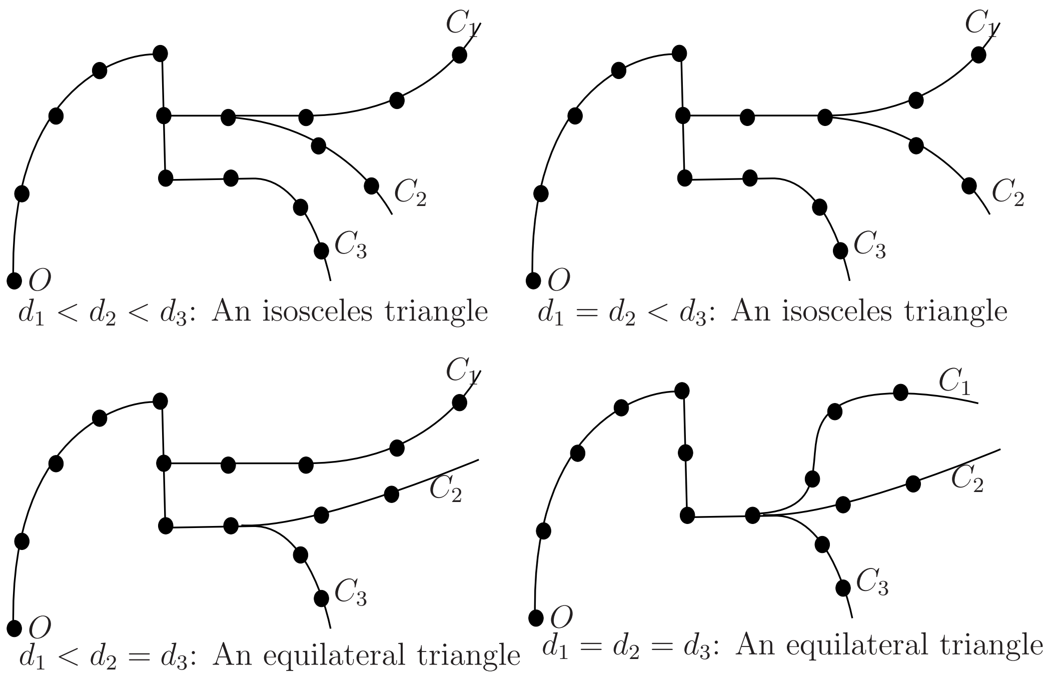


Figure 1.8: Different kinds of triangles formed by curves sharing the same common free points.

Summarizing the previous results we have:

Corollary 1.4.8. *Let C_1, C_2, C_3 three curves sharing the same common free points, and let p_k be the last common free point. Suppose that $n_k(C_1) \geqslant n_k(C_2) \geqslant n_k(C_3)$. Then, the three curves form an equilateral triangle if and only if $n_k(C_2) = n_k(C_3)$. Otherwise, C_1 and C_2 are closer than C_3.*

To conclude the study of the case of three curves sharing the same common free points, we will give a method for comparing the distances d_1, d_2 and d_3 at *first sight* on the Enriques diagrams of the singularities of the curves. Let C_1, C_2 be two curves splitting up at a satellite point q. Let p_k be the last common free point of C_1 and C_2. Let C be a curve such that passes through p_k with multiplicity $n_k(C) = b_k(C)$, and we define $d_i = d_{\mathcal{C}}(C, C_i)$.

We give a is a simple rule to know whether $d_1 < d_2$, $d_1 > d_2$ or $d_1 = d_2$: "*going right* is nearer to C than *going free*, which is nearer to C than *going down*". This must be read on the drawing of the Enriques diagrams of these curves. Assume that the drawing of the stair at which the curves split up starts going down. At the splitting point q there are three possibilities for the Enriques diagram of C_i to go on, depending on the nature (satellite or free) of the point in the first neighbourhood of q which C_i passes through: either go to a free point (and we say C_i is *going free*), or go to one of the two satellite points, which, due the convention on the drawing of the Enriques diagrams, one lies on a straight segment going to the right (and we say C_i is *going right*), and the other on a straight segment going down (and we say C_i is *going down*).

Therefore our rule says that, when the curves C_1 and C_2 split up, if one of them is going right, then it is nearer from C than the other; if there is a curve going down, it is farer from C than the other; if the two curves have a free point just after the last common point, then they are equidistant from C. Figure 1.9 illustrates all these cases.

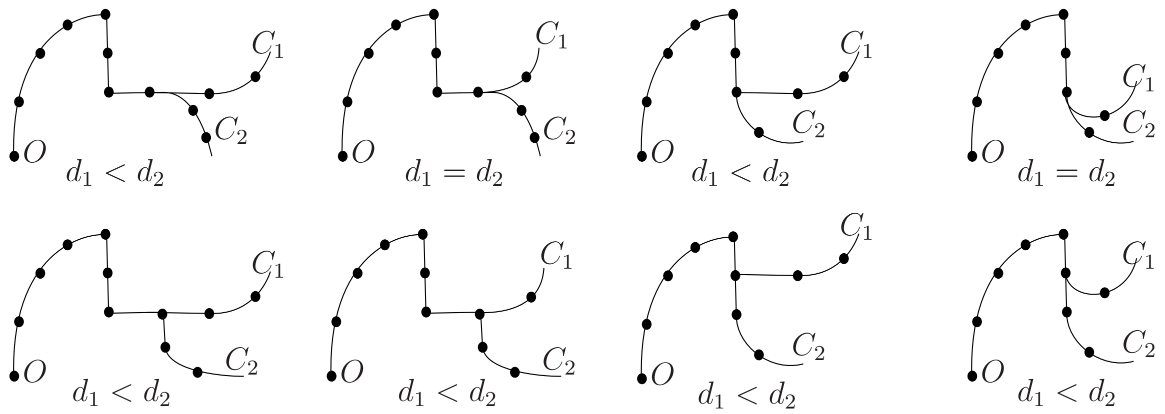


Figure 1.9: All the different ways of splitting up at a satellite point.

The proof of this rule is based in basic properties of continued fraction.s It is known that the structure of the stair of a Enriques diagram of a curve C that starts at a free point p_i is given by the continued fraction of the rational number $n_i(C)/b_i(C)$ (see Theorem 1.1.18), so this result is obtained by applying Proposition 1.5.11 and Noether Formula.

Let us reconsider the case of three curves C, D, E which $F(D) \cap F(E) \supsetneq F(C) \cap F(D) \cap F(E)$. A generalization of Proposition 1.4.4 will be given:

Theorem 1.4.9. *Let $C, D, E \in \mathcal{C}$ be curves. Suppose that $F(D) \cap F(E) \supsetneq F(C) \cap F(D) \cap F(E)$. Then $d_{\mathcal{C}}(C, D) = d_{\mathcal{C}}(C, E) > d_{\mathcal{C}}(D, E)$.*

Proof. The case where $n_p(C) \leqslant n_p(D) = n_p(E)$ for all $p \in F(C) \cap F(D) \cap F(E)$ is proved in Proposition 1.4.4. Let us prove the other case.

Suppose now that $n_N(D) = n_N(E) < n_N(C)$, where p_N is the last common free point of C, D and E (see Example 3 or Example 4 of Figure 1.6). Let us consider two auxiliary curves F and G (see Figure 1.10): take F a curve which passes through p_N with normalized multiplicity $n_N(F) = n_N(D) = n_N(E)$ and sharing no other free point with D or E after p_N, and take G a curve which passes through p_N with normalized multiplicity $n_N(F) = b_N(D) = b_N(E) = b_N(C)$ and sharing no other free point with C after p_N.

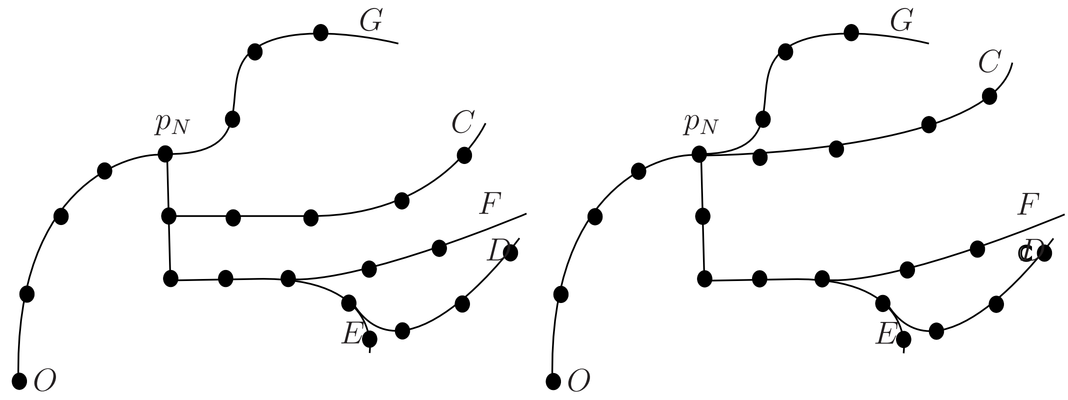


Figure 1.10: Examples of the auxiliary curves F and G.

Now C, F and D share the same common free points and we are under the hypothesis of Lemma 1.4.6 and Proposition 1.4.7. Let us define $d_C = d_{\mathcal{C}}(C, G), d_F = d_{\mathcal{C}}(F, G), d_D = d_{\mathcal{C}}(D, G)$. By Lemma 1.4.6, it holds that $d_C < d_F = d_D$. Notice that G is play the role of the curve C of Lemma 1.4.6. By Proposition 1.4.7, C, F, D form an equilateral triangle.

Now let us compare the distance $d_{\mathcal{C}}(E, C)$ with the distances $d_{\mathcal{C}}(E, D)$ and $d_{\mathcal{C}}(D, C)$. We know that $d_{\mathcal{C}}(D, C) = d_{\mathcal{C}}(D, F)$, but, owing to Proposition 1.4.7, $d_{\mathcal{C}}(D, F) > d_{\mathcal{C}}(D, E)$. Therefore

$$d_{\mathcal{C}}(D, C) > d_{\mathcal{C}}(D, E) \Rightarrow d_{\mathcal{C}}(E, C) = \max\{d_{\mathcal{C}}(D, C), d_{\mathcal{C}}(D, E)\} = d_{\mathcal{C}}(D, C) > d_{\mathcal{C}}(D, E).$$

Hence C, D, E form an isosceles triangle, and DE is the shortest side, as we wanted to show. □

1.5 The inverse distances $t_i(C)$

This section 1.5 is devoted to describe the set of inverse distances of one fixed irreducible curve, which will be denoted by $T(C)$. This set is a topological invariant, and it is related to most other invariants. We will give some methods for computing this set and we also prove that this set determines the equisingularity class of the curve C.

1.5.1 Continued fractions

In this section we recall some basic results about continued fractions. The reader is referred to [6], Chapter I or in [5], Chapter X for their proof. Continued fractions will be a key tool to work in the space $\mathcal{C}$, c.f. Theorem 1.1.18.

Let $\alpha = \alpha_0$ be a real number. Let a_0 be the integral part of α_0 (i.e., the highest integer less or equal than α_0). If α is an integer, then $\alpha = a$. Otherwise, there exists another real number $\alpha_1 > 1$ such that $\alpha_0 = a_0 + \frac{1}{\alpha_1}$. Inductively, we let

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}},$$

where a_n is the integral part of α_n, and $\alpha_{n+1} > 1$ is a real number (if $a_n \neq \alpha_n$).

It will be written as

$$[a_0, a_1, \ldots, a_n] := a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}}.$$

It is clear that the process will finish if and only if $\alpha = \alpha_0$ is rational. In this case, it holds $\alpha = [a_0, a_1, \ldots, a_n]$.

Let $a_0, \ldots, a_n, \ldots$ be integers such that $a_i > 0$ for all $i > 0$. We define

$$p_n(a_0, \ldots, a_n) := \begin{cases} 0 & \text{if } n = -2, \\ 1 & \text{if } n = -1, \\ a_n p_{n-1}(a_0, \ldots, a_{n-1}) + p_{n-2}(a_0, \ldots, a_{n-2}) & \text{if } n \geqslant 0. \end{cases}$$

Similarly we define

$$q_n(a_0, \ldots, a_n) := \begin{cases} 1 & \text{if } n = -2, \\ 0 & \text{if } n = -1, \\ a_n q_{n-1}(a_0, \ldots, a_{n-1}) + q_{n-2}(a_0, \ldots, a_{n-2}) & \text{if } n \geqslant 0. \end{cases}$$

It will be written $p_n(\alpha)$ and $q_n(\alpha)$ or just p_n and q_n instead of $p_n(a_0, \ldots, a_n)$ and $q_n(a_0, \ldots, a_n)$ when no confusion is possible.

Proposition 1.5.1. *For any $n \geqslant 0$, p_n and q_n are integers, relatively primes, and it holds*

$$\frac{p_n}{q_n} = [a_0, \ldots, a_n].$$

Corollary 1.5.2. *Let x be any real number. Then*

$$[a_0, a_1, \ldots, a_{n-1}, x] = \frac{x p_{n-1} + p_{n-2}}{x q_{n-1} + q_{n-2}}.$$

Let α be a real number. We construct the finite or infinite sequence $\{a_0, a_1, \ldots\}$. Then the sequence of fractions $\{p_n/q_n\}$ is called the *continued fraction* of α.

Proposition 1.5.3. *For any $n \geqslant 1$ we have*

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n.$$

Corollary 1.5.4.

$$[a_0, \ldots, a_{n-1}] - [a_0, \ldots, a_n] = \frac{(-1)^n}{q_n q_{n-1}}.$$

Corollary 1.5.5. *$\{q_1, q_2, \ldots\}$ is a strictly increasing sequence of positive integers, i.e., $0 < q_1 < q_2 < \cdots$*

Proposition 1.5.6. *For any $n \geqslant 2$ we have*

$$q_n p_{n-} - p_n q_{n-} = (-1)^{n-1} a_n.$$

Corollary 1.5.7.

$$[a_0, \dots, a_{n-2}] - [a_0, \dots, a_n] = \frac{(-1)^{n-1} a_n}{q_n q_{n-2}}.$$

Proposition 1.5.8. *For any $n \geqslant 1$ we have*

$$\frac{q_n}{q_{n-1}} = [a_n, \dots, a_1].$$

Proposition 1.5.9. *The sequence $\{p_{2n}/q_{2n}\}$ is strictly increasing and converge to α, and the sequence $\{p_{2n-1}/q_{2n-1}\}$ is strictly decreasing and also converge to α. Furthermore, we have*

$$\frac{1}{2q_{n+1}} < \frac{1}{q_n + q_{n+1}} < |q_n \alpha - p_n| < \frac{1}{q_{n+1}}.$$

Proposition 1.5.10. *Let $[a_0, \dots, a_n] = [b_0, \dots, b_m]$ be two continued fractions with a_n, $b_m > 1$. Then $n = m$ and $a_i = b_i$ for all i.*

Proposition 1.5.11. *Let $[a_0, \dots, a_{r-1}, b_r, \dots, b_n], [a_0, \dots, a_{r-1}, c_r, \dots, c_m]$ two continued fractions with $b_n > 1, c_m > 1$, and suppose that $b_r > c_r$. Then*

$$\begin{cases} [a_0, \dots, a_{r-1}, b_r, \dots, b_n] > [a_0, \dots, a_{r-1}, c_r, \dots, c_m] & \text{if } r \text{ is even,} \\ [a_0, \dots, a_{r-1}, b_r, \dots, b_n] < [a_0, \dots, a_{r-1}, c_r, \dots, c_m] & \text{if } r \text{ is odd.} \end{cases}$$

1.5.2 The inverse distances at satellite points. The set $T(C)$

Let C be a germ of an irreducible curve at O. Given $p \in K(C)$ (free or satellite), we define $t_p(C)$ as

$$t_p(C) = \frac{1}{d_{\mathcal{C}}(C, D_p)},$$

where D_p is a curve which passes through $p$and satisfying that the point in the first neighbourhood of p lying on D_p is free and that D_p share no points with C after p. We can also define $n_p(C), b_p(C)$ in all (free and satellite) points of $K(C)$: $n_p(C)$ is the normalized multiplicity of the curve C at the point p, and $b_p(C)$ is the normalized multiplicity of C at the immediate predecessor of p. In this section we will write t_p, b_p, n_p instead of $t_p(C), b_p(C), n_p(C)$, since no confusion may arise.

It is clear that these definitions extend the previous ones for free points on C. It was seen in Corollary 1.3.5 a recursive formula for computing the inverse distances at free points:

$$t_p = t_{p'} + n_p b_p,$$

where p is a free point and p' is the last free point which precedes p.

If C is a smooth germ, then $n_p = b_p = 1$ for all $p \in K(C)$. Therefore, $\{t_p \mid p \in K(C)\} = \mathbb{N}$.

Suppose now that C is not a smooth germ. Let p be the first point such that $n_p < b_p$. Then $b_p = 1, n_p < 1$. So $n_p = [0, a_1, \dots, a_k]$ for some $a_1, \dots, a_k \in \mathbb{N}$.

Proposition 1.5.12. *Let C be a smooth germ and keep the notations of this section. Take $r, i \in \mathbb{N}$ such that $0 < r \leqslant k$ and $0 \leqslant i < a_r$. Let q be the point in $K(C)$ in the $a_1 + a_2 + \cdots + a_{r-1} + i + 1$-th neighbourhood of p, and p' be the immediate predecessor point of p (which is free as we have taken p). Then*

$$t_q = \begin{cases} t_p = t_{p'} + [0, a_1, \ldots, a_k] = [a_0, a_1, \ldots, a_k] & \text{if } r \text{ is odd,} \\ t_{p'} + [0, a_1, \ldots, a_{r-1}, i+1] = [a_0, a_1, \ldots, a_{r-1}, i+1] & \text{if } r \text{ is even,} \end{cases}$$

where a_0 is the number of free points preceding p.

Proof. Let D be an irreducible curve which passes through q such that the point on D in the first neighbourhood of q is free and D and C has not more common points after q. It is clear (see Theorem 1.1.18) that $n_p(D) = [0, a_1, \ldots, a_r, i+1]$, and $b_p(D) = b_p(C) = 1$. By Proposition 1.3.3,

$$t_q = d_{\mathcal{C}}(C, D)^{-1} = t_p + \min\{n_p(C), n_p(D)\},$$

now the result follows in virtue of Proposition 1.5.11. □

Proposition 1.5.13. *Let C be a non-smooth germ. Let p be the first point on C such that $n_p < b_p$, and write $n_p = [0, a_1, \ldots, a_k]$ for some $a_1, \ldots, a_k \in \mathbb{N}$. Suppose $\tilde{O}$ is the first free point in $K(C)$ after p. Then*

$$\begin{aligned}\{t_q \mid q \in K(C), q < \tilde{O}\} = \{t_q \mid q \in K(C), t_q \leqslant t_p\} = \\ = \{[a_0, a_1, \ldots, a_{r-1}, i+1] \mid 0 \leqslant r \leqslant k, r \text{ even }, 0 \leqslant i < a_r\},\end{aligned}$$

where a_0 is the number of free points preceding p.

Proof. Let $p_0, p_1, \ldots, p_{a_0-1}$ be the points on C before p ($p_0 = O$, p_i in the i-th neighbourhood of O). It is clear that $t_{p_i} = i + 1$. The set $\{q \in K(C), q < \tilde{O}\}$ is the (disjoint) union of the set $\{p_0, p_1, \ldots, p_{a_0-1}\}$, the point p, and the set $S = \{q \in K(C) \mid p < q < \tilde{O}\}$. Notice that any point of S satisfies the conditions of Proposition 1.5.12.

Hence, the set $\{t_q \mid q \in K(C), q < \tilde{O}\}$ is the union of $\{t_{p_0}, \ldots, t_{p_{a_0}-1}\} = \{1, 2, \ldots, a_0\}$, $t_p = [a_0, a_1, \ldots, a_k]$ and $\{t_q \mid q \in S\}$. Applying Proposition 1.5.12, the last set is

$$\{[a_0, a_1, \ldots, a_{r-1}, i+1] \mid 0 < r \leqslant k, r \text{ even }, 0 \leqslant i < a_r\}.$$

Therefore

$$\{t_q \mid q \in K(C), q < \tilde{O}\} = \{[a_0, a_1, \ldots, a_{r-1}, i+1] \mid 0 \leqslant r \leqslant k, r \text{ even }, 0 \leqslant i < a_r\}.$$

In order to conclude the proof we have to see that $\{t_q \mid q \in K(C), q < \tilde{O}\} = \{t_q \mid q \in K(C), t_q \leqslant t_p\}$. It is enough to prove that at any $q \in K(C), q \geqslant \tilde{O}$ it holds $t_q > t_p$. Let D_p, D_q be curves such that D_p passes through p, the point of D_p in the first neighbourhood of p is free and does not belong to $K(C)$ (the same with D_q). By definition, $t_p = 1/d_{\mathcal{C}}(C, D_p)$ and $t_q = 1/d_{\mathcal{C}}(C, D_q)$. The proof ends by applying Theorem 1.4.9. □

These results allow to describe the set $\{t_q \mid q \in K(C), q < \tilde{O}\}$ (where $\tilde{O}$ is the first free point on C which has an satellite point as a predecessor) in terms of n_p (where p is the first free point such that $n_p < b_p$). The goal is to generalize these results and to describe the set $\{t_q \mid q \in K(C)\}$ in terms of the set $\{n_p \mid p \text{ is a free point such that } n_p < b_p\}$.

Applying Noether Formula (theorem 3.3.1, of [1]) the computation of $\{t_q \mid q \in K(C), q \geqslant \tilde{O}\}$ can be done in the following way:

1. Compute t_p (see Proposition 1.5.12), where p is the first free point on C such that $n_p < b_p$.
2. Let $\tilde{O}$ be the first free point on C after p. In $K_{\tilde{O}}(C)$, let $\tilde{n}_q = n_q/b_{\tilde{O}}$, $\tilde{b}_q = b_q/b_{\tilde{O}}$, and compute $\{\tilde{t}_q\}$, the values of a curve with Enriques Diagram as $K_{\tilde{O}}(C)$ and normalized multiplicities n_q. The computation of these values can be done applying Proposition 1.5.12 and Proposition 1.5.13.
3. By the Noether formula, $t_q = t_p + b_p^2 \tilde{t}_q$.

Remark 1.5.14. *The value $b_{\tilde{O}}$ is determined from n_p. Namely, $b_{\tilde{O}}$ is the inverse of the denominator of n_p.*

Proof. Notice that $b_p = 1$. Therefore, as consequence of Lemma 1.2.1, 3,

$$b_{\tilde{O}} = \frac{\gcd(n_p m_O(C), b_p m_O(C))}{m_O(C)} = \frac{\gcd(n_p m_O(C), m_O(C))}{m_O(C)}.$$

□

This algorithm and the previous results prove the following:

Theorem 1.5.15. *Let C be a non-smooth curve and let $\{p_1, \ldots, p_N\}$ be the free points on C for which $n_{p_i} < b_{p_i}$. Define*

$$t_n = \begin{cases} 0 & \text{if } n = 0, \\ t_{p_1} & \text{if } n = 1, \\ d_{n-1}^2 (t_{p_n} - t_{p_{n-1}}) & \text{if } 2 \leqslant n \leqslant N, \\ \infty & \text{if } n = N+1. \end{cases}$$

Write $t_n = [a_0^n, \ldots, a_{k^n}^n]$ (take $t_{N+1} = \infty = [\infty]$). Define

$$d_n = \begin{cases} 1 & \text{if } n = 0, \\ d_{n-1} q_{r_n}(a_0^n, \ldots, a_{k^n}^n) & \text{if } 1 \leqslant n \leqslant N. \end{cases}$$

Compute

$$T_n = \{[a_0^n, a_1^n, \ldots, a_{r-1}^n, i+1] \mid 0 \leqslant r \leqslant k^n, r \text{ even}, 0 \leqslant i < a_r\} \text{ for } 1 \leqslant n \leqslant N+1.$$

Then

$$T(C) = \{t_p(C) \mid p \in K(C)\} = \bigcup_{n=1}^{N+1} \{t_{p_{n-1}} + \frac{x}{d_{n-1}^2} \mid x \in T_n\}.$$

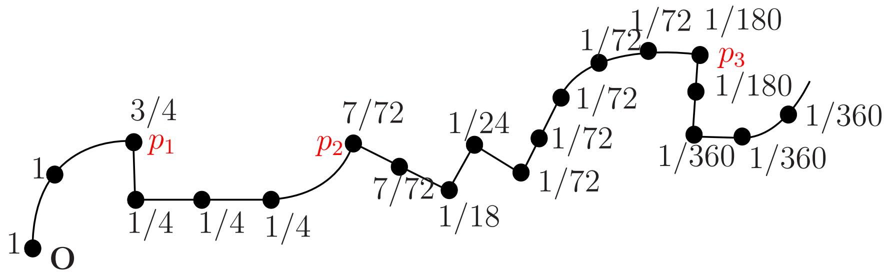


Figure 1.11: Enriques Diagram of a curve C of Example 1.5.16. Normalized multiplicities are indicated.

Example 1.5.16. *Take C a curve with Enriques diagram as in Figure 1.11. A simple computation shows that $t_0 = 0$, $t_{p_1} = 11/4$, $t_{p_2} = 799/288$, $t_{p_3} = 11987/4320$ and $t_4 = \infty$, and $d_0 = 1$. Therefore,*

$$\begin{array}{ll} t_1 = t_{p_1} = 11/4 = [2,1,3] & \Rightarrow\ d_1 = 1 \cdot q_2(2,1,3) = 4, \\ t_2 = d_1^2(t_{p_2} - t_{p_1}) = 7/18 = [0,2,1,1,3] & \Rightarrow\ d_2 = 4 \cdot q_2(0,2,1,1,3) = 4 \cdot 18 = 72, \\ t_3 = d_2^2(t_{p_3} - t_{p_2}) = 12/5 = [2,2,2] & \Rightarrow\ d_3 = 172 \cdot q_2(2,2,2) = 72 \cdot 5 = 360. \end{array}$$

Then

$$\begin{array}{l} T_1 = \{[1],[2],[2,1,1],[2,1,2],[2,1,3]\} = \{1,2,5/2,8/3,11/4\}, \\ T_2 = \{[0,2,1],[0,2,1,1,1],[0,2,1,1,2],[0,2,1,1,3]\} = \{1/3,3/8,5/13,7/18\}, \\ T_3 = \{[1],[2],[2,2,1],[2,2,2]\} = \{1,2,7/3,12/5\}, \\ T_4 = \{[1],[2],\ldots\} = \mathbb{N}. \end{array}$$

Therefore

$$\begin{array}{l} T(C) = \{1,2,5/2,8/3,11/4\} \cup \\ \quad \{11/4 + 1/4^2 \cdot 1/3, 11/4 + 1/4^2 \cdot 3/8, 11/4 + 1/4^2 \cdot 5/13, 11/4 + 1/4^2 \cdot 7/18\} \\ \cup \{799/288 + 1/72^2 \cdot 1, 799/288 + 1/72^2 \cdot 2, 799/288 + 1/72^2 \cdot 7/3, 799/288 + 1/72^2 \cdot 12/5\} \\ \qquad \cup \{11987/4320 + 1/360^2 \cdot x \mid x \in \mathbb{N}\}. \end{array}$$

1.5.3 The set $T(C)$ and the equisingularity class of C

In this section we will prove the next that for an irreducible curve C, $T(C)$ determines the equisingularity class of C.

Notice that the value $t_p(C)$ depends in general on the curve C. If q is a satellite point, the value of t_q depends on the nature (free or satellite) of the point in the first neighbourhood of q in C:

Proposition 1.5.17. *Let C_1 be an irreducible curve, and let q be a satellite point of $K(C_1)$. Suppose that q_1, the point on C_1 in the first neighbourhood of q, is a free point.*

Let q_2, q_3 be the two different satellite points in the first neighbourhood of q, and let C_2, C_3 be irreducible curves passing through q_2 and q_3 respectively. Suppose that C_1 and C_2 are closer than C_1 and C_3 (see Figure 1.12).

Then $t_p(C_1) = t_p(C_2) < t_p(C_3)$.

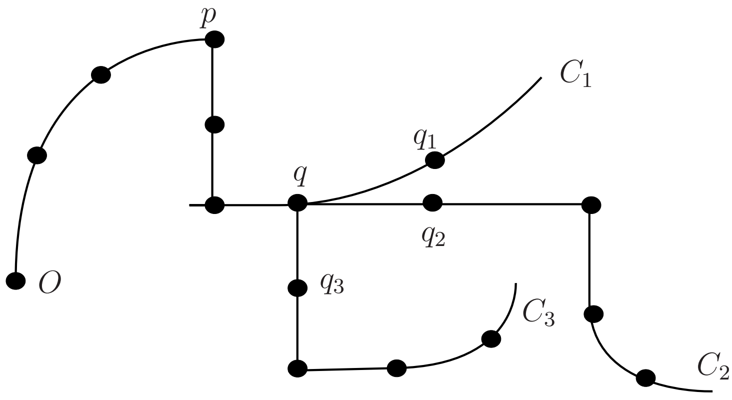


Figure 1.12: An example illustrating the Enriques diagrams of some curves C_1, C_2 and C_3 satisfying the hypothesis of Proposition 1.5.17.

Proof. Let p be the first point on C_1 such that $n_p(C_1) < b_p(C_1) = 1$. By Theorem 1.5.15, we can suppose that all the points between p and q are satellite points. Let $n_p(C_1) = [0, a_1, \ldots, a_n]$. We can suppose that n is even, because if n is odd, we can take $n_p(C_1) = [0, a_1, \ldots, a_n] = [0, a_1, \ldots, a_n - 1, 1]$. Then (see Theorem 1.1.18) $n_p(C_i) = [0, a_1, \ldots, a_{n-1}, b_n, \ldots]$ for some $b_n > a_n$, and $n_p(C_j) = [0, \ldots, a_n, \ldots]$ with either $i = 2, j = 3$ or $i = 3, j = 2$.

By Proposition 1.5.9 $n_p(C_j) > n_p(C_1)$, because $n_p(C_1)$ is, for an even n, equal to $p_n(0, \ldots, a_n, \ldots)/q_n(0, \ldots, a_n, \ldots)$. And, in virtue of Proposition 1.5.11, $n_p(C_i) < n_p(C_1)$. So by applying Proposition 1.3.3,

$$d_{\mathcal{C}}(C_i, C_1) = \frac{1}{t + n_p(C_1)} > \frac{1}{t + n_p(C_j)} = d_{\mathcal{C}}(C_j, C_1)$$

Therefore, $i = 2, j = 3$. By Proposition 1.5.12, $t_q(C_1) = t_q(C_2) = t + [0, a_1, \ldots, a_n]$, but $t_q(C_3) = t + [0, \ldots, a_n, \ldots]$, and this ends the proof. □

Remark 1.5.18. *In virtue of Proposition 1.5.17, all the curves C such that passes through q and such that the curve goes right or goes free in q (see the definition of* going right, going free *and* going down *in Section 1.4.2) have the same value $t_q(C)$. In particular, this value does not depends on the form of the stair.*

On the other hand, in the proof of that proposition, it has been seen that if C goes down on q, then $t_q(C) = t_p(C)$, where p is the first free point on C such that $n_p(C) < 1$. In particular, $t_p(C)$ depends on $n_p(C)$ and, thence, on the form of the stair.

The value $t_p(C_1)$ (which depends only on the point p, not on the curve C_1) will be denoted by τ_p. Furthermore, if q is a point (free or satellite) infinitely near to O, we define τ_q as the value $t_q(D)$, where D is an irreducible curve which passes through q and the point in the first neighbourhood of q on D is a free point.

Given $p \in \mathcal{N}$, we denote by s_p the value $m_O(D_p)$, where D_p is an irreducible curve which does not have any satellite point following p.

With the same notations of the proof of Proposition 1.5.17, $\tau_p = a_0 + \frac{p_k(a_1, \ldots, a_{k-1}, i)}{q_k(a_1, \ldots, a_{k-1}, i)}$ for some k, i (where a_0 is the number of points preceding p'). It is easy to observe that

$s_p = q_k(a_1, \ldots, a_{k-1}, i)$. But this fact is not true in general: if p has two or more free preceding points for which $n_{p'} < b_{p'}$, it is false.

Lemma 1.5.19. *Let C be a curve, and let p, q be two points of $K(C)$, such that $t_p(C) < t_q(C)$ and $(t_p(C), t_q(C)) \cap T(C) = \emptyset$ (i.e., there is not a point $p' \in K(C)$ such that $t_p(C) < t_{p'}(C) < t_q(C)$). Suppose that p is a satellite point, and suppose that p', the last free point of $K(C)$ preceding p, is the first free point for which $n_{p'} < 1$. Then*

$$t_q(C) - t_p(C) = \frac{1}{s_q}\frac{1}{s_p}.$$

Proof. Let $e = [a_0, \ldots, a_k]$ be the value $t_{p'}(C)$. Two different cases are considered:

- $t_p(C) = e$: In this case we can suppose that the point on C in the first neighbourhood of p is free. So q is the last point on C proximate to p. Furthermore, $n_p(C) = s_p$ by definition of s_p.

  Let x be the number of points proximate to p in C. Therefore, $s_q = s_p x$. By Theorem 1.5.15, $t_q(C) = e + \frac{1}{s_p^2}\frac{1}{x}$, and the result is proved in this case.

- $t_p(C) < e$: Then it is clear that $t_p(C) = [a_0, \ldots, a_{r-1}, i]$, and $t_q(C) = [a_0, \ldots, a_{r-1}, i, x]$ with $x \geqslant 1$ (see Proposition 1.5.13).

  Using the results of Section 1.5.1, we have:

  $$t_q(C) = \frac{x p_r(a_0, \ldots, a_{r-1}, i) + p_{r-1}(a_0, \ldots, a_{r-1})}{x q_r(a_0, \ldots, a_{r-1}, i) + q_{r-1}(a_0, \ldots, a_{r-1})},$$
  $$t_p(C) = \frac{p_r(a_0, \ldots, a_{r-1}, i)}{q_r(a_0, \ldots, a_{r-1}, i)}.$$

  It will be written p_r, p_{r-1}, q_r and q_{r-1}, for short, and we obtain

  $$t_q(C) - t_p(C) = \frac{p_{r-1}q_r - p_r q_{r-1}}{q_r(x q_r + q_{r-1})} = \frac{(-1)^r}{s_p s_q}.$$

  But r must be even by hypothesis, therefore the proof is completed. The reader can observe that $s_q = x s_p y$, with $0 < y < s_p$ (in fact, $y = s_{p''}$ for some $p'' < p$).

□

Remark 1.5.20. *In both cases of the proof of Lemma 1.5.19, $t_p(C) = e$ and $t_p(C) < e$, the value x is exactly the number of points on C proximate to p, i.e., $x - 1$ is the number of points $\tilde{p}$ such that $p < \tilde{p} < q$.*

Furthermore, s_p divides s_q if and only if the point on C in the first neighbourhood of the point of p is a free point.

Proposition 1.5.21. *Let C_1 be an irreducible curve, and let p be a satellite point of $K(C_1)$. Suppose that q_1, the point on C_1 in the first neighbourhood of p, is a free point.*

Let C_2 be an irreducible curve which passes through p and satisfies $t_p(C_2) = \tau_p$, and such that q_2, the point on C_2 in the first neighbourhood of p, is a satellite point. Then

$$\min\{t_q(C_1) \mid q' \in K(C_1), q > p\} \neq \min\{t_q(C_2) \mid q \in K(C_2), q > p\}.$$

Proof. Let p' be the first point on C_1 such that $n_{p'}(C_1) < b_{p'}(C_1) = 1$. By Theorem 1.5.15, we can suppose that all the points between p and p' are satellite points.

Let p_1 be the point on C_2 such that $t_{p_1}(C_1) = \min\{t_{q'}(C_1) \mid q' \in K(C_1), q' > p\}$. By Proposition 1.5.12 and Theorem 1.5.15, p_1 is the last point on C_1 proximate to p ($p_1 = q_1$ if and only if the point on C_1 in the neighbourhood of q_1 is free).

Let p_2 be the point on C_2 such that $t_{p_2}(C_2) = \min\{t_{q'}(C_2) \mid q' \in K(C_2), q' > p\}$. By Proposition 1.5.12, p_2 is the last point on C_2 proximate to p ($p_2 = q_2$ if and only if $t_{q_2}(C_2) = \tau_{q_2}$).

It is enough to prove that $t_{p_1}(C_1) \neq t_{p_2}(C_2)$. By Lemma 1.5.19, it is enough to prove that $s_{p_1} \neq s_{p_2}$. By Remark 1.5.20, s_p divides s_{p_1} but s_p does not divide s_{p_2}. Therefore, it is clear that $s_{p_1} \neq s_{p_2}$, and the proof ends. ☐

Theorem 1.5.22. *The set $T(C)$ determines the equisingularity class of the curve C, that is, given C_1, C_2 two irreducible curves such that $T(C_1) = T(C_2)$ then C_1 and C_2 have the same equisingularity class.*

Proof. Let C be an irreducible curve. It will be seen that the set $T(C)$ determines the proximity relations in $K(C)$, which proves this theorem.

We will give an algorithm such that in every step we will compute all the points proximate to the last point which we have determined. The algorithm will work until we find a point which is a free point preceding a satellite point (notice we can apply Lemma 1.5.19 only in these cases). Let us see the algorithm

Of course, the first point in $K(C)$ is O, and $t_O(C) = 1$. We call $p_0 = O, t_0 = 1, s_0 = 1$ (Step 0) (where $t_i = t_{p_i}$ and $s_i = s_{p_i}$).

Step i: Take $t_i = \min\{t \in T(C) \mid t > t_{i-1}\}$. By Lemma 1.5.19, $t_i - t_{i-1} = s_{i-1}^{-1} s_i^{-1}$. So we proceed to compute s_i. Let x, y be two natural numbers such that $s_i = s_{i-1}x + y$. By Remark 1.5.20, x is equal to the number of points proximate to p_{i-1} and $y > 0$ if and only if the point in the first neighbourhood of p_{i-1} is free.

If there is not a free point preceding a satellite point on C (this is, is there is no satellites points on C), this algorithm will compute the equisingularity class of C.

On the other hand, suppose that $\tilde{O}$ is the first free point on C in the first neighbourhood of a satellite point, and suppose that $\tilde{O}'$ is the immediate predecessor of $\tilde{O}$. This algorithm determines the proximity relations of the points $\{p \in K(C) \mid p \leqslant \tilde{O}$. We define

$$T_1 = \{\frac{t - t_{\tilde{O}'}}{s_{\tilde{O}}} \mid t > t_{\tilde{O}'}\}.$$

Let C_1 be a curve such that $T(C_1) = T_1$. By Theorem 1.5.15, this curve exists and the points of $K(C_1)$ has the same proximity relations than the points of $K_{\tilde{O}}$. Therefore, we can apply another time this algorithm at C_1, and we will obtain a set T_2 and a curve C_2. And this will end because there are a finite number of singular points in $K(C)$. ☐

Example 1.5.23. *Take*

$$T(C) = \{1, 2, \frac{7}{3}, \frac{19}{8}, \frac{31}{13}, \frac{43}{18}, \frac{1549}{648}, \frac{1162}{486}, \frac{3486+n}{1458} \mid n \in \mathbb{N}\}.$$

Let us determine the equisingularity type of C.

Step 0. $p_0 = O, t_0 = 1, s_0 = 1$.

Step 1. $t_1 = \min\{t \in T(C) \mid t > 1\} = 2$. *Then* $t_1 - t_0 = 1 = s_0 s_1$ *so* $s_1 = 1$. *Therefore,* $x = 1, y = 0$, *and there is only one point on* C *proximate to* O, *which is* p_1.

Step 2. $t_2 = \min\{t \in T(C) \mid t > 2\} = \frac{7}{3}$. *Then* $t_2 - t_1 = \frac{1}{3} = s_1 s_2$ *so* $s_2 = 3$. *Therefore,* $x = 3, y = 0$, *and there are three points on* C *proximate to* p_1, *which are* q_2^1, q_2^2 *and* p_2.

Step 3. $t_3 = \min\{t \in T(C) \mid t > \frac{7}{3}\} = \frac{19}{8}$. *Then* $t_3 - t_2 = \frac{1}{24} = s_2 s_3$ *so* $s_3 = 8$. *Therefore,* $x = 2, y = 2$. *This means that* q_3^1, *the point on* C *in the first neighbourhood of* p_2, *is proximate to* q_2^1 *(because* $y > 0$*), and there are two points on* C *proximate to* p_2, *which are* q_3^1 *and* p_3.

Step 4. $t_4 = \min\{t \in T(C) \mid t > \frac{19}{8}\} = \frac{31}{13}$. *Then* $t_4 - t_3 = \frac{1}{104} = s_3 s_4$ *so* $s_4 = 13$. *Therefore,* $x = 1, y = 5$. *This means that* p_4, *the point on* C *in the first neighbourhood of* p_3, *is proximate to* q_3^1 *(because of* $y > 0$*), and to* p_3, *and it is the only point on* C *proximate* p_3.

Step 5. $t_5 = \min\{t \in T(C) \mid t > \frac{31}{13}\} = \frac{43}{18}$. *Then* $t_5 - t_4 = \frac{1}{234} = s_4 s_5$ *so* $s_5 = 18$. *Therefore,* $x = 1, y = 5$. *This means that* p_5, *the point on* C *in the first neighbourhood of* p_4, *is proximate to* q_3^1 *(because* $y > 0$*), and to* p_4, *and is the only point on* C *proximate* p_4.

Step 6. $t_6 = \min\{t \in T(C) \mid t > \frac{43}{18}\} = \frac{1549}{648}$. *Then* $t_6 - t_5 = \frac{1}{648} = s_5 s_6$ *so* $s_6 = 36$. *Therefore,* $x = 2, y = 0$. *This means that* q_6^1, *the point on* C *in the first neighbourhood of* p_5, *is free (because* $y = 0$*), and there are two points on* C *proximate to* p_5, *which are* q_6^1 *and* p_6.

Now q_6^1 *is a free point which precedes* p_5, *a satellite point. We apply Theorem 1.5.15:*

$$\tilde{T} = \{s_5^2(t - t_5) \mid t > t_5\} = \{\frac{1}{2}, \frac{2}{3}, \frac{6+n}{9} \mid n \in \mathbb{N}\}$$

and we go on applying the algorithm, but now $\tilde{t}_6 = s_5^2(t_6 - t_5) = \frac{1}{2}$, $\tilde{s}_6 = s_6/s_5 = 2$.

Step 7. $\tilde{t}_7 = \min\{t \in \tilde{T} \mid t > \frac{1}{2}\} = \frac{2}{3}$. *Then* $\tilde{t}_7 - \tilde{t}_6 = \frac{1}{6} = \tilde{s}_6 \tilde{s}_7$ *so* $\tilde{s}_7 = 3$. *Therefore,* $x = 1, y = 1$. *This means that* p_7, *the point on* C *in the first neighbourhood of* p_6, *is proximate to* q_6^1 *(because* $y > 0$*), and to* p_6, *and is the only point on* C *proximate* p_6.

Step 8. $\tilde{t}_8 = \min\{t \in \tilde{T} \mid t > \frac{2}{3}\} = \frac{7}{9}$. *Then* $\tilde{t}_8 - \tilde{t}_7 = \frac{1}{9} = \tilde{s}_7 \tilde{s}_8$ *so* $\tilde{s}_8 = 3$. *Therefore,* $x = 1, y = 0$. *This means that* p_8, *the point on* C *in the first neighbourhood of* p_7, *is free (because* $y = 0$*), and it is the only point on* C *proximate to* p_7.

We apply Theorem 1.5.15 another time. But this time the set obtained will be $\mathbb{N}$*: this means that the curve is smooth after* p_7.

In Figure 1.13 the construction of the Enriques diagram of C *is done step by step.*

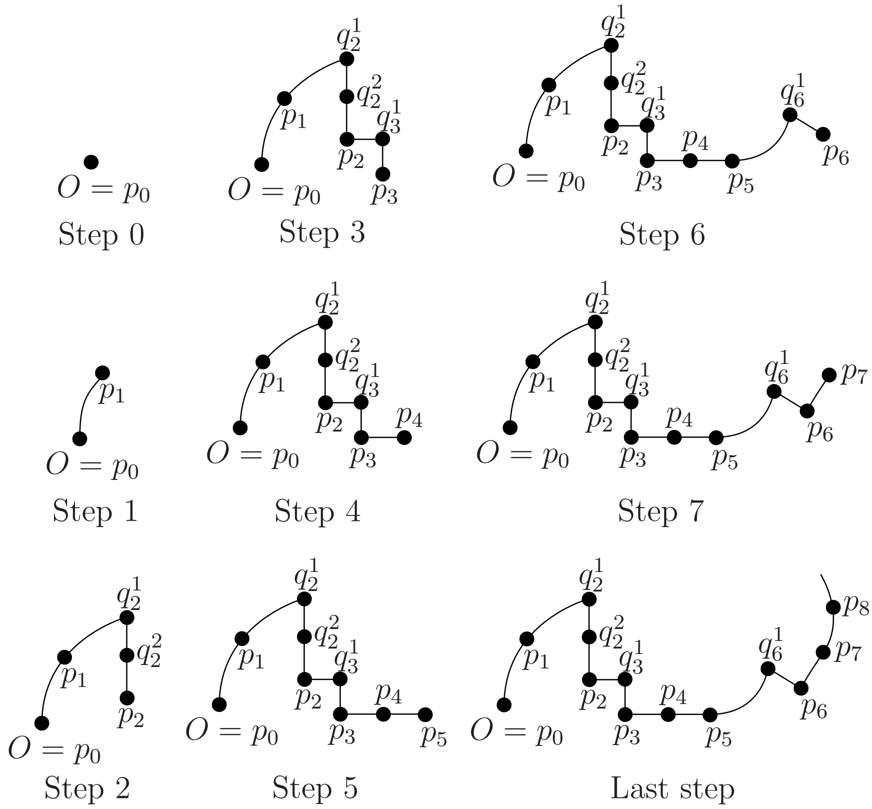


Figure 1.13: The construction of the Enriques Diagram of a curve given $T(C)$.

1.5.4 Connection of inverse distances to other singularity invariants

In this section we will show some relations between the inverse distance $t_p(C)$, the invariant introduced in Section 1.5.2, and some other singularity invariants.

Another invariant considered in this memory is the *skewness*, introduced by Favre and Jonsson in [4]. It will be defined in Section 2.4.3.

Proposition 1.5.24. *Let C be an irreducible germ an p be a point on C. Suppose that the point in the first neighbourhood of p on C is a free point. Then $t_p(C) = \alpha(\nu_p)$, where ν_p is the divisorial valuation such that its cluster K has p as a maximal point, i.e., such that $K(\nu_p) = \{q \in \mathcal{N}_O \mid q \leqslant p\}$.*

Proof. By definition, $\alpha(\nu_p) = \sup\{\nu(D)/m_O(D) \mid D \in R\}$. By Noether formula on valuations, this supremum is obviously satisfied by an irreducible curve D passing through all the points on the cluster of ν_p and being free in the point that follows p.

On the other hand, by definition, $t_p(C) = d_{\mathcal{C}}(C, D)^{-1}$, where D is an irreducible curve passing through p, the point on the first neighbourhood of p on D is free and without sharing other points with C after p.

Therefore, we can take the same curve D in both definitions.

$$t_p(C) = d_{\mathcal{C}}(C, D)^{-1} = \frac{m_O(C)m_O(D)}{C \cdot D} = \frac{\nu_p(D)}{m_O(D)} = \alpha(\nu_p)$$

□

Corollary 1.5.25. *Let p be a point infinitely near to O. Then $\tau_p = \alpha(\nu_p)$, where ν_p is the divisorial valuation such that its cluster K has p as a maximal point.*

Let C be an irreducible curve, and let p be a point infinitely near to the origin O on C. Consider the following rational number, which is a multiple of the inverse distance:

$$m_O(C)t_p(C) = \frac{m_O(C)}{d_{\mathcal{C}}(C, D_p)} = \frac{C \cdot D_p}{m_O(D_p)},$$

where D_p is an irreducible curve passing through p such that the point on the first neighbourhood of p on D is free and without sharing other points with C after p.

The set $\{m_O(C)t_p(C)\}_{p \in I}$ for a distingish subset I of the singular points of a reduced singular germ of a curve C (see [1] 6.11) is known as the *polar invariants* or *polar quotients* of C (see also [7]).

The extension of this notion at each singular point of C appears in [1] 7.6, where some properties about the growing of these quotients are established in [1] 7.6.5 and 7.6.8. However, the treatment of the whole sequence for $C \in \mathcal{C}$, as it has been carried in this memory, and specially Result 1.5.22 are novel.

Chapter 2

The valuative tree

In this section we deal with the valuations and their properties. We will also present the set $\mathcal{V}$ of all centered real normalized valuations on the ring of the germs at O of the holomorphic functions in O, R, and its structure: the valuative tree. We will study the valuations from two different points of view: from the Favre and Jonsson's point of view, using the ultrametric space $\mathcal{C}$, and from the Casas' point of view, using clusters and Enriques diagrams.

2.1 Valuation theory

In this section we will give the most important classical results of the valuation theory. More results can be found in [9] and [8].

Let K be a field and let K^* be its multiplicative group. Let Γ be an additive abelian totally ordered group. A *valuation* is a map ν of K^* into Γ such that

- $\nu(xy) = \nu(x) + \nu(y)$.
- $\nu(x + y) \geqslant \min\{\nu(x), \nu(y)\}$.

Given $x \in K^*$, $\nu(x)$ is called the *value* of x. The subgroup $\nu(K^*)$ of Γ is called the *value group*. It will be supposed to be Γ in the sequel. A valuation ν is called *non-trivial* if its value group is non-trivial, that is, has cardinal greater than one.

Let us state some basic properties about the valuations. All these results can be found in [9].

Proposition 2.1.1. *Let $\nu : K^* \to \Gamma$ be a valuation. Then:*

1. *If $x \in K$ is a n-th root of the unit, then $\nu(x) = 0$. In particular, $\nu(1) = \nu(-1) = 0$.*
2. *$\nu(x - y) \geqslant \min\{\nu(x), \nu(y)\}$.*
3. *$\nu(1/x) = -\nu(x)$, where $x \neq 0$.*
4. *$\nu(y/x) = \nu(y) - \nu(x)$, where $x \neq 0$.*
5. *If $\nu(x) < \nu(y)$, then $\nu(x + y) = \nu(x)$.*

Two valuations $\nu : K^* \to \Gamma$ and $\nu' : K^* \to \Gamma'$ are said *equivalent* or *isomorphic* if and only if there exists an order preserving isomorphism φ of Γ to Γ' such that $\varphi \circ \nu = \nu'$. We are interested in the study of non-equivalent valuations, so two equivalent valuations will be considered as the same valuation.

If ν is a valuation, for convention $\nu(0) = \infty$.

The set of all elements of K such that $\nu(x) \geqslant 0$ is called the *valuation ring of* ν, and it is denoted by R_ν.

The following results can also be found in [9].

Proposition 2.1.2. *Let ν be a valuation. The valuation ring is in fact a ring, and for every $x \in K^*$, either x or x^{-1} belong to R_ν. Furthermore, the set of all the units of this ring is $\{x \in K \mid \nu(x) = 0\}$.*

Proposition 2.1.3. *Let ν be a valuation non-trivial. Then R_ν is a local ring, and its maximal ideal is the set $\{x \in K \mid \nu(x) > 0\}$. Furthermore, for all $x \in K^*$, x belongs to the maximal ideal if and only if $1/x$ does not belong to R_ν.*

This maximal ideal is called the *prime ideal of* ν, and is denoted by $\mathfrak{m}_\nu$. The field $R_\nu/\mathfrak{m}_\nu$ is called the *residue field of* ν and is denoted by D_ν[1].

Theorem 2.1.4. *Two valuations are equivalent if and only if they have the same valuation ring.*

Theorem 2.1.5. *Let R be an integral domain and let K be its quotient field. Let ν_0 be a map of $R \setminus \{0\}$ into an additive abelian totally orderer group Γ such that*

- $\nu_0(xy) = \nu_0(x) + \nu_0(y)$
- $\nu_0(x + y) \geqslant \min\{\nu_0(x), \nu_0(y)\}$

Then there is a unique valuation ν in K that extends ν_0.

Proof. The uniqueness of ν is proved using property 4 of Proposition 2.1.1:

$$\nu\left(\frac{x}{y}\right) = \nu(x) - \nu(y).$$

It is clear that ν is defined with this property. For the existence, it is sufficient to check that a map defined in this way is a well-defined valuation which extends ν_0, and it can be proved easily. □

Theorem 2.1.6. *Let ν be a valuation. The valuation ring R_ν is Noetherian if and only if the value group Γ is isomorphic to $\mathbb{Z}$.*

The valuations with value ring $\mathbb{Z}$ are called *discrete* valuations.

Valuations have the following numerical invariants:

- The *rank*, which is the Krull dimension of the ring R_ν.
- The rational rank, defined as $\dim_{\mathbb{Q}}(\nu(K^*) \otimes_{\mathbb{Z}} \mathbb{Q})$.
- The transcendence degree, which is the transcendence degree of the field extension $K \subset D_\nu$.

These invariants are usefull for classify the valuations of some field K.

[1]Sometimes the residue field of a valuation is denoted by k_ν. See for example [4].

2.2 $\mathbb{R}$-Trees

In this section we will introduce the concept of $\mathbb{R}$-tree. The study of $\mathbb{R}$-trees are important in the study of plane curves because the set of centered real normalized valuations is an $\mathbb{R}$-tree (see Section 2.4 for their definition). A more extensive study of $\mathbb{R}$-trees can be found in Section 3.1 of [4].

Let $(\mathcal{T}, \leqslant)$ be a partially ordered set. $\mathcal{T}$ is called *rooted nonmetric* $\mathbb{R}$*-tree* or simply *rooted nonmetric tree* if and only if it satisfies:

1. $\mathcal{T}$ has a unique minimal element τ_0, called the root of $\mathcal{T}$.

2. For any $\tau \in \mathcal{T}$, the set $\{\sigma \in \mathcal{T} \mid \sigma \leqslant \tau\}$ is isomorphic to a real interval.

3. Every maximal totally ordered subset of $\mathcal{T}$ is isomorphic to a real interval,

where *isomorphic* means that there is an order-preserving bijection.

In the definition, $\mathbb{R}$ can be changed by any totally ordered set. For example, rooted nonmetric $\mathbb{N}$-trees can be defined similarly.

Let $\mathcal{T}$ be a rooted nonmetric tree, and $S \subset T$ any subset of it. By the completeness of $\mathbb{R}$, S admits a unique maximal element less than every element in S. This element is called *infimum*, and it is denoted by $\wedge_{\tau \in S} \tau$.

Let $\mathcal{T}$ be a rooted nonmetric tree, and let τ_1, τ_2 be two elements of $\mathcal{T}$. The set $\{\tau \in \mathcal{T} \mid \tau_1 \wedge \tau_2 \leqslant \tau \leqslant \tau_1\} \cup \{\tau \in \mathcal{T} \mid \tau_1 \wedge \tau_2 \leqslant \tau \leqslant \tau_2\}$ is called *segment* and is denoted by $[\tau_1, \tau_2]$.

Let $(\mathcal{T}, \tau_0), (\mathcal{S}, \sigma_0)$ be rooted nonmetric trees, and let $\Phi : \mathcal{T} \to \mathcal{S}$ be a map. Φ is called *morphism of rooted nonmetric trees* if

$$\Phi|_{[\tau_0, \tau]} : [\tau_0, \tau] \to [\sigma_0, \Phi(\tau)]$$

is an order-preserving bijection.

A rooted nonmetric tree $\mathcal{T}$ is *complete* if every increasing sequence has an upper bound in $\mathcal{T}$. Any rooted nonmetric tree has a *completion*, denoted by $\bar{\mathcal{T}}$, obtained by adding maximal upper bounds for any unbounded increasing sequence.

Let $\mathcal{T}$ be a rooted nonmetric tree. A subset $\mathcal{S} \subset \mathcal{T}$ is called a *subtree* if for every $\sigma \in S$ it holds $\{\tau \in \mathcal{T} \mid \tau < \sigma\} \subset \mathcal{S}$. Any subtree of a roted nonmetric tree is a rooted nonmetric tree with the same root.

Let $\tau \in \mathcal{T}$ be a point of a rooted nonmetric tree. An equivalence relation can be defined in $\mathcal{T} \setminus \tau$ as follows: $\sigma_1 \sim_\tau \sigma_2$ if and only if $(\tau, \sigma_1] \cap (\tau, \sigma_2] \neq \emptyset$.

Lemma 2.2.1. *Let $\mathcal{T}$ be a rooted nonmetric tree, and let $\tau \in \mathcal{T}$. Then:*

$$\sigma_1 \sim_\tau \sigma_2 \Leftrightarrow \ or \begin{cases} \sigma_1 \not> \tau \text{ and } \sigma_2 \not> \tau, \\ \sigma_1 \wedge \sigma_2 > \tau. \end{cases}$$

Proof. Observe that if $\sigma > \tau$, then

$$[\sigma, \tau) = \{\tau' \in \mathcal{T} \mid \tau < \tau' \leqslant \sigma\}.$$

In particular, every $\tau' \in [\sigma, \tau)$ is greater than τ.

On the other hand, if $\sigma \not> \tau$, there are no $\tau' \in [\sigma, \tau)$ greater than τ.

So it is obvious that if $\sigma_1 \sim_\tau \sigma_2$ and $\sigma_1 \not> \tau$, then $\sigma_2 \not> \tau$.

Let us suppose that $\sigma_1 \sim_\tau \sigma_2$ and $\sigma_1, \sigma_2 > \tau$. Notice that the condition $\sigma_1 \wedge \sigma_2 > \tau$ is satisfied if and only if there exists τ' such that $\tau < \tau' < \sigma_1$ and $\tau < \tau' < \sigma_2$.

Let τ' be a point in $[\sigma_1, \tau) \cap [\sigma_2, \tau)$. Then $\tau < \tau' \leqslant \sigma_1$ and $\tau < \tau' \leqslant \sigma_2$.

Reciprocally, if $\tau < \tau' \leqslant \sigma_1$ and $\tau < \tau' \leqslant \sigma_2$, then τ' belongs to $(\tau, \sigma_1]$ and to $(\tau, \sigma_2]$.

Finally, we must prove that if σ_1 and σ_2 are not greater than τ, then $\sigma_1 \sim_\tau \sigma_2$.

Let us observe that if σ is not greater than τ, then $\sigma \sim_\tau \tau_0$ because $\sigma \wedge \tau$ belongs to $(\tau, \sigma]$ and to $(\tau, \tau_0]$.

Therefore if both σ_1 and σ_2 are not greater than τ, $\sigma_1 \sim_\tau \sigma_2$, because $\sim_\tau$ is an equivalence relation. $\square$

An equivalence class is called *tangent vector* at τ, and the quotient set is called *tangent space* at τ, and it is denoted by $T_\tau\mathcal{T}$[2]. Notice that $T_\tau\mathcal{T}$ is not a usual tangent space, since it is not a vectorial space; $T_\tau\mathcal{T}$is in fact a projectivized tangent space.

A point τ of $\mathcal{T}$ is an end if and only if $T_\tau\mathcal{T}$ has only one element. If $T_\tau\mathcal{T}$ has exactly two elements, τ is called *regular point*, and if it has more than two elements, τ is called *branch point*.

Let $\mathcal{T}$ be a rooted nonmetric tree. A *parameterization* of $\mathcal{T}$ is an increasing (or decreasing) function $\alpha : \mathcal{T} \to [-\infty, \infty]$ such that its restriction to any maximal totally ordered subtree of T is a bijection onto an interval.

A rooted nonmetric tree $\mathcal{T}$ that admits a parameterization α is called *parameterizable*, and $(\mathcal{T}, \alpha)$ is called *parameterized tree*. A *morphism of parameterized trees* is a morphism of rooted nonmetric trees that commutes with the parameterizations.

A parameterized tree induces a distance: we can suppose that $\alpha : \mathcal{T} \to [0, 1]$, composing alpha with a suitable homeomorphism from $[-\infty, \infty]$ to $[0, 1]$. Then $d(\sigma, \tau) = \alpha(\sigma) + \alpha(\tau) - 2\alpha(\sigma \wedge \tau)$ is a distance.

A rooted nonmetric tree with a distance that, restricted to any segment, gives an isometry to a real interval, is called *metric trees*. A parameterized tree with the distance induced from a parameterization is a metric tree of *finite diameter*. The *diameter* of any metric space is the supremum of the distances of the points of that space. Therefore, a metric tree of finite diameter is a metric tree such that the distances of their elements are bounded.

Reciprocally, if $\mathcal{T}$ is a metric tree (with diameter not necessarily finite), then $\alpha : \mathcal{T} \to [0, \infty)$ defined as $\alpha(\tau) = d(\tau, \tau_0)$ gives a parameterization.

The following result can be found in [4].

Proposition 2.2.2. *Let $\mathcal{T}$ be a metric tree. If $\mathcal{T}$ is complete as a rooted nonmetric tree, then it is complete as a metric space. Reciprocally, if $\mathcal{T}$ has finite diameter and it is complete as a metric space, then it is complete as a rooted nonmetric tree.*

Furthermore, if $\mathcal{T}$ is a metric space with finite diameter, then its completion as a metric space agrees with its completion as a rooted nonmetric tree.

---

[2]This is not standard notation but it will be used in this memory because of its clarity. In particular, it is not the notation used in [4].

Let $\vec{v}_\tau \in T_\tau \mathcal{T}$ be a tangent vector in τ. The *weak topology* of $\mathcal{T}$ is defined as the topology with semibasis

$$\{\vec{v}_\tau \mid \vec{v}_\tau \in T_\tau \mathcal{T}, \tau \in \mathcal{T}\} = \bigcup_{\tau \in \mathcal{T}} T_\tau \mathcal{T}.$$

Rooted nonmetric trees are Hausdorff spaces with the weak topology. Any subtree $\mathcal{S}$ of $\mathcal{T}$ is a closed set of $\mathcal{T}$, and the inclusion $\mathcal{S} \hookrightarrow \mathcal{T}$ is an embedding. In particular, the segments $[\tau, \tau']$ are closed sets of $\mathcal{T}$, and any segment is homeomorphic (with the induced topology) to a real closed segment. Any complete rooted metric tree is compact. Furthermore, if $\mathcal{T}$ is a metric tree, the completion $\bar{\mathcal{T}}$ is a compactification of $\mathcal{T}$.

Remark 2.2.3. *Let $\mathcal{T}$ be a metric tree. Then the topology of $\mathcal{T}$ induced by the metric does not agrees with the weak topology of $\mathcal{T}$ in general.*

2.3 Classification of valuations in the ring of plane germs of curves

In this section we will study the valuations on the ring $R = \mathbb{C}\{x, y\}$ of plane germs of curves. Namely a cluster will be assigned to any valuation such that two valuations are isomorphic if and only if they have the same cluster. Finally, we will give a classification of the valuations according to the structure of their clusters. This study has been developed following [1] (see also [8]).

Let ν be a valuation defined in the local ring $(R, \mathfrak{m})$. Then the ideal $\{\psi \in R \mid \nu(\psi) = 0\}$ is called the *center* of ν. Obviously, the center of ν is a prime ideal of R. Since $\operatorname{Spec} R = \{(\psi) \mid \psi \text{ irreducible }\} \cup \{\mathfrak{m}\}$, if the center of ν is the maximal ideal $\mathfrak{m}$, ν is called *0-dimensional* valuation; otherwise, it is called *1-dimensional* valuation.

Proposition 2.3.1. *For any $\psi \in R$ irreducible, there exists a unique valuation ν (up to isomorphism) with center (ψ).*

Proof. Take $\phi \in R$, and write $\phi = \psi^n \phi'$, with $\phi' \notin (\psi)$. By definition of center, $\nu(\phi') = 0$. Therefore, $\nu(\phi) = n\nu(\psi)$. □

Let us study the 0-dimensional valuations. An example of 0-dimensional valuation is the *multiplicity valuation* or $\mathfrak{m}$-*adic valuation*, which will be denoted by $\nu_\mathfrak{m}$. It is defined by $\nu_\mathfrak{m}(\phi) = m_O(\phi)$.

The value

$$\min\{\nu(\phi) \mid \phi \in \mathfrak{m}\}$$

is denoted by $m_O(\nu)$ and it is called *multiplicity of the valuation ν* at the point O. Notice that this minimum is achieved since the ring R is a Noetherian one. Therefore, $m_O(\nu) > 0$ for any 0-dimensional valuation.

Proposition 2.3.2. *Let ν be a 0-dimensional valuation, not isomorphic to the $\mathfrak{m}$-adic valuation. Then there is a tangent line l at O such that for any element $\phi \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$, it is satisfied that $\nu(\phi) > n m_O(\nu)$ if and only if the germ of curve $\phi = 0$ is tangent to l.*

Proof. Write $e = m_O(\nu)$. If ν is not the $\mathfrak{m}$-adic valuation, there exists an homogeneous form ϕ of some degree n such that $\nu(\phi) > ne$. But since ϕ is homogeneous, $\phi = \prod l_i$, where l_i are forms of degree 1. Therefore, $\nu(\phi) = \sum \nu(l_i) > ne$. It is clear that $\nu(l_i) \geqslant e$ for any i and that there exists l_i such that $\nu(l_i) > e$.

If l_i and l_j are two independent forms of degree 1 such that $\nu(l_i), \nu(l_j) > e$, then for all l of degree one it is satisfied $\nu(l) > e$; but it is a contradiction with the definition of the multiplicity. Let l be the unique form of degree 1 such that satisfies $\nu(l) > e$, and take $\phi \in \mathfrak{m}^n \setminus \mathfrak{m}^{n+1}$. ϕ can be decomposed as $\phi = \phi_n + \phi'$, such that ϕ_n is an homogeneous form of degree n and $\phi' \in \mathfrak{m}^{n+1}$. It is clear that $\nu(\phi) > ne$ if and only if $\nu(\phi_n) > ne$, because $\nu(\phi') \geqslant (n+1)e > ne$. And $\nu(\phi_n) > ne$ if and only if it is multiple of l. Then, the proof is completed. □

The line l is called the *tangent line* of ν. For any point p in the exceptional divisor E of blowing-up O, let R_p be the local ring induced by the blowing-up.

Theorem 2.3.3 (Theorem 8.1.3 of [1])**.** *Let ν be a 0-dimensional valuation not isomorphic to the multiplicity valuation. Let l be the tangent line of ν. Then, we can extend ν to a valuation of the ring R_p if and only if $p = \tau(l)$. Furthermore, in that case the extension is unique, has the same value group than ν, and will be denoted also by ν.*

In this case, p is called *the center of ν in the first neighbourhood of O*. The multiplicity of ν in the ring R_p will be denoted by $m_p(\nu)$.

Let us suppose that ν is a 0-dimensional valuation of R. If ν is not the $\mathfrak{m}$-adic valuation, there exist a point p in the first neighbourhood of O which is a center of ν, and ν is extended at the ring R_p. If ν is not the $\mathfrak{m}_p$-adic valuation, then we can iterate this process, and we will obtain a sequence of points $O = p_0, p_1, \ldots$, with p_i at the i-th neighbourhood of O. p_i is called *the center of ν in the i-th neighbourhood of O*, and we can define $m_{p_i}(\nu)$ as the multiplicity of ν in the ring R_{p_i}. The sequence of centers is called *the cluster of ν*, and it is denoted by $K(\nu)$.

Lemma 2.3.4. *Let K be a totally ordered cluster. Then*

- *If there is a curve which contains infinitely many points of K, then $K = K(C)$ for some irreducible curve C and a curve D contains infinitely many points of K if and only if D has C as a branch.*
- *If there is a point $p \in K$ that has infinitely many points proximate to it, this point is the unique with this property. Furthermore, all the points greater than p of K are proximate to p.*

Proof.
- If there is a curve with infinitely many points on K, there is a branch C of that curve with infinitely many points on K. Then, for all point $p \in C$, there is a point $q > p$ which belongs to $K(C) \cap K$. This implies that $K = K(C)$, because K and $K(C)$ are unramified clusters.

  Let D be a curve. It is clear that if D has C as a branch, then $K(D)$ contain all points of K. Suppose now that D contains infinitely many points of $K = K(C)$. By the Noether Formula, it implies that C is a branch of D.

- Notice that, in virtue of Remark 1.1.14 (1) for any two points p_1, p_2 of K, if $p_2 \to p_1$ then $q \to p_1$ for all $p_1 < q < p_2$. Therefore, if p is a point on K with infinitely many points on K proximate to it, then all the points greater than p are proximate to p. But since any point can only be proximate to two points, and it is always proximate to its immediate predecessor, p is the unique point of K with that property.

□

Theorem 2.3.5 (Theorem 8.1.6 of [1])**.** *Let $K(\nu)$ be the cluster of some valuation ν. Then for any $\phi \in R$ such that ϕ not share infinitely many points with $K(\nu)$ it is satisfied*

$$\nu(\xi) = \sum_{p \in K(\nu)} m_p(\phi) m_p(\nu).$$

This formula is called the *Noether Formula by valuations.*

Theorem 2.3.6 (Theorem 8.1.7 of [1])**.** *Let $K(\nu)$ be the cluster of some valuation ν. Let p be a point of $K(\nu)$, and let q_1, $q_2, \ldots, q_r$ be points of $K(\nu)$ in the first, the second, etc. neighbourhood of p respectively. Suppose that every q_i is proximate to p. Then*

$$m_p(\nu) \geqslant \sum_{i=1}^{r} m_{q_i}(\nu),$$

and the inequality is not strict if and only if there exists a point in the $r+1$-th neighbourhood of p on $K(\nu)$ and it is proximate to p.

There are some immediate consequences of that theorem:

Corollary 2.3.7. *If p and q belong to $K(\nu)$ and q is in the first neighbourhood of p, then $m_p(\nu) \geqslant m_q(\nu)$, with equality if and only if the point in the first neighbourhood of q proximate to p does not belong to $K(\nu)$.*

Corollary 2.3.8. *With the same hypothesis as in Theorem 2.3.6. Then $m_{q_1}(\nu) = m_{q_2}(\nu) = \ldots = m_{q_{r-1}}(\nu) \geqslant m_{q_r}(\nu)$.*

Corollary 2.3.9. *Let us suppose that p and q_1 belong to $K(\nu)$ and q_1 is in the first neighbourhood of p.*

- *If there exists an integer h such that $hm_{q_1}(\nu) < m_p(\nu) \leqslant (h+1)m_{q_1}(\nu)$, then there are points $q_2, \ldots q_{h+1}$ points in $K(\nu)$, q_i in the i-th neighbourhood of p, such that $m_{q_1}(\nu) = m_{q_2}(\nu) = \ldots = m_{q_h}(\nu) \geqslant m_{q_{h+1}}(\nu)$, $m_p(\nu) = hm_{q_1}(\nu) + m_{q_{h+1}}(\nu)$ and $q_1, \ldots q_{h+1}$ are the unique points in $K(\nu)$ proximate to p.*

- *Otherwise, there are infinitely many points proximate to p, and for any h, $m_{q_1}(\nu) = m_{q_h}(\nu)$, where q_h is the point in the h-th neighbourhood of p in $K(\nu)$ (which is proximate to p). In particular, the valuation is non-Archimedean.*

Division algorithm: Let us suppose that p and q belong to $K(\nu)$ and q is in the first neighbourhood of p. Write $e_0 = m_p(\nu)$, and $e_1 = m_q(\nu)$.

If $he_1 < e_0$ for all $h \in \mathbb{N}$, then we say that the *algorithm is obstructed at q*. Otherwise, let $h_1 \in \mathbb{N}$ be the value such that $h_1 e_1 \leqslant e_0 < (h_1+1)e_1$, and write

$e_2 = e_0 - e_1 h_1$. If $e_2 = 0$, then there are in $K(\nu)$ exactly h_1 points proximate to p with multiplicity e_1. Otherwise, there are in $K(\nu)$ $h_1 + 1$ points proximate to p, all with multiplicity e_1 but the lust, with multiplicity e_2. In this case, we can repeat the algorithm with the two last points.

There are three possibilities with the division algorithm:

- The algorithm is obstructed at any point q. In this case, it is know that there are infinitely many point proximate to the immediate predecessor of q, and the valuation is non-Archimedean.

- The algorithm ends. Notice that this implies that e_0 and e_1 are $\mathbb{Q}$-dependent. In this case, if q' is the last point obtained the division algorithm, either q' is the last point of $K(\nu)$ or the point on the first neighbourhood of q' is a free point.

  It is easy to prove that the points found by the algorithm in this case are satellite points, and the proximity relations are defined by the finite continued fraction of e_0/e_1.

- The algorithm is not obstructed but does not end. Notice that this implies that e_0 and e_1 are not $\mathbb{Q}$-dependent. In this case, the algorithm find all the points greater than p of $K(\nu)$.

  It is easy to prove that the points found by the algorithm in this case are satellite points, and the proximity relations are defined by the infinite continued fraction of e_0/e_1.

In short, given a valuation ν, we can assign at ν a cluster $K(\nu)$ satisfying Theorem 2.3.5. Notice that, by Theorem 2.3.5, the map $\nu \to K(\nu)$ is injective.

Theorem 2.3.10 (Theorem 8.2.6 of [1])**.** *If K is a totally ordered cluster, there is a unique valuation (up to isomorphism) ν such that $K(\nu) = K$.*

The next result is explain with more detail in the section 8.2 of [1].

Theorem 2.3.11 (Classification of valuations)**.** *The valuations can be classified as follows:*

1. *Divisorial valuations. It correspond to the valuations ν with a finitely many centers. Let p be the last center of ν. Then ν is the $\mathfrak{m}_p$-adic valuation in the ring R_p. Equivalently, let E be the exceptional divisor of blowing-up p. Then for any germ ϕ, $\nu(\phi)$ corresponds to the multiplicity of E in the total transform of ϕ.*

2. *Analytic curve valuations. It correspond to the valuations ν with infinitely many centers in a germ of a curve. Let $\psi = 0$ be an irreducible curve such that $K(\nu) = K(\psi)$. The value group is $\mathbb{Z} \oplus \mathbb{Z}$ with the lexicographically order, which is non-Archimedean. For any $\phi \in R$, write $\phi = \psi^n \phi'$, with $\phi' \notin (\psi)$. Then, $\nu(\phi) = (n, [\psi, \phi'])$.*

3. *Formal curve valuations. It correspond to the valuations ν with infinitely many centers, but with a finitely many of them are satellites and finitely many of them lies in the same germ of curve. This valuations can be computed using the Noether Formula. On the other hand, the value group is $\mathbb{Z}$. Furthermore, there exists a non-analytic curve $D \in \mathcal{C}$ such that $K(D) = K(\nu)$, and for any $\phi \in R$, $\nu(\phi) = [D, \phi]$.*

4. *Infinitely singular valuations. It correspond to the valuations ν with infinitely many satellite centers, but not infinitely many of them consecutive. The valuation can be computed using the Noether Formula. It is an Archimedean valuation, but it is not a discrete one: the value group is a subgroup of $\mathbb{Q}$ not isomorphic to $\mathbb{Z}$. In some way, infinitely singular valuations can be thought as curve valuations for some "curve" of infinite multiplicity.*

5. *Irrational valuations. It correspond to the valuations ν with infinitely many consecutive satellite points, but not infinitely many of them proximate to the same point. This valuations are obtained when the division algorithm does not obstructed but does not end. The value group is isomorphic to $\mathbb{Z} \oplus Ze \subset \mathbb{R}$ for some $e \in \mathbb{R} \setminus \mathbb{Q}$. They are Archimedean non-discrete valuations.*

6. *Exceptional curve valuations. It correspond to the valuations ν with infinitely many centers proximate to the same point. The value group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ with the lexicographically order, which is non-Archimedean, and it is genered by $\nu(p), \nu(q)$, where p is the point with infinitely many centers proximate to it, and q is the center of ν in the first neighbourhood of p.*

The next result can be found in [4].

Type of valuation	**Rank**	**Rational rank**	**Transcendence degree**
1	1	1	1
2	2	2	0
3	2	2	0
4	1	1	0
5	1	2	0
6	2	2	0

2.4 The valuative tree

In this section we will construct a structure of $\mathbb{R}$-tree to a set of some valuations in the ring $R = \mathbb{C}\{x, y\}$ of plane germs of curves. The study of this tree has allowed to prove some important results, such as an Eigenvaluation Theorem (see [3]).

In this section, the concept of valuation will be slightly different from that defined in Section 2.1. A *real valuation* is a map $\nu : R \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, such that

- $\nu(fg) = \nu(f) + \nu(g)$,
- $\nu(f + g) \geqslant \min\{\nu(f), \nu(g)\}$.

The valuation is called *centered* if and only if the set $\{x \in R \mid \nu(x) \neq 0\}$ is $\mathfrak{m} = (x, y)$. The valuation is called *normalized* if and only if $\min\{\nu(\psi) \mid \nu(\psi) > 0\} = 1$.

Let $\mathcal{V}$ be the set of centered real normalized valuations on R, i.e., the valuations $\nu : R \to [0, \infty]$ such that $\nu(\mathfrak{m}) := \min\{\nu(\psi) \mid \psi \in \mathfrak{m}\} = 1$. $\mathcal{V}$ is called the *valuative tree*. Notice that these valuations can take the value ∞, unlike the valuations defined in Section 2.1.

We define in $\mathcal{V}$ a partial ordering: $\nu \leqslant \mu$ if and only if $\nu(\psi) \leqslant \mu(\psi)$ for all $\psi \in R$. Let $\nu_{\mathfrak{m}}$ be the *multiplicity valuation*, defined as $\nu_{\mathfrak{m}}(\psi) = \max\{k \mid \psi \in \mathfrak{m}^k\} = m_O(\psi)$. Notice that $\nu_{\mathfrak{m}}$ is the unique minimal valuation in $\mathcal{V}$.

Proposition 2.4.1 (Theorem 3.14 of [4])**.** *$(\mathcal{V}, \geqslant)$ is a complete rooted nonmetric tree.*

Let us study the elements of $\mathcal{V}$.

For any $C \in \mathcal{C}$, recall the *curve valuation* ν_C defined in Section 1.2.1, $\nu_C(\psi) = \frac{C\cdot(\psi=0)}{m_O(C)}$. This maps are in fact valuations of $\mathcal{V}$.

If C is a formal curve, ν_C is called *formal curve valuation*. Formal curve valuations can be identified to valuations of Type 3 (see 2.3.11). Otherwise ν_C is called *analytic curve valuation*. Analytic curve valuations are isomorphic to valuations of Type 2 (see 2.3.11).

Given $C \in \mathcal{C}$ and $t \in \mathbb{R}, t \geqslant 1$, we define the *quasimonomial valuation* $\nu_{C,t}$ as

$$\nu_{C,t}(\psi) = \min\{\nu_D(\psi) \mid d_{\mathcal{C}}(C, D) \leqslant t^{-1}\}.$$

It is easy to see that $\nu_{C,t} \leqslant \nu_{C',t'}$ if and only if $d_{\mathcal{C}}(C, C') \leqslant t^{-1}$ and $t \leqslant t'$, with the equality if and only if $d_{\mathcal{C}}(C, C') \leqslant t^{-1}$ and $t = t'$.

Quasimonomial valuations are valuations which, after a finite number of blowing-ups (see 2.3), are *monomial valuations*. In the forthcoming Section 2.4.5, monomial valuations will be defined and characterized. In Proposition 2.4.28 we will prove that result.

Lemma 2.4.2. *The set of quasimonomial valuations, called $\mathcal{V}_{qm}$, is a rooted nonmetric tree non-complete. Furthermore, $\mathcal{V}$ is the completion of $\mathcal{V}_{qm}$.*

Proof. $\mathcal{V}_{\text{qm}}$ has $\nu_{\mathfrak{m}}$ as a root, because $\nu_{\mathfrak{m}}$ is a quasimonomial valuation. In fact, $\nu_{\mathfrak{m}} = \nu_{C,1}$ for any irreducible curve C.

Given $\nu, \nu' \in \mathcal{V}_{\text{qm}}$ with $\nu < \nu'$, we can write $\nu = \nu_{C,t}$ and $\nu' = \nu_{C,t'}$, with $t < t'$. So $[\nu, \nu'] = \{\mu \in \mathcal{V}_{\text{qm}} \mid \nu \leqslant \mu \leqslant \nu'\} = \{\nu_{C,r} \mid t \leqslant r \leqslant t'\} \simeq \{r \in \mathbb{R} \mid t \leqslant r \leqslant t'\} = [t, t']$.

Similarly, it can be seen that every maximal totally ordered subset of $\mathcal{V}_{\text{qm}}$ is isomorphic to $[0, \infty]$.

It is clear that $\mathcal{V}_{\text{qm}}$ is not a complete tree, because the sequence $(\nu_{C,n})_{n\in\mathbb{N}}$ has not a limit in $\mathcal{V}_{\text{qm}}$ for any irreducible curve C. The last statement, $\mathcal{V}$ is the completion of $\mathcal{V}_{\text{qm}}$, will be proved next. ☐

Quasimonomial valuations are either *divisorial* if $t \in \mathbb{Q}$, or *irrational* if t is irrational. Divisorial valuations are isomorphic to valuations Type 1 (see 2.3.11), and Irrational valuations are isomorphic to valuations Type 5 (see 2.3.11).

The space $\mathcal{V}$ is the tree completion of $\mathcal{V}_{\text{qm}}$: the maximal elements in $\mathcal{V}$, $\mathcal{V} \setminus \mathcal{V}_{\text{qm}}$, are the curve valuations and some others, called *infinitely singular* valuations. These valuations are isomorphic to valuations Type 4 (see 2.3.11).

In short, the valuative tree contains all of the valuations except valuations of Type 6 (see 2.3.11). By Noether formula, it is clear that a curve valuation ν_C has as cluster the cluster of infinitely near points of the curve C, $K(C)$, i.e., $\nu_C = \nu(K(C))$. An infinitely singular valuation ν has a cluster with infinitely many singular points, no infinitely many of them consecutive. This cluster $K(\nu)$ can be thought as the "limit of clusters" of irreducible curves C_n such that $(\nu(C_n))_{n\in\mathbb{N}}$ is an increasing (no necessarily divergent) sequence. To the cluster of quasimonomial valuations is devoted the following section.

2.4.1 Quasimonomial valuations

In this section we will study the quasimonomial valuation, and we will assign a cluster to each valuation, such that $\nu_{C,t} = \nu(K(\nu_{C,t}))$. This will allow to make a correspondence to these valuations with the valuations studied in Section 2.3.11.

Let C be an irreducible curve and let $t \geqslant 1$ be a real number. We define the set

$$\mathcal{D}_{C,t} = \{D \in \mathcal{C} \mid \frac{1}{d_{\mathcal{C}}(C,D)} \geqslant t\} = \{D \in \mathcal{C} \mid \frac{C \cdot D}{m(D)} \geqslant tm(C)\} = \{D \mid \nu_D(C) \geqslant tm(C)\}.$$

Then

$$\nu_{C,t}(\psi) = \inf\{\nu_D(\psi) \mid D \in \mathcal{D}_{C,t}\}.$$

We also define the subset $\widetilde{\mathcal{D}_{C,t}} \subset \mathcal{C}$

$$\widetilde{\mathcal{D}_{C,t}} = \{D \in \mathcal{C} \mid \frac{1}{d_{\mathcal{C}}(C,D)} = t\} = \{D \in \mathcal{C} \mid \frac{C \cdot D}{m(D)} = tm(C)\} = \{D \mid \nu_D(C) = tm(C)\},$$

By Theorem 1.3.1, this set is nonempty if and only if $t \in \mathbb{Q}$. In this case,

$$\nu_{C,t}(\psi) = \inf\{\nu_D(\psi) \mid D \in \widetilde{\mathcal{D}_{C,t}}\}.$$

Before the study of the general case, let us discuss the easiest case, which will help us to understand the quasimonomial valuations. Let $\nu_{C,t}$ be a quasiomonomial valuation, such that $t \in \mathbb{N}$, $C \in \mathcal{C}$ smooth. In this case $\mathcal{D}_{C,t}$ is the set of the curves which pass through the first t points with normalized multiplicity 1. For any $\psi \in R$ irreducible, we can compute, at least, $\nu_{C,t}(\psi)$ as follows:

1. If $(\psi = 0)$ does not belong to $\mathcal{D}_{C,t}$ then, by the ultrametric inequality, $\nu_{C,t}(\psi) = \nu_D(\psi)$ for all $D \in \mathcal{D}_{C,t}$. In particular, $\nu_{C,t}(\psi) = \nu_C(\psi) = \frac{C\cdot(\psi=0)}{m_O(C)}$.
2. Otherwise, if $(\psi = 0)$ belongs to $\mathcal{D}_{C,t}$, then let $D \in \mathcal{D}_{C,t}$ be an irreducible curve such that $K(D) \cap K(\psi)$ has only the first t points of $K(D)$. Therefore, by Noether formula, $\nu_{C,t}(\psi) = tm_O(\psi)$.

Therefore, by Noether formula on valuations, $K(\nu_{C,t})$ is the cluster which consists of the first t points of $K(C)$.

Let us discuss now the case $t \in \mathbb{Q}$ and $C \in \mathcal{C}$ (no necessarily irreducible). Let n be a natural number such that $t_n(C) < t \leqslant t_{n+1}(C)$, and write $p = p_n(C)$ (in this case, $t_n(C) = t_p(C)$), and $q = p_{n+1}(C)$.

Lemma 2.4.3. *Let $D \in \mathcal{C}$ be an irreducible curve. Then D belongs to $\widetilde{\mathcal{D}_{C,t}}$ if and only if D passes through every point less or equal than p with the same normalized multiplicity as C and through q with multiplicity $n_q(D) = \frac{t-t_p(C)}{b_q(C)}$, and does not pass through $p_{n+2}(C)$.*

This lemma is a corollary of Proposition 1.3.3. As a corollary, the cluster of $\nu_{C,t}$ can be computed as follows:

Corollary 2.4.4. *$K(\nu_{C,t})$ contain every point less or equal than q, and the satellite points defined by the continued fraction of $\frac{t-t_p(C)}{b_p(C)^2}$.*

As before, we can compute $\nu_{C,t}(\psi)$ for any irreducible $\psi \in R$ as follows:

1. If $(\psi = 0)$ does not belong to $\mathcal{D}_{C,t}$, then, by the ultrametric inequality, $\nu_{C,t}(\psi) = \nu_D(\psi)$ for all $D \in \mathcal{D}_{C,t}$. In particular, $\nu_{C,t}(\psi) = \nu_C(\psi) = \frac{C\cdot(\psi=0)}{m_O(C)}$.

2. If $(\psi = 0)$ belongs to $\mathcal{D}_{C,t} \setminus \widetilde{\mathcal{D}_{C,t}}$, then, by Proposition 1.3.3, $\nu_{C,t}(\psi) = \nu_D(\psi)$ for all $D \in \widetilde{\mathcal{D}_{C,t}}$. Therefore, $\nu_{C,t}(\psi) = tm_O(\psi = 0)$.

3. Otherwise, if $(\psi = 0)$ belongs to $\widetilde{\mathcal{D}_{C,t}}$, then let $D \in \widetilde{\mathcal{D}_{C,t}}$ be an irreducible curve such that $d_{\mathcal{C}}(D, \psi) = 1/t$ (it exists, see Remark 1.3.4). Therefore, by Noether formula, $\nu_{C,t}(\psi) = \nu_D(\psi) = tm_O(\psi = 0)$.

Remark 2.4.5. *It is easy to describe the set $\mathcal{D}_{C,t}$. In virtue of Proposition 1.3.3: A curve $D \in \mathcal{C}$ belongs to $\mathcal{D}_{C,t}$ if and only if $q \in K(D)$ and $n_q(D) \geqslant \frac{t-t_p(C)}{b_q(C)}$.*

Example 2.4.6. *Let C be a curve with Enriques Diagram as in Figure 2.1, and let $\psi = \psi_1 \cdot \psi_2 \cdot \psi_3 \in R$, where any ψ_i is irreducible and $\psi_i = 0$ has the Enriques Diagram as in Figure 2.1, and $t = \frac{455}{162}$.*

The next values can be computed easily: $t_p(C) = \frac{25}{9}, t_q(C) = \frac{26}{9}$, $b_p(C) = n_p(C) = \frac{1}{3}$ and $b_q(C) = n_q(C) = \frac{1}{3}$.

Every curve of $\widetilde{\mathcal{D}_{C,t}}$ passes through p with normalized multiplicity $\frac{1}{3}$ and through q with normalized multiplicity $\frac{t-t_p(C)}{b_q(C)} = 3\frac{5}{162} = \frac{5}{54}$. For example, D is a curve of $\widetilde{\mathcal{D}_{C,t}}$ (see Figure 2.1). Therefore, the cluster of $\nu_{C,t}$ is formed from the common points of all the curves of $\widetilde{\mathcal{D}_{C,t}}$.

Now let us compute $\nu_{C,t}(\psi)$. We should compute $\nu_{C,t}(\psi_i)$ for all $i = 1, 2, 3$.

$(\psi_1 = 0)$ does not belongs to $\mathcal{D}_{C,t}$, so $\nu_{C,t}(\psi_1) = \frac{C\cdot(\psi_1=0)}{m_O(C)} = \frac{8}{3}$.

$(\psi_2 = 0$ belongs to $\mathcal{D}_{C,t} \setminus \widetilde{\mathcal{D}_{C,t}}$, because $n_q(\psi_2) = \frac{2}{15}$. Therefore $\nu_{C,t}(\psi_2) = tm_0(\psi_2 = 0) = \frac{455}{162}15 = \frac{2275}{54}$.

Finally, $(\psi_3 = 0)$ belongs to $\widetilde{\mathcal{D}_{C,t}}$, because $n_q(\psi_3) = \frac{5}{18}$. Notice that $d_{\mathcal{C}}(D, \psi_3) = 1/t$, so $\nu_{C,t}(\psi_3) = tm_0(\psi_3 = 0) = \frac{455}{162}54 = \frac{244}{3}$.

Therefore:

$$\nu_{C,t}(\psi) = \frac{8}{3} + \frac{2275}{54} + \frac{244}{3} = \frac{6811}{54}$$

Let us consider the case $t \notin \mathbb{Q}$. In this case the set $\widetilde{\mathcal{D}_{C,t}}$ is empty. Let n be the natural number such that $t_n(C) < t < t_{n+1}(C)$, and put $p = p_n(C), q = p_{n+1}(C)$. Suppose that $\frac{t-t_p(C)}{b_q(C)^2} = [0, a_1, a_2, \ldots]$. Then

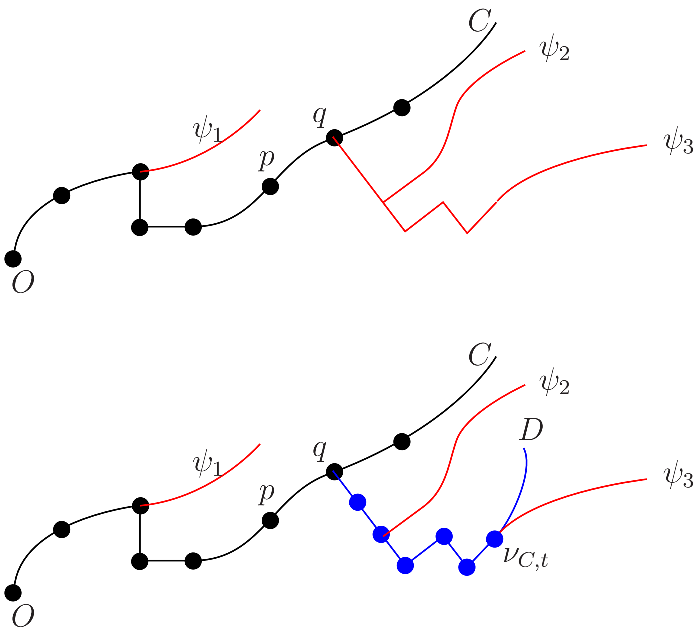


Figure 2.1: An example of computing $K(\nu_{C,t})$ and $\nu_{C,t}(\psi)$, where $\psi = \psi_1\psi_2\psi_3$.

Proposition 2.4.7. *Keep the above notations. The set $K(\nu_{C,t})$ contains every point less or equal than q and a stair of satellite points defined by the infinite continued fraction $[0, a_1, a_2, \ldots]$, namely, a_1 points proximate to the immediate predecessor of q, a_2 points proximate to the point in the $(a_1 - 1)$-th neighbourhood of q, etc.*

Proof. Let us consider the sequence of rational numbers $n_k = b_q(C)[0, a_1, a_2, \ldots, a_{2k}]$. By Proposition 1.5.9, it is clear that this is an increasing sequence with limit

$$b_q(C)[0, a_1, a_2, \ldots] = \frac{t_p(C) - t}{b_q(C)}.$$

Any irreducible curve D_k that passes through q with normalized multiplicity $n_q(D_k) = n_k$ (by hypothesis $n_k < [0, a_1, a_2, \ldots] < n_q(C)$, so there are curves which passes through q with that multiplicity), by Proposition 1.3.3, satisfies that

$$d_{\mathcal{C}}(C, D_k)^{-1} = t_p(C) + b_q(C)\min\{n_k, n_q(C)\} = t_p(C) + b_q(C)n_k =: t_k.$$

Therefore, $(d_{\mathcal{C}}(C, D_k)^{-1} = t_k)_{k\in\mathbb{N}}$ is clearly an increasing sequence which tends to t. Furthermore, for any $\psi \in R$,

$$(\min\{\nu_D(\psi) \mid n_q(D) = n_k\})_{k\in\mathbb{N}} = (\nu_{C,t_k}(\psi))_{k\in\mathbb{N}}$$

is an increasing sequence. Hence

$$\nu_{C,t}(\psi) = \lim_k\{\min\{\nu_D(\psi) \mid n_q(D) = n_k\}\} = \lim_k\{\nu_{C,t_k}(\psi)\}.$$

The values $\nu_{C,t_k}(\psi)$ can be computed by Noether formula. Notice that $K(\nu_{C,t_k})$ is an increasing sequence of sets, because $K(\nu_{C,t_{k+1}})$ contains all points of $K(\nu_{C,t_k})$ and $a_{2k+1} + a_{2k+2}$ satellite points. Therefore the limit is a valuation with cluster the union of all these clusters, and the proof is completed. ☐

Example 2.4.8. *Let C be an smooth curve an take $t = [3, 2, 1, 2, 3, 2, 3, 5, 1, 3, 2, 5, 2, \ldots] \notin \mathbb{Q}$. Some elements of the sequence of clusters $K(\nu_{C,t_k})$ are represented in Figure 2.2.*

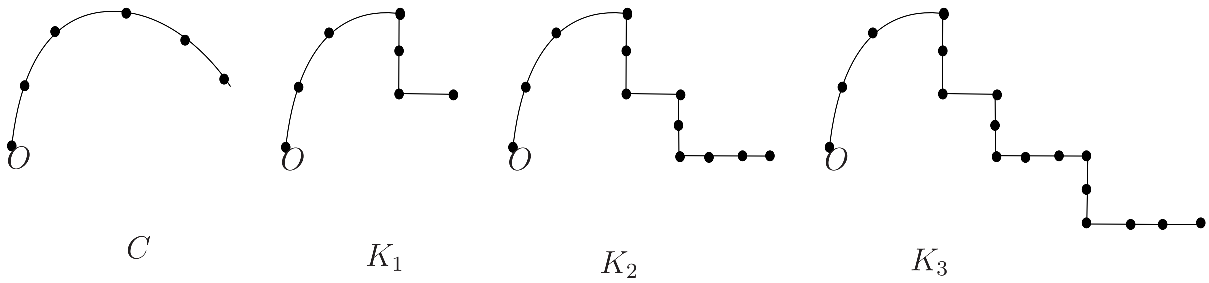


Figure 2.2: The case $t \notin \mathbb{Q}$. The curve C and first three elements of the sequence $\{K_k = K(\nu_{C,t_k})\}$.

The computation of $\nu_{C,t}(\psi)$ for any irreducible $\psi \in R$ can carried out as before:

1. If $(\psi = 0)$ does not belong to $\mathcal{D}_{C,t}$, then, by the ultrametric inequality, $\nu_{C,t}(\psi) = \nu_D(\psi)\ \forall D \in \mathcal{D}_{C,t}$. In particular, $\nu_{C,t}(\psi) = \nu_C(\psi) = \frac{C\cdot(\psi=0)}{m_O(C)}$.
2. Otherwise, if $(\psi = 0)$ belongs to $\mathcal{D}_{C,t}$, then take $\{D_k\}$ a sequence of curves such that $d_{\mathcal{C}}(D_k, C) = 1/t_k$ and such that all D_k share the same free points with ψ. Therefore, $\nu_{C,t_k}(\psi) = \nu_{D_k}(\psi) = t_k m_O(\psi = 0)$. When k tends to infinity, $\nu_{C,t}(\psi) = t m_O(\psi = 0)$.

2.4.2 Comparison of valuations

In this section we will see alternative ways to decide whether two valuations are comparable. Let ν be a valuation. We have seen that every valuation has an associated cluster $K(\nu)$. We consider $F(\nu) = F(K(\nu))$, in other words, the set of free points of $K(\nu)$, and we write $p_i(\nu), n_i(\nu)$, $b_i(\nu)$ and $t_i(\nu)$ for mean $p_i(K(\nu)), n_i(K(\nu))$, $b_i(K(\nu))$ and $t_i(K(\nu))$ respectively.

Comparing valuations and comparing distances between curves are very close problems. For example, let $\nu_{C,t_1}, \nu_{C,t_2} \in \mathcal{V}_{\mathrm{qm}}$ be two divisorial valuations, and let $D_i \in \mathcal{C}$ be a curve such that $d_{\mathcal{C}}(C, D_i) = t_i^{-1}$. Then $\nu_{C,t_1} > \nu_{C,t_2}$ if and only if $d_{\mathcal{C}}(C, D_1) < d_{\mathcal{C}}(C, D_2)$.

Notice that for any $t \in \mathbb{Q}$ and $D \in \mathcal{C}$, D belongs to $\widetilde{\mathcal{D}_{C,t}}$ if and only if $K(D) \supset K(\nu_{C,t})$ and the minimal point of $K(D) \setminus K(\nu_{C,t})$ is a free point.

We look for a way for comparing two valuations using their clusters $K(\nu)$. The first result allow us to compare valuations by comparing the free points of their clusters and the normalized valuation of the last free common point.

Proposition 2.4.9. *Let ν_1, ν_2 be two valuations and let $p_i(\nu_1) = p_i(\nu_2)$ be the last free common point. Then*

$$\nu_1 \leqslant \nu_2 \iff F(\nu_1) \subset F(\nu_2) \text{ and } n_i(\nu_1) \leqslant n_i(\nu_2)$$

Proof. $\Rightarrow$ By reductio ad absurdum: let $r \in \mathbb{N}$ be the smallest number satisfying $p_r \in F(\nu_1) \setminus F(\nu_2)$ or $n_r(\nu_1) > n_r(\nu_2)$. Let C be a curve that passes through $p_i(\nu_1) = p_i(\nu_2)$ with multiplicity equal to $n_i(\nu_1) = n_i(\nu_2)$ for $0 \leqslant i < r$ and that passes through $p_r(\nu_1)$ with multiplicity $n_r(\nu_1)$. By the Noether formula, we have that $\nu_1(C) > \nu_2(C)$ (so $\nu_1 \nleqslant \nu_2$).

$\Leftarrow$ It is an immediate consequence of the Noether formula on valuations.

□

Proposition 2.4.10. *Let $\nu_1, \nu_2 \in \mathcal{V}_{qm}$ be two different quasimonomial valuations such that $F(\nu_1) = F(\nu_2) = \{p_1, \ldots, p_N\}$. Let q be the maximal point in $K(\nu_1) \cap K(\nu_2)$. Therefore, one of this cases holds, after renaming the valuations if needed:*

- *There exist q_1 and q_2 in the first neighbourhood of q and in $K(\nu_1)$ and $K(\nu_2)$ respectively. In this case, $\nu_1 > \nu_2$ if and only if $K(\nu_1)$ goes right in q and $K(\nu_2)$ goes down in q (see the definition of* going right, going free *and* going down *in Section 1.4.2).*

- *There exists q_1 in the first neighbourhood of q and in $K(\nu_1)$ and the maximal point of $K(\nu_2)$ is q. In this case, $\nu_1 > \nu_2$ if and only if $K(\nu_1)$ goes right in q.*

Proof. Let C be a curve such that $F(C) = \{p_1, \ldots, p_N, p_{N+1}(C), \ldots\}$, with $p_{N+1}(C)$ in the first neighbourhood of p_N. Then $\nu_i = \nu_{C,t_i}$, where $t_i = d_{\mathcal{C}}(C, D_i)^{-1}$, where D_i is a curve which passes through all the points on $K(\nu_i)$ and is free after these points.

By previous Proposition 1.3.3, $\nu_1 > \nu_2$ if and only if $n_N(\nu_1) > n_N(\nu_2)$. Then the result follows in virtue of Theorem 1.5.11. □

These two propositions allow us to compare quasimonomial valuations from the Enriques Diagram of their clusters. Next result will let us to identify a quasimonomial valuation by only computing one value:

Proposition 2.4.11. *Let ν be a quasimonomial valuation. Then:*

1. *Suppose that $\nu_C > \nu$. Then $\nu = \nu_{C,t}$, where $t = \frac{\nu(C)}{m_O(C)}$.*

2. *Reciprocally, if $\nu = \nu_{C,t}$ and D is an irreducible curve such that $\nu(D) = t m_O(D)$, then $\nu_D > \nu$.*

Proof.

1. By definition, $\nu_{C,t'}(C) = \min\{\nu_D(C) \mid d_{\mathcal{C}}(C, D) \leqslant t'^{-1}\}$. But $d_{\mathcal{C}}(C, D) = \nu_D(C) m_O(C)$. By Theorem 1.3.1, $\nu_{C,t'}(C) = t' m_O(C)$. Therefore, $\nu_{C,t'}(C) = t m_O(C)$ implies that $t' = t$.

2. If $\nu = \nu_{C,t}$, then $\nu_D > \nu$ if and only if $D \in \mathcal{D}_{C,t}$. Suppose that D does not belong to $\mathcal{D}_{C,t}$. Then $d_{\mathcal{C}}(C, D) > t^{-1}$. But in this case, $\nu_{C,t}(D) = \nu_C(D) = \frac{m_O(D)}{d_{\mathcal{C}}(C,D)} < t m_O(D)$. By reductio ad absurdum, $\nu_D > \nu$.

□

2.4.3 Two remarkable parameterization of the valuative tree

In this section we will give two parameterizations of the valuative tree: skewness and thinness. We will define them, we will give some methods for their computation and we will give some relations between these parameterizations and other invariants.

Let ν be a valuation of the valuative tree. The value

$$\sup\{\nu(D)/m_O(D) \mid D \in R\} \in [1, \infty]$$

is called the *skewness* of ν, and it is denoted by $\alpha(\nu)$.

Lemma 2.4.12. *Let $\nu_{C,t}$ be a quasimonomial valuation. Then $\alpha(\nu_{C,t}) = t$.*

Proof. In virtue of Proposition 2.4.11, $\nu_{C,t}(C) = tm_O(C)$. Then, $\alpha(\nu_{C,t}) \geqslant t$. But by Proposition 2.4.11 again, $\nu_{C,t}(D) \leqslant tm_O(D)$ for all irreducible curve D. Therefore, the equality holds. ☐

Theorem 2.4.13. *α is a parameterization of the Valuative Tree $\mathcal{V}$.*

Proof. In virtue of Lemma 2.4.12, α is a strictly increasing function. The restriction of α in any maximal totally ordered subtree is a bijection: is injective because is strictly increasing and is exhaustive because if $\nu_{C,t}$ and $\nu_{C',t'}$ belong to a maximal totally ordered subtree, with $t < t'$, then $\nu_{C,t} < \nu_{C',t'}$ and $\nu_{C',r}$ belongs to that subtree for any r in (t, t'). ☐

Proposition 2.4.14. *The skewness satisfies the following statements:*

- *For any $\nu \in \mathcal{V}$ and $\phi \in R$ irreducible, $\nu(\phi) = \alpha(\nu \wedge \nu_\phi)m_O(\phi)$.*
- *For any irreducible curves $C, D \in \mathcal{C}$, $\alpha(\nu_C \wedge \nu_D) = \frac{1}{d_{\mathcal{C}}(C,D)}$.*

Proof.
- It is clear that for any irreducible curve ϕ, $\nu(\phi) = (\nu \wedge \nu_\phi)(\phi)$. But $(\nu \wedge \nu_\phi) < \nu_\phi$ for definition of infimum. In virtue of Proposition 2.4.11, it implies that $(\nu \wedge \nu_\phi) = \nu_{\phi,t}$ for some t. And by Lemma 2.4.12, $t = \alpha(\nu \wedge \nu_\phi) = \frac{(\nu \wedge \nu_\phi)(\phi)}{m_O(\phi)}$.
- In virtue of the previous part of that Proposition, $\alpha(\nu_C \wedge \nu_D) = \frac{\nu_D(C)}{m_O(D)}$. And by definition of curve valuation $\nu_D(C) = \frac{C \cdot D}{m_O(C)}$.

☐

It is easy to compute the skewness of any quasimonomial valuation if we have its Enriques Diagram. In virtue of Corollary 1.5.25, the skewness and the inverse distances can be computed in the same way. Therefore, we can use the results of Section 1.3 for computing the skewness. In particular, Proposition 1.4.1 is very usefull.

Example 2.4.15. *Let ν be a quasimonomial valuation with Enriques Diagram as in Figure 2.3. Let C be a curve such that $K(\nu) \subset K(C)$ and without any satellite point in $K(C) \setminus K(\nu)$. Let p be the maximal point in $K(\nu)$. Let D be a smooth curve such that $F(\nu) \subset F(D)$.*

Therefore

$$\alpha(\nu) = \tau_p = t_p(C) = \frac{C \cdot D}{m_O(C)m_O(D)} = \frac{55}{12}$$

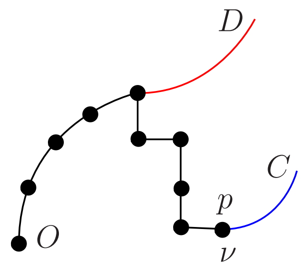


Figure 2.3: Enriques Diagram of a valuation ν of Example 2.4.15.

Let ν be an element of the Valuative Tree. We define the *multiplicity* of ν, denoted by $m_O(\nu)$, by $m_O(\nu) = \min\{m_O(C) \mid \nu_C \geqslant \nu\}$. By convention, $\min \emptyset = \infty$. Therefore

- $m_O(\nu) = \infty$ if and only if ν is infinitely singular.
- $m_O(\nu_C) = m_O(C)$.

Remark 2.4.16. *m_O is an increasing function of $\mathcal{V}$ to $\mathbb{N} \cup \{\infty\}$.*

Let τ be a valuation, and let $\vec{v}_\tau \in T_\tau\mathcal{T}$ be a tangent vector. If $\nu_\mu \in \vec{v}_\tau$, we define $m_O(\vec{v}_\tau) = m_O(\tau)$. Otherwise, we define $m_O(\vec{v}_\tau) = \min\{m_O(\nu) \mid \nu \in \vec{v}_\tau\}$.

Let τ be a divisorial valuation, and let p the maximal point of $K(\tau)$. The value s_p is called *generic multiplicity* of τ, and is denoted by $b(\tau)$.

Proposition 2.4.17. *Let τ be a quasimonomial valuation. Let q be last free point of $K(\tau)$. Then $m_O(\tau) = s_q$.*

Proof. It is clear that $m_O(\tau) = m_O(C)$, where C is a curve which passes through q and it is smooth in q. Therefore, it is sufficient to prove that $m_O(C) = s_q$. But s_q was defined as $m_O(C)$. □

Proposition 2.4.18. *Let τ be a quasimonomial valuation. Let p be the last point of $K(\tau)$ and let q be the last free point of $K(\tau)$. Then:*

- *If $p = q$, for any tangent vector $\vec{v}_\tau \in T_\tau\mathcal{T}$ it holds that $m_O(\vec{v}_\tau) = m_O(\tau) = b(\tau)$.*
- *If $p > q$, for any tangent vector $\vec{v}_\tau \in T_\tau\mathcal{T}$ but exactly two it holds that $m_O(\vec{v}_\tau) = b(\tau)$. For the other two vectors, $m_O(\vec{v}_\tau) = m_O(\tau)$.*

  Furthermore, if $\nu \neq \tau$ is a valuation such that p belongs to $K(\nu)$, then $m_O(\vec{v}_\tau) = b(\tau)$ if and only if the point of $K(\nu)$ in the first neighbourhood of p is a free point.

Proof. Suppose that $p = q$ and $\vec{v}_\tau \in T_\tau\mathcal{T}$. The result is clear if $\nu_\mu \in \vec{v}_\tau$, so let us suppose that the elements of $\vec{v}_\tau$ are greater than τ (see Lemma 2.2.1). Let $\nu \in \vec{v}_\tau$. $\nu > \tau$, but $n_q(\tau) = b_q(\tau)$ because q is the last point of the cluster of τ. It implies that $n_q(\nu) = b_q(\nu)$, because $n_q(\nu) \geqslant n_q(\tau)$ (see Proposition 2.4.9). Therefore, the point of $K(\nu)$ in the first neighbourhood of q, say q', is free.

Let C be a curve such that passes through q' and it is free in q'. Then it is obvious that $m_O(C) = m_O(\tau)$. Hence, $m_O(\vec{v}_\tau) = m_O(\nu_C) = m_O(\tau)$.

Let us suppose now that $p > q$. Let p_1, p_2 be the two satellite points in the first neighbourhood of p, and let ν_1, ν_2 be the valuations such that p_i is the maximal point of their cluster. It is clear that ν_1 and ν_2 belong to two different tangent vectors, because one of them is greater than τ and the other is less. Suppose that ν_1 is greater than τ. $[\nu_2]_\tau$, the vector represented by ν_2, satisfies that $m_O([\nu_2]_\tau) = m_O(\tau)$ by definition of multiplicity, because $\nu_{\mathfrak{m}}$ belongs to $[\nu_2]_\tau$ in virtue of Lemma 2.2.1.

Let C be a curve such that passes through q and it is free in q. Then $\nu_C > \nu_1$, and, hence, ν_C belongs to $[\nu_2]_\tau$. But $m_O(\nu_C) = m_O(C) = m_O(\tau)$.

On the other hand, take a valuation such that p belongs to $K(\nu)$, and that the point on $K(\nu)$ in the first neighbourhood of p, say p', is a free point. Take C a curve such that passes through p' and that it is free in p'. Therefore, ν_C and ν belongs to the same tangent vector, and $m_O([\nu_C]_\tau) = m_O(C) = b(\tau)$ by definition of b. □

Remark 2.4.19. *Let τ be a quasimonomial valuation. Let p be last point of $K(\tau)$ and let q be the last free point of $K(K)$. Then, $m_O(\tau) = s_q$ and $b(\tau) = s_p$. Therefore, $b(\tau)$ is a multiple of $m_O(\tau)$.*

Lemma 2.4.20. *Let τ be a quasimonomial valuation. Let p be last point of $K(\tau)$. Then $b(\tau)$ is the value at p of the weight cluster with points $K(\tau)$ and value 1 at the origin and 0 at the other points.*

Proof. It is a consequence of the proximity equalities (Theorem 3.5.3 of [1]). □

Let $\nu \in \mathcal{V}$ be an element of the Valuative Tree. We define the *thinness* of ν as

$$A(\nu) = 2 + \int_{\nu_{\mathfrak{m}}}^{\nu} m_O(\mu) d\alpha(\mu).$$

Proposition 2.4.21 (Proposition 3.46 of [4])**.** *The thinness is a parameterization of the Valuative Tree. Furthermore, $A(\nu)$ is rational for the divisorial valuations, it is irrational for the irrational valuations, and it is equal to infinity for the curve valuations.*

Proposition 2.4.22. *Let ν be a divisorial valuation, and let p be the maximal point of $K(\nu)$. Then $A(\nu) = \frac{a_p}{b(\nu)}$, where a_p is the value at p of a weighted cluster with maximal point p and value 1 at every point.*

A proof of this fact can be found in [3].

2.4.4 On the weak topology in the valuative tree

In Section 2.2 we defined a topology on any rooted nonmetric tree. In this Section we will show some properties of this topology in the case of the Valuative Tree.

Let C be the set of functions from a set D to a metric space M. The *weak convergence topology* of C is the topology defined as follows: for any sequence $(f_n)_{n\in\mathbb{N}}$ of C and $f \in C$, $f_n \to f$ if and only if $f_n(d) \to f(d)$ for all $d \in D$.

Proposition 2.4.23. *The weak tree topology of $\mathcal{V}$ coincides with the induced weak convergence topology of $\mathcal{V}$ as a set of functions from R to $\mathbb{R}$.*

Proof. It is sufficient to prove that if $(\nu_n)_{n\in\mathbb{N}}$ is a sequence of elements of $\mathcal{V}$ and ν belongs to the Valuative Tree, then $\nu_n \to \nu$ in the weak tree topology if and only if $\nu_n(\psi) \to \nu(\psi)$ for all $\psi \in R$.

$\Rightarrow$ By reductio ad absurdum. Suppose that $\nu_n \to \nu$ in the weak tree topology but there exists an irreducible curve ψ such that $\nu_n(\psi) \not\to \nu(\psi)$. Write $M = \nu(\psi)$. There are two possibilities:

- $M = \infty$: Notice that it implies that $\nu = \nu_\psi$, and $\nu_n(\psi) \not\to \nu(\psi) = \infty$ implies that there is a bounded subsequence of the sequence (ν_n). Let ν_{n_k} be that subsequence, and take $K > 0$ such that $\nu_{n_k}(\psi) < K$. Write $N = K/m_o(\psi)$. By Proposition 2.4.11, $\nu_{\psi,N}(\psi) = Nm_O(\psi) = K$. Therefore, $\nu_{n_k}(\psi) < \nu_{\psi,N}(\psi)$ which implies that $\nu_{n_k} \not> \nu_{\psi,N}$ for all k. Take U the vector of $T_{\nu_{\psi,N}}\mathcal{V}$ that contains ν. On the other hand, $\nu = \nu_\psi > \nu_{\psi,N}$. In virtue of Lemma 2.2.1, any element of the subsequence ν_{n_k} belongs to U. But this fact contradicts that $\nu_n \to \nu$.
- $M < \infty$: Then, by definition of limit in $\mathbb{R}$, there exists $\varepsilon > 0$ and a subsequence of the sequence (ν_n) such that any element of that subsequence belongs to $(M-\varepsilon, M+\varepsilon)$. Therefore, there are two possibilities:
  - There is a subsequence of the sequence (ν_n), namely ν_{n_k}, such that $\nu_{n_k}(\psi) < M-\varepsilon$ for all k. In this case, write $N = (M-\varepsilon/2)/m_O(\psi)$. By Proposition 2.4.11, $\nu_{\psi,N}(\psi) = Nm_O(\psi) = M-\varepsilon/2$. Therefore, $\nu_{n_k}(\psi) < \nu_{\psi,N}(\psi)$ which implies that $\nu_{n_k} \not> \nu_{\psi,N}$ for all k. Take U the vector of $T_{\nu_{\psi,N}}\mathcal{V}$ that contains ν. $\nu(\psi) = M$. By Proposition 2.4.11, $\nu > \nu_{\psi,N}$. In virtue of Lemma 2.2.1, any element of the subsequence ν_{n_k} belongs to U. But this fact contradicts that $\nu_n \to \nu$.
  - There is a subsequence of the sequence (ν_n), namely ν_{n_k}, such that $\nu_{n_k}(\psi) > M+\varepsilon$ for all k. In this case, write $N = (M+\varepsilon/2)/m_O(\psi)$. By Proposition 2.4.11, $\nu_{\psi,N}(\psi) = Nm_O(\psi) = M+\varepsilon/2$. Therefore, $\nu_{n_k}(\psi) > \nu_{\psi,N}(\psi)$ which implies, in virtue of Proposition 2.4.11, that $\nu_{n_k} > \nu_{\psi,N}$ for all k. Take U the vector of $T_{\nu_{\psi,N}}\mathcal{V}$ that contains ν. On the other hand, $\nu(\psi) = M$ implies that $\nu \not> \nu_{\psi,N}$. In virtue of Lemma 2.2.1, any element of the subsequence ν_{n_k} belongs to U. But this fact contradicts that $\nu_n \to \nu$.

$\Leftarrow$ Let $\tau \neq \nu$ be a valuation, and let U be the tangent vector at τ that contains ν. It is sufficient to prove that there exists $N > 0$ such that ν_n belongs to U for all $n > N$. If τ is maximal in $\mathcal{V}$, then $U = \mathcal{V} \setminus \{\tau\}$, and the result is trivially true. Let us suppose that τ is not maximal. Therefore, $\tau = \nu_{\psi,t}$ for some irreducible $\psi \in R$ and $t \geqslant 1$. Two different cases are considered:

- $\nu(\psi) < tm_O(\psi)$. Hence, $\nu \not> \tau$. But there exists N such that $\nu_n(\psi) < tm_O(\psi)$ for all $n > N$, so $\nu_n \not> \tau$. By Lemma 2.2.1, ν_n and ν belong to the same tangent vector at τ for all $n > N$.
- $\nu(\psi) \geqslant tm_O(\psi)$. Then, by Proposition 2.4.11, $\nu > \tau$. By definition of tree, (τ, ν) is isomorphic to a real segment. Take $\tau' \in (\tau, \nu)$. We can write $\tau' = \nu_{\psi',t'}$ and $\tau = \nu_{\psi',t}$ with $t < t'$. And in virtue of Proposition 2.4.11, $\nu(\psi') \geqslant t'm_O(\psi')$.

Take $\varepsilon > 0$ such that $t' - \varepsilon > t$. There exists N such that $\nu_n(\psi') > t' - \varepsilon$ for any $n > N$. By Proposition 2.4.11, it implies that all ν_n are in the same tangent vector at τ that τ'. But ν is in that vector.

□

Notice that $\mathcal{V}$ is a metric space with the distance induced by the skewness:

$$d_\alpha(\nu, \mu) = \left(\frac{1}{\alpha(\mu \wedge \nu)} - \frac{1}{\alpha(\mu)}\right) + \left(\frac{1}{\alpha(\mu \wedge \nu)} - \frac{1}{\alpha(\nu)}\right).$$

The topology defined with this metric is called *strong tree topology* on $\mathcal{V}$.

But we can define another distance in $\mathcal{V}$.

$$d_{\mathcal{V}}^{\text{str}}(\nu_1, \nu_2) = \sup_{\phi \in \mathfrak{m} \text{ irreducible}} \left| \frac{m_O(\phi)}{\nu_1(\phi)} - \frac{m_O(\phi)}{\nu_2(\phi)} \right|.$$

The *strong topology* of $\mathcal{V}$ is the topology defined by this distance.

Proposition 2.4.24 (Theorem 5.7 of [4])**.** *The strong topology of $\mathcal{V}$ coincides with the strong tree topology. Furthermore, the two distances are equivalents:*

$$d_{\mathcal{V}}^{\text{str}}(\nu_1, \nu_2) \leqslant d_\alpha(\nu_1, \nu_2) \leqslant 2 d_{\mathcal{V}}^{\text{str}}(\nu_1, \nu_2).$$

Remark 2.4.25. *We will write simply* weak topology *and* strong topology *instead of* weak tree topology *and* strong tree topology. *Notice that in virtue of Proposition 2.4.23 and Proposition 2.4.24 no confusion is possible.*

Proposition 2.4.26 (Proposition 5.8 of [4])**.** *The strong topology on $\mathcal{V}$ is strictly stronger than the weak topology. Furthemore, $\mathcal{V}$ is not locally compact with the strong topology.*

2.4.5 Monomial valuations

This section is devoted to Monomial valuations. These valuations are some quasimonomial valuations, and they are important because the computations with them are very easy. In this section we will define the concept of monomial valuation and we will give some characterizations and properties of them.

A valuation $\nu \in \mathcal{V}$ is called *monomial valuation* if there exists some coordinates (x, y) and some $\alpha \geqslant 1$, $\alpha \in \mathbb{R}$, such that for every ϕ of R, it holds

$$\nu(\phi) = \min\{i + \alpha j \mid \alpha_{ij} \neq 0\},$$

where

$$\phi = \sum_{i,j} \alpha_{ij} x^i y^j.$$

As we can notice, the computation of values with a monomial valuation is easy: it is reduced to the computation of a minimum.

Proposition 2.4.27. *Let $\nu \in \mathcal{V}$ be a quasimonomial valuation. Then, the following are equivalent:*

1. *ν is monomial.*

2. *There are not any free point preceding a satellite point in $K(\nu)$.*

3. *$m_O(\nu) = 1$.*

Proof. $1 \Rightarrow 2$**.** Take x, y the coordinates which make ν a monomial valuation. It is clear that $\nu(y = 0) = \alpha$. It is sufficient to prove that $\nu_y > \nu$, because then $\nu = \nu_{y,\alpha}$ (the cluster of $\nu_{y,\alpha}$ has a_0 free points and then $a_1 + a_2 + \ldots$ satellite points, where $\alpha = [a_0, a_1, \ldots])$.

Let ϕ be an element of R. If y divides ϕ, then $\nu_y(\phi) = \infty$, and $\nu_y(\psi) \geqslant \nu(\psi)$. Let us suppose that y does not divide ϕ. Write $\phi = \sum \phi_i y^i$, where $\phi_i \in \mathbb{C}\{x\}$, and $\phi_0 \neq 0$. Let c_i be the minimum degree of ϕ_i. Therefore, $\nu_y(\phi) = c_0$, and $\nu(\phi) = \min\{\alpha i + c_i\}$. It is clear that $\nu_y(\phi) \geqslant \nu(\phi)$, quod erat demonstrandum.

$2 \Rightarrow 1$**.** Let p be the last free point of $K(\nu)$. Let C be a smooth curve such that passes through p. This curve exists because p does not precede any satellite point by hypothesis. Let D be a smooth curve at distance 1 to C, that is, such that $K(C) \cap K(D) = \{O\}$. Take a coordinates defined by C and D, this is, a coordinates in which C is defined by $y = 0$ and D is the curve $x = 0$. Then ν is monomial with this coordinates. The prove of this fact is analogous at the proof of $\nu_y \geqslant \nu$ of the previous point of this proof.

$2 \Rightarrow 3$**.** Let p be the last free point of $K(\nu)$. Therefore, $s_p = 1$ by hypothesis. And $m_O(\nu) = 1$ in virtue of Proposition 2.4.17.

$2 \Rightarrow 3$**.** Let p be the last free point of $K(\nu)$. Therefore, p precedes a satellite point if and only if $s_p > 1$. But $m_O(\nu) = 1 = s_p$ in virtue of Proposition 2.4.17.

□

Proposition 2.4.28. *Let ν be a valuation of $\mathcal{V}$. Then ν is quasimonomial if and only if there exists a modification $\pi : S \to (\mathbb{C}^2, O)$ such that $\pi^*\nu$ is a monomial valuation.*

Proof. This result is a consequence of Proposition 2.4.27. For any quasimonomial valuation ν, let p be the last free point of $K(\nu)$. The modification π in such that p is a proper point turns ν onto a monomial valuation, because there are not any free point preceding a satellite point in $K(\pi^*\nu) = K_p(\nu)$. The converse is trivial in virtue that ν is quasimonimal if and only if $K(\nu)$ has a finite number of free points (see Proposition 2.4.9). □

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