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Application of Aboodh Transform Iterative Method for Solving Time – Fractional Partial Differential Equations

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Abstract

In this paper, the Aboodh transform iterative method is used to obtain approximate analytical solution of time-fractional partial differential equations. The fractional derivative are considered in Caputo sense, this method is a combination of the Aboodh transform and the new iterative method. Illustrative examples are considered and the comparison between the exact and approximate solutions are presented for different values of alphas. Also, the surface plots are provided in order to comprehend the effect of the fractional order. The major advantage of this method is the reduced computational effort and complexity without involving the tedious calculations of Adomian polynomials. In general, the method is efficient, precise, easy to implement and yield good results.

Keywords: Iterative method; Fractional derivative; Partial differential equation; Integral transform; Aboodh transform.

1. Introduction

The conceptual idea of fractional calculus can be traced back to the question asked by L'Hospital in 1695 [1]. Since then, several researchers have proposed variety of new fractional operators with different contributions and growth to the field of fractional calculus [2-7].

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Fractional calculus is concerned with the integrals and derivatives of arbitrary non-integer order of functions. It has gained much attention because it provides more practical models [8-10], for some essential properties of fractional calculus see [11-13]. Many integral transform methods have been used to solve linear and nonlinear differential equations of fractional order. In the case of nonlinear partial differential equations of fractional order, asymptotic, perturbation and iterative methods have been used to handle the nonlinear term [14-16]. Though this methods are simple in principle but they involve complex and so much computational effort which is costly. In pursuit of a reduced complex and computational method, integral transform methods are now been coupled together with the new iterative method introduced by Daftardar-Gejji and Jafari [17-20]. In this paper, we proposed a new iterative method called Aboodh transform iterative method for solving fractional partial differential equations without the need of Lagrange multipliers and Adomian polynomials. The rest of the paper is structured as follows: Definitions and preliminary ideas are considered in Section 2, the main concept of the proposed method is summarized in Section 3. In Section 4, we illustrate the efficiency of the method by considering some examples of practical importance. Finally, conclusion is presented in Section 5.

2. Preliminary Ideas

In this section, we presents some known definitions and results.

Definition 1. Aboodh transform is defined over the set of exponential function as [18]

$$A=\left\{Q:|Q(t)|<Pe^{k_i|t|},t\in(-1)^i\times[0,\infty),i=1,2;(p,k_1,k_2>0)\right\}, \tag{1}$$

$Q(t)$ is denoted by

$$A[Q(t)]=H(\vartheta), \tag{2}$$

And defined as

$$A[Q(t)]=\frac{1}{\vartheta}\int_0^\infty Q(t)e^{-\vartheta t}dt=H(\vartheta),t\le 0,k_1\le\vartheta\le k_2\ . \tag{3}$$

Definition 2. Inverse Aboodh transform of function $Q(t)$, if

$$A[Q(t)]=H(\vartheta),$$

then the inverse Aboodh transform is defined as

$$Q(t)=A^{-1}[H(\vartheta)],t\in(0,\infty). \tag{4}$$

Definition 3. One parameter Mittag-Leffler function is given as [14]

$$E_{\alpha}(z)=\sum_{r=0}^{\infty}\frac{z^{r}}{\Gamma(r\alpha+1)},\alpha,z\in\mathbb{C},\mathrm{Re}(\alpha)\geq 0. \tag{5}$$

Lemma 1. Aboodh transform of Caputo fractional derivative is given as [18]

$$A[(D_{t}^{\alpha}Q(t);\vartheta)]=\vartheta^{\alpha}A[Q(t)]-\sum_{r=0}^{n-1}\frac{Q^{(s)}(\vartheta)}{\vartheta^{2-\alpha+r}},\ n-1<\alpha\leq n,\ n\in\mathbb{N}. \tag{6}$$

Lemma 2. If Aboodh transform of $Q_1(t),Q_2(t),\cdots Q_k(t)$ are $H_1(\vartheta),H_2(\vartheta),\cdots,H_k(\vartheta)$ respectively, k $=1,2,\cdots,n$, then [18]

$$\begin{aligned}&A[\lambda_1Q_1(t)+\lambda_2Q_2(t)+\cdots+\lambda_kQ_k(t)]\\&=A[\lambda_1Q_1(t)]+A[\lambda_2Q_2(t)]+\cdots+A[\lambda_kQ_k(t)]\\&=\lambda_1H_1(\vartheta)+\lambda_2H_2(\vartheta)+\cdots+\lambda_kH_k(\vartheta),\end{aligned} \tag{7}$$

Where $\lambda_1,\lambda_2,\cdots,\lambda_k$ are arbitrary constants. Aboodh transform of some functions are given in Table 1.

Table 1: Aboodh transform of some functions [18]

$Q(t)$	$A[Q(t)]=H(\vartheta)$
1	$\frac{1}{\vartheta}$
t	$\frac{1}{\vartheta^3}$
t^n	$\frac{n!}{\vartheta^{n+2}}$
t^α	$\frac{\Gamma(\alpha+1)}{\vartheta^{\alpha+2}}$

3. Aboodh Transform Iterative Method

In this section, we demonstrate the idea of Aboodh transform iterative method. Consider the fractional partial differential equation of the form:

$$D^{\alpha}Q(x,t)=L(Q(x,t))+N(Q(x,t))+g(x,t),\ n-1<\alpha\leq n, \tag{8}$$

with initial condition

$$Q^{(r)}(x,0)=Q_r(x),\ r=0,1,2,\ldots,n-1. \tag{9}$$

$D^{\alpha}Q(x,t)$ is the Caputo fractional derivative of the fractional function $Q(x,t)$ while L, N and $g(x,t)$ are linear operator, nonlinear operator and the source function respectively. Applying the Aboodh transform to eq.(8) and using the initial condition, we have:

$$A[Q(x,t)]=\tfrac{1}{\vartheta^{\alpha}}\left(\sum_{r=0}^{n-1}\tfrac{Q^{(r)}(x,0)}{\vartheta^{2-\alpha+r}}+A\big[g(x,t)\big]\right)+\tfrac{1}{\vartheta^{\alpha}}\Big(A\big[L(Q(x,t))+N(Q(x,t))\big]\Big). \tag{10}$$

Taking the inverse Aboodh transform of eq.(10), we have:

$$Q(x,t)=A^{-1}\left[\tfrac{1}{\vartheta^{\alpha}}\left(\sum_{r=0}^{n-1}\tfrac{Q^{(r)}(x,0)}{\vartheta^{2-\alpha+r}}+A\big[g(x,t)\big]\right)+\tfrac{1}{\vartheta^{\alpha}}\Big(A\big[L(Q(x,t))+N(Q(x,t))\big]\Big)\right]. \tag{11}$$

The nonlinear term is decomposed as in [18]:

$$N(Q(x,t))=N\left(\sum_{r=0}^{\infty}(Q_r(x,t))\right)=N(Q_0(x,t))+\sum_{r=1}^{\infty}\left\{N\left(\sum_{j=0}^{r}Q_j(x,t)\right)-N\left(\sum_{j=0}^{r-1}Q_j(x,t)\right)\right\}. \tag{12}$$

Similarly, the linear term can be decomposed as:

$$L(Q(x,t))=L\left(\sum_{r=0}^{\infty}(Q_r(x,t))\right)=L(Q_0(x,t))+\sum_{r=1}^{\infty}\left\{L\left(\sum_{j=0}^{r}Q_j(x,t)\right)-L\left(\sum_{j=0}^{r-1}Q_j(x,t)\right)\right\} \tag{13}$$

Now, we define the k-th order approximate series as:

$$Q^{(k)}(x,t)=\sum_{m=0}^{k}Q_m(x,t)=Q_0(x,t)+Q_1(x,t)+Q_2(x,t)+\ldots+Q_k(x,t),k\in\mathbb{N}. \tag{14}$$

Assume that the solution of eq.(8) is:

$$Q(x,t)=\lim_{k\to\infty}Q^{(k)}(x,t)=\sum_{m=0}^{k}Q_m(x,t). \tag{15}$$

The series solution of eq.(15) absolutely and uniformly converges to a unique solution for eq.(8) if L and N are contractions, see [17]. Applying the linearity property after substituting eq.(12) and eq.(14) into eq.(11), we have:

$$\sum_{r=0}^{\infty} Q_r(x,t) = A^{-1}\left[\tfrac{1}{\vartheta^{\alpha}}\left(\sum_{r=0}^{n-1}\tfrac{Q^{(r)}(x,0)}{\vartheta^{2-\alpha+r}} + A\left[g(x,t)\right]\right)\right] +$$
$$A^{-1}\left[\tfrac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_0(x,t)) + N(Q_0(x,t))\right]\right) + \sum_{r=1}^{\infty}\left(L(Q_r(x,t)) + \left\{N\left(\sum_{j=0}^{r} Q_j(x,t)\right) - N\left(\sum_{j=0}^{r-1} Q_j(x,t)\right)\right\}\right)\right].$$

(16)

Now, we generate the following iterations:

$$Q_0(x,t) = A^{-1}\left[\tfrac{1}{\vartheta^{\alpha}}\left(\sum_{r=0}^{n-1}\tfrac{Q^{(r)}(x,0)}{\vartheta^{2-\alpha+r}} + A\left[g(x,t)\right]\right)\right],\ n-1<\alpha\le n, \tag{17}$$

$$Q_1(x,t) = A^{-1}\left[\tfrac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_0(x,t)) + N(Q_0(x,t))\right]\right)\right] \tag{18}$$

$\vdots$

$$Q_{r+1}(x,t) = A^{-1}\left[\tfrac{1}{\vartheta^{\alpha}}\left(A\left[\sum_{r=1}^{\infty}\left(L(Q_r(x,t)) + \left\{N\left(\sum_{j=0}^{r} Q_j(x,t)\right) - N\left(\sum_{j=0}^{r-1} Q_j(x,t)\right)\right\}\right)\right]\right)\right], \tag{19}$$

4. Application

In this section, some examples were considered in order to demonstrate the efficiency of the method.

Example 1. Consider the time-fractional gas dynamics equation:

$$D_t^{\alpha} Q + \tfrac{1}{2}(Q^2)_x - Q(1-Q) = 0,\ 0<\alpha\le 1, \tag{20}$$

with the initial condition:

$$Q_0(x) = e^{-x}, \tag{21}$$

from example 1, we set: $L(Q(x,t)) = -Q$,

$N(Q(x,t)) = \tfrac{1}{2}(Q^2)_x + Q^2$,

$Q(x,0) = e^{-x}$.

Now, using the iterative procedure described in the previous section, we have:

$$Q_0 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(\sum_{r=0}^{n-1}\frac{Q^{(r)}(x,0)}{\vartheta^{2-\alpha+r}}\right)\right]$$

$$= A^{-1}\left[\frac{Q(x,0)}{\vartheta^{2}}\right] \tag{22}$$

$$= e^{-x}.$$

$$Q_1 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q(x,t)) + N(Q(x,t))\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[Q_0 - \left(\frac{1}{2}(Q_0^2)_x + Q_0^2\right)\right]\right)\right] \tag{23}$$

$$= \frac{e^{-x}t^{\alpha}}{\Gamma(\alpha+1)}.$$

$$Q_2 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_1(x,t)) + \left\{N\left(Q_0(x,t)+Q_1(x,t)\right) - N(Q_0(x,t))\right\}\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[Q_1 - \left\{\left(\frac{1}{2}((Q_0+Q_1)^2)_x + (Q_0+Q_1)^2\right) + \left(\frac{1}{2}(Q_0^2)_x + Q_0^2\right)\right\}\right]\right)\right] \tag{24}$$

$$= \frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)}.$$

$$\vdots$$

$$Q_k = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L\left(Q_{k-1}(x,t)\right) + \left\{N\left(\sum_{j=0}^{r-1}Q_j(x,t)\right) - N\left(\sum_{j=0}^{r-2}Q_j(x,t)\right)\right\}\right]\right)\right] \tag{25}$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[Q_{k-1}(x,t)\right]\right)\right]$$

$$= \frac{e^{-x}t^{k\alpha}}{\Gamma(k\alpha+1)}.$$

We get the k-th order approximate series as:

$$Q^{k}(x,t) = \sum_{m=0}^{k} Q_m(x,t) = Q_0(x,t) + Q_1(x,t) + Q_2(x,t) + \cdots + Q_k(x,t)$$

$$= e^{-x} + \frac{e^{-x}t^{\alpha}}{\Gamma(\alpha+1)} + \frac{e^{-x}t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{e^{-x}t^{k\alpha}}{\Gamma(k\alpha+1)} \qquad (26)$$

$$= e^{-x}\left(1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \cdots + \frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\right)$$

$$= e^{-x}\sum_{m=0}^{k}\frac{t^{m\alpha}}{\Gamma(m\alpha+1)} .$$

The approximate series solution approach the exact solution as $k \to \infty$,

$$Q(x,t) = \lim_{k\to\infty} Q^{(k)}(x,t)$$

$$= e^{-x}\lim_{k\to\infty}\sum_{m=0}^{k}\frac{t^{m\alpha}}{\Gamma(m\alpha+1)}$$

$$= e^{-x}E_{\alpha}(t^{\alpha}). \qquad (27)$$

If α =1, then the exact solution to eq.(13) is:

$$Q(x,t) = e^{-x}E_{1}(t)$$

$$= e^{t-x}. \qquad (28)$$

In Table 2, we calculated the absolute error $E_a = \left\|Q - Q_{10}\right\|$ when α =0.5, 0.7, 0.9 and 1, Figure 1 displays the surface plot when α = 0.5, 0.7, 0.9 and 1.

Table 2:. Absolute error for Example 1.

x	t	$\alpha = 0.50$	$\alpha = 0.70$	$\alpha = 0.90$	$\alpha = 1.00$
0.25	0.25	1.5205×10^{-6}	9.2193×10^{-10}	3.0753×10^{-13}	4.7740×10^{-15}
	0.50	8.2890×10^{-5}	2.0323×10^{-7}	3.0295×10^{-10}	9.9392×10^{-12}
	0.75	8.4239×10^{-4}	4.8597×10^{-6}	1.7279×10^{-8}	8.7867×10^{-10}
	1.00	4.4421×10^{-3}	4.6780×10^{-5}	3.0698×10^{-7}	2.1271×10^{-8}

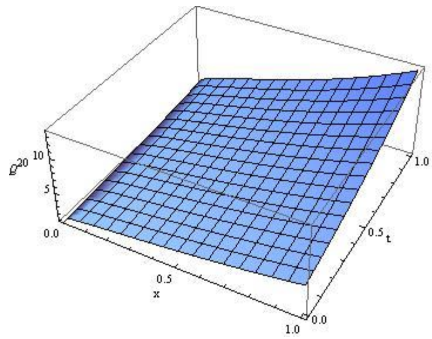


Figure 1(a): α = 0.5

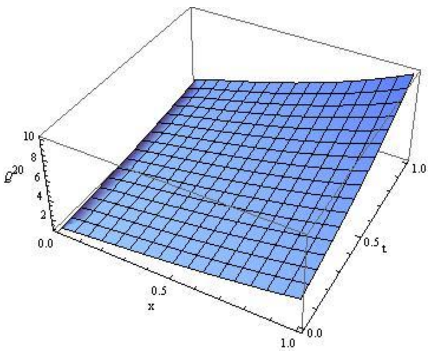


Figure 1(b): α = 0.7

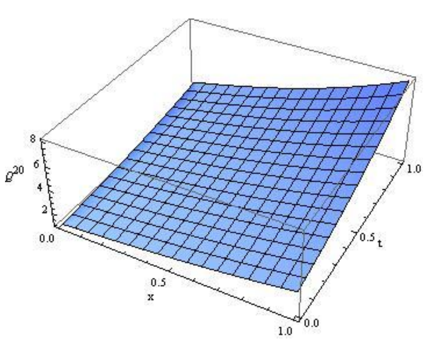


Figure 1(c): α = 0.9

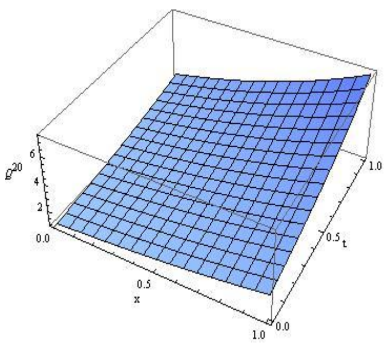


Figure 1(d): α = 1.0

Figure 1: Surface plot for example 1 with different values of alpha.

Example 2. Consider the time-fractional Fokker-Plane equation:

$$D_t^{\alpha} Q + \left(\tfrac{4}{x} Q^2\right)_x - \left(\tfrac{x}{3} Q\right)_x - (Q^2)_{xx} = 0,\ 0 < \alpha \le 1, \tag{29}$$

with the initial condition

$$Q_0(x) = x^2. \tag{30}$$

From example 2, we set:

$$L(Q(x,t)) = -\left(\tfrac{x}{3} Q\right)_x,$$

$$N(Q(x,t)) = -(Q^2)_{xx} + \left(\tfrac{4}{x} Q^2\right)_x,$$

$$Q(x,0) = x^2.$$

Now, using the iterative procedure described in the previous section, we have:

$$Q_0 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(\sum_{r=0}^{n-1}\frac{Q^{(r)}(x,0)}{\vartheta^{2-\alpha+r}}\right)\right]$$

$$= A^{-1}\left[\frac{Q(x,0)}{\vartheta^{2}}\right] \tag{31}$$

$= x^2$.

$$Q_1 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_0(x,t)) + N(Q_0(x,t))\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[\left(\frac{x}{3}Q_0\right)_x + \left((Q_0^2)_{xx} - \left(\frac{4}{x}Q_0^2\right)_x\right)\right]\right)\right] \tag{32}$$

$= \frac{x^2 t^{\alpha}}{\Gamma(\alpha+1)}$.

$$Q_2 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_1(x,t)) + \left\{N(Q_0(x,t) + Q_1(x,t)) - N(Q_0(x,t))\right\}\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[\left(\frac{x}{3}Q_1\right)_x + \left\{((Q_0+Q_1)^2)_{xx} - \left(\frac{4}{x}(Q_0+Q_1)^2\right)_x - (Q_0^2)_{xx} + \left(\frac{4}{x}Q_0^2\right)_x\right\}\right]\right)\right] \tag{33}$$

$= \frac{x^2 t^{2\alpha}}{\Gamma(2\alpha+1)}$.

$\vdots$

$$Q_k = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_{k-1}(x,t)) + \left\{N\left(\sum_{j=0}^{r-1}Q_j(x,t)\right)\right\} - N\left(\sum_{j=0}^{r-2}Q_j(x,t)\right)\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[\left(\frac{x}{3}Q_{k-1}\right)_x\right]\right)\right] \tag{34}$$

$= \frac{x^2 t^{k\alpha}}{\Gamma(k\alpha+1)}$.

We get the k-th order approximate series as:

$$Q^k(x,t)=\sum_{m=0}^{k}Q_m(x,t)=Q_0(x,t)+Q_1(x,t)+Q_2(x,t)+\cdots+Q_k(x,t)$$

$$=x^2+\frac{x^2t^{\alpha}}{\Gamma(\alpha+1)}+\frac{x^2t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+\frac{x^2t^{k\alpha}}{\Gamma(k\alpha+1)} \tag{35}$$

$$=x^2\left(1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\cdots+\frac{t^{k\alpha}}{\Gamma(k\alpha+1)}\right)$$

$$=x^2\sum_{m=0}^{k}\frac{x^2t^{m\alpha}}{\Gamma(m\alpha+1)}\,.$$

The approximate series solution approach the exact solution as $k\rightarrow\infty$,

$$Q(x,t)=\lim_{k\to\infty}Q^{(k)}(x,t)$$

$$=x^2\lim_{k\to\infty}\sum_{m=0}^{k}\frac{x^2t^{m\alpha}}{\Gamma(m\alpha+1)} \tag{36}$$

$$=x^2E_{\alpha}(t^{\alpha})\,.$$

If α =1, the exact solution to eq.(29) is:

$$Q(x,t)=x^2E_1(t)$$

$$=x^2e^t. \tag{37}$$

In Table 3, we calculated the absolute error $E_a=\|Q-Q_{10}\|$ when α = 0.5, 0.7, 0.9 and 1, Figure 1 displays the surface plot when α = 0.5, 0.7, 0.9 and 1.

Table 3: Absolute error for Example 2.

x	t	$\alpha = 0.50$	$\alpha = 0.70$	$\alpha = 0.90$	$\alpha = 1.00$
	0.25	1.3220×10^{-7}	7.3986×10^{-11}	2.4689×10^{-14}	3.8858×10^{-16}
0.25	0.50	6.6521×10^{-6}	1.6310×10^{-8}	2.4313×10^{-11}	7.9764×10^{-13}
	0.75	6.7603×10^{-5}	3.9000×10^{-7}	1.3866×10^{-9}	7.0515×10^{-11}
	1.00	3.5649×10^{-4}	3.7542×10^{-6}	2.4636×10^{-8}	1.7070×10^{-9}

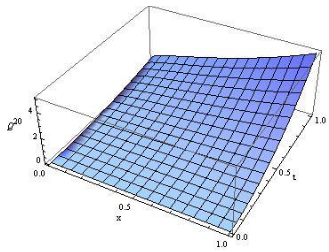


Figure 2(a): α = 0.5

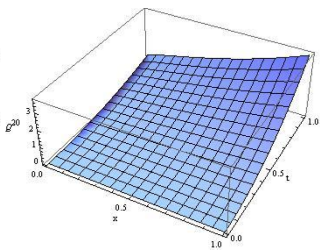


Figure 2(b): α = 0.7

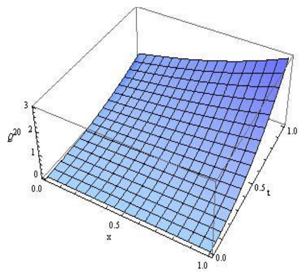


Figure 2(c): α = 0.9

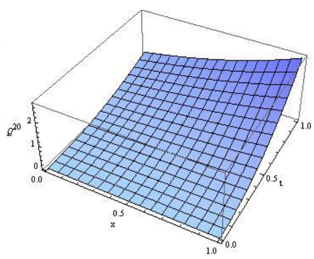


Figure 2(d): α = 1.0

Figure 2: Surface plot for example 2 with different values of alpha.

Example 3. Consider the nonhomogenous time-fractional Klomogorov equation:

$$D_t^{\alpha} Q + x^2 e^t Q_{xx} - (x+1)Q_x = xt\,,\; 0 < \alpha \le 1, \tag{38}$$

with the initial condition

$$Q_0(x) = x+1, \tag{39}$$

from example 3, we set:

$$L(Q(x,t)) = -x^2 e^t Q_{xx} + (x+1)Q_x\,,$$

$N(Q(x,t))=0$,

$g(x,t)=xt$.

Now, using the iterative procedure described in the previous section we have:

$$Q_0 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(\sum_{r=0}^{n-1}\frac{Q^{(r)}(x,0)}{\vartheta^{2-\alpha+r}}+A\left[g(x,t)\right]\right)\right]$$

$$= A^{-1}\left[\sum_{r=0}^{n-1}\frac{Q(x,0)}{\vartheta^{2}}+\frac{x}{\vartheta^{3+\alpha}}\right] \tag{40}$$

$$=(x+1)+\frac{xt^{\alpha+1}}{\Gamma(\alpha+2)}.$$

$$Q_1 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_0(x,t))+N(Q_0(x,t))\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[-x^2e^t(Q_0)_{xx}+(x+1)(Q_0)_x\right]\right)\right] \tag{41}$$

$$=(x+1)\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)}+\frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right).$$

$$Q_2 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_1(x,t))+\left\{N(Q_0(x,t)+Q_1(x,t))-N(Q_0(x,t))\right\}\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[-x^2e^t(Q_1)_{xx}+(x+1)(Q_1)_x\right]\right)\right] \tag{42}$$

$$=(x+1)\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)}+\frac{t^{3\alpha+1}}{\Gamma(3\alpha+3)}\right).$$

$\vdots$

$$Q_k = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_{k-1}(x,t))+\left\{N\left(\sum_{j=0}^{r-1}Q_j(x,t)\right)-N\left(\sum_{j=0}^{r-2}Q_j(x,t)\right)\right\}\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[(x+1)(Q_{k-1})_x\right]\right)\right] \tag{43}$$

$$= (x+1)\left(\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{t^{(k+1)\alpha+1}}{\Gamma((k+1)\alpha+2)}\right).$$

We get the k-th order approximate series as:

$$Q^k(x,t) = \sum_{m=0}^{k} Q_m(x,t) = Q_0(x,t) + Q_1(x,t) + Q_2(x,t) + \cdots + Q_k(x,t)$$

$$= (x+1) + ((x+1)-1)\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x+1)\left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}\right) +$$
$$(x+1)\left(\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)}\right) + \cdots + (x+1)\left(\frac{t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{t^{(k+1)\alpha+1}}{\Gamma((k+1)\alpha+2)}\right)$$

$$= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x+1)\left(\sum_{m=0}^{k}\frac{t^{m\alpha}}{\Gamma(m\alpha+1)} + \sum_{m=0}^{k}\frac{t^{(m+1)\alpha+1}}{\Gamma((m+1)\alpha+2)}\right). \quad (44)$$

The approximate series solution approach the exact solution as $k \to \infty$,

$$Q(x,t) = \lim_{k\to\infty} Q^{(k)}(x,t)$$

$$= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x+1)\lim_{k\to\infty}\left(\sum_{m=0}^{k}\frac{t^{m\alpha}}{\Gamma(m\alpha+1)} + \sum_{m=0}^{k}\frac{t^{(m+1)\alpha+1}}{\Gamma((m+1)\alpha+2)}\right) \quad (45)$$

$$= -\frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + (x+1)\left(E_\alpha(t^\alpha) + \lim_{k\to\infty}\sum_{m=0}^{k}\frac{t^{(m+1)\alpha+1}}{\Gamma((m+1)\alpha+2)}\right).$$

If α =1, then the exact solution to eq.(38) is:

$$Q(x,t) = -\frac{t^2}{2} + (x+1)\left(E_1(t) + \lim_{k\to\infty}\sum_{m=0}^{k}\frac{t^{(m+1)\alpha+1}}{\Gamma((m+1)\alpha+2)}\right)$$

$$= -\frac{t^2}{2} + (x+1)(2e^t - t - 1). \quad (46)$$

In Table 4, we calculated the absolute error $E_a = \|Q - Q_{10}\|$ when α = 0.5, 0.7, 0.9 and 1, Figure 1 displays the surface plot when α = 0.5, 0.7, 0.9 and 1.

Table 4: Absolute error for Example 3.

x	t	$\alpha = 0.50$	$\alpha = 0.70$	$\alpha = 0.90$	$\alpha = 1.00$
	0.25	2.6439×10^{-6}	1.4797×10^{-9}	4.9361×10^{-13}	7.9936×10^{-15}
0.25	0.50	1.3304×10^{-4}	3.2619×10^{-7}	4.8625×10^{-10}	1.5953×10^{-11}
	0.75	1.3521×10^{-3}	7.7100×10^{-6}	2.7736×10^{-8}	1.4103×10^{-9}
	1.00	7.1297×10^{-3}	7.5083×10^{-5}	4.9271×10^{-7}	3.4141×10^{-8}

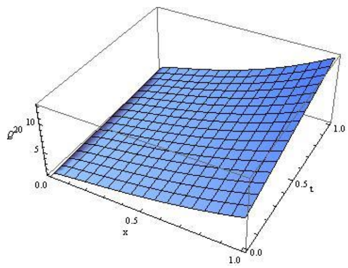


Figure 3(a): α = 0.5

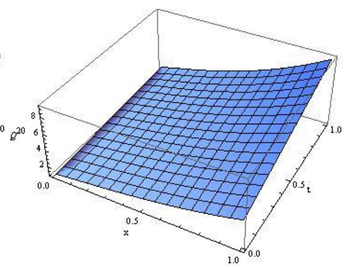


Figure 3(b): α = 0.7

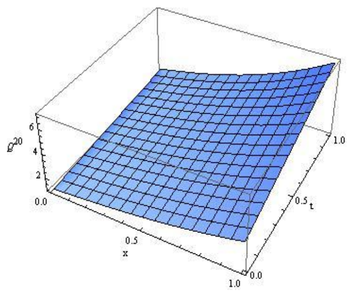


Figure 3(c): α = 0.9

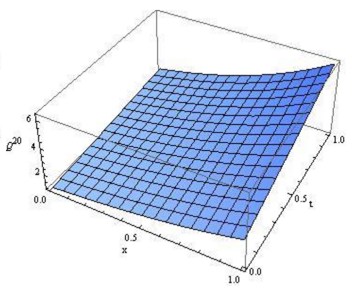


Figure 3(d): α = 1.0

Figure 3: Surface plot for example 3 with different values of alpha.

Example 4. Consider the time-fractional Klein-Gordon type equation:

$$D_t^{\alpha} Q = Q_{xx} - Q + 2c\,\mathrm{os}(x)\,,\ 1 < \alpha \le 2, \tag{47}$$

with initial conditions

$$Q_0(x) = \cos(x)\,, Q_0^{/}(x) = 1, \tag{48}$$

from example 5, we set:

$$L(Q(x,t)) = Q_{xx} - Q\ ,$$

$N(Q(x,t)) = 0$,

$g(x,t) = 2\cos(x)$,

$Q(x,0) = \cos(x)$,

$Q^{/}(x,0) = 1$.

Now, using the iterative procedure described in the previous section we have:

$$Q_0 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(\sum_{r=0}^{n-1}\frac{Q^{(r)}(x,0)}{\vartheta^{2-\alpha+r}} + A\left[g(x,t)\right]\right)\right], n = 2$$

$$= A^{-1}\left[\frac{Q^{(0)}(x,0)}{\vartheta^{2}} + \frac{Q^{(1)}(x,0)}{\vartheta^{3}} + \frac{g(x,t)}{\vartheta^{\alpha+2}}\right] \qquad (49)$$

$$= \cos(x) + t + \frac{2\cos(x)t^{\alpha}}{\Gamma(\alpha+1)}.$$

$$Q_1 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_0(x,t)) + N(Q_0(x,t))\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[(Q_0)_{xx} - Q_0\right]\right)\right] \qquad (50)$$

$$= \frac{-2\cos(x)t^{\alpha}}{\Gamma(\alpha+1)} - \frac{4\cos(x)t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

$$Q_2 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_1(x,t)) + \left\{N\left(Q_0(x,t) + Q_1(x,t)\right) - N(Q_0(x,t))\right\}\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[(Q_1)_{xx} - Q_1\right]\right)\right] \qquad (51)$$

$$= \frac{4\cos(x)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{8\cos(x)t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}.$$

$\vdots$

$$Q_k = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_{k-1}(x,t)) + \left\{N\left(\sum_{j=0}^{r-1}Q_j(x,t)\right) - N\left(\sum_{j=0}^{r-2}Q_j(x,t)\right)\right\}\right]\right)\right]$$

$$= A^{-1}\left[\tfrac{1}{\vartheta^{\alpha}}\left(A\left[(Q_{k-1})_{xx} - Q_{k-1}\right]\right)\right] \tag{52}$$

$$= \frac{(-1)^k 2^k \cos(x) t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{(-1)^k 2^{k+1} \cos(x) t^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)} + \frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}.$$

We get the k-th order approximate series as:

$$Q^k(x,t) = \sum_{m=0}^{k} Q_m(x,t) = Q_0(x,t) + Q_1(x,t) + Q_2(x,t) + \cdots + Q_k(x,t)$$

$$= \cos(x) + t - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)} - \frac{t^{3\alpha+1}}{\Gamma(3\alpha+2)} + \cdots + \frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)} \tag{53}$$

$$= \cos(x) + \sum_{m=0}^{k} \frac{(-1)^m t^{m\alpha+1}}{\Gamma(m\alpha+2)}.$$

The approximate series solution approach the exact solution as $k \to \infty$,

$$Q(x,t) = \lim_{k\to\infty} Q^{(k)}(x,t)$$

$$= \cos(x) + \lim_{k\to\infty} \sum_{m=0}^{k} \frac{(-1)^m t^{m\alpha+1}}{\Gamma(m\alpha+2)}. \tag{54}$$

If α = 2, then the exact solution to eq.(47) is:

$$Q(x,t) = \cos(x) + \sin(t). \tag{55}$$

In Table 4, we calculated the absolute error $E_a = \|Q - Q_{10}\|$ when α =1.95, 1.97, 1.99 and 2 with x = 0.1, Figure 4 displays the surface plot when α =1.4, 1.6, 1.8 and 2.

Table 5: Absolute error for Example 4.

t	$\alpha = 1.95$	$\alpha = 1.97$	$\alpha = 1.99$	$\alpha = 2.00$
0.01	5.6674×10^{-8}	3.2015×10^{-8}	1.0058×10^{-8}	0
0.02	3.9252×10^{-7}	2.2340×10^{-7}	7.0697×10^{-8}	0
0.03	1.3521×10^{-3}	6.9042×10^{-7}	2.1942×10^{-7}	0
0.04	2.6695×10^{-6}	1.5307×10^{-6}	4.8793×10^{-7}	0

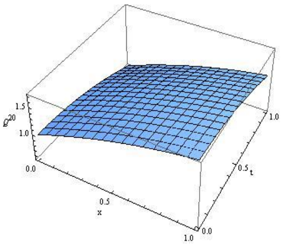
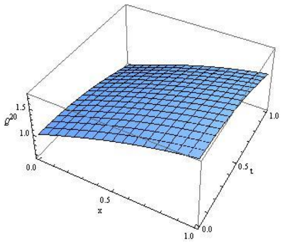



Figure 4(a): α =1.4 **Figure 4(b):** α =1.6

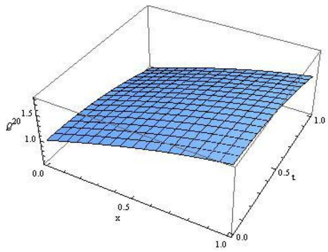
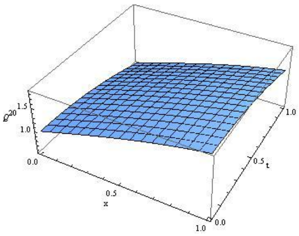



Figure 4(c): α =1.8 **Figure 4(d):** α =2

Figure 4: Surface plot for example 4 with different values of alpha.

Example 5: Consider the time-fractional Klein-Gordon type equation:

$$D_t^{\alpha} Q = Q_{xx} - Q + \eta \sin(x),\ 1 < \alpha \le 2, \tag{56}$$

with the initial conditions:

$$Q_0(x) = \sin(x),\ \ Q_0^{/}(x) = 1, \tag{57}$$

from example 5, we set:

$$L(Q(x,t)) = Q_{xx} - Q\ ,$$

$$N(Q(x,t)) = 0\ ,$$

$$g(x,t) = \eta \sin(x)\ ,$$

$$Q(x,0) = \sin(x)\ ,$$

$$Q^{/}(x,0) = 1\ .$$

Now, using the iterative procedure described in the previous section, we have:

$$Q_0 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(\sum_{r=0}^{n-1}\frac{Q^{(r)}(x,0)}{\vartheta^{2-\alpha+r}} + A\left[g(x,t)\right]\right)\right], n = 2$$

$$= A^{-1}\left[\frac{Q^{(0)}(x,0)}{\vartheta^{2}} + \frac{Q^{(1)}(x,0)}{\vartheta^{3}} + \frac{g(x,t)}{\vartheta^{\alpha+2}}\right] \tag{58}$$

$$= \sin(x) + t + \frac{\eta \sin(x) t^{\alpha}}{\Gamma(\alpha+1)}.$$

$$Q_1 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_0(x,t)) + N(Q_0(x,t))\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[(Q_0)_{xx} - Q_0\right]\right)\right] \tag{59}$$

$$= \frac{-2\sin(x)t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2\eta\sin(x)t^{\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)}.$$

$$Q_2 = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_1(x,t)) + \left\{N\left(Q_0(x,t) + Q_1(x,t)\right) - N(Q_0(x,t))\right\}\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[(Q_1)_{xx} - Q_1\right]\right)\right] \tag{60}$$

$$= \frac{4\sin(x)t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{4\eta\sin(x)t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+2)}.$$

$$\vdots$$

$$Q_k = A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[L(Q_{k-1}(x,t)) + \left\{N\left(\sum_{j=0}^{r-1}Q_j(x,t)\right) - N\left(\sum_{j=0}^{r-2}Q_j(x,t)\right)\right\}\right]\right)\right]$$

$$= A^{-1}\left[\frac{1}{\vartheta^{\alpha}}\left(A\left[(Q_{k-1})_{xx} - Q_{k-1}\right]\right)\right] \tag{61}$$

$$= \frac{(-1)^k 2^k \sin(x)t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{(-1)^k 2^k \eta\sin(x)t^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)} + \frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)}.$$

We get the k-th order approximate series as:

$$Q^k(x,t) = \sum_{m=0}^{k} Q_m(x,t) = Q_0(x,t) + Q_1(x,t) + Q_2(x,t) + \cdots + Q_k(x,t)$$

$$= \sin(x) + t + \frac{\eta \sin(x) t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2\sin(x) t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2\eta \sin(x) t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{\alpha+1}}{\Gamma(\alpha+2)} + \frac{4\sin(x) t^{2\alpha}}{\Gamma(2\alpha+1)} + \frac{4\eta \sin(x) t^{3\alpha}}{\Gamma(3\alpha+1)} + \frac{t^{2\alpha+1}}{\Gamma(2\alpha+1)} + \cdots + \frac{(-1)^k 2^k \sin(x) t^{k\alpha}}{\Gamma(k\alpha+1)} + \frac{(-1)^k t^{k\alpha+1}}{\Gamma(k\alpha+2)} + \frac{(-1)^k 2^k \eta \sin(x) t^{(k+1)\alpha}}{\Gamma((k+1)\alpha+1)} \quad (62)$$

$$= \sum_{m=0}^{k} \left(\frac{(-1)^m 2^m \sin(x) t^{m\alpha}}{\Gamma(m\alpha+1)} + \frac{(-1)^m t^{m\alpha+1}}{\Gamma(m\alpha+2)} + \frac{(-1)^m 2^m \eta \sin(x) t^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)} \right).$$

The approximate series solution approach the exact solution as $k \to \infty$,

$$Q(x,t) = \lim_{k\to\infty} Q^{(k)}(x,t)$$

$$= \lim_{k\to\infty} \sum_{m=0}^{k} \left(\frac{(-1)^m 2^m \sin(x) t^{m\alpha}}{\Gamma(m\alpha+1)} + \frac{(-1)^m t^{m\alpha}}{\Gamma(m\alpha+2)} + \frac{(-1)^m 2^m \eta \sin(x) t^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)} \right). \quad (63)$$

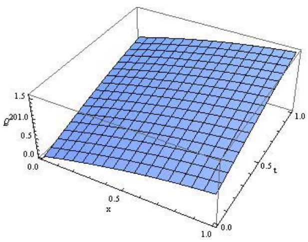
If α = 2 with η set to 2, then the exact solution to eq.(56) is:

$$Q(x,t) = \sin(x) + \sin(t), \quad (64)$$

which agrees with the solution obtained by perturbation iteration transform in [21]. In Table 5, we calculated the absolute error $E_a = \|Q - Q_{10}\|$ when α =1.95, 1.97, 1.99 and 2 for x = 0.1, Figure 5 displays the surface plot when α =1.4, 1.6, 1.8 and 2.

Table 6: Absolute error for Example 5.

t	$\alpha = 1.95$	$\alpha = 1.97$	$\alpha = 1.99$	$\alpha = 2.00$
0.01	2.0718×10^{-8}	3.2015×10^{-8}	1.0058×10^{-8}	0
0.02	1.4510×10^{-7}	2.2340×10^{-7}	7.0697×10^{-8}	1.3878×10^{-17}
0.03	4.4939×10^{-7}	6.9042×10^{-7}	2.1942×10^{-7}	0
0.04	9.9784×10^{-7}	1.5307×10^{-6}	4.8793×10^{-7}	0



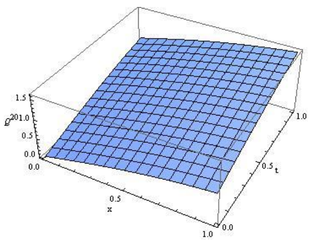
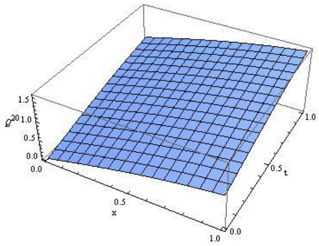


Figure 5(a): α = 1.4 **Figure 6(b):** α = 1.6



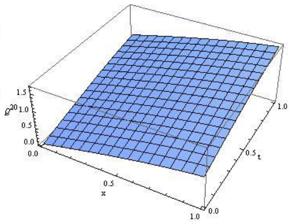


Figure 5(c): α = 1.8 **Figure 5(d):** α = 2

Figure 5: Surface plot for example 5 with different values of alpha.

5. Conclusion

In this paper, we proposed the Aboodh transform iterative method for the approximate analytical solutions of time fractional partial differential equations. The fractional order are considered in Caputo sense, we obtained both approximate and exact solutions. It was observed that the series solutions converges rapidly to the exact solutions. Also, the graphical solutions in Figures 1-5 and the absolute error in Tables 2-6 shows that the solution depends on time t and the fractional order. The proposed method is easy to implement without requirement for perturbation, discretization, linearization or any restrictive assumptions. In constrast to other approximate analytical methods, the proposed method provides the exact solutions without the need for Largrange multipliers or Adomian's polynomials. Future research work can extend the purposed method to solve boundary value problems.

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