Two-sided estimates for the Green's function of fractional evolution equations

by

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Diolch yn fawr hefyd i Gruffydd, Lowri, Dafydd a'u teuluoedd am bopeth. Ac yn ola, mae angen diolch i'r Hwligang.

Er cof am Siân Myfanwy Johnston.

Declarations

The main results presented in this thesis are from the articles Johnston and Kolokoltsov (2019a) and Johnston and Kolokoltsov, 2019b. Both articles are joint work with Vassili Kolokoltsov.

Abstract

In this thesis, fractional calculus is investigated from a probabilistic point of view. We begin by exploring the current development of generalised fractional integrals and derivatives, before extending these generalisations to higher dimension. We then look at two-sided estimates for the Green's function associated to various fractional evolution equations, as well as generalised fractional evolutions. The main tools here are *generalised operator-valued Mittag-Leffler functions*, and in particular their representation as Laplace transforms of the densities of inverse subordinators. Finally we look at boundary value problems involving the *generalised multidimensional* fractional derivatives of Caputo and Riemann-Liouville type, which are introduced in the first part.

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Chapter 1

Introduction

Fractional evolution equations has been a rapidly developing area over the last few decades. One of the reasons is their ability to better model real-world phenomena compared to their non-fractional counterpart, which usually model local behaviour. The nature of fractional in time operators (respectively in space) allow us to model, for example, processes that exhibit some kind of memory (resp. non-local interactions). The processes associated with time-fractional evolution models possess some remarkable properties. For some motivation, let us compare the difference between the standard and the time-fractional heat equation,

$$D^{\beta}_{0+*}u = \frac{1}{2}\Delta u, \quad \partial_t u = \frac{1}{2}\Delta u,$$

where D^{β}_{*} is the Captuo-Dzhbrayashan fractional derivative in time, $\beta \in (0,1)$ and Δ is the Laplacian operator (a second order uniformly elliptic operator). In the standard heat equation, the fundamental solution is given by the transition density of a standard Brownian motion. In the time-fractional heat equation, the fundamental solution is given by a standard Brownian motion, *time-changed by an inverted stable subordinator*. An inverted stable subordinator is obtained as the right-inverse of stable subordinator (which is an increasing jump process), which means that it is an increasing continuous process which is constant precisely whenever the subordinator jumps. For this reason, time fractional diffusion equation is widely used to model anomalous diffusions which exhibit subdiffusive behaviour, due to the diffusive particles being trapped. Such fractional time diffusion equations also arise as a scaling limit of random conductance models (random walks in random environments). This point of view is particularly interesting, since the limiting process is a non-Markovian process which arises as the scaling limit of Markovian process, see Barlow and Cerny (2011) and Meerschaert and Sikorskii (2012). The authors in Hairer et al. (2018) discussed how a fractional kinetic process (with $\beta = \frac{1}{2}$) emerges as the intermediate time behaviour of perturbed cellular flows.

Recently much attention has been given to the Green's function of fractional differential equations. In Z.-Q. Chen et al. (2018), the authors obtain two-sided estimates for the Green's functions of fractional evolution equations, under the assumption that the Green's function of the spatial operator satisfies global (in time) two-sided estimates. In Grigoryan and Kumagai (2008), the authors explore the general structure of two-sided estimates for the transition probabilities associated to local or non-local Dirichlet forms. They show that the bounds for transition probabilities associated to local Dirichlet forms will always be of exponential type, and for those associated to non-local Dirichlet forms the bounds will be of polynomial type. Even more recently, the authors in Deng and Schilling (2018) give some exact asymptotic formulas for the Green's function of fractional evolution equations. The authors in Kelbert et al. (2016) study error estimates for continuous time random walk (CTRW) approximation of classical fractional evolution equations, for which the heat kernel estimates for $D^\beta u = \Delta u$ and $D^\beta u = -\psi(-i\nabla)u$, where $-\psi(-i\nabla)$ generates a symmetric stable process, are obtained as a by-product.

In Eidelman and Kochubei (2004) the authors use the parametrix method (or Levi method) to study the equation $D^\beta u(t,x) - Bu(t,x) = f(t,x)$, where the operator B is a uniformly elliptic second order differential operator (which we look at in Theorem 4.2.1) with bounded continuous real-valued coefficients. They do this by using the machinery of the parametrix method (Levi method), looking first at the constant coefficient case then using these estimates to study the variable coefficient case. In the articles Kochubei et al. (2018, 2019), the authors study the Cesaro mean of the heat kernel of subordinated processes and for this they use a version of a Karamata-Tauberian theorem. Diffusion processes in random environments are also closely related objects, and in fact there are many works looking at estimates for the heat kernels of such processes, for example in Cabezas et al. (2015) the authors obtain sub-Gaussian bounds for the transition kernel of a random walk in a random environment.

Boundary value problems on $\mathbb{R}_+^k \times \mathbb{R}^n$ of the form

$$\sum_{i=1}^{k} {}_{t_i}D_{0+*}^{\beta_i} u(t,x) = L_x u(t,x), \quad (t,x) \in \mathbb{R}_+^k \times \mathbb{R}^n,$$
$$u(t,x)|_{t_i=0} = \phi_i(z), \quad z \in \mathbb{R}_+^{k-1} \times \mathbb{R}^n$$

where each CD derivative acts on a different coordinate, arise in many areas of mathematics. A particularly noteworthy application can be found in the mathematics of insurance. Consider k processes $(X_{t_1}^{\beta_1}(s), \cdots, X_{t_k}^{\beta_k}(s))$, where each $X_{t_i}^{\beta_i}(s)$ is a process started at $t_i > 0$ generated by $-{}_{t_i}D_{0+*}^{\beta_i}$. If each process corresponds to the wealth of a company, then whenever one of the coordinates hit zero, one of the companies have defaulted. Insurance companies are interested in the *ruin probability*, which is the probability of one of the companies defaulting before a finite time horizon T. That is, if $\tau_0^{\beta_i}(t_i)$ denotes the first time the process $X_{t_i}^{\beta_i}(s)$ hits zero,

$$\tau_0^{\beta_i}(t_i) := \inf\{s > 0 : X_{t_i}^{\beta_i}(s) \leq 0\},$$

then the ruin probability is the quantity

$$\Psi(t_i, T) = \mathbb{P}[\tau_0^{\beta_i}(t_i) < T].$$

See Y. Chen et al. (2013), Djehiche (1993), Konstantinides and J. Li (2016), X. Li et al. (2015), and Ramasubramanian (2016) for ruin probabilities of multidimensional risk models, or Asmussen and Albrecher (2010) for a broader treatment of ruin probabilities. Similar kinds of questions also appear when looking at barrier options under one-dimensional Markov models, see Mijatović and Pistorius (2013). It is natural to consider multi-dimensional versions of these, Leccadito et al. (2016), as investors often deal with basket options. A further natural appearance comes when considering portfolios of credit derivative obligations (CDO), which can be described by a Markov process in $\mathbb{R}_+^k$. Reaching a boundary of dimension $k-n$ means that n out of d bonds underlying the portfolio of CDOs have defaulted. It is natural in this setting to consider spatially non-homogeneous processes, since the behaviour of the processes should feel the approach to the boundary, which is not the case for

Lévy processes. This would then mean looking at evolutions of the form

$$\sum_{i=1}^{k} {}_{t_i}D_{0+*}^{(\nu_i)} u(t,x) = L_x u(t,x),$$

where each ν_i is a Lévy-type kernel which may depend on t_i. The series of articles Scalas, Gorenflo, and Mainardi (2000a,b) and Scalas, Gorenflo, Mainardi, and Raberto (2001), give a nice overview of the usage of fractional calculus and jump-diffusion processes in finance. Another popular model these days is the so called Pearson diffusion, and also the fractional version, which are diffusion processes with polynomial diffusion coefficients, see Leonenko et al. (2013). Fractional models are also finding new footing in theoretical physics, via fractional and non-local Schrödinger operators, see for example Kaleta, Kwaśnicki, et al. (2018) and Kaleta and Lörinczi (2019). Also for a broader scientific development, see Herrmann (2014), Mainardi (2010), Meerschaert and Sikorskii (2012), and Tarasov (2011).

Of more general interest in finance are *affine processes* which live in $\mathbb{R}_+^k \times \mathbb{R}^d$, see Duffie et al. (2003). Our final motivation for considering stable processes on $\mathbb{R}_+^2$ (i.e, (5.0.1) without the spatial operator L_x), is the topic of limit order books. A simplified model would be that one coordinate of $X_{t_1,t_2}^{\beta_1,\beta_2}(s)$ is the volume of trades available at the best buy price while the other is the volume at the best sell price. The event that this process hits the boundary means that there are no more trades offered at that price and thus a price change occurs. We discuss this problem in more detail in Chapter 6. See Cont and De Larrard (2012) and Hambly et al. (2018) and references therein for related attempts at modelling order books using reflected Brownian motions on the orthant and reflected SPDEs.

1.1 Structure

The main aim of this thesis is to explore two-sided estimates for the Green's function of fractional evolution equations. In Chapter 2, we recall some background material from classical fractional calculus, probability theory, operator semigroups and asymptotic analysis. In Chapter 3, we begin by defining generalised fractional derivatives in dimension one, before making our way to d-dimensional generalisations. We also discuss some general results from evolution equations involving fractional-type operators.

Chapters 4 and 5 make up the main focus of the thesis, where we obtain two-sided estimates for the Green's function of a wide range of fractional evolution equations. We wrap up the story with a discussion on a possible financial application of our estimates - limit order books.

1.2 General notation and function spaces

For an open or closed convex subset S of $\mathbb{R}^d$, $C(S)$ is the Banach space of continuous functions on S equipped with the sup-norm. $C^k(S)$ is a Banach space of k times continuously differential functions with bounded derivatives on S with the norm being the sum of the sup norms of the function itself and all its partial derivatives up to and including order k. For a subset $A \subset S$, we define the spaces

$$C_{constA}(S) = \{f \in C(S) \ : \ f|_A \text{ is a constant}\},$$

$$C_{killA}(S) = \{f \in C(S) \ : \ f|_A = 0\},$$

$$C_\infty(\mathbb{R}^d) = \{f \in C(\mathbb{R}^d) \ : \ \lim_{x\to\infty} f(x) = 0\},$$

$$C_c^2(\mathbb{R}^d) = \{f \in C^2(\mathbb{R}^d) \ : \ \text{f has compact support }\}$$

$$B(S) = \{f : S \to \mathbb{R} \text{ bounded and measurable }\}.$$

We denote by $\mathbb{R}^d, \mathcal{O}, \ \mathbb{S}^{d-1}, \mathbb{N}$, a.e., $a \vee b$ and $a \wedge b$, the d-dimensional Euclidean space, the positive orthant $\{x \in \mathbb{R}^d, x \geq 0\}$, the surface of the d-dimensional sphere, the positive integers, the statement almost everywhere with respect to Lebesgue measure, the maximum and the minimum between $a, b \in \mathbb{R}$, respectively. The space $\mathcal{M}^+(\mathbb{R}^d \setminus \{0\})$ is the space of positive Borel measures on $\mathbb{R}^d \setminus \{0\}$.

Chapter 2

Preliminaries

In this chapter we introduce relevant background material which is used throughout the thesis. We begin with some general facts about Lévy (and Feller) processes and their associated semigroups. Following this we discuss some well-known estimates for the transition densities of such processes, before going on to describe some methods from asymptotic analysis which are used at various points.

2.1 Lévy processes, Feller processes and operator semigroups

Our main references here are Kolokoltsov (2011), Böttcher et al. (2014) and Sato (1999). For $x \in E \subset \mathbb{R}^d$, we use the notation $X_x(s) = (X_x(s))_{s\geq 0}$ to mean an E-valued stochastic process which is started at x. When $E = \mathbb{R}_+$ (or $\mathbb{R}_+^k$ for $k \geq 1$), we will mostly use the letter t to denote the starting point of the process, while keeping x as the starting point for processes living in $\mathbb{R}^d$. Recall that a Lévy process $X = (X(s))_{s\geq 0}$ is a stochastic process which has *stationary and independent increments*, $X_0 = 0$ almost surely (a.s) and is continuous in probability. Such a process always has a modification which has a.s cádlág (from the French for continuous from the right, limits from the left) sample paths. We always work with such a modification.

Let B be a Banach space. Then a (one-parameter) family of linear operators $(T_t)_{t\geq 0}$ on B is a *semigroup* if $T_{t+s} = T_s T_r$ for every $s, r \geq 0$ and T_0 is the identity mapping in B. A semigroup of operators is *strongly continuous* if

$\lim_{t\to 0}\|T_t f - f\| = 0$ for any $f \in B$. A semigroup of operators is a *contraction semigroup* if each T_t is a contraction: $\|T_t\| \le 1$. If $f \ge 0$ implies $T_t f \ge 0$, then the operator T_t is *positivity preserving.* A strongly continuous semigroup of positivity preserving linear contractions on the Banach space $B = C_\infty(S)$, where S is a locally compact metric space, is called a *Feller semigroup*.

We only deal with cases $S = \mathbb{R}^d$ or $S \subset \mathbb{R}^d$. A *Feller process* is a time-homogeneous Markov process whose transition semigroup $T_t f(x) = \mathbb{E} f(X_x(t))$ is a Feller semigroup. Again, any Feller process has a cádlág modification and we always work with such a modification, see Böttcher et al. (2014, Theorem 1.19). A consequence of the Riesz-Markov theorem is that for an arbitrary Feller semigroup T_t on $C_\infty(S)$, there exists a uniquely defined family of positive Borel measures $p_t(x, \mathrm{d}y)$ on S with norm not exceeding one, depending vaguely continuous on x such that

$$T_t f(x) = \int_S p_t(x, \mathrm{d}y) f(y). \tag{2.1.1}$$

A Feller semigroup is called *conservative* if such measures $p_t(x, \cdot)$ are probability measures.

The *infinitesimal generator* of a Feller semigroup $(T_t)_{t\ge 0}$ (or of a Feller process $(X_t)_{t\ge 0}$) is the linear operator $(L, \mathcal{D}(L))$ defined by

$$\mathcal{D}(L) := \left\{ u \in C_\infty(S) : \lim_{t\to 0} \frac{T_t u - u}{t} \text{ exists as uniform limit} \right\},$$

$$Lu := \lim_{t\to 0} \frac{T_t u - u}{t}, \quad u \in \mathcal{D}(L).$$

A classical result on the structure of generators of Feller semigroups is due to Courrege (1965), which says that if the domain of a Feller generator in $C_\infty(\mathbb{R}^d)$ contains the space $C_c^2(\mathbb{R}^d)$, then on that space it has the following *Lévy-Khintchine* form with variable coefficients:

$$\begin{aligned} Lf(x) = {} & \frac{1}{2}(G(x)\nabla, \nabla) f(x) + (b(x), \nabla f(x)) + c(x) f(x) \\ & + \int_{\mathbb{R}^d} [f(x+y) - f(x) - (\nabla f(x), y)\mathbf{1}_{B_1}(y)] \nu(x, \mathrm{d}y), \quad f \in C_c^2(\mathbb{R}^d), \end{aligned} \tag{2.1.2}$$

where B_1 is a ball of radius 1, $G(x)$ is a symmetric non-negative matrix, and

$\nu(x, \cdot)$ is a Lévy measure on $\mathbb{R}^d$:

$$\int_{\mathbb{R}^d} \min(1, |y|^2)\nu(x, \mathrm{d}y) < \infty, \ \ \nu(x, \{0\}) = 0,$$

depending measurably on x. If additionally L is the generator of a *conservative* Feller semigroup, then the term $c(x)$ vanishes.

In the above, note that for a fixed $x \in \mathbb{R}^d$, $(b(x), G(x), \nu(x, \cdot))$ is a Lévy triplet in the sense that $b(x) \in \mathbb{R}^d$ is the *drift* coefficient, $G(x) \in \mathbb{R}^{d\times d}$ is the *diffusion* coefficient and $\nu(x, \cdot) \in \mathcal{M}^+(\mathbb{R}^d \setminus \{0\})$ is the Lévy jump measure. The term $c(x)$ is the *killing rate* of the associated process, and if it is present, the associated process is a sub-Markov process (because the measure in the representation (2.1.1) will be a sub-probability measure). The operator L (2.1.2) (with vanishing c) is a pseudo-differential operator, with symbol

$$\psi(x, \xi) = \frac{1}{2}\xi \cdot G(x)\xi - ib(x) \cdot \xi - \int_{\mathbb{R}^d\setminus\{0\}} \left(e^{i\xi\cdot x} - 1 - iy \cdot \xi \mathbf{1}_{B_1}(y)\right) \nu(x, \mathrm{d}y).$$

The resolvent (or λ-potential) of a Feller semigroup T_t generated by an operator $(L, \mathcal{D}(L))$ is defined as the Laplace transform of the semigroup,

$$R_\lambda f(x) = (\lambda - L)^{-1} f(x) = \int_0^\infty e^{-\lambda s} T_s f(x) \ \mathrm{d}s.$$

The image of the resolvent operator (of the semigroup T_t) coincides with the domain of the generator $\mathcal{D}(L)$. Note that for a Feller process with transition probabilities $p_s(x, \mathrm{d}y)$, the resolvent operator can be written as

$$\begin{aligned} R_\lambda f(x) &= \int_0^\infty e^{-\lambda s} \mathbb{E} f(X_x(s)) \ \mathrm{d}s \\ &= \int_0^\infty \int_{\mathbb{R}^d} e^{-\lambda s} f(y) \ p_s(x, \mathrm{d}y) \ \mathrm{d}s \\ &= \int_{\mathbb{R}^d} f(y) U_\lambda(x, \mathrm{d}y), \end{aligned}$$

where the integral kernel $U_\lambda(x, \cdot)$ is the λ-potential measure of the process $X_x(t)$. The potential operator (i.e, the 0-potential operator) is given by

$$R_0 f(x) = \mathbb{E} \int_0^\infty f(X_x(s)) \ \mathrm{d}s,$$

whenever it exists. The potential operator is in general unbounded, however when it is bounded on the Banach space B, it follows that $R_0 : B \to \mathcal{D}(L)$ is a bijection and $LR_0 g = -g$, see Dynkin (1965, Theorem 1.1').

2.2 Stable processes

A particularly important class of Lévy processes for us are those that are *α-stable processes*[1], whose basic properties we recall now. Our standard references for stable processes are Zolotarev (1986) and Samorodnitsky and Taqqu (1994). A random variable X is said to have a stable distribution if there are parameters[2] $0 < \alpha \le 2$, $\sigma \ge 0$, $-1 \le \gamma \le 1$ and $\mu \in \mathbb{R}$ such that its characteristic function has the following form

$$\phi(y) := \mathbb{E}[e^{iyX}] = \exp\left\{-\sigma^\alpha |y|^\alpha (1 - i\gamma(\text{sign } y) \tan\frac{\pi\alpha}{2}) + i\mu y\right\}.$$

In which case we write $X \sim W_\alpha(\sigma, \gamma, \mu)$. The parameter σ is the *scale*, γ the *skewness* and μ is the *location* parameter of the distribution. Of particular interest to us are the random variables which are *totally positively skewed* ($\gamma = 1$), centred ($\mu = 0$) and are stable of order $\alpha \in (0, 1)$. Such random variables are called *stable subordinators*. A Lévy process $(X(s))_{s\ge 0}$ is a standard α-stable Lévy process if $X(s) - X(t) \sim W_\alpha((s-t)^{1/\alpha}, \gamma, 0)$ for any $0 \le t < s < \infty$ and for some $0 < \alpha \le 2$, $-1 \le \gamma \le 1$.

The transition density of an increasing α-stable Lévy subordinator corresponding to the characteristic function ϕ, which we denote by $p^{+\alpha}(t, x)$ (i.e, $\gamma = 1$, $\sigma = t^{\frac{1}{\alpha}}$ and $\mu = 0$), is given by the following Fourier transform:

$$\begin{aligned} p_{+\alpha}(t, x) &= \frac{1}{2\pi} \int_{\mathbb{R}} \exp\left\{-ixy - t|y|^\alpha \exp\left\{\frac{i\pi}{2}\alpha \text{ sgn } y\right\}\right\} \, \mathrm{d}y \\ &= \frac{1}{\pi} \Re \int_0^\infty \exp\left\{-ixy - ty^\alpha \exp\left\{\frac{i\pi}{2}\alpha\right\}\right\} \, \mathrm{d}y, \end{aligned} \tag{2.2.1}$$

where $\Re(z)$ is the real part of $z \in \mathbb{C}$. Similarly, the transition density of a

[1] also sometimes called *α-stable Lévy motions*

[2] We exclude the case $\alpha = 1$ here for simplicity as this special case, which involves logarithmic behaviour, does not come up in the following chapters.

decreasing α-stable Lévy subordinator is given by

$$p_{-\alpha}(t,x) = \frac{1}{\pi}\Re\int_0^\infty \exp\left\{-ixy - ty^\alpha \exp\left\{-\frac{i\pi}{2}\alpha\right\}\right\} \,\mathrm{d}y.$$

Remark 1. *From the analytical point of view, the functions $p_{\pm\alpha}(t,x)$ for $\alpha \in (0,1)$ are the Green's functions of the left and right fractional derivatives (cf. (3.1.1)). Thus $p_{\pm\alpha}(t,x)$ solves*

$$\partial_t p_{\pm\alpha}(t,x) = -D^\alpha_\pm p_{\pm\alpha}(t,x), \quad t \geq 0, \quad p^{\pm\alpha}(0,x) = \delta(x).$$

For this reason we may write $G_\alpha(t,x)$ sometimes instead of $p_\alpha(t,x)$, and when we omit the $\pm$ sign, we usually mean $p_{+\alpha}(t,x)$.

The function $p_\alpha(t,x)$ has the following scaling property,

$$p_\alpha(t,x) = t^{-\frac{1}{\alpha}} p_\alpha(1, t^{-\frac{1}{\alpha}}x),$$

and for this reason we write $p_\alpha(1,z) := w_\alpha(z)$, which we use throughout the thesis.

For $\alpha \in (0,2)$, the characteristic function of a symmetric stable distribution in $\mathbb{R}^d$ (up to a shift) has the form

$$\phi_\alpha(p) = \exp\left\{-|p|^\alpha \int_{\mathbb{S}^{d-1}} |(p/|p|,s)|^\alpha \mu(\mathrm{d}s)\right\}, \tag{2.2.2}$$

where the (finite Borel) measure μ on $\mathbb{S}^{d-1}$ is called the spectral measure, see Samorodnitsky and Taqqu (1994, Theorem 2.3.1). Let S_μ be the function on $\mathbb{S}^{d-1}$ given by

$$S_\mu(p) = \int_{\mathbb{S}^{d-1}} |(p,s)|^\alpha \mu(\mathrm{d}s), \tag{2.2.3}$$

so that

$$\psi_\alpha(p) := -\log\phi_\alpha(p) = |p|^\alpha S_\mu(p/|p|), \quad p \in \mathbb{R}^d. \tag{2.2.4}$$

Note that ψ_α is the symbol of a pseudo-differential operator $\Psi_\alpha(-i\nabla)$ which we will study later. When μ is the uniform measure on $\mathbb{S}^{d-1}$ the operator $\Psi_\alpha(-i\nabla)$ is just the fractional Laplacian $-(-\Delta)^{\frac{\alpha}{2}}$ with symbol $\psi_\alpha(\xi) = |\xi|^\alpha$, which generates a symmetric α-stable Lévy process in $\mathbb{R}^d$.

2.3 Estimates

For a domain $D \subset \mathbb{R}^d$, the notation $f(x) \asymp g(x)$ in D means that there exists constants $C, c > 0$ such that f satisfies the following two-sided estimate,

$$cg(x) \leq f(x) \leq Cg(x), \quad \forall x \in D,$$

The notation $f(x) \sim g(x)$ for $x \to \infty$ means that

$$\frac{f(x)}{g(x)} \to 1, \qquad \text{as } x \to \infty.$$

Then for each $M > 0$ there exists a constant $C > 0$ such that

$$C^{-1}g(x) \leq f(x) \leq Cg(x), \quad x \in (M, \infty).$$

Similarly, the notation $f(x) \sim g(x)$ for $x \to 0$ means

$$\frac{f(x)}{g(x)} \to 1, \qquad \text{as } x \to 0.$$

Then for each $m > 0$ there exists a $c > 0$ such that

$$c^{-1}g(x) \leq f(x) \leq cg(x), \quad x \in (0, m).$$

If both f and g on $\mathbb{R}_+$ are positive, bounded and satisfy $f(x) \sim g(x)$ for $x \to \infty$ (resp. $x \to 0$), then $f(x) \asymp g(x)$ in (M, ∞) for any $M > 0$ (resp. in $(0, m)$ for any $m < \infty$). See Bruijn (1981) for more details on asymptotic analysis.

Aronson estimates

An operator $H = \partial_t - L$, where $L = \{a_{ij}(t,x)\partial_{x_i}\partial_{x_j}u + b_i(t,x)\partial_{x_i}u + c(t,x)u\}$, is said to be uniformly parabolic if the operator L is elliptic (see (2.3.4)) for each $(t, x) \in (0, T] \times \mathbb{R}^d$ for some fixed $T > 0$. Let $Z(t, x; s, \xi)$ be the fundamental solution of the Cauchy problem for the uniformly parabolic equations

$$\partial_t u - \{a_{ij}(t,x)\partial_{x_i}\partial_{x_j}u + b_i(t,x)\partial_{x_i}u + c(t,x)u\} = 0, \quad u(0,x) = \delta(x-\xi). \tag{2.3.1}$$

Under the assumption that the coefficients are bounded and uniformly Hölder continuous in x defined on $(0,T] \times \mathbb{R}^d$ for some fixed $T > 0$, the fundamental solutions Z are known to satisfy the following two-sided estimates, see for example Porper and Èidel'man (1984),

$$Z(t,x;s,\xi) \asymp (s-t)^{-d/2} \exp\left\{-c\frac{(x-\xi)^2}{s-t}\right\}. \tag{2.3.2}$$

Remark 2. *Let us comment on the link between fundamental solutions and Feller processes. The spatial operator in (2.3.1) generates a diffusion process in $\mathbb{R}^d$, whose transition function is given by $p(s-t,x,\mathrm{d}y) = Z(t,x;s,\mathrm{d}y)$ and the corresponding transition semigroup $T_s f(x)$ is the integral operator with integral kernel $Z(0,x;s,\mathrm{d}y)$.*

On the other hand, Aronson (Aronson, 1967) obtained global (i.e, for all $t > 0$) two-sided estimates for the fundamental solution (or Green's function) $G(t,x,y)$ of the divergence equation

$$\partial_t u = \nabla \cdot (a(x)\nabla u). \tag{2.3.3}$$

Assuming that the coefficients $a(x) = (a_{ij}(x))_{1\le i,j\le d}$ are continuous, symmetric and uniformly elliptic, i.e. there exists $\mu \ge 1$ such that

$$\mu^{-1}|\xi|^2 \le a_{ij}(x)\xi_i\xi_j \le \mu|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^d, \tag{2.3.4}$$

then there exists a constant C such that for $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$G(t,x,y) \asymp t^{-d/2} \exp\left\{-C\frac{|x-y|^2}{t}\right\}. \tag{2.3.5}$$

We call the estimates (2.3.2) and (2.3.5) the *local* and *global* Aronson estimates respectively. From the probabilistic point of view, Aronsons estimates show that (non-degenerate) diffusion processes, whose generators are of the form $Lu = \nabla \cdot (a(x)\nabla u)$ [3], are comparable to standard Brownian motion in the sense that their transition densities are comparable. From the analytical point of view, Aronson estimates show that the fundamental solution to second order uniformly elliptic PDE's are estimated above and below by the fundamental

[3] or more generally the spatial operator in (2.3.1)

solution to the standard heat equation

$$\partial_t u = \frac{1}{2}\Delta u.$$

Stable and stable-like estimates

Next we recall the estimates for the fundamental solution to the pseudo-differential evolution,

$$\partial_t u = -\Psi_\alpha(-i\nabla)u, \tag{2.3.6}$$

where Ψ_α is a pseudo-differential operator which is homogeneous of order $\alpha \in (0, 2)$. That is, the symbol of Ψ_α is of the form

$$\psi_\alpha(p) = |p|^\alpha S_\mu(p/|p|), \quad p \in \mathbb{R}^d,$$

where $S_\mu(p)$ is given by

$$S_\mu(p) = \int_{\mathbb{S}^{d-1}} |(p, s)|^\alpha \mu(\mathrm{d}s),$$

and μ is called the *spectral measure* (cf. 2.2.4). The operator Ψ_α is the generator of an α-stable process which lives on $\mathbb{R}^d$, with characteristic exponent ψ_α. See for example Kolokoltsov (2019a, Theorem 4.5.1) or Eidelman, Ivasyshen, et al. (2004) for the following estimates. Assuming that

- The function S_μ belongs to $C^{d+1+[\alpha]}(\mathbb{S}^{d-1})$,
- The spectral measure μ has a density which is strictly positive,
- $\alpha \in (0, 2)$,

then the Green's function $G_{\psi_\alpha}(t, x - y)$ of the evolution (2.3.6) satisfies the following two-sided estimates for $(t, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$G_{\psi_\alpha}(t, x - y) \asymp \min\left(\frac{t}{|x - y|^{d+\alpha}}, t^{-\frac{d}{\alpha}}\right). \tag{2.3.7}$$

Note that the restriction $\alpha \in (0, 2)$ and the positivity of the density of the spectral measure is required for the lower bound of G_{ψ_α} - the upper bound still holds if we drop the strict positivity of the density μ and take any $\alpha > 0$.

If additionally S_μ is $(d+1+[\alpha]+l)$-times continuously differentiable, then $G_{\psi_\alpha}(t,x)$ is l-times continuously differentiable in x and for $(t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$,

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G_{\psi_\alpha}(t,x-y)\right| \leq C\min\left(\frac{t}{|x-y|^{d+\alpha+k}}, t^{-(d+k)/\alpha}\right), \tag{2.3.8}$$

for all $k \leq l$ and all indicies $i_1,\cdots,i_k$.

Next we have the case when Ψ_α may have variable coefficients, which are known as *stable-like* operators. Let $G_{\psi_\alpha,x}$ denote the fundamental solution to the pseudo-differential evolution equation

$$\partial_t u = -\Psi_\alpha(x,-i\nabla)u,$$

with homogeneous symbol $\psi_\alpha(x,p) = |p|^\alpha S_\mu(x,p/|p|)$, where

$$S_\mu(x,p) = \int_{\mathbb{S}^{d-1}} |(p,s)|^\alpha \mu(x,\mathrm{d}s).$$

Theorem 2.3.1. *Assume that $S_\mu(x,p)$ is a γ-Hölder continuous function in the variable x taking values in a compact subset of $(0,\infty)$ and $\gamma\in(0,1]$. Assume further that for all $x\in\mathbb{R}^d$, μ has a strictly positive density. Then for some fixed $T>0$, there exists a constant $C>0$ such that for $t\in(0,T)$ and $x,y\in\mathbb{R}^d$,*

$$\frac{1}{C}G_{\psi_\alpha}(t,x-y) \leq G_{\psi_\alpha,x}(t,x,y) \leq CG_{\psi_\alpha}(t,x-y).$$

What this means is that the global in time estimates (4.2.17) for the Green's function G_{ψ_α}, also serve as a small-time estimate for the Green's function $G_{\psi_\alpha,x}$. Indeed one would hope that operators with variable coefficients can be approximated by the method of freezing coefficients. So we have the following small-time estimate for $t\in(0,T)$, $x,y\in\mathbb{R}^d$

$$\frac{1}{C}\min\left(\frac{t}{|x-y|^{d+\alpha}}, t^{-\frac{d}{\alpha}}\right) \leq G_{\psi_\alpha,x}(t,x,y) \leq C\min\left(\frac{t}{|x-y|^{d+\alpha}}, t^{-\frac{d}{\alpha}}\right), \tag{2.3.9}$$

for some fixed $0<T<\infty$. We also have the following estimates for the spatial derivatives of the $G_{\psi_\alpha,x}$, see Kolokoltsov (2019a) (Theorem 5.8.3).

Theorem 2.3.2. *Let $\alpha>0$, and denote by l the maximal integer less than*

α. Assume that $\mu \geq \mu_0 > 0$, for some positive number μ_0, and for all p $S_\mu(x,p)$ is q-times differentiable in x and each of these derivatives are, for all x, $(d+1+(l+q)(\alpha+1))$-times continuously differentiable in p. Then for a fixed $T>0$ and any $k \leq l$,

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G_{\psi_\alpha,x}(t,x,y)\right| \leq C\min\left(\frac{t}{|x-y|^{d+k+\alpha}}, t^{-\frac{(d+k)}{\alpha}}\right) \tag{2.3.10}$$

for $(t,x,y) \in (0,T)\times\mathbb{R}^d\times\mathbb{R}^d$.

Stable subordinators

For $\beta \in (0,1)$ a *β-stable subordinator* $X_t^\beta(s)$ is a β-stable process with transition density $p_\beta(s,x)$ of the form (2.2.1). We define the following first passage times,

$$\tau_0^\beta(t) := \inf\{s>0 : X_0^\beta(s) \geq t\}, \quad t \in \mathbb{R}_+.$$

Note that this is the same as the process $t - X_0^\beta(s)$ exiting the interval $\mathbb{R}_+$. Recall that the transition density of $X_t^\beta(s)$ can be written as $p_\beta(s,x) = s^{-\frac{1}{\beta}} w_\beta(s^{-\frac{1}{\beta}}x)$ where $w_\beta(\cdot)$ is the density of a standard β-stable random variable $W_\beta(1,1,0)$. For $\beta \in (0,1)$ the density $w_\beta(r)$ has the following asymptotic behaviour in a neighbourhood of 0, Uchaikin and Zolotarev (1999, Theorem 5.4.1)

$$w_\beta(r) \sim C_\beta r^{-\frac{2-\beta}{2(1-\beta)}} \exp\{-c_\beta r^{-\frac{\beta}{1-\beta}}\} := f_\beta(r), \quad r \to 0, \tag{2.3.11}$$

and in a neighbourhood of ∞,

$$w_\beta(r) \sim \tilde{C}_\beta r^{-1-\beta}, \quad r \to \infty.$$

Remark 3. *One can see (2.3.11) directly from (2.2.1) by applying the saddle point method of Proposition 2.4.1.*

Due to the positivity of $w_\beta(x)$, we can combine these behaviours so that there exists constants $C, \tilde{C} > 0$ such that

$$w_\beta(r) \asymp C\mathbf{1}_{\{r<1\}} f_\beta(r) + \tilde{C}\mathbf{1}_{\{r\geq 1\}} r^{-1-\beta}. \tag{2.3.12}$$

We will also be using the asymptotic behaviour of the density of $\tau_0^\beta(t)$, which we denote by $\mu^{\beta}_{0,t}(s)$. This density is given by

$$\mu^{\beta}_{0,t}(s) = \frac{t}{\beta} s^{-1-\frac{1}{\beta}} w_{\beta}(ts^{-\frac{1}{\beta}}), \tag{2.3.13}$$

see Meerschaert and Scheffler (2004, Corollary 3.1). Thus we have

Lemma 2.3.3. For $\beta \in (0,1)$ the density $\mu^{\beta}_{0,t}(s)$ of τ^{β}_{0} has the following asymptotic behaviour at 0 and ∞,

$$t^{\beta}\mu^{\beta}_{0,t}(t^{\beta}s) \sim \begin{cases} c_{\beta}, & s \to 0, \\ c_{\beta} s^{-1+\frac{1}{2(1-\beta)}} \exp\{-cs^{\frac{1}{1-\beta}}\}, & s \to \infty. \end{cases}$$

for some constants $c_{\beta} > 0$.

Proof. Since $w_{\beta}(r) \sim r^{-1-\beta}$ as $r \to \infty$, then $w_{\beta}(r^{-\frac{1}{\beta}}) \sim r^{1+\frac{1}{\beta}}$ as $r \to 0$. Thus using (2.3.13), we have for $s \to 0$,

$$t^{\beta}\mu^{\beta}_{0}(t^{\beta}s) = c_{\beta} s^{-1-\frac{1}{\beta}} w_{\beta}(s^{-\frac{1}{\beta}}) \sim c_{\beta} s^{-1-\frac{1}{\beta}} s^{1+\frac{1}{\beta}} = c_{\beta}.$$

Using (2.3.11), note that $w_{\beta}(r^{-\frac{1}{\beta}}) \sim f_{\beta}(r^{-\frac{1}{\beta}})$ for $r \to \infty$. Thus for $s \to \infty$,

$$\begin{aligned} t^{\beta}\mu^{\beta}_{0}(t^{\beta}s) = c_{\beta} s^{-1-\frac{1}{\beta}} w_{\beta}(s^{-\frac{1}{\beta}}) &\sim c_{\beta} s^{-1-\frac{1}{\beta}} f_{\beta}(s^{-\frac{1}{\beta}}) \\ &= c_{\beta} s^{-1+\frac{1}{2(1-\beta)}} \exp\left\{-c_{\beta} s^{\frac{1}{1-\beta}}\right\}, \end{aligned}$$

as claimed. □

2.4 Asymptotic Methods

We describe here some methods from asymptotic analysis, namely variants of the Laplace method and its application to the incomplete gamma function. Our main references for asymptotic analysis are Bruijn (1981), Fedoryuk (1987), and Murray (1984).

Laplace method

The main goal of the Laplace method is to estimate integrals of the form

$$\int_a^b g(x) \exp\{-\lambda h(x)\} \, \mathrm{d}x.$$

As a motivating example, let $a = 1, b = \infty$, $h(x) = x$ and $g(x) = x^N$ for some integer $N > 0$[4]. In this case, one could integrate by parts N times, until the x^N term vanishes, and one is left with a final integral

$$\int_1^\infty \exp\{-\lambda x\} \, \mathrm{d}x = \lambda^{-1} \exp\{-\lambda\},$$

so that, for sufficiently large λ,

$$\int_1^\infty x^N \exp\{-\lambda x\} \, \mathrm{d}x = O(1)\lambda^{-1} \exp\{-\lambda\} + O(\lambda^{-N-1} \exp\{-\lambda\}).$$

Now the main idea is that largest contribution to the asymptotic behaviour of

$$\int_a^b g(x) \exp\{-\lambda h(x)\} \, \mathrm{d}x, \tag{2.4.1}$$

comes from a neighbourhood around the point (or neighbourhoods around the points) at which the function $h(x)$ in the exponent attains its minimum value. Outside this neighbourhood the contribution is exponentially small, and so when one proves asymptotic formulas using Laplace methods, the integrals are split up into the neighbourhood around which the major contribution occurs (or around each such neighbourhood, if $-h(x)$ is not unimodal) and the regions for which the approximation error is exponentially small. Although we focus on integrals over some interval (a, ∞), the point is that extending the interval only introduces exponentially small errors and so the value of the integral over a larger interval is essentially the same. Let us assume that in 2.4.1 h is a real continuous function which attains a minimum at the boundary point b, that $h'(b)$ exists and $h'(b) > 0$. Moreover assume that $h(x) > h(b)$ (for $x > b$) and $h(x) \to \infty$ as $x \to \infty$. Then we have the following asymptotic formula, see for

[4] This is just the upper incomplete gamma function, which we look at next.

example Bruijn (1981, Sections 4.2, 4.3)

$$\int_b^\infty g(x)\exp\{-\lambda h(x)\}\ \mathrm{d}x \sim g(b)(\lambda h'(b))^{-1}\exp\{-\lambda h(b)\}, \quad \lambda \to \infty. \quad (2.4.2)$$

On the other hand, if the function h has a minimum on the interior of the interval (b,∞), say at the point $\tilde{b} \in (b,\infty)$. Finally, assume that the derivative $h'(x)$ exists in some neighbourhood of $x = \tilde{b}$, that $h''(\tilde{b})$ exists and that $h''(\tilde{b}) > 0$. Then

$$\int_b^\infty g(x)\exp\{-\lambda h(x)\}\ \mathrm{d}x \sim g(\tilde{b})\sqrt{\frac{2\pi}{\lambda h''(\tilde{b})}}\exp\{-\lambda h(\tilde{b})\}, \quad \lambda \to \infty. \quad (2.4.3)$$

Using these formulas, we now prove an asymptotic formula for a particular type of integrals which comes up in our estimates in later chapters.

Proposition 2.4.1. Let $a > 0$, $N \in \mathbb{R}$, $c > 0$ and $\Omega \geq 1$. Then the following asymptotic formula holds as $\Omega \to \infty$,

$$\int_0^1 w^N \exp\{-\Omega w - cw^{-a}\}\ \mathrm{d}w \sim C_1(a,N,c)\Omega^{-\frac{2(N+1)+a}{2(a+1)}}\exp\left\{-C_2(c,a)\Omega^{\frac{a}{a+1}}\right\},$$

where $C_1(a,N,c) = (ac)^{\frac{2(N+1)-1}{2(a+1)}}\sqrt{\frac{2\pi}{a+1}}$, and $C_2(c,a) = (ac)^{\frac{1}{a+1}}[1+a^{-1}]$.

Proof. Define

$$J(\Omega) := \int_0^1 w^N \exp\{-w\Omega - cw^{-a}\}\ \mathrm{d}w,$$

and let $h(w) = -w\Omega - cw^{-a}$. Differentiating h with respect to w, one finds the maximum of h at

$$w = w_* := \left(\frac{\Omega}{ac}\right)^{-\frac{1}{a+1}}.$$

Now the trick is to make the substitution $w = w_* s$ in the integral $J(\Omega)$, to obtain

$$\begin{aligned} J(\Omega) &= w_*^{N+1}\int_0^{w_*^{-1}} s^N \exp\{-w_* s\Omega - c(w_* s)^{-a}\}\ \mathrm{d}s \\ &= w_*^{N+1}\int_0^{w_*^{-1}} s^N \exp\{-(\Omega^a ac)^{\frac{1}{a+1}}[s + a^{-1}s^{-a}]\}\ \mathrm{d}s. \end{aligned}$$

Now we are in a position to apply the asymptotic formula (2.4.3), with

$g(s) = s^N$, $h(s) = s + a^{-1}s^{-a}$ and $\lambda = (\Omega^a ac)^{\frac{1}{a+1}}$. For this we need some derivatives of h,

$$h'(s) = 1 - s^{-a-1},$$

$$h''(s) = (a+1)s^{-a-2},$$

thus h has a minimum at $s = 1 \in (0, w_*^{-1})$. Finally applying (2.4.3) we have

$$\begin{aligned} J(\Omega) &\sim \left(\frac{\Omega}{ac}\right)^{-\frac{N+1}{a+1}} \sqrt{\frac{2\pi}{(\Omega^a ac)^{\frac{1}{a+1}}(a+1)}} \exp\{-(\Omega^a ac)^{\frac{1}{a+1}}[1+a^{-1}]\} \\ &= C_1(a,N,c)\Omega^{-\frac{2(N+1)+a}{2(a+1)}} \exp\left\{-C_2(c,a)\Omega^{\frac{a}{a+1}}\right\}, \end{aligned}$$

as required. □

We further have the slight extension of the above calculations.

Corollary 2.4.2. Let $a, b > 1$, $n \in \mathbb{R}$, and $c := \min(a,b)$. Then as $\Omega A^{-1} \to \infty$,

$$\int_1^\infty z^n \exp\{-\Omega z^{-1} - A^{-a}z^a - z^b\}\,\mathrm{d}z \sim C_1 \Omega^{\frac{2(n+1)-c}{2(c+1)}} A^{\frac{2c(n+1)+c}{2(c+1)}} \exp\left\{-C_2\left(\Omega A^{-1}\right)^{\frac{c}{c+1}}\right\}.$$

Proof. This formula is a consequence of the previous proposition after estimating the terms in the exponential:

$$\begin{aligned} \int_1^\infty z^n \exp\left\{-\Omega z^{-1} - A^{-a}z^a - z^b\right\}\,\mathrm{d}z &\sim \int_1^\infty z^n \exp\left\{-\Omega z^{-1} - A^{-c}z^c - z^c\right\}\,\mathrm{d}z \\ &\sim \int_1^\infty z^n \exp\left\{-\Omega z^{-1} - CA^{-c}z^c\right\}\,\mathrm{d}z. \end{aligned}$$

After a change of variables, the formula follows from an application of the previous Proposition. □

Incomplete Gamma function

Here we describe the asymptotic behaviour of the upper incomplete gamma function, which is defined by

$$\Gamma(s, A) = \int_A^\infty y^{s-1}\exp\{-y\}\,\mathrm{d}y.$$

Equivalently after a change of variables $y = Aw$,

$$\Gamma(s, A) = A^s \int_1^\infty w^{s-1} \exp\{-Aw\} \, \mathrm{d}w.$$

We have the following asymptotic behaviour of $\Gamma(s, A)$ for $A \to 0$,

$$\Gamma(s, A) \sim \begin{cases} -s^{-1}A^s, & s < 0, \\ (|\log A| + 1), & s = 0, \\ 1 - s^{-s}A^s, & s > 0. \end{cases}$$

Thus, for $A \leq 1$,

$$A^{-s}\Gamma(s, A) \asymp C_s \begin{cases} 1, & s < 0, \\ (|\log A| + 1), & s = 0, \\ A^{-s}, & s > 0. \end{cases}$$

For $A \to \infty$, we use the Laplace method (2.4.2) with $h(x) = x$, $b = 1$, $g(x) = x^{s-1}$,

$$A^{-s}\Gamma(s, A) \sim A^{-1} \exp\{-A\}, \quad A \to \infty.$$

Chapter 3

Caputo-Dzherbashyan and Riemann-Liouville type operators

In this chapter we first discuss the operators arising in standard fractional calculus (or 'classical' fractional calculus, by now), mentioning also the probabilistic interpretation along the way. This will then take us to the naturally defined *generalised fractional operators* motivated by the probabilistic interpretation of the classical operators. We then discuss the various extensions of these operators, whose foundations were laid in Kolokoltsov (2015). In particular we discuss various multidimensional extensions, while keeping in mind the probabilistic meaning of such operators.

3.1 Standard fractional derivatives

For a function $f \in C([a,b])$, the *iterated Riemann integral* of order $n \in \mathbb{N}$ is given by the formula

$$I^n_{a+}f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\ \mathrm{d}t, \quad x \in [a,b].$$

From this a natural definition of the (left) fractional integral of order $\beta > 0$ is given by:

$$I^\beta_{a+}f(x) = \frac{1}{\Gamma(\beta)}\int_a^x (x-t)^{\beta-1} f(t)\ \mathrm{d}t.$$

The integral I^{β}_{a+} is called the *left Riemann-Liouville* (RL) fractional integral. Now we are presented with two ways of defining a corresponding notion of *fractional derivatives*, either by differentiating the fractional integral of a function, or by taking the fractional integral of the derivative of a function.

Definition 3.1.1. For $n \in \mathbb{N}_+$ and $\beta \in (n, n+1)$, the (left) *Riemann-Liouville* fractional derivative of a function $f \in C^{n+1}([a,b])$ is given by

$$D^{\beta}_{a+}f(x) = \frac{d^{n+1}}{dx^{n+1}} I^{n+1-\beta}_{a+} f(x) = \frac{1}{\Gamma(n+1-\beta)} \frac{d^{n+1}}{dx^{n+1}} \int_a^x (x-t)^{n-\beta} f(t)\ \mathrm{d}t,$$

for $x > a$. The (left) *Caputo-Dzherbashyan* (CD) fractional derivative is given by

$$D^{\beta}_{a+*}f(x) = I^{n+1-\beta}_{a+} \frac{d^{n+1}}{dx^{n+1}} f(x) = \frac{1}{\Gamma(n+1-\beta)} \int_a^x (x-t)^{n-\beta} \left[\frac{d^{n+1}}{dt^{n+1}} f\right](t)\ \mathrm{d}t,$$

for $x > a$.

In this thesis we are mostly interested in the fractional derivatives of order $\beta \in (0,1)$, since for such β they represent the generators of (spectrally one-sided) Lévy process.

Remark 4. *Also of interest to probabilists are derivatives of order $\beta \in (1,2)$, which represent generators of (two-sided) β-stable processes. We do not study their generalisations in this thesis. One also defines*

Notice that the RL fractional integral I^{β}_{a+} is obtained from $I^{\beta}_{-\infty+}$ by restricting its action to the space $C_{kill(-\infty,a]}(\mathbb{R})$,

$$I^{\beta}_{a+}f(x) = I^{\beta}_{-\infty+}f(x), \quad f \in C_{kill(-\infty,a]}(\mathbb{R}).$$

For $\beta \in (0,1)$, after integrating by parts (here we need f to be β-Hölder continuous) one can write the RL and CD derivatives as

$$D^{\beta}_{a+}f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{x-a} (f(x-y) - f(x)) y^{-1-\beta}\ \mathrm{d}y + \frac{f(x)}{\Gamma(1-\beta)(x-a)^{\beta}}, \quad x > a,$$

and

$$D^{\beta}_{a+*}f(x) = \frac{1}{\Gamma(-\beta)} \int_0^{x-a} (f(x-y) - f(x)) y^{-1-\beta}\ \mathrm{d}y + \frac{f(x) - f(a)}{\Gamma(1-\beta)(x-a)^{\beta}}, \quad x > a.$$

The *right* versions of RL and CD derivatives of order $\beta \in (0,1)$ can be analogously defined by

$$D^{\beta}_{a-}f(x) = \frac{1}{\Gamma(-\beta)}\int_0^{a-x}(f(x+y)-f(x))y^{-1-\beta}\,\mathrm{d}y+\frac{f(x)}{\Gamma(1-\beta)(a-x)^{\beta}}, \quad x<a,$$

and

$$D^{\beta}_{a-*}f(x) = \frac{1}{\Gamma(-\beta)}\int_0^{a-x}(f(x+y)-f(x))y^{-1-\beta}\,\mathrm{d}y+\frac{f(x)-f(a)}{\Gamma(1-\beta)(a-x)^{\beta}}, \quad x<a.$$

Comparing the derivatives D^{β}_{a+} and D^{β}_{a+*} with (2.1.2), we note that $-D^{\beta}_{a+}$ generates a sub-Markov Feller process (which is killed upon crossing a) with Lévy measure $\nu(y) = -y^{-1-\beta}/(\Gamma(-\beta))$ and killing rate $-1/(\Gamma(1-\beta)(x-a)^{\beta})$. On the other hand $-D^{\beta}_{a+*}$ generates a decreasing Feller process which is absorbed upon crossing a, see for example Kolokoltsov (2015, Section 3.1) or Böttcher et al. (2014) for the probabilistic interpretation of fractional derivatives. Notice that for smooth bounded integrable functions, the RL and CD derivatives coincide for $a=-\infty$, $\beta \in (0,1)$. We call their common value the *fractional derivative in generator form*[1],

$$D^{\beta}_{+}f(x) := D^{\beta}_{-\infty+}f(x) = D^{\beta}_{-\infty+*}f(x) = \frac{1}{\Gamma(-\beta)}\int_0^{\infty}(f(x-y)-f(x))y^{-1-\beta}\,\mathrm{d}y. \tag{3.1.1}$$

From the theory of Lévy processes in Section 2.1, we recognise $-D^{\beta}_{+}$ as the generator of a decreasing β-stable Lévy process with Lévy measure $\nu(y) = -y^{-1-\beta}/(\Gamma(-\beta))$ and thus generates a strongly continuous semigroup of positivity preserving contractions on $C_{\infty}(\mathbb{R})$.

It is clear from the definition that the composition $D^{\beta}_{+}\circ I^{\beta}_{-\infty}$ acts like the identity operator on functions with compact support. Thus $I^{\beta}_{-\infty}$ represents the potential operator of the strongly continuous semigroup of linear operators in $C_{\infty}(\mathbb{R})$ which is generated by $-D^{\beta}_{+}$. Note that $I^{\beta}_{-\infty}$ is unbounded in $C_{\infty}(\mathbb{R})$, but becomes bounded when restricted to $C_{kill(-\infty,a]}(\mathbb{R})$. That is,

$$R_{\lambda} = (\lambda + D^{\beta}_{+})^{-1} \to I^{\beta}_{-\infty}, \quad \text{as } \lambda \to 0. \tag{3.1.2}$$

[1] also known as the Marchaud derivative. This is the *left* derivative, with the *right* version denoted by D^{β}_{-} which is given by changing the variable of integration $y \mapsto -y$. This also applies to D^{β}_{a-} and D^{β}_{a-*}.

Thus the operator I^{β}_{a+} is the potential operator of the semigroup generated by $-D^{\beta}_{+}$ restricted to the space $C_{kill(-\infty,a]}(\mathbb{R})$. For a background on potential operators and measures, see Schilling et al. (2012) or Van Den Berg and Forst (2012) (or from a probabilistic point of view, Feller (2008)). We can also obtain a path integral representation of I^{β}_{a+}. For this we need Dynkin's martingale, see Dynkin (1965, Theorem 5.1).

Theorem 3.1.2. *Let $(A, D(A))$ be the generator of a Feller process $X_x(t)$. Then for $f \in D(A)$,*

$$f(X_x(t)) - f(X_x(0)) - \int_0^t Af(X_x(s))\,\mathrm{d}s,$$

is a martingale.

Assuming that τ is a stopping time such that $\mathbb{E}[\tau] < \infty$, we apply Doob's optimal stopping theorem to the above which gives *Dynkin's formula*: for $f \in D(A)$,

$$f(x) = \mathbb{E}[f(X_x(\tau))] - \mathbb{E}\int_0^{\tau} g(X_x(s))\,\mathrm{d}s$$

where $g = Af$, and τ is the first time $X_x(t)$ exits an interval $(a, x']$ for some $x' > x$. The solution to the boundary value problem

$$\begin{aligned} D^{\beta}_{a+*}f(x) = D^{\beta}_{a+}f(x) = D^{\beta}_{+}f(x) = g(x), \quad & x \in (a, \infty], \\ f(x) = 0, \quad & x \in (-\infty, a], \end{aligned}$$

is given by $I^{\beta}_{a+}g$ for $g \in C_{kill(-\infty,a]}(\mathbb{R})$. Thus, seeing as $-D^{\beta}_{+}$ is the generator of Feller process $X^{\beta}_x(t)$, we use Dynkin's formula to get

$$\begin{aligned} f(x) &= \mathbb{E}[f(X^{\beta}_x(\tau_a))] + \mathbb{E}\int_0^{\tau_a} D^{\beta}f(X^{\beta}_x(s))\,\mathrm{d}s \\ &= \mathbb{E}\int_0^{\tau_a} g(X^{\beta}_x(s))\,\mathrm{d}s. \end{aligned}$$

A final interpretation of I^{β}_{a+} is given by noting that the fundamental solution (supported on $\mathbb{R}_+$) of D^{β}_{+} is given by $U^{\beta}(z) = z^{\beta-1}_{+}/\Gamma(\beta)$, which is precisely the integral kernel of $I^{\beta}_{-\infty+}$. These three facets of the operator I^{β}_{a+} lead us naturally to the generalised fractional operators.

Finally, we recall one of the most important tools from fractional calculus

- the *Mittag-Leffler function*. It is defined by the power series

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}, \quad z \in \mathbb{C},\ \beta > 0,$$

where Γ is the Gamma function:

$$\Gamma(z) = \int_0^\infty x^{z-1} e^{-x}\, \mathrm{d}x, \quad \Re(z) > 0.$$

The fundamental importance of the Mittag-Leffler function is due to the fact that solutions of the simplest fractional linear equations of the form

$$D^\beta_{a+*} f(x) = -\lambda f(x) + g(x),\ f(a) = f_a, \quad x > a,$$

for $\beta \in (0,1)$, can be given by

$$f(x) = E_\beta(-\lambda(x-a)^\beta) f_a + \beta \int_0^{x-a} g(x-y) y^{\beta-1} \frac{d}{dy} E_\beta(-\lambda y^\beta)\, \mathrm{d}y,$$

see Diethelm (2010, Theorem 7.2). Another important fact about Mittag-Leffler functions is that they can be represented in terms of the transition densities of β-stable subordinators,

$$E_\beta(s) = \frac{1}{\beta} \int_0^\infty e^{sx} x^{-1-\frac{1}{\beta}} w_\beta(x^{-\frac{1}{\beta}})\, \mathrm{d}x, \quad \beta \in (0,1),\ s \in \mathbb{C}.$$

We return to this remarkably important formula in the next section.

3.2 Generalised fractional operators

In view of the fractional derivative in generator form D^β_+, a natural generalisation from the probabilistic point of view is to replace the kernel $y^{-1-\beta}/(-\Gamma(-\beta))$ by some general Lévy-type kernel $\nu(t, \mathrm{d}y)$. That is, consider the operator $D^{(\nu)}_+$ on $\mathbb{R}$ defined by:

$$D^{(\nu)}_+ f(t) = -\int_0^\infty (f(t-r) - f(t)) \nu(t, \mathrm{d}r).$$

We will always assume that ν has a density $\nu(t,r)$ which satisfies:

Assumption 3.2.1 ((L0))**.** $\nu(t, s)$ is continuous as a function of both variables, and is continuously differentiable in the first. Furthermore,

$$\sup_t \int (1 \wedge r)\nu(t, r)\ \mathrm{d}r < \infty, \quad \sup_t \int (1 \wedge r) \left| \frac{\partial}{\partial t} \nu(t, r) \right|\ \mathrm{d}r < \infty,$$

and for any $\epsilon > 0$ there exists a $K > 0$ such that

$$\sup_t \int_{\mathbb{R}\setminus(-K,K)} \nu(t, s)\ \mathrm{d}s < \epsilon, \quad \sup_t \int_{\mathbb{R}\setminus(-K,K)} \left| \frac{\partial}{\partial t} \nu(t, r) \right|\ \mathrm{d}r < \epsilon,$$

$$\sup_t \int_{(-1/K,1/K)} |r|\nu(t, r)\ \mathrm{d}r < \epsilon.$$

Under the assumption (L0), the operator $-D_+^{(\nu)}$ generates a conservative Feller semigroup T_t in $C_\infty(\mathbb{R})$ with invariant core $C_\infty^1(\mathbb{R})$, see Kolokoltsov (2019a). We may occasionally also make the following assumptions.

Assumption 3.2.2 ((L1))**.** There exists $\epsilon > 0$ and $\delta > 0$ such that $\nu(t, r) \geq \delta > 0$ for all t and all $|r| < \epsilon$.

Assumption 3.2.3 ((L2))**.** The transition probabilities of the process $X^{+(\nu)}$ are absolutely continuous with respect to the Lebesgue measure, and we denote by $p^{+(\nu)}(s, r, y)$ the transition densities.

Assumption 3.2.4 ((L3))**.** The transition density $p^{+(\nu)}(s, r, y)$ is continuously differentiable in s.

The classical CD derivative D_{a+*}^β is obtained from D_+^β by the restriction of its action to the space $C_{const(-\infty,a]}(\mathbb{R})$ considered as the subspace of $C(\mathbb{R})$ by extending their values as constants to the left of a, see Kolokoltsov (2019a, Proposition 1.8.2). Then looking for a generalised CD derivative arising from $D_+^{(\nu)}$ we define

$$D_{a+*}^{(\nu)} f(t) = -\int_0^{t-a} (f(t-s) - f(t))\nu(t, s)\ \mathrm{d}s - (f(a) - f(t)) \int_{t-a}^\infty \nu(t, s)\ \mathrm{d}s.$$

Analogously, the generalised RL derivative arising from $D_+^{(\nu)}$ is obtained by the restriction of its action to the space $C_{kill(-\infty,a]}(\mathbb{R})$ considered as the subspace of $C(\mathbb{R})$ by extending their values as 0 to the left of a:

$$D_{a+}^{(\nu)} f(t) = -\int_0^{t-a} (f(t-s) - f(s))\nu(t,s)\ \mathrm{d}s + f(t)\int_{t-a}^{\infty} \nu(t,\mathrm{s})\ \mathrm{d}s.$$

Under assumption (L0), the operator $D_{a+*}^{(\nu)}$ (respectively $D_{a+}^{(\nu)}$) generates a Feller semigroup on $C_{const(-\infty,a]}(\mathbb{R})$ (resp. a sub-Feller semigroup on $C_{kill(-\infty,a]}(\mathbb{R})$). The corresponding generalised fractional integrals arising from the generalised fractional derivative $D_{+}^{(\nu)}$ can be defined in a few different ways depending on which point of view one chooses: probability, semigroup theory or generalised functions. In view of (3.1.2) and the discussion thereafter, the operator $I_{-\infty+}^{\beta}$ is the potential operator of the semigroup generated by $-D_{+}^{\beta}$. Thus we define the *generalised fractional integral* I_{a+}^{ν} as the potential operator of the semigroup generated by $-D_{+}^{(\nu)}$ restricted to the space $C_{kill(a)}([a,\infty))$.

Let us denote by $(T_t^{\nu})_{t\geq 0}$ the semigroup generated by the operator $-D_{+}^{(\nu)}$. Then for $f \in C_{kill(a)}(\mathbb{R}) \cap C_{\infty}(\mathbb{R})$, the potential operator $U^{(\nu)}$ of the semigroup T_t^{ν} is given by

$$\begin{aligned}
U^{(\nu)} f(t) = \int_0^{\infty} T_r^{\nu} f(t)\mathrm{d}r &= \int_0^{\infty}\int_a^{\infty} f(s)p^{(\nu)}(r,t,\mathrm{d}s)\ \mathrm{d}r \\
&= \int_0^{\infty}\int_a^{t} f(s)p^{(\nu)}(r,t,\mathrm{d}s)\ \mathrm{d}r \\
&= \int_0^{t-a} f(t-s')\left(\int_0^{\infty} p^{(\nu)}(r,t,\mathrm{d}s')\mathrm{d}r\right),
\end{aligned}$$

where $p^{(\nu)}(r,t,\mathrm{d}s)$ are the (transformed) transition probabilities of the process generated by $-D_{+}^{(\nu)}$. The potential measure is defined as the integral kernel of the potential operator, and by an abuse of notation, we denote this measure by $U^{(\nu)}(t,\mathrm{d}s)$. Thus the generalised fractional integral $I_{a+}^{(\nu)}$ is given by

$$I_{a+}^{(\nu)} f(t) = \int_0^{t-a} f(t-s)U^{(\nu)}(t,\mathrm{d}s)$$

where the potential measure $U^{(\nu)}(t,\mathrm{d}s)$ is equal to the vague limit

$$U^{(\nu)}(t,M) = \int_0^{\infty} p^{(\nu)}(r,t,M)\mathrm{d}r,$$

of the measures $\int_0^K p^{(\nu)}(r,t,\cdot)\mathrm{d}r,\ K\to\infty$ (see Schilling et al. (2012) (p. 63)) for any compact set M. Furthermore the λ-potential measure is defined by

$$U_\lambda^{(\nu)}(t,M) = \int_0^\infty e^{-\lambda r} p^{(\nu)}(r,t,M)\mathrm{d}r,$$

so that if $\lambda > 0$ and $g \in C_{kill(-\infty,a]}(\mathbb{R}) \cap C_\infty(\mathbb{R})$, the convolution $(U_\lambda^{(\nu)} \star g)(t)$, which is given by

$$(U_\lambda^{(\nu)} \star g)(t) = \int_0^{t-a} g(t-s) \int_0^\infty e^{-\lambda r} p^{(\nu)}(r,t,\mathrm{d}s)\mathrm{d}r, \tag{3.2.1}$$

is the resolvent operator of the semigroup generated by $-D_+^{(\nu)}$ restricted to $C_{kill(-\infty,a]}(\mathbb{R})$. That is, $f(x) = (U_\lambda^{(\nu)} \star g)(x)$ for $g \in C_{kill(-\infty,a]}(\mathbb{R}) \cap C_\infty(\mathbb{R})$ is the classic solution to the equation

$$D_+^{(\nu)} f = D_{a+}^{(\nu)} f = D_{a+*}^{(\nu)} f = -\lambda f + g.$$

This also holds for $\lambda = 0$, and so the potential operator with kernel $U^{(\nu)}(t,\mathrm{d}y)$,

$$(U^{(\nu)} \star g)(x) = I_{a+}^{(\nu)} g(x),$$

represents the classical solution to the equation

$$D_{a+*}^{(\nu)} f = g,$$

on $C_{kill(-\infty,a]}(\mathbb{R})$.

Example 3.2.5. *For the case $\nu(t,\mathrm{d}y) = -1/[\Gamma(-\beta)y^{1+\beta}]\mathrm{d}y$, (3.2.1) says that*

$$f(t) = \int_0^{t-a} g(t-s) \int_0^\infty e^{-\lambda r} p^\beta(r,s)\mathrm{d}s\mathrm{d}r,$$

where $p^\beta(r,s)$ are the transition densities of a β-stable subordinator, is the solution to the linear fractional equation

$$D_+^\beta f(t) = -\lambda f(t) + g(t), \quad f(a) = 0.$$

On the other hand, it is well known that the solution to such linear *fractional equations are given by*

$$f(t) = \beta \int_a^t g(z)(t-z)^{\beta-1} E'_\beta(-\lambda(t-z)^\beta)\mathrm{d}z$$
$$= \beta \int_0^{t-a} g(t-y) y^{\beta-1} E'_\beta(-\lambda y^\beta)\mathrm{d}y,$$

where $E_\beta(z)$ is the Mittag-Leffler function

$$E_\beta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\beta+1)}, \quad z \in \mathbb{C}. \tag{3.2.2}$$

Thus we have

$$U_\lambda^\beta(t) = \int_0^\infty e^{-\lambda r} p^\beta(r,t)\mathrm{d}r = \beta t^{\beta-1} E'_\beta(-\lambda t^\beta),$$

which is equivalent to the Zolotarev-Pollard formula for the Mittag-Leffler function in terms of the transition densities of stable subordinators,

$$E_\beta(s) = \frac{1}{\beta} \int_0^\infty e^{sx} x^{-1-1/\beta} p^\beta(1, x^{-1/\beta})\ \mathrm{d}x. \tag{3.2.3}$$

This representation of the Mittag-Leffler function is the key starting point for all estimates of the Green's function associated to fractional evolution equations that we obtain in later chapters.

As noted in Kolokoltsov (2017), we can extrapolate from the case

$$D_{0+*}^\beta u(t) = -\lambda u(t), \quad \lambda > 0, \quad u(0) = u_0,$$

to the Banach-valued version

$$D_{0+*}^\beta u(t,x) = Lu(t,x), \quad u(0,x) = Y(x),$$

where L is some operator generating a Feller semigroup. One can expect that the solution to this equation can be written in terms of an operator-valued Mittag-Leffler function,

$$u(t,x) = E_\beta\left(Lt^\beta\right) Y(x), \tag{3.2.4}$$

where $E_\beta(s)$ are Mittag-Leffler functions defined by (3.2.2). However, this series representation does not allow one to define $E_\beta(L)$ for an unbounded operator L. In both Kolokoltsov and Veretennikova (2014) and Kolokoltsov (2017) the authors find that the most convenient way to overcome this difficulty is to use the formula (3.2.3) for the Mittag-Leffler function. This connection between Mittag-Leffler functions, Laplace transforms and stable densities is due to Zolotarev (1957, 1961, 1986)—although preliminary versions of this formula were also noted almost a decade earlier by Pollard (1948). Thus formula (3.2.3) could be called the Pollard-Zolotarev formula. Notice that if an operator L generates a Feller semigroup with transition densities $G(t,x,y)$, then

$$e^{Lt^\beta z}Y(\cdot) = \int_{\mathbb{R}^d} G(t^\beta z, \cdot, y)Y(y)\,\mathrm{d}y.$$

With the help of Fubini's theorem, the solution (3.2.4) can be written as

$$\begin{aligned}
u(t,x) &= E_\beta(t^\beta L)Y(x)\\
&= \frac{1}{\beta}\int_0^\infty e^{Lt^\beta z}Y(x)z^{-1-\frac{1}{\beta}}w_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z\\
&= \frac{1}{\beta}\int_0^\infty \int_{\mathbb{R}^d} G(t^\beta z, x, y)Y(y)z^{-1-\frac{1}{\beta}}w_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}y\,\mathrm{d}z\\
&= \int_{\mathbb{R}^d}\left(\frac{1}{\beta}\int_0^\infty G(t^\beta z, x, y)z^{-1-\frac{1}{\beta}}w_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z\right)Y(y)\,\mathrm{d}y\\
&=: \int_{\mathbb{R}^d} G^{(\beta)}(t,x,y)Y(y)\,\mathrm{d}y.
\end{aligned}$$

In Chapter 4, we obtain estimates for the Green's function given by,

$$G_L^{(\beta)}(t,x,y) := \frac{1}{\beta}\int_0^\infty G_L(t^\beta z, x, y)z^{-1-\frac{1}{\beta}}w_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z, \tag{3.2.5}$$

where $G_L(z,x,y)$ is the Green's function associated with the spatial operator L, i.e., the fundamental solution of

$$\partial_t u = Lu.$$

3.3 Mixed RL and CD-type operators on the orthant

In this section we look at mixtures of generalised CD and RL derivatives on the d-dimensional orthant, i.e, on the domain $\mathcal{O} \subset \mathbb{R}^d$,

$$\mathcal{O} = \{(t_1, \cdots, t_d) \in \mathbb{R}^d \ : \ t_i \geq 0, \ i = 1, \cdots, d\}.$$

We define also the space

$$\mathcal{O}^{n,d} = \{(t_1, \cdots, t_d) \in \mathbb{R}^d \ : \ t_i > 0, \ i = 1, \cdots, n, \ t_j \geq 0, \ j = n+1, \cdots d\},$$

for $0 \leq n \leq d$ with the convention that $\mathcal{O}^{0,d} = \mathcal{O}$ and $\mathcal{O}^{d,d} = \mathcal{O} \setminus \{\mathbf{0}\}$. In this section we will use a boldface letter to denote an element living in a subset of $\mathbb{R}^d$, for example an $\mathcal{O}$-valued process $\mathbf{X}_{\mathbf{r}}(s)$ starting from $\mathbf{r} \in \mathcal{O}$. We denote by $\mathcal{O}_{i,0}$ the i-th face of the boundary (at zero) of $\mathcal{O}$, that is for $i = 1, \cdots, d$,

$$\mathcal{O}_{i,0} = \{\mathbf{t} \in \mathcal{O}; \ t_i = 0\}.$$

Define $h_{i,0}(t)$ to be the projection of $\mathcal{O}_{i,0}$ onto the subspace $\mathcal{O}_i \subset \mathbb{R}^{d-1}$ by removing the coordinate which is zero, that is, $h_{i,0}(t) : \mathcal{O}_{i,0} \mapsto \mathcal{O}_i$,

$$h_{i,0}(t) = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_d). \tag{3.3.1}$$

Let $\underline{\nu} = (\nu_1, \cdots, \nu_d)$ be a collection of Lévy kernels which each satisfy the assumptions (L0)-(L3) (thus for $i = 1, \cdots, d$ the corresponding CD or RL type operators generate a Feller process with a continuously differentiable transition density). Define the operators

$$D^{(\underline{\nu})}_{mix} = \sum_{i=1}^{n} {}_{t_i}D^{(\nu_i)}_{0+} + \sum_{i=n+1}^{d} {}_{t_i}D^{(\nu_i)}_{0+*}, \quad 0 \leq n \leq d, \tag{3.3.2}$$

$$D^{(\underline{\nu})}_{free} = \sum_{i=1}^{d} {}_{t_i}D^{(\nu_i)}_{+}.$$

We denote by

$$\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s) := \left(X_{t_1}^{0+(\nu_1)}(s), \cdots, X_{t_n}^{0+(\nu_n)}(s), X_{t_{n+1}}^{0+*(\nu_{n+1})}(s), \cdots, X_{t_d}^{0+*(\nu_d)}(s)\right),$$

the Feller process generated by the operator $-D_{mix}^{(\underline{\nu})}$, which lives on $\mathcal{O}^{n,d} \cup \{\delta\}$ where δ is a cemetery state. This process is obtained from the process

$$\mathbf{X_t}^{+(\underline{\nu})}(s) := \left(X_{t_1}^{+(\nu_1)}(s), \cdots X_{t_n}^{+(\nu_n)}(s), X_{t_{n+1}}^{+(\nu_{n+1})}(s), \cdots, X_{t_d}^{+(\nu_d)}(s)\right)$$

which is generated by $-D_{free}^{(\underline{\nu})}$, by either killing it whenever any of the first n coordinates attempt to cross the boundary points $r_i = 0$, $1 \leq i \leq n$, or by stopping it if any of the last $d-n$ coordinates does the same with the boundary points $r_j = 0$, $n+1 \leq j \leq d$.

Remark 5. *Since each $-D_+^{\nu_i}$ generates an independent Feller process, the Lie-Trotter theorem implies that $-D_{free}^{(\nu)}$ also generates a Feller process whose coordinates are independent Feller processes generated by $-D_+^{(\nu_i)}$. Note that $\mathbf{X_t}^{+(\underline{\nu})}(s)$ is an $\mathbb{R}^d$-valued process.*

Let $p^{+(\nu_i)}(s, t_i, r_i)$ denote the transition density function of the process $X_{t_i}^{+(\nu_i)}$. Due to the independence between the coordinates of the process $\mathbf{X_t}^{+(\underline{\nu})}(s)$, its transition density function, denoted by $\mathbf{p}^{+(\underline{\nu})}(s, \mathbf{t}, \mathbf{r})$, satisfies

$$\mathbf{p}^{+(\underline{\nu})}(s, \mathbf{t}, \mathbf{r}) = \prod_{i=1}^{d} p^{+(\nu_i)}(s, t_i, r_i).$$

If $\tau_0^{t_i,(\nu_i)}$ is the first exit time from $\mathbb{R}_+$ of the process $X_{t_i}^{+(\nu_i)}$, then denote the first exit time via the CD-type boundary by $\tilde{\tau} = \min_{n+1 \leq i \leq d} \tau_0^{t_i,(\nu_i)}$, and the first exit time via the RL-type boundary $\tau' = \min_{1 \leq j \leq n} \tau_0^{t_j,(\nu_j)}$. Then the first exit time of $\mathbf{X_t}^{+(\underline{\nu})}(s)$ from $\mathcal{O}$ is

$$\begin{aligned} \tau_0^{\mathbf{t},(\underline{\nu})} &:= \min(\tau', \tilde{\tau}) \\ &:= \min\left(\min_{1 \leq j \leq n} \tau_0^{t_j,(\nu_j)}, \min_{n+1 \leq i \leq d} \tau_0^{t_i,(\nu_i)}\right) \end{aligned} \tag{3.3.3}$$

Lemma 3.3.1. Let $\mathbf{t} \in \mathcal{O}^{n,d}$ for some $1 \leq n \leq d$. Let $\underline{\nu} = (\nu_1, \nu_2, \cdots, \nu_d)$ be a collection of functions such that each ν_i satisfy assumptions (L0) and (L1). Then:

i) The sets $\mathcal{O}_{i,0}$ for $i = 1, \cdots d$ are regular in expectation for both operators $-\mathbf{D}_{\mathbf{0}+}^{(\nu)}$ and $-\mathbf{D}_{\mathbf{0}+*}^{(\nu)}$. Moreover, $\mathbb{E}\left[\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}\right] < \infty$.[2]

Further assuming that each ν_i satisfies (L2)-(L3),

ii) If $\mu_{\mathbf{0}}^{\mathbf{t},(\nu)}(ds)$ denotes the law of $\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}$, then its density function $\mu_{\mathbf{0}}^{\mathbf{t},(\nu)}(s)$ is given by

$$\mu_{\mathbf{0}}^{\mathbf{t},(\nu)}(s) = \sum_{i=1}^{d} \mu_0^{t_i,(\nu_i)}(s) \prod_{j\neq i} \int_0^{t_j} p^{+(\nu_j)}(s, t_j, r)\, \mathrm{d}r, \quad s \geq 0, \tag{3.3.4}$$

where $\mu_0^{t_i,(\nu_i)}(s)$ is the density of $\tau_0^{t_i,(\nu_i)}$.

Proof.

i) The regularity in expectation of the boundary $\mathcal{O}_{i,0}$ is a consequence of assumption (L1) and the method of Lyapunov functions. Namely, to show that the boundary is regular, it is sufficient to find a continuous function f in a neighbourhood of $\mathcal{O}$ such that f is differentiable for $\mathbf{x} > 0$, $f(\mathbf{y}) = 0$ for each $\mathbf{y} \in \mathcal{O}_{i,0}$, and for $\mathbf{x} \in (\mathbf{0}, \mathbf{c})$ with some $\mathbf{c} \in \mathcal{O} \setminus \{\mathcal{O}_{\mathbf{i},\mathbf{0}}\}$ one has $f(\mathbf{x}) > \mathbf{0}$, $-\mathbf{D}_{\mathbf{0}+}^{(\nu)}\mathbf{f}(\mathbf{x}) < \mathbf{0}$ (similarly for the CD-type operator). One can take the function

$$f_\omega(\mathbf{x}) = \prod_{i=1}^{d} x_i^{\omega_i}, \quad \omega_i \in (0,1) \text{ for } 1 \leq i \leq d.$$

Clearly $f_\omega(\mathbf{y}) = 0$ for each $\mathbf{y} \in \mathcal{O}_{i,0}$ since such a $\mathbf{y}$ has a 0 in atleast 1 of the coordinates. For $\mathbf{x} \in \mathcal{O} \setminus \{\mathbf{0}\}$, $f_\omega(\mathbf{x}) > \mathbf{0}$ since each coordinate is non-zero. For $\mathbf{x}$ approaching $\mathbf{0}$ (in any of its coordinates) from the right, $-\mathbf{D}_{\mathbf{0}+}^{(\nu)} f_\omega(\mathbf{x}) < 0$ due to (L1). To show that $\mathbb{E}[\tau_0^{\mathbf{t},(\nu)}] < \infty$, it suffices to show that the exit time for each coordinate has finite expectation. For this, compare each process $(X_{t_i}^{(\nu_i)}(s))_{s\geq 0}$ with a process $(t_i - X_0^{(\tilde{\nu})}(s))_{s\geq 0}$ where $X_0^{(\tilde{\nu})}(s)$ is a (non-decreasing) compound Poisson process with Lévy density $\tilde{\nu}_i(dy) = \gamma \mathbf{1}_{[0,\epsilon]}(y)$, where γ and ϵ are chosen from Assumption (L1). See for example the comparison principle in Zhang (2000).

[2] The operators $-\mathbf{D}_{\mathbf{0}+}^{(\nu)}$ and $-\mathbf{D}_{\mathbf{0}+*}^{(\nu)}$ correspond to the mixed operator $-D_{mix}^{(\nu)}$ with $n = d$ and $n = 0$ respectively

ii) This follows by differentiating

$$\mathbb{P}[\tau_{\mathbf{0}}^{\mathbf{t},(\underline{\nu})} > s] = \prod_{i=1}^{d} \mathbb{P}[\tau_{0}^{t_i,(\nu_i)} > s]$$

with respect to s and using the chain rule:

$$\begin{aligned}
\frac{\partial}{\partial s}\mathbb{P}[\tau_{\mathbf{0}}^{\mathbf{t},(\nu)} > s] &= \frac{\partial}{\partial s}\prod_{i=1}^{d}\int_0^{t_i} p^{+(\nu_i)}(s,t_i,r)\ \mathrm{d}r \\
&= \left[\sum_{i=1}^{d}\frac{\partial}{\partial s}\int_0^{t_i} p^{+(\nu_i)}(s,t_i,r)\ \mathrm{d}r\right]\prod_{j\neq i}\int_0^{t_i} p^{+(\nu_j)}(s,t_j,r)\ \mathrm{d}r \\
&= \sum_{i=1}^{d}\mu_0^{t_i,(\nu_i)}(s)\prod_{1\le j\le d, j\neq i}\int_0^{t_j} p^{+(\nu_j)}(s,t_j,r)\ \mathrm{d}r.
\end{aligned}$$

□

We will use the shorthand $(-D_{mix}^{(\underline{\nu})}, \lambda, g, \phi)$ to mean the problem

$$\begin{aligned}
D_{mix}^{(\underline{\nu})}u(\mathbf{t}) &= -\lambda u(\mathbf{t}) + g(\mathbf{t}), \quad \text{in } \mathcal{O}, \\
u(\mathbf{t}) &= \phi_i(h_{i,0}(\mathbf{t})), \quad \text{in } \mathcal{O}_{i,0}.
\end{aligned}$$

For $t \in \mathcal{O}$ we denote by $B_i(t)$ the subset of $\mathcal{O}_i$ (cf. (3.3.1)) which is defined by

$$B_i(t) := \{r \in \mathcal{O}_i, \quad r_j \le h_{i,0}(t),\ j \neq i\}.$$

Theorem 3.3.2. *Let $\underline{\nu} = (\nu_1, \nu_2, \cdots, \nu_d)$ be a vector such that each ν_i is a function satisfying conditions (H0)-(H1). Suppose $\lambda > 0$ and $\phi_j \in C_{kill(\partial\mathcal{O}_j)}[\mathbb{R}_+^{d-1}]$ where $n+1 \le j \le d$.*

1. *If $g \in C[\mathcal{O}]$ satisfies $g(t_i, \cdot)|_{t_i=0} \equiv 0$ when $0 \le i \le n$ and $g(\cdot, t_j)|_{t_j=0} = \lambda\phi_j(\cdot)$ for $n+1 \le j \le d$, then the mixed problem $(-D_{mix}^{(\underline{\nu})}, \lambda, g, \phi)$ has a unique solution in the domain of the generator given by $u = \mathcal{R}_\lambda^{\mathbf{0}+(\underline{\nu})*}g$, the resolvent operator of the process $\mathbf{X}_{\mathbf{t}}^{\mathbf{0}+(\underline{\nu})*}$.*

2. *For any $g \in B[\mathcal{O}]$ the mixed linear problem $(-D_{mix}^{(\underline{\nu})}, \lambda, g, \phi)$ is well-posed in the generalized sense and the solution admits the stochastic representation*

$$u(\mathbf{t}) = \mathbb{E}\left[\int_0^{\tau_{\mathbf{0}}^{\mathbf{t},(\nu)}} e^{-\lambda s} g\left(\mathbf{X_t}^{\mathbf{0}+(\nu)*}(s)\right) \mathrm{d}s\right]$$

$$+\mathbb{E}\left[\sum_{j=n+1}^{d} e^{-\lambda \tau_0^{t_j,(\nu_j)}} \phi_j\left(h_{j,0}\left(X_{t_1}^{0+(\nu_1)}\left(\tau_0^{t_j,(\nu_j)}\right),\cdots,\right.\right.\right.$$

$$\left.\left.\left. X_{t_d}^{0+*(\nu_d)}\left(\tau_0^{t_j,(\nu_j)}\right)\right)\right)\mathbf{1}_{\{\tilde{\tau}=\tau_0^{t_j,(\nu_j)}\}}\right]$$

Moreover if each $\nu_i, i = 1, \cdots, d$ satisfies condition (L2)-(L3), then the solution has the further representation

$$u(\mathbf{t}) = \int_{\mathbf{0}}^{\mathbf{t}} g(\mathbf{t}-\mathbf{r}) \int_0^{\infty} e^{-\lambda s} \mathbf{p}^{+(\underline{\nu})}(s, \mathbf{t}, \mathbf{t}-\mathbf{r}) \ \mathrm{d}s \ \mathrm{d}\mathbf{r}$$

$$+\sum_{j=n+1}^{d} \int_{B_j(t)} \phi_j(h_{j,0}(\mathbf{t})-\mathbf{r}) \left(\int_0^{\infty} e^{-\lambda s} \mu_0^{t_j,(\nu_j)}(s) \prod_{\substack{i=1 \\ i\neq j}}^{d} p^{+(\nu_i)}(s, t_i, t_i - r_i) \ \mathrm{d}s\right) \mathrm{d}\mathbf{r} \tag{3.3.5}$$

Proof.

1. Since $-D_{mix}^{(\underline{\nu})}$ generates a Feller process, we can apply Theorem 1.1 from Dynkin (1965). Then if g is a continuous function on $\mathcal{O}$ such that $g(t_i,\cdot)|_{t_i=0} \equiv 0,\ 0 \le i \le n$ then the function $u(\mathbf{t}) = \mathcal{R}_{\lambda}^{\mathbf{0}+(\underline{\nu})*}\mathbf{g}(\mathbf{t})$ solves the mixed equation without any boundary conditions. Also

$$\begin{aligned} u(t_1,\cdots,t_n,0,\cdots,0) &= \mathcal{R}_{\lambda}^{\mathbf{0}+(\underline{\nu})*} g(t_1,\cdots,t_n,0,\cdots,0) \\ &= g(t_1,\cdots,t_n,0,\cdots,0)/\lambda, \end{aligned}$$

implies that, under the condition $g(\cdot,t_j)|_{t_j=0} = \lambda\phi_j(\cdot)$ for $n+1 \le j \le d$, the function u solves the mixed problem.

2. For the generalized solution, take a function $g \in B[\mathcal{O}]$, and a function ψ in the domain of $-D_{mix}^{(\underline{\nu})}$ satisfying $\psi(t_i,\cdot)|_{t_i=0} = 0$, for $1 \le i \le n$, and $\psi(\cdot,t_j)|_{t_j=0} = \phi_j(\cdot)$, for $n+1 \le j \le d$. Then set $w := u - \psi$, and since w vanishes on the boundary $\partial\mathcal{O}$,

$$w(t) = \mathbb{E}\left[\int_0^{\tau_0^{\mathbf{t},(\underline{\nu})}} e^{-\lambda s}\tilde{g}\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)\right)\mathrm{d}s\right]$$

where $\tilde{g} = g - \lambda\psi - (D_{mix}^{(\underline{\nu})})\psi$. So this rewrites as

$$
\begin{aligned}
w(\mathbf{t}) :=& \mathbb{E}\left[\int_0^{\tau_0^{\mathbf{t},(\underline{\nu})}} e^{-\lambda s} g\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)\right) \mathrm{d}s\right] \\
&- \mathbb{E}\left[\int_0^{\tau_0^{\mathbf{t},(\underline{\nu})}} e^{-\lambda s}(\lambda + D_{mix}^{(\underline{\nu})})\psi\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)\right) \mathrm{d}s\right]
\end{aligned}
$$

Now Doob's stopping theorem applied to the martingale

$$
e^{-\lambda r}\psi\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(r)\right) + \int_0^r e^{-\lambda s}(\lambda + D_{mix}^{(\underline{\nu})}\psi\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)\right)\ \mathrm{d}s
$$

with the stopping time $\tau_{\mathbf{0}}^{\mathbf{t},(\underline{\nu})}$ implies that the second term is

$$
\psi(\mathbf{t}) - \mathbb{E}\left[\mathbf{e}^{-\lambda\tau_{\mathbf{0}}^{\mathbf{t},(\underline{\nu})}}\psi\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(\tau_{\mathbf{0}}^{\mathbf{t},(\underline{\nu})})\right)\right]
$$

Now using $u = w + \psi$, we have

$$
\begin{aligned}
u(\mathbf{t}) = w(\mathbf{t}) + \psi(\mathbf{t}) =& \mathbb{E}\left[\int_0^{\tau_0^{\mathbf{t},(\underline{\nu})}} e^{-\lambda s} g\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)\right) \mathrm{d}s\right] \\
&+ \mathbb{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{\mathbf{t},(\underline{\nu})}}\psi\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(\tau_{\mathbf{0}}^{\mathbf{t},(\underline{\nu})})\right)\right].
\end{aligned}
$$

Finally using $\psi(\cdot, t_j)|_{t_j=0, n+1\le j\le d} = \phi_j(\cdot)$ and (3.3.3), we have

$$
\begin{aligned}
&\mathbb{E}\left[e^{-\lambda\tau_{\mathbf{0}}^{\mathbf{t},(\underline{\nu})}}\psi\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(\tau_{\mathbf{0}}^{\mathbf{t},(\underline{\nu})})\right)\right] \\
&\quad = \mathbb{E}\left[\sum_{j=n+1}^{d} e^{-\lambda\tau_0^{t_j,(\nu_j)}}\phi_j\left(h_{j,0}\left(X_{t_1}^{0+(\nu_1)}\left(\tau_0^{t_j,(\nu_j)}\right), \cdots,\right.\right.\right. \\
&\qquad\qquad \left.\left.\left. X_{t_d}^{0+*(\nu_d)}\left(\tau_0^{t_j,(\nu_j)}\right)\right)\right) \mathbf{1}_{\{\tilde{\tau}=\tau_0^{t_j,(\nu_j)}\}}\right].
\end{aligned}
$$

For the stochastic representation when (L2)-(L3) holds, we begin with the inhomogeneous term. Notice first that $\tau_0^{\mathbf{t},(\underline{\nu})}$ and $\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)$ are *not* independent, however we can rewrite as follows,

$$
\mathbb{E}\left[\int_0^{\tau_0^{\mathbf{t},(\underline{\nu})}} e^{-\lambda s} g\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)\right) \mathrm{d}s\right]
$$

$$
\begin{aligned}
&= \mathbb{E}\left[\int_0^\infty e^{-\lambda s} g\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)\right)\mathbf{1}_{\{s<\tau_0^{\mathbf{t},(\underline{\nu})}\}}\mathrm{d}s\right] \\
&= \mathbb{E}\left[\int_0^\infty e^{-\lambda s} g\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)\right)\mathbf{1}_{\{\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)>0\}}\mathrm{d}s\right].
\end{aligned}
$$

Now conditioning on $\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)$ and making a substitution we have

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^\infty e^{-\lambda s} g\left(\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)\right)\mathbf{1}_{\{\mathbf{X_t}^{\mathbf{0}+(\underline{\nu})*}(s)>0\}}\mathrm{d}s\right] \\
&\qquad = \int_0^\infty e^{-\lambda s}\int_0^{\mathbf{t}} g(\mathbf{r})\mathbf{1}_{\{\mathbf{r}>0\}}\mathbf{p}^{+(\underline{\nu})}(s,\mathbf{t},\mathbf{r})\ \mathrm{d}\mathbf{r}\mathrm{d}s \\
&\qquad\qquad = \int_0^{\mathbf{t}} g(\mathbf{t}-\mathbf{r})\int_0^\infty e^{-\lambda s}\mathbf{p}^{+(\underline{\nu})}(s,\mathbf{t},\mathbf{t}-\mathbf{r})\ \mathrm{d}s\mathrm{d}\mathbf{r}.
\end{aligned}
$$

Turning to the homogeneous term, let us focus only on the first term of the summation. Recall that the projection $h_{n+1,0}$ removes the coordinate which is zero. The first term of the sum corresponds to the process $X_{t_{n+1}}^{0+(\nu_{n+1})*}(s)$ (i.e, the $(n+1)$-th coordinate) being the first process to hit the boundary. So as to not make the notation too cumbersome, let us assume that $n+1=1$, which corresponds to having only CD-type derivatives.

$$
\mathbb{E}\left[e^{-\lambda\tau_0^{t_1,(\nu_1)}}\phi_1\left(X_{t_2}^{0+*(\nu_2)}\left(\tau_0^{t_1,(\nu_1)}\right),\cdots,X_{t_d}^{0+*(\nu_d)}\left(\tau_0^{t_1,(\nu_1)}\right)\right)\mathbf{1}_{\{\tilde\tau=\tau_0^{t_1,(\nu_1)}\}}\right].
$$

Note that the event $\{\tilde\tau=\tau_0^{t_1,(\nu_1)}\}$ is equivalent to the event

$$
\left\{\min_{2\le j\le d}\tau_0^{t_j,(\nu_j)}>\tau_0^{t_1,(\nu_1)}\right\}.
$$

Thus we can first condition on $\tau_0^{t_1,(\nu_1)}$, whose density is denoted by $\mu_0^{t_1,(\nu_1)}(s)$, to get

$$
\int_0^\infty e^{-\lambda s}\mathbb{E}\left[\phi_1\left(X_{t_2}^{0+*(\nu_2)}(s),\cdots,X_{t_d}^{0+*(\nu_d)}(s)\right)\mathbf{1}_A\right]\mu_0^{t_1,(\nu_1)}(s)\ \mathrm{d}s, \quad (3.3.6)
$$

where $A=\left\{\min_{2\le j\le d}\tau_0^{t_j,(\nu_j)}>s\right\}$. Now since the event A is equivalent to

$$\{X_{t_j}^{0+*(\nu_j)}(s) > 0,\ 2 \leq j \leq d\},$$

we now condition on each $X_{t_j}^{0+*(\nu_j)}$. Due to their independence, the transition density is given by

$$\prod_{j=2}^{d} p^{(\nu_j)}(s, t_j, r_j).$$

Thus after a transformation of $r_i \mapsto t_i - r_i$ and rearranging, (3.3.6) is equal to

$$\int_0^\infty e^{-\lambda s} \int_0^{t_2} \cdots \int_0^{t_d} \phi_1(r_2, \cdots, r_d) \mu_0^{t_1,(\nu_1)}(s) \prod_{j=2}^{d} p^{(\nu_j)}(s, t_j, r_j)\, \mathrm{d}r_2 \cdots \mathrm{d}r_d \mathrm{d}s$$

$$= \int_{B_1(t)} \phi_1\left(h_{1,0}(\mathbf{t}) - \mathbf{r}\right) \int_0^\infty e^{-\lambda s} \mu_0^{t_1,(\nu_1)}(s) \prod_{j=2}^{d} p^{(\nu_j)}(s, t_j, t_j - r_j)\, \mathrm{d}s \mathrm{d}\mathbf{r},$$

which is precisely the first term of the sum appearing in the homogeneous term in (3.3.5). The other terms of the sum are obtained in the same way.

□

3.4 Mixed Linear equations: stable case

Let us specialise the results of the previous section to the case of stable processes, since we focus on this case in Chapter 5. That is, in the set up of (3.3.2), let $n = 0$, $d = 2$ and

$$\nu_1(x, y) = \nu_1(y) = y^{-1-\beta}/(-\Gamma(-\beta)),$$

$$\nu_2(x, y) = \nu_2(y) = y^{-1-\gamma}/(-\Gamma(-\gamma)),$$

where $\beta, \gamma \in (0, 1)$. Then the operator $D_{mix}^{\nu_1,\nu_2}$ is the sum of standard CD derivatives $D_{0+*}^{(\beta,\gamma)} := {}_{t_1}D_{0+*}^{\beta} + {}_{t_2}D_{0+*}^{\gamma}$. The operator $-D_{0+*}^{(\beta,\gamma)}$ generates an $\mathcal{O}^2$-valued Feller process given by

$$X_{\mathbf{t}}^{(\beta,\gamma)}(s) := (X_{t_1}^{\beta}(s), X_{t_2}^{\gamma}(s)),$$

where each coordinate is a decreasing stable process, absorbed at 0 on an attempt to cross it. The transition density of the process on the orthant is given by

$$p_s^{(\beta,\gamma)}(\mathbf{t},\mathbf{r}) = p_s^{\beta}(t_1,r_1)p_s^{\gamma}(t_2,r_2).$$

Let $\tau_0^{\mathbf{t}}$ denote the first time process $X_{\mathbf{t}}^{\beta,\gamma}$ hits ∂O,

$$\tau_0^{\mathbf{t}} := \inf\{s > 0\ ,\ X_{\mathbf{t}}^{\beta,\gamma}(s) \notin (0,\infty)\times(0,\infty)\}.$$

Using (3.3.3), this exit time is

$$\tau_0^{\mathbf{t}} = \tau_0^{t_1,\beta} \wedge \tau_0^{t_2,\gamma},$$

where $\tau_0^{t_1,\beta}$ and $\tau_0^{t_2,\gamma}$ are the exit times of X^β and X^γ from $(0,\infty)$. Plugging in the density (2.3.13) of these exit times into (3.3.4), $\tau_0^{\mathbf{t}}$ then has a density given by

$$\begin{aligned}
\mu_0^{\beta,\gamma}(s) &= \mu_0^{\beta}(s)\int_0^{t_1} p_s^{\beta}(t_1,r)\ \mathrm{d}r + \mu_0^{\gamma}(s)\int_0^{t_2} p_s^{\gamma}(t_2,r)\ \mathrm{d}r\\
&= \frac{t_1}{\beta}s^{-1-\frac{1}{\beta}}w_\beta(t_1 s^{-\frac{1}{\beta}})\int_0^{t_2} s^{-\frac{1}{\gamma}}w_\gamma(rs^{-\frac{1}{\gamma}})\mathrm{d}r\\
&\quad + \frac{t_2}{\gamma}s^{-1-\frac{1}{\gamma}}w_\gamma(t_2 s^{-\frac{1}{\gamma}})\int_0^{t_1} s^{-\frac{1}{\beta}}w_\beta(rs^{-\frac{1}{\beta}})\mathrm{d}r
\end{aligned}$$

Finally consider the following problem on the orthant, for $g \in C[\mathcal{O}]$ such that $g(t_1,0) = g(0,t_2) = 0$

$$(-\,{}_{t_1}D_{0+*}^{\beta} - {}_{t_2}D_{0+*}^{\gamma})u(t_1,t_2) = -\lambda u(t_1,t_2) + g(t_1,t_2), \quad t_1,t_2 > 0, \lambda > 0,$$

$$u(0,t_2) = \phi_1(t_2), \quad u(t_1,0) = \phi_2(t_1).$$

Using Theorem 3.3.2, the solution u has the following stochastic representation,

$$\begin{aligned}
u(\mathbf{t}) = &\int_0^{t_2} \phi_1(t_2-r_2)\left(\int_0^\infty e^{-\lambda s}\mu_0^{\beta}(s)p_s^{\gamma}(t_2-r_2)\ \mathrm{d}s\right)\ \mathrm{d}r_2\\
&+ \int_0^{t_1} \phi_2(t_1-r_1)\left(\int_0^\infty e^{-\lambda s}\mu_0^{\gamma}(s)p_s^{\beta}(t_1-r_1)\ \mathrm{d}s\right)\ \mathrm{d}r_1\\
&+ \int_0^{t_1}\int_0^{t_2} g(\mathbf{t}-\mathbf{r})\left(\int_0^\infty e^{-\lambda s}p_s^{\beta,\gamma}(\mathbf{t}-\mathbf{r})\ \mathrm{d}s\right)\mathrm{d}r_1\mathrm{d}r_2
\end{aligned}$$

$$
\begin{aligned}
= & \int_0^{t_2} \phi_1(t_2 - r_2) \left(\frac{t_1}{\beta} \int_0^\infty e^{-\lambda s} s^{-1-\frac{1}{\beta}-\frac{1}{\gamma}} w_\beta(t_1 s^{-\frac{1}{\beta}}) w_\gamma(r_2 s^{-\frac{1}{\gamma}}) \, \mathrm{d}s \right) \mathrm{d}r_2 \\
& + \int_0^{t_1} \phi_2(t_1 - r_1) \left(\frac{t_2}{\gamma} \int_0^\infty e^{-\lambda s} s^{-1-\frac{1}{\beta}-\frac{1}{\gamma}} w_\gamma(t_2 s^{-\frac{1}{\gamma}}) w_\beta(r_1 s^{-\frac{1}{\beta}}) \, \mathrm{d}s \right) \mathrm{d}r_1 \\
& + \int_0^{t_1} \int_0^{t_2} g(t_1 - r_1, t_2 - r_2) \left(\int_0^\infty e^{-\lambda s} s^{-\frac{1}{\beta}-\frac{1}{\gamma}} w_\beta(r_1 s^{-\frac{1}{\beta}}) w_\gamma(r_2 s^{-\frac{1}{\gamma}}) \mathrm{d}s \right) \mathrm{d}r_1 \mathrm{d}r_2.
\end{aligned}
$$

We return to a related problem in Chapter 5, with vanishing g and the scalar λ replaced with the generator of a Feller semigroup.

3.5 Two-sided fractional derivatives on the band in $\mathbb{R}^d$

The ideas of interrupting and killing processes to define generalised fractional derivatives extends naturally to higher dimension. Compared to the last two sections, where we focused on monotone processes with independent coordinates on the orthant, here we outline the case of a general Feller process in $\mathbb{R}^d$ and focusing in particular on the domain

$$
B = \{x \in \mathbb{R}^d, \ x_1 \in (a,b), x_2 \in \mathbb{R}^{d-1}\}. \tag{3.5.1}
$$

Consider the following operator acting on smooth bounded functions f,

$$
L_\nu f(x) = \int_{\mathbb{R}^d} (f(x+y) - f(x)) \nu(x,y) dy,
$$

where $\nu(\cdot,\cdot) : \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to \mathbb{R}_+$ is a Lévy kernel, i.e a function satisfying

$$
\sup_x \int_{\mathbb{R}^d} \min(1, |y|) \nu(x,y) dy < \infty.
$$

Operators of the form L_ν generate Feller processes X_x^{free} on $\mathbb{R}^d$. The analogue of RL derivative arising from a process X_x^{free} in $\mathbb{R}^d$ and domain $D \subset \mathbb{R}^d$ is the generator of the process killed upon leaving D. The case of the CD-type derivative is more delicate. We need to specify a point where a process jumps across a boundary. A natural method is to assume that a trajectory of a jump follows shortest path (i.e, in the Euclidean case $\mathbb{R}^d$ just a straight line). In the case $d = 1$, there is only one way to specify where the process crosses the boundary, so that the two-sided CD-type operator $D^{(\nu)}_{[a,b]*}$ is defined for $f \in C^1([a,b])$ by

$$D^{(\nu)}_{[a,b]*}f(x) = \int_{a-x}^{b-x} (f(x+y) - f(x))\,\nu(x,y)dy + (f(a) - f(x))\int_{-\infty}^{a-x} \nu(x,y)dy \\ + (f(b) - f(x))\int_{b-x}^{\infty} \nu(x,y)dy,$$

where $a < 0 < b$. Note that this operator coincides with $L_{(\nu)}$ when $a = -\infty$ and $b = \infty$.

For $d > 1$, the corresponding operator on the band (3.5.1) is given by restricting the operator to functions that are constant (in the first variable) outside of the interval (a, b) in the first variable:

$$\begin{aligned} D^{(\nu)}_{B*}f(x) = &\int_{\mathbb{R}^{d-1}} \int_{a-x_1}^{b-x_1} (f(x+y) - f(x))\,\nu(x,y)\ \mathrm{d}y_1\mathrm{d}y_2 \\ &+ \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{a-x_1} \left(f\left(a, x_2 + \frac{a-x_1}{y_1} \cdot y_2 \right) - f(x) \right) \nu(x,y)\ \mathrm{d}y_1\mathrm{d}y_2 \\ &+ \int_{\mathbb{R}^{d-1}} \int_{b-x_1}^{\infty} \left(f\left(b, x_2 + \frac{b-x_1}{y_1} \cdot y_2 \right) - f(x) \right) \nu(x,y)\ \mathrm{d}y_1\mathrm{d}y_2, \end{aligned}$$

which is the multidimensional extension of the CD operator on the band in $\mathbb{R}^d$. See Figure 3.1 for an illustration of the 'interruption' procedure.

The extension of the RL derivative on the band in $\mathbb{R}^d$ is given by

$$D^{(\nu)}_{B}f(x) = \int_{\mathbb{R}^{d-1}} \int_{a-x_1}^{b-x_1} (f(x+y) - f(x))\nu(x,y)\ \mathrm{d}y_1\mathrm{d}y_2 - K_{a,b}f(x),$$

where $-K_{a,b}$ is the operator of multiplication by $-k_{a,b}(x)$, defined as

$$k_{a,b}(x) := \left(\int_{\mathbb{R}^{d-1}} \int_{-\infty}^{a-x_1} + \int_{\mathbb{R}^{d-1}} \int_{b-x_1}^{\infty} \right) \nu(x,y)\ \mathrm{d}y_1\mathrm{d}y_2.$$

Remark 6. *Let us make some remarks about the well-posedness of these generalised derivatives. In the case $d = 1$ (i.e, when $B = [a,b]$), it is shown in Kolokoltsov (2019b, Theorem 5.1) that $-D^{(\nu)}_{[a,b]}$ generates a Feller semigroup in $C_{kill\{a,b\}}([a,b])$ and a bounded semigroup in $\{f \in C_{kill\{a,b\}}([a,b]) : f' \in C_{kill\{a,b\}}([a,b])\}$. The main idea is to take a bounded approximation of the integral over B in $D^{(\nu)}_B$ and this approximated operator is just $-K_{a,b}$ perturbed by*

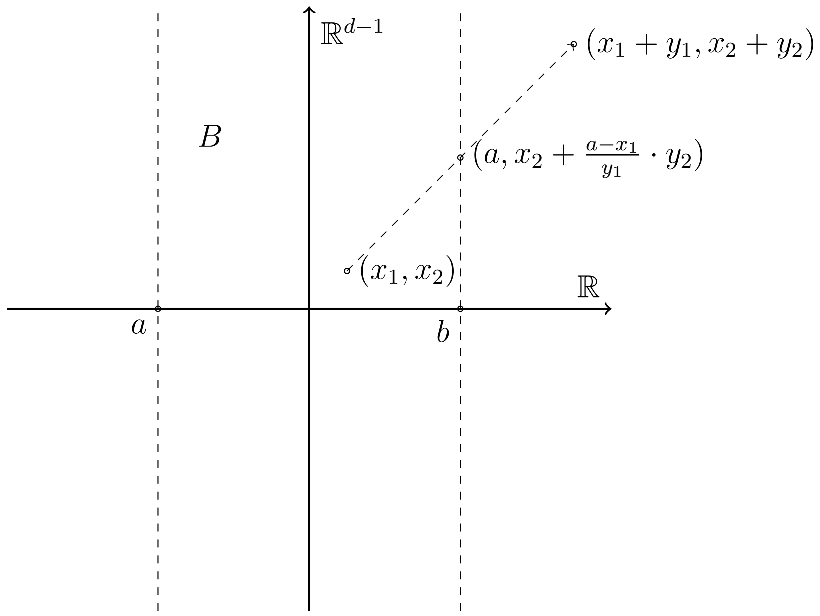

Figure 3.1: Illustration of interruption procedure on the band in $\mathbb{R}^d$. A process which tries to jump from $(x_1, x_2) \in B$ to $(x_1 + y_1, x_2 + y_2) \notin \overline{B}$ gets placed at the point where the boundary intersects with the straight line between (x_1, x_2) and $(x_1 + y_1, x_2 + y_2)$.

a bounded operator which, by standard perturbation theory arguments, generates a family of bounded semigroups. Perturbation theory also provides a series representation of the approximated semigroup which allows one to deduce the regularity required to show that the approximation semigroup converges to a Feller semigroup whose generator is $D_B^{(\nu)}$. Further, Kolokoltsov (2019b, Theorem 5.2) shows that the CD-type derivative $D_{[a,b]*}^{(\nu)}$ generates a Feller semigroup in $C([a,b])$ and a strongly continuous semigroup in $\{f \in C^1([a,b]) : \ f'(a) = f'(b) = 0\}$. In Kolokoltsov (2015, Theorem 4.4), it is proven that the CD-type derivative $D_{B*}^{(\nu)}$ generates a Feller process on B and a Feller semigroup on $C_\infty(B)$ with invariant core $C_\infty^1(B)$. When looking at boundary value problems involving CD-type operators (with non-zero boundary conditions), one can either work directly with the resolvent of the semigroup generated by $D_*^{(\nu)}$ (which requires first proving that it generates a semigroup), or alternatively one can first shift the unknown function to obtain the equivalent boundary value problem involving RL-type derivatives with zero boundary values.

Next we consider an important property of the interrupted process $X_x^{(\nu)*}(s)$ generated by $D_{B*}^{(\nu)}$, which is the regularity of the boundary ∂B. Recall that a point $x_0 \in \partial B$ is regular if $\tau_B(x) \to 0$ in probability as $B \ni x \to x_0$, where $\tau_B(x)$ is first time the process $(X_x(t))_{t\geq 0}$ enters ∂B defined by

$$\tau_B(x) := \inf\{s > 0 : X_x^{(\nu)*}(s) \in \partial B\}.$$

In order to prove the regularity of the boundary ∂B, we need some additional assumptions of the behaviour of the jump kernel $\nu(\cdot, y)$ close to the boundary.

Assumption 3.5.1. There exists a constant $C > 0$ and $q \in (0, 1)$ such that

$$\int_{\mathbb{R}^{d-1}} \int_{-\infty}^{0} \min(|y_1|, \epsilon)\nu(a, x_2; y_1, y_2)dy_1dy_2 > C\epsilon^q$$

and

$$\int_{\mathbb{R}^{d-1}} \int_{0}^{\infty} \min(y_1, \epsilon)\nu(b, x_2; y_1, y_2)dy_1dy_2 > C\epsilon^q,$$

for all $x_2 \in \mathbb{R}^{d-1}$.

Proposition 3.5.2. The set ∂B is regular in expectation for $D_{B*}^{(\nu)}$. Further, $\tau_B(x)$ for $x \in B$ has finite expectation.

Proof. We use the method of Lyapunov functions (see Kolokoltsov, 2011, Proposition 6.3.1). For this, we need to find a function f from the domain of the generator which is strictly positive in the interior of B, zero on the boundary of B and for which $D_{B*}^{(\nu)} f(x) \leq -c < 0$. Let us deal with the ∂B_a (the part of the boundary in which the first coordinate is $x_1 = a$), and take as the Lyapunov function $f_\omega(x_1, x_2) = (x_1 - a)^\omega$, where $\omega \in (0, 1)$. Then clearly for $x_0 \in \partial B_a$, $f_\omega(x_0) = f_\omega(a, x_2) = 0$, and for $x \in \overline{B} \setminus \{x_0\}$, $f(x) > 0$. Applying $D_{B*}^{(\nu)}$ to the Lyapunov function f_ω,

$$\begin{aligned}
-D_{B*}^{(\nu)} f(x) = &\int_{\mathbb{R}^{d-1}} \int_{a-x_1}^{b-x_1} f(x+y) - f(x)\nu(x; y)dy_1dy_2 + \\
&+ \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{a-x_1} f\left(a, x_2 + \frac{a - x_1}{y_1} y_2\right) - f(x)\nu(x; y)dy_1dy_2 \\
&+ \int_{\mathbb{R}^{d-1}} \int_{b-x_1}^{\infty} f\left(b, x_2 + \frac{b - x_1}{y_1} y_2\right) - f(x)\nu(x; y)dy_1dy_2 \\
&= \int_{\mathbb{R}^{d-1}} \int_{a-x_1}^{b-x_1} (x_1 + y_1 - a)^\omega - (x_1 - a)^\omega \nu(x; y)dy_1dy_2 +
\end{aligned}$$

$$
\begin{aligned}
&- \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{a-x_1} (x_1 - a)^\omega \nu(x;y) dy_1 dy_2 \\
&+ \int_{\mathbb{R}^{d-1}} \int_{b-x_1}^{\infty} (b-a)^\omega - (x_1 - a)^\omega \nu(x;y) dy_1 dy_2 \\
&\leq -C, \quad x_1 \to a,
\end{aligned}
$$

due to Assumption 3.5.1 on the behaviour of ν close to the boundary. Thus the boundary ∂B_a is regular in expectation. The regularity of the other boundary point works in the same way with the Lyapunov function $f_\omega(x_1, x_2) = (b - x_1)^\omega$. □

Example 3.5.3. *Consider the CD derivative on the band in $\mathbb{R}^d$, with the Lyapunov function $f(x_1, x_2) = (x_1 - a)^\omega$ for some $\omega \in (0, 1)$. Then*

$$
\begin{aligned}
-D^\beta_{B*} f(x) = & \int_{\mathbb{R}^{d-1}} \int_{a-x_1}^{b-x_1} \left[(x_1 + y_1 - a)^\omega - (x_1 - a)^\omega\right] \nu(x;y) dy_1 dy_2 \\
& - \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{a-x_1} (x_1 - a)^\omega \nu(x;y) dy_1 dy_2 \\
& + \int_{\mathbb{R}^{d-1}} \int_{b-x_1}^{\infty} \left[(b-a)^\omega - (x_1 - a)^\omega\right] \nu(x;y) dy_1 dy_2 \\
= & \int_{\mathbb{R}^{d-1}} \int_{a-x_1}^{b-x_1} \left[(x_1 + y_1 - a)^\omega - (x_1 - a)^\omega\right] \frac{dy_1 dy_2}{|y|^{d+\beta}} \\
& - \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{a-x_1} (x_1 - a)^\omega \frac{dy_1 dy_2}{|y|^{d+\beta}} \\
& + \int_{\mathbb{R}^{d-1}} \int_{b-x_1}^{\infty} \left[(b-a)^\omega - (x_1 - a)^\omega\right] \nu(x;y) \frac{dy_1 dy_2}{|y|^{d+\beta}} \\
= & I_1 + I_2 + I_3.
\end{aligned} \tag{3.5.2}
$$

The first term I_1 in (3.5.2) splits into two parts $I_1 = I_{1,+} + I_{1,-}$ which are the positive jumps towards the boundary point b in the first coordinate, and the negative jumps towards the boundary point a in the first coordinate. Considering the negative jumps first:

$$
I_{1,-} := \int_{\mathbb{R}^{d-1}} \int_{a-x_1}^{0} \left[(x_1 + y_1 - a)^\omega - (x_1 - a)^\omega\right] \frac{d\mathbf{y}}{|y|^{d+\beta}}
$$

$$
I_{1,-} = \int_{\mathbb{R}^{d-1}} \int_{-1}^{0} \left[(y_1(x_1 - a) - (a - x_1))^\omega - (x_1 - a)^\omega\right] \frac{d\mathbf{y}(x_1 - a)}{(y_1^2 (x_1 - a)^2 + \sum_{i=2}^d y_i^2)^{\frac{1}{2}(d+\beta)}}
$$

$$= \int_{\mathbb{R}^{d-1}} \int_0^1 [(1-y)^\omega - 1] \frac{d\mathbf{y}(x_1-a)^{1+\omega}}{(x_1-a)^{d+\beta}(y_1^2 + \sum_{i=2}^d y_i^2 (x_1-a)^{-2})^{\frac{1}{2}(d+\beta)}}$$

Making the substitution $y_i \mapsto \tilde{y}_i(x_1-a)$, we have

$$\begin{aligned} I_{1,-} =& \int_{\mathbb{R}^{d-1}} \int_0^1 [(1-y_1)^\omega - 1] \frac{dy_1 d\tilde{\mathbf{y}}(x_1-a)^{1+\omega-d-\beta+d-1}}{(y_1^2 + \sum_{i=2}^d \tilde{y}_i^2)^{\frac{1}{2}(d+\beta)}} \\ =& (x_1-a)^{\omega-\beta} \int_{\mathbb{R}^{d-1}} \int_0^1 [(1-y_1)^\omega - 1] \frac{dy_1 dy_2}{|y|^{d+\beta}}, \end{aligned}$$

which approaches 0 *from below as $x_1 \to 0$. The term containing the positive jumps between* 0 *and b and the jumps above b are dealt with similarly. The integral I_2 is clearly negative as $x_1 \to a$.*

3.6 Two-sided equations on the band

Now we will consider some problems on the band $B \subset \mathbb{R}^d$ involving the CD and RL-type operators that we have described in the previous section,[3]

$$\begin{aligned} D_{B*}^{(\nu)} u(x) &= -\lambda u(x) + g(x), \quad x \in B \\ u(a, x_2) = u_a(x_2), &\quad u(b, x_2) = u_a(x_2), \end{aligned} \tag{3.6.1}$$

where $\lambda \geq 0$, $u_a(\cdot), u_b(\cdot) \in C(\mathbb{R}^d)$ and g is some given function on B. We refer to these problems with the short hand notation $(-D_{B*}^{(\nu)}, \lambda, g, u_a(\cdot), u_b(\cdot))$. In order to solve problems involving $D_{B*}^{(\nu)}$ we shift the unknown function and look at the equivalent zero-boundary value problem of the form $(-D_B^{(\nu)}, \lambda, \tilde{g}, 0, 0)$, which is the corresponding RL-type problem. Let us consider some notion of solutions to the two sided RL-type problem,

$$\begin{aligned} D_B^{(\nu)} u(x) &= -\lambda u(x) + g(x), \quad x \in B \\ u(a, x_2) = & \ u(b, x_2) = 0, \end{aligned} \tag{3.6.2}$$

for $\lambda \geq 0$. We denote by $\mathcal{D}_B^{kill}$ and $\mathcal{D}_B^{stop}$ the domain of the generators $D_B^{(\nu)}$ and $D_{B*}^{(\nu)}$ respectively.

[3]For full details in the case of two-sided operators on the interval $B = [a, b]$, see Hernández-Hernández and Kolokoltsov (2016). The proofs work largely in the same way in the setting of the band, so we only sketch them here.

Definition 3.6.1. Let $g \in B[B]$ and $\lambda \geq 0$. A function $u \in C_{kill\partial B}[B]$ is said to solve the linear equation of RL type $(-D_B^{(\nu)}, \lambda, g, 0, 0)$ as:

- a *solution in the domain of the generator* if u is a solution belonging to $\mathcal{D}_B^{kill}$;
- a *generalized solution* if for all sequences of functions $g_n \in C_{kill\partial B}[B]$ such that $\sup_n \|g_n\| < \infty$ and $\lim_{n\to\infty} g_n \to g$ a.e., it holds that $u(x) = \lim_{n\to\infty} w_n(x)$ for all $x \in B$ where w_n is the unique solution (in the domain of the generator) to the RL type problem $(-D_B^{(\nu)}, \lambda, g_n, 0, 0)$.

Since $D_B^{(\nu)}$ generates a Feller process $X_x^{(\nu)}(s)$ which is killed upon crossing ∂B, we can use Dynkin's formula to obtain a unique solution in the domain of the generator to the RL-type problem (3.6.2). Recall that Dynkin's formula says that for a function f from the domain of a generator L of a Feller process $X_x(s)$,

$$f(x) = \mathbb{E}[f(X_x(\tau))e^{-\lambda\tau}] + \mathbb{E}\int_0^{\tau} e^{-\lambda s}(\lambda - L)f(X_x(s))\, \mathrm{d}s,$$

where τ is a stopping time with finite expectation. Thus if u is a solution to (3.6.2) in the domain of the generator, then recalling that $\tau_B(x)$ has finite expectation and $X_x^{(\nu)}(\tau_B(x)) = 0$ and $u(a, x_2) = u(b, x_2) = 0$, it can be written as

$$\begin{aligned} u(x) &= \mathbb{E}\left[\int_0^{\tau_B(x)} e^{-\lambda s}(\lambda - D_B^{(\nu)})u(X_x^{(\nu)}(s))\, \mathrm{d}s\right] \\ &= \mathbb{E}\left[\int_0^{\tau_B(x)} e^{-\lambda s} g(X_x^{(\nu)}(s))\, \mathrm{d}s\right]. \end{aligned} \tag{3.6.3}$$

Equations on the band involving CD-type operators

Take the CD-type operator on the band B given by

$$\begin{aligned} D_{B*}^{(\nu)} f(x) = &\int_{\mathbb{R}^{d-1}} \int_{a-x_1}^{b-x_1} [f(x+y) - f(x)]\, \nu(x;y) dy_1 dy_2 + \\ &+ \int_{\mathbb{R}^{d-1}} \int_{-\infty}^{a-x_1} \left[f\left(a, x_2 + \frac{a-x_1}{y_1} y_2\right) - f(x)\right] \nu(x;y) dy_1 dy_2 \\ &+ \int_{\mathbb{R}^{d-1}} \int_{b-x_1}^{\infty} \left[f\left(b, x_2 + \frac{b-x_1}{y_1} y_2\right) - f(x)\right] \nu(x;y) dy_1 dy_2. \end{aligned}$$

The CD-type equation is

$$D_{B*}^{(\nu)} u(x) = -\lambda u(x) + g(x), \quad x \in B$$

$$u(a, x_2) = u_a(x_2), \quad u(b, x_2) = u_b(x_2).$$

Let u be a function that solves the CD-type equation with the correct boundary values $u(a, x_2) = u_a(x_2)$, $u(b, x_2) = u_b(x_2)$. Now take any function $\phi \in \mathcal{D}_{D*}^{stop}$ that satisfies the correct boundary condition $\phi(a, x_2) = u_a(x_2)$ and $\phi(b, x_2) = u_b(x_2)$. For such a function we could take $\phi \in C^2[B]$ such that $\phi' \in C_0[B]$ with $(-D_{B*}^{(\nu)}\phi)(x_1, x_2) = 0$ for $x \in \partial B$ and $\phi(a, x_2) = u_a(x_2)$ and $\phi(b, x_2) = u_b(x_2)$. Now define $w(x_1, x_2) := u(x_1, x_2) - \phi(x_1, x_2)$ for $x \in B$. Then we have

$$D_B^{(\nu)} w(x) = D_{B*}^{(\nu)} w(x) = D_{B*}^{(\nu)} u(x) - D_{B*}^{(\nu)} \phi(x),$$

because w is a function that vanishes on the boundary ∂B. Thus

$$\begin{aligned} -D_B^{(\nu)} w(x_1, x_2) &= -\lambda u(x_1, x_2) + g(x_1, x_2) - D_{B*}^{(\nu)} \phi(x_1, x_2) \\ &= -\lambda w(x_1, x_2) - \lambda \phi(x_1, x_2) + g(x_1, x_2) - D_{B*}^{(\nu)} \phi(x_1, x_2). \end{aligned}$$

And so we have arrived at the RL type equation $(-D_B^{(\nu)}, \lambda, g - \lambda\phi - D_{B*}^{(\nu)}\phi, 0, 0)$ for the function w. Thus if this w solves the RL problem, then $u = w + \phi$ can be considered as a generalized solution to the CD type problem.

Definition 3.6.2. Let $g \in B[B]$ and $\lambda \geq 0$. A function $u \in C[B]$ is said to solve the CD-type equation as

1. A solution in the domain of the generator if u is a solution belonging to $\mathcal{D}_{D*}^{stop}$;

2. a generalized solution if u can be written as $u = \phi + w$, where w is the (possibly generalized) solution to the RL type problem

$$(-D_B^{(\nu)}, \lambda, g - D_{B*}^{(\nu)}\phi - \lambda\phi, 0, 0),$$

   with $\phi \in C^2[\overline{B}]$ satisfying that $\phi' \in C_0[B]$, $(-D_{B*}^{(\nu)}\phi)(x_1, x_2) = 0$ for $x_1 \in \{a, b\}$, $\phi(a, x_2) = u_a(x_2)$ and $\phi(b, x_2) = u_b(x_2)$.

Theorem 3.6.3. *If $u = w + \phi$ exist for the CD-type linear equation, with w, ϕ are as the above definition, then the solution u is unique and independent of ϕ.*

Now we can state the well-posedness result for equations involving CD-type operators.

Theorem 3.6.4. *Let $\lambda \geq 0$. Suppose that $-D_B^{(\nu)}$ generates a Feller process on B.*

1. *For any $g \in B[B]$, the two-sided equation of CD-type is well-posed in the generalized sense. The solution admits the stochastic representation*

$$\begin{aligned} u(x) = &\ \mathbb{E}\left[u_a(X_x^{(2)}(\tau_D(x)))e^{-\lambda\tau_D(x)}\mathbf{1}_{\{X_x(\tau_D(x))\in D_a^c\}}\right] \\ &+ \mathbb{E}\left[u_b(X_x^{(2)}(\tau_D(x)))e^{-\lambda\tau_D(x)}\mathbf{1}_{\{X_x(\tau_D(x))\in D_b^c\}}\right] \\ &+ \mathbb{E}\left[\int_0^{\tau_D(x)} e^{-\lambda t} g(X_x(t))dt\right]. \end{aligned} \tag{3.6.4}$$

2. *The solution to the Caputo-type equation depends continuously on the function g and on the boundary conditions $\{u_a(\cdot), u_b(\cdot)\}$*

Proof.

1. Already we know that $(-D_B^{(\nu)}, \mathcal{D}^{kill})$ generates a killed Feller process $X_x^{(\nu)}$ on B and this also ensures that $\tau_B(x)$ has finite expectation. Let us take any function $\phi \in C^2[B]$ satisfying the conditions of Definition 3.6.2. After recasting the CD problem as a RL-type one, we can use (3.6.3) to get that the generalized solution w to the RL-type problem

$$(-D_B^{(\nu)}, g - D_{B*}^{(\nu)}\phi - \lambda\phi, \lambda, 0, 0),$$

is given by $w = I - II$, where

$$I := \mathbb{E}\left[\int_0^{\tau_B(x)} e^{-\lambda t} g(X_x^{(\nu)}(t))\ \mathrm{d}s\right],$$

$$II := \mathbb{E}\left[\int_0^{\tau_B(x)} e^{-\lambda t}(\lambda + D_{B*}^{(\nu)})\phi(X_x^{(\nu)}(s))\ \mathrm{d}s\right].$$

Then by definition, $u = w + \phi$ is the generalized solution to the Caputo-type equation. Now we will making use of Dynkin's martingale again,

$$Y(r) := e^{-\lambda r}\phi\left(X_x^{(\nu)}(r)\right) + \int_0^r e^{-\lambda s}(\lambda + D_{B*}^{(\nu)})\phi(X_x^{(\nu)}(s))ds,$$

along with the stopping time $\tau_B(x)$. The idea is to note that $\mathbb{E}[Y(\tau_B(x))] = \mathbb{E}[Y(0)]$ (by Doob's stopping theorem), which gives us

$$\begin{aligned}\mathbb{E}\left[e^{-\lambda\tau_B(x)}\phi\left(X_x^{(\nu)}(\tau_B(x))\right)\right] + \int_0^{\tau_B(x)} e^{-\lambda s}(\lambda + D_{B*}^{(\nu)})\phi(X_x^{(\nu)}(s))ds \\ = \mathbb{E}\left[e^{-\lambda 0}\phi(X_x^{(\nu)}(0))\right] = \phi(x),\end{aligned}$$

since $X_x(0) = x$, and so we have

$$\begin{aligned}II &= \mathbb{E}\left[\int_0^{\tau_B(x)} e^{-\lambda s}(\lambda + D_{B*}^{(\nu)})\phi(X_x^{(\nu)}(s))ds\right] \\ &= \phi(x) - \mathbb{E}\left[e^{-\lambda\tau_B(x)}\phi\left(X_x^{(\nu)}(\tau_B(x))\right)\right].\end{aligned}$$

Now recall that $u = w + \phi$ and $w = I - II$, thus combining these expressions we get

$$\begin{aligned}u(x) &= \mathbb{E}\left[\int_0^{\tau_B(x)} e^{-\lambda t} g\left(X_x^{(\nu)}(t)\right)dt\right] \\ &\quad - \phi(x) + \mathbb{E}\left[e^{-\lambda\tau_B(x)}\phi\left(X_x^{(\nu)}(\tau_B(x))\right)\right] + \phi(x) \\ &= \mathbb{E}\left[\int_0^{\tau_B(x)} e^{-\lambda t} g\left(X_x^{(\nu)}(t)\right)dt\right] \\ &\quad + \mathbb{E}\left[e^{-\lambda\tau_B(x)}u\left(X_x^{(\nu)}(\tau_B(x))\right)\right],\end{aligned} \tag{3.6.5}$$

since by assumption, ϕ agrees with u on the boundary of B (i.e, at $X_x^{(\nu)}(\tau_B(x))$). Now since at the random time $\tau_B(x)$ the process $X_x^{(\nu)}$ takes values either along (a, x_2) or (b, x_2), the last term in (3.6.5) can be written as

$$\begin{aligned}\mathbb{E}\left[e^{-\lambda\tau_B(x)}u\left(X_x^{(\nu)}(\tau_B(x))\right)\right] = \mathbb{E}\left[u_a(X_x^{(2)}(\tau_B(x)))e^{-\lambda\tau_B(x)}\mathbf{1}_{\{X_x^{(\nu)}(\tau_B(x))\in D_a^c\}}\right] \\ + \mathbb{E}\left[u_b(X_x^{(2)}(\tau_B(x)))e^{-\lambda\tau_B(x)}\mathbf{1}_{\{X_x(\tau_B(x))\in D_b^c\}}\right],\end{aligned}$$

which yields the first result.

2. The continuous dependency follows from the stochastic representation and the estimate

$$\|u - u_n\| \leq (\|u_a\| + \|u_b\|) \sup_{x \in B} \mathbb{E}[e^{-\lambda \tau_B(x)}] + \|g - g_n\| \sup_{x \in B} \mathbb{E}[\tau_B(x)],$$

where u_n and g_n are as in Definition 3.6.1.

$\square$

Chapter 4

Two-sided estimates for Green's function of fractional evolution equations

The work in this chapter is adapted from the article Johnston and Kolokoltsov (2019a). The main aim is to obtain two-sided estimates for the Green's function of fractional evolution equations of the form

$$\begin{aligned} {}_tD^{\beta}_{0+*}u(t,x) &= L_x u(t,x), \\ u(0,x) &= Y(x). \end{aligned}$$

Under suitable conditions on the operator L_x, the solution to such equations is given by the *operator valued Mittag-Leffler* function

$$u(t,x) = E_\beta\left[t^\beta L\right]Y(x) = \int_0^\infty e^{sL}Y(x)\mu_0^\beta(s)\ \mathrm{d}s,$$

where μ_0^β is the density of $\tau_0^\beta := \inf\{s > 0 : X_t^\beta(s) \leq 0\}$, where $(X_t^\beta(s))_{s\geq 0}$ is the β-stable subordinator (with inverted direction) generated by $-D^\beta_{0+*}$. Denoting by $G_L(s,x,y)$ the transition densities of the process generated by L, the solution u can be written in the form

$$\begin{aligned} u(t,x) &= \int_{\mathbb{R}^d} Y(y)\left(\int_0^\infty G_L(s,x,y)\mu_0^\beta(s)\ \mathrm{d}s\right)\ \mathrm{d}y \\ &=: \int_{\mathbb{R}^d} Y(y)G_L^{(\beta)}(t,x,y)\ \mathrm{d}y \end{aligned}$$

This section is dedicated to obtaining two-sided estimates for the Green's function $G_L^{(\beta)}$ when L is the generator of the following $\mathbb{R}^d$-valued processes:

- Diffusion processes whose transition densities satisfy global Aronson estimates (2.3.5)
- Non-degenerate diffusion processes whose transition densities satisfy local Aronson estimates (2.3.2)
- Non-isotropic $\alpha \in (0, 2)$ stable processes whose transition densities satisfy the *global stable estimates* (2.3.7)
- Non-isotropic stable-like processes whose transition densities satisfy the *local stable-like estimates* (2.3.9)

4.1 Global Estimates

We first look at global in time two-sided estimates for $G_L^{(\beta)}$ in two special cases. Notice in (3.2.5) that the integral over the time variable z ranges from 0 to ∞, and so in order to perform any estimates on the term $G_L(z, x, y)$ one can only use estimates that hold for all $z \in (0, \infty)$. We begin with two such cases, when one has global in time estimates for G_L. Namely, when L is a second order uniformly elliptic operator in divergence form or when L is a homogeneous pseudo-differential operator (with constant coefficients).

4.1.1 Divergence Structure

In this section we consider the time-fractional diffusion equation given by

$$D_{0+*}^{\beta} u(t,x) = Lu(t,x) := \nabla \cdot (A(x)\nabla u(t,x)), \quad u(0,x) = Y(x), \qquad (4.1.1)$$

where D_{0+*}^{β} is the Caputo fractional derivative acting on the time variable, and the spatial operator is a second order elliptic operator in divergence form which was discussed in Section 2.3. Recall that the solution of (4.1.1) is given by

$$u(t,x) = E_\beta(Lt^\beta)Y(x),$$

and the associated Green's function is given by

$$G^{(\beta)}(t,x,y) = \frac{1}{\beta}\int_0^\infty G(t^\beta z,x,y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z, \tag{4.1.2}$$

where $G(t,x,y)$ is the Green's function associated with the second order elliptic operator in divergence form, (2.3.3). We have the following two-sided estimates for the Green's function $G^{(\beta)}$, which are global in time. In the following, we use the notation $\Omega := |x-y|^2 t^{-\beta}$.

Theorem 4.1.1. *Assume that the function $A(x)$ is measurable, symmetric and satisfies (2.3.4) for some $\mu \geq 1$. Then there exists a constant C such that for $(t,x,y) \in (0,\infty)\times\mathbb{R}^d\times\mathbb{R}^d$, the Green's function $G^{(\beta)}(t,x,y)$ for the time-fractional diffusion Equation (4.1.1) satisfies the following two-sided estimates,*

- *For $\Omega \leq 1$,*

$$G^{(\beta)}(t,x,y) \asymp C\begin{cases} t^{-\frac{\beta}{2}}, & d=1,\\ t^{-\beta}(|\log\Omega|+1), & d=2,\\ t^{-\frac{d\beta}{2}}\Omega^{1-\frac{d}{2}}, & d\geq 3.\end{cases} \tag{4.1.3}$$

- *For $\Omega \geq 1$,*

$$G^{(\beta)}(t,x,y) \asymp Ct^{-\frac{d\beta}{2}}\Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)}\exp\left\{-C_\beta \Omega^{\frac{1}{2-\beta}}\right\}. \tag{4.1.4}$$

Proof. Let us begin by using the asymptotic behaviour of the stable density w_β in (4.1.2),

$$\begin{aligned} G^{(\beta)}(t,x,y) &= \frac{1}{\beta}\int_0^\infty G(t^\beta z,x,y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z\\ &\asymp c_\beta\int_0^1 G(t^\beta z,x,y)\,\mathrm{d}z + c_\beta\int_1^\infty G(t^\beta z,x,y)z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z,\end{aligned}$$

where $f_\beta(x) = x^{-\frac{2-\beta}{2(1-\beta)}}\exp\{-c_\beta x^{-\frac{\beta}{1-\beta}}\}$. Next we apply Aronsons estimates (2.3.5) to $G(t^\beta z,x,y)$,

$$G^{(\beta)}(t,x,y) \asymp Ct^{-\frac{d\beta}{2}}\int_0^1 z^{-\frac{d}{2}}\exp\{-\Omega z^{-1}\}\,\mathrm{d}z$$

$$+ Ct^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1-\frac{1}{\beta}} \exp\{-\Omega z^{-1}\} f_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z,$$

where $\Omega := |x-y|^2 t^{-\beta}$. Making a change of variables $z = w^{-1}$ so that $\mathrm{d}z = -w^{-2}\mathrm{d}w$,

$$\begin{aligned} G^{(\beta)}(t,x,y) \asymp Ct^{-\frac{d\beta}{2}} \int_1^\infty w^{\frac{d}{2}-2} \exp\{-\Omega w\}\ \mathrm{d}w \\ + Ct^{-\frac{d\beta}{2}} \int_0^1 w^{\frac{d}{2}-1+\frac{1}{\beta}} \exp\{-\Omega w\} f_\beta(w^{\frac{1}{\beta}})\ \mathrm{d}w \\ =: I_1 + I_2. \end{aligned} \tag{4.1.5}$$

We now estimate I_1 and I_2 in two different cases, depending on the behaviour of Ω.

<u>**Case 1:** $\Omega \le 1$.</u> Making a further substitution of $V = \Omega w$ in the integral I_1 gives us the simpler form of

$$I_1 = Ct^{-\frac{d\beta}{2}} \Omega^{1-\frac{d}{2}} \int_\Omega^\infty V^{\frac{d}{2}-2} \exp\{-V\}\ \mathrm{d}V.$$

Now if $d = 1$, then we have the asymptotic behaviour

$$I_1 = t^{-\frac{\beta}{2}} \Omega^{\frac{1}{2}} \int_\Omega^\infty V^{-\frac{3}{2}} \exp\{-V\}\ \mathrm{d}V \sim t^{-\frac{\beta}{2}} \Omega^{\frac{1}{2}} \Omega^{-\frac{1}{2}} = Ct^{-\frac{\beta}{2}}, \quad \text{as } \Omega \to 0,$$

and in particular for $\Omega \le 1$ there exists a constant $C > 0$ such that

$$I_1 \asymp Ct^{-\frac{\beta}{2}}.$$

If $d = 2$, then we see logarithmic behaviour,

$$I_1 = Ct^{-\beta} \int_\Omega^\infty V^{-1} \exp\{-V\}\ \mathrm{d}V \sim t^{-\beta}(|\log \Omega| + 1), \quad \Omega \to 0, \tag{4.1.6}$$

and in particular for $\Omega \le 1$ there exists a constant $C > 0$ such that

$$I_1 \asymp Ct^{-\beta}(|\log \Omega| + 1).$$

If $d \ge 3$, then the integral is the so-called upper incomplete gamma function, and has the asymptotic behaviour

$$I_1 = Ct^{-\frac{d\beta}{2}}\Omega^{1-\frac{d}{2}} \int_\Omega^\infty V^{\frac{d}{2}-1}\exp\{-V\}\,\mathrm{d}V \sim Ct^{-\frac{d\beta}{2}}\Omega^{1-\frac{d}{2}}\Gamma\left(\frac{d}{2}-1\right), \qquad \text{as } \Omega \to 0,$$

and in particular for $\Omega \leq 1$ there exists a constant $C > 0$ such that the two-sided estimate

$$I_1 \asymp Ct^{-\frac{d\beta}{2}}\Omega^{1-\frac{d}{2}},$$

holds. Thus we have the following two-sided estimate for I_1,

$$I_1 \asymp C \begin{cases} t^{-\frac{\beta}{2}}, & d = 1, \\ t^{-\beta}(|\log \Omega| + 1), & d = 2, \\ t^{-\frac{d\beta}{2}}\Omega^{1-\frac{d}{2}}, & d \geq 3. \end{cases}$$

Turning to the integral I_2,

$$\begin{aligned} I_2 &= Ct^{-\frac{d\beta}{2}} \int_0^1 w^{-\frac{d}{2}-1+\frac{1}{\beta}}\exp\{-\Omega w\} f_\beta(w^{\frac{1}{\beta}})\,\mathrm{d}w \\ &= Ct^{-\frac{d\beta}{2}} \int_0^1 w^{-\frac{d}{2}-1-\frac{1}{2(1-\beta)}}\exp\{-\Omega w - c_\beta w^{-\frac{1}{1-\beta}}\}\,\mathrm{d}w \\ &\asymp C_{d,\beta}t^{-\frac{d\beta}{2}}, \end{aligned} \tag{4.1.7}$$

due to the fast decay of f_β in a neighbourhood of 0. Thus combining the estimates for I_1 and I_2 gives (4.1.3).

Case 2: $\Omega \geq 1$. In this case we use the Laplace method as described in Section 2.4. Firstly for I_1, using $g(w) = w^{\frac{d}{2}-1}$, $h(w) = w$ and $b = 1$ in (2.4.2) we have

$$I_1 = Ct^{-\frac{d\beta}{2}} \int_1^\infty w^{\frac{d}{2}-1}\exp\{-\Omega w\}\,\mathrm{d}w \sim t^{-\frac{d\beta}{2}}\Omega^{-1}\exp\{-\Omega\},$$

and in particular the estimate

$$I_1 \asymp Ct^{-\frac{d\beta}{2}}\Omega^{-1}\exp\{-\Omega\}, \quad \Omega \geq 1.$$

For the second integral, we use Proposition 2.4.1 with $N = \frac{d}{2} - 1 - \frac{1}{2(1-\beta)}$

and $a = \frac{1}{1-\beta}$,

$$
\begin{aligned}
I_2 &= Ct^{-\frac{d\beta}{2}} \int_0^1 w^{-\frac{d}{2}-1+\frac{1}{\beta}} \exp\{-\Omega w\} f_\beta(w^{\frac{1}{\beta}})\, \mathrm{d}w \\
&\sim Ct^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\},
\end{aligned}
$$

and again in particular, the two-sided estimate

$$
I_2 \asymp Ct^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\}.
$$

Combining the estimates for I_1 and I_2 shows (4.1.4), and we are done. $\square$

If one additionally assumes that the diffusion coefficients $A(x)$ of (2.3.3) are twice continuously differentiable, the following estimates hold for the spatial derivatives of the fundamental solution of (2.3.3),

$$
\left| \frac{\partial}{\partial x} G(t,x,y) \right| \leq Ct^{-\frac{d+1}{2}} \exp\left\{ -C\frac{|x-y|^2}{t} \right\}, \tag{4.1.8}
$$

for $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$. We next have estimates for the spatial derivative of $G^{(\beta)}(t,x,y)$.

Proposition 4.1.2. Under the same assumptions as Theorem 4.1.1, assume additionally that $A(x)$ is twice continuously differentiable, then the following estimates for the spatial derivatives of the Green's function $G^{(\beta)}(t,x,y)$ holds for all $(t,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

- For $\Omega \leq 1$,

$$
\left| \frac{\partial}{\partial x} G^{(\beta)}(t,x,y) \right| \leq C \begin{cases} t^{-\beta}(|\log \Omega| + 1), & d = 1, \\ t^{-\frac{(d+1)\beta}{2}} \Omega^{1-\frac{d+1}{2}}, & d \geq 2. \end{cases}
$$

- For $\Omega \geq 1$,

$$
\left| \frac{\partial}{\partial x} G^{(\beta)}(t,x,y) \right| \leq Ct^{-\frac{(d+1)\beta}{2}} \Omega^{-\frac{(d+1)}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\left\{ -C_\beta \Omega^{-\frac{1}{2-\beta}} \right\}.
$$

Proof. Recall that

$$G^{(\beta)}(t,x,y) = \frac{1}{\beta}\int_0^\infty G(t^\beta z,x,y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z,$$

where G satisfies the global estimate (4.1.8). Using the triangle inequality after taking the derivative inside the integral,

$$\begin{aligned}
\left|\frac{\partial}{\partial x}G^{(\beta)}(t,x,y)\right| &= \left|\frac{\partial}{\partial x}\frac{1}{\beta}\int_0^\infty G(t^\beta z,x,y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z.\right| \\
&\leq C\int_0^\infty \left|\frac{\partial}{\partial x}G(t^\beta z,x,y)\right| z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z \\
&\leq Ct^{-\frac{(d+1)\beta}{2}}\int_0^\infty z^{-\frac{(d+1)}{2}}\exp\{-\Omega z^{-1}\} z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z \\
&\leq C_\beta t^{-\frac{(d+1)\beta}{2}}\int_0^1 z^{-\frac{(d+1)}{2}}\exp\{-\Omega z^{-1}\}\ \mathrm{d}z \qquad (4.1.9) \\
&\quad + C_\beta t^{-\frac{(d+1)\beta}{2}}\int_1^\infty z^{-\frac{d+1}{2}}\exp\{-\Omega z^{-1}\} z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z \\
&= C_\beta t^{-\frac{(d+1)\beta}{2}}\int_1^\infty w^{\frac{d+1}{2}-2}\exp\{-\Omega w\}\ \mathrm{d}w \\
&\quad + C_\beta t^{-\frac{(d+1)\beta}{2}}\int_0^1 w^{\frac{d+1}{2}-1-\frac{1}{2(1-\beta)}}\exp\{-\Omega w - c_\beta w^{-\frac{1}{1-\beta}}\}\ \mathrm{d}w \\
&:= I_1 + I_2,
\end{aligned}$$

where in the above calculations, after using the estimates (4.1.8) and (2.3.12), we made the substitution $z = w^{-1}$. Note that the integrals I_1 and I_2 differ from those appearing in (4.1.5) only by replacing d with $d+1$. Thus the only change in the calculations is where the dimension dictates the behaviour of the estimate, namely in the integral I_1 under the regime $\Omega \leq 1$. In this case, make the substitution $w\Omega = V$,

$$I_1 = Ct^{-\frac{(d+1)\beta}{2}}\Omega^{1-\frac{d+1}{2}}\int_\Omega^\infty V^{\frac{d+1}{2}-2}\exp\{-V\}\ \mathrm{d}V.$$

For $d = 1$, we are in the same situation as (4.1.6), thus

$$I_1 \sim Ct^{-\beta}(|\log\Omega| + 1), \quad \Omega \to 0,$$

and in particular

$$I_1 \leq Ct^{-\beta}(|\log\Omega|+1), \qquad \text{for } \Omega \leq 1.$$

Otherwise for $d \geq 2$ we have

$$I_1 \leq Ct^{-\frac{(d+1)\beta}{2}}\Omega^{1-\frac{d+1}{2}}\Gamma\left(\frac{d+1}{2}-1\right) = C_d t^{-\frac{(d+1)\beta}{2}}\Omega^{1-\frac{d+1}{2}}.$$

For the integral I_2, replacing d with $d+1$ in (4.1.7) does not spoil the estimate, thus

$$I_2 \leq C_{d,\beta}t^{-\frac{(d+1)\beta}{2}}, \qquad \text{for } \Omega \leq 1.$$

This shows

$$\left|\frac{\partial}{\partial x}G^{(\beta)}(t,x,y)\right| \leq C\begin{cases} t^{-\beta}(|\log\Omega|+1), & d=1, \\ t^{-\frac{(d+1)\beta}{2}}\Omega^{1-\frac{d+1}{2}}, & d \geq 2. \end{cases}$$

for $\Omega \leq 1$ as required. For $\Omega \geq 1$, the estimates follow again by using the Laplace method. Namely taking $g(w) = w^{\frac{d+1}{2}-1}$, $h(w) = w$ and $b = 1$ in (2.4.2) we have

$$I_1 \leq t^{-\frac{(d+1)\beta}{2}}\Omega^{-1}\exp\{-\Omega\}, \qquad \text{for } \Omega \geq 1.$$

Finally using $N = \frac{d+1}{2} - 1 - \frac{1}{2(1-\beta)}$ and $a = \frac{1}{1-\beta}$ in Proposition 2.4.1 gives us

$$I_2 \leq Ct^{-\frac{(d+1)\beta}{2}}\Omega^{-\frac{d+1}{2}\left(\frac{1-\beta}{2-\beta}\right)}\exp\left\{-C\Omega^{\frac{1}{2-\beta}}\right\}, \qquad \text{for } \Omega \geq 1.$$

Thus

$$\begin{aligned}\left|\frac{\partial}{\partial x}G^{(\beta)}(t,x,y)\right| &\leq I_1 + I_2 \\ &\leq Ct^{-\frac{(d+1)\beta}{2}}\Omega^{-\frac{d+1}{2}\left(\frac{1-\beta}{2-\beta}\right)}\exp\{-C\Omega^{\frac{1}{2-\beta}}\},\end{aligned}$$

as required. □

4.1.2 Pseudo-differential Operators: Constant Coefficients

Next we turn our attention to another class of problems, where the spatial operator is a homogeneous (constant coefficient) pseudo-differential operator.

That is, for $\beta \in (0,1)$ and $\alpha > 0$,

$$D_0^\beta u(t,x) = -\Psi_\alpha(-i\nabla)u(t,x), \quad u(0,x) = Y(x), \tag{4.1.10}$$

where Ψ_α is a pseudo-differential operator whose symbol is of the form

$$\psi_\alpha(p) = |p|^\alpha S_\mu(p/|p|),$$

where S_μ is a positive function on $\mathbb{S}^{d-1}$, see (2.2.4). To this end, we use known properties of the Green's function $G_{\psi_\alpha}(t,x)$ associated with Ψ_α, namely that it satisfies the stable estimates (2.3.7)

As discussed in the introduction of this chapter, the solution of (4.1.10) is given by

$$u(t,x) = E_\beta(-\Psi_\alpha(-i\nabla)t^\beta)Y(x),$$

where

$$E_\beta(s) = \frac{1}{\beta}\int_0^\infty e^{sz} z^{-1-\frac{1}{\beta}} w_\beta(z^{-1/\beta})\, \mathrm{d}z.$$

Thus the corresponding Green's function $G_{\psi_\alpha}^{(\beta)}(t,x,y)$ of (4.1.10) is given by

$$G_{\psi_\alpha}^{(\beta)}(t,x,y) := \frac{1}{\beta}\int_0^\infty G_{\psi_\alpha}(t^\beta z, x-y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z.$$

In keeping with the previous section, we denote $\Omega := |x-y|^\alpha t^{-\beta}$. We have the following two-sided estimate for the Green's function $G_{\psi_\alpha}^{(\beta)}(t,x,y)$.

Theorem 4.1.3. *Let $\alpha \in (0,2)$ and $\beta \in (0,1)$. Assume that $w \in C^{(d+1+[\alpha])}(\mathbb{S}^{d-1})$, and that $S_\mu \geq S_0 > 0$ for some constant S_0. Further assume that the spectral measure μ of the stable operator Ψ_α has a strictly positive density. Then there exists a constant $C > 0$ such that the Green's function for the fractional evolution equation (4.1.10) satisfies the following two-sided estimates for $(t, x-y) \in (0,\infty) \times \mathbb{R}^d$.*

- *For $\Omega \leq 1$,*

$$G_{\psi_\alpha}^{(\beta)}(t,x-y) \asymp C \begin{cases} t^{-\frac{d\beta}{\alpha}}, & d < \alpha, \\ t^{-\beta}(|\log \Omega| + 1), & d = \alpha, \\ t^{-\frac{d\beta}{\alpha}} \Omega^{1-\frac{d}{\alpha}}, & d > \alpha. \end{cases} \tag{4.1.11}$$

- *For* $\Omega \geq 1$,

$$G^{(\beta)}_{\psi_\alpha}(t, x-y) \asymp Ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}. \tag{4.1.12}$$

Proof. We again begin by using the asymptotic behaviour of the stable density,

$$\begin{aligned} G^{(\beta)}_{\psi_\alpha}(t^\beta z, x-y) \asymp & c_\beta \int_0^1 G_{\psi_\alpha}(t, x-y)\ \mathrm{d}z \\ & + c_\beta \int_1^\infty G_{\psi_\alpha}(t^\beta z, x-y) z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z, \end{aligned}$$

where $f_\beta(z) = z^{-\frac{2-\beta}{2(1-\beta)}} \exp\{-cz^{-\frac{\beta}{1-\beta}}\}$. Before using the estimates (2.3.7) for G_{ψ_α} (with $t = t^\beta z$), note that using the notation $\Omega = |x-y|^\alpha t^{-\beta}$, we have

$$\min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} z, t^{-\frac{d\beta}{\alpha}} z^{-\frac{d}{\alpha}}\right) = \begin{cases} t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} z, & \text{for } z < \Omega, \\ t^{-\frac{d\beta}{\alpha}} z^{-\frac{d}{\alpha}}, & \text{for } z \geq \Omega. \end{cases} \tag{4.1.13}$$

Thus we have,

$$\begin{aligned} G^{(\beta)}_{\psi_\alpha}(t, x-y) \asymp c & \int_0^1 \min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} z, t^{-\frac{d\beta}{\alpha}} z^{-\frac{d}{\alpha}}\right)\ \mathrm{d}z \\ & + c\int_1^\infty \min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} z, t^{-\frac{d\beta}{\alpha}} z^{-\frac{d}{\alpha}}\right) z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z \\ & := I_1 + I_2. \end{aligned} \tag{4.1.14}$$

Now we deal with two cases.

<u>**Case 1: $\Omega \leq 1$.**</u> Using (4.1.13), in this case the integral I_1 equals

$$\begin{aligned} I_1 &= ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} \int_0^\Omega z\ \mathrm{d}z + ct^{-\frac{d\beta}{\alpha}} \int_\Omega^1 z^{-\frac{d}{\alpha}}\ \mathrm{d}z \\ &= \frac{c}{2} t^{-\frac{d\beta}{\alpha}}\Omega^{1-\frac{d}{\alpha}} + ct^{-\frac{d\beta}{\alpha}} \int_\Omega^1 z^{-\frac{d}{\alpha}}\ \mathrm{d}z. \end{aligned}$$

Note that for $d = \alpha$, the integral over the interval $(\Omega, 1)$ is

$$t^{-\beta}\int_\Omega^1 z^{-1}\ \mathrm{d}z = t^{-\beta}|\log\Omega|.$$

On the other hand, for $d \neq \alpha$ we have

$$\begin{aligned} t^{-\frac{d\beta}{\alpha}} \int_{\Omega}^{1} z^{-\frac{d}{\alpha}}\, dz &= \frac{1}{1-\frac{d}{\alpha}} t^{-\frac{d\beta}{\alpha}} (1 - \Omega^{1-\frac{d}{\alpha}}) \\ &\asymp C \begin{cases} t^{-\frac{d\beta}{\alpha}}, & d < \alpha, \\ t^{-\frac{d\beta}{\alpha}} \Omega^{1-\frac{d}{\alpha}}, & d > \alpha. \end{cases} \end{aligned}$$

Thus in this case we have,

$$I_1 \asymp C \begin{cases} t^{-\frac{d\beta}{\alpha}}, & d < \alpha, \\ t^{-\beta}(|\log \Omega| + 1), & d = \alpha, \\ t^{-\frac{d\beta}{\alpha}} \Omega^{1-\frac{d}{\alpha}}, & d > \alpha. \end{cases}$$

Turning to I_2, note that the integral does not involve Ω, and is convergent since $f_\beta(z^{-\frac{1}{\beta}})$ is bounded and vanishes as $z \to \infty$. Thus

$$I_2 = C t^{-\frac{d\beta}{\alpha}} \int_{1}^{\infty} z^{-\frac{d}{\alpha}-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z \asymp C_{\beta,d,\alpha} t^{-\frac{d\beta}{\alpha}}.$$

Combining the estimates for I_1 and I_2 shows (4.1.11).

<u>**Case 2:** $\Omega \geq 1$.</u> In this case, the integral I_1 is simply

$$I_1 = c t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} \int_0^1 z\, \mathrm{d}z = \frac{C}{2} t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}}.$$

For the second integral, we have

$$I_2 = c t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} \int_1^{\Omega} z^{-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z + c t^{-\frac{d\beta}{\alpha}} \int_{\Omega}^{\infty} z^{-\frac{d}{\alpha}-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z.$$

Note that the integral in the first term approaches a convergent integral (for large Ω), while the second can be dealt with by using the Laplace method, see 2.4.2,

$$\begin{aligned} I_2 &\asymp c t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} \int_1^{\infty} z^{-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z + c t^{-\frac{d\beta}{\alpha}} \int_{\Omega}^{\infty} z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{-c_\beta z^{\frac{1}{1-\beta}}\}\, \mathrm{d}z \\ &\asymp C_\beta t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} + c t^{-\frac{d\beta}{\alpha}} \Omega^{-\frac{d}{\alpha}-\frac{1}{2(1-\beta)}} \exp\{-c\Omega^{\frac{1}{1-\beta}}\} \\ &\asymp C t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}}, \end{aligned}$$

where we have used (2.4.2) with $g(x) = x^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}}$ and $h(x) = x^{\frac{1}{1-\beta}}$. Combining the estimates for I_1 and I_2 proves (4.1.12), which completes the proof. $\square$

Proposition 4.1.4. Under the assumptions of Theorem 4.1.3, assume additionally that $w \in C^{d+1+[\alpha]+l}(\mathbb{S}^{d-1})$. Then the following estimates hold,

- For $\Omega \leq 1$,

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha}(t, x-y)\right| \leq C \begin{cases} t^{-\frac{(d+k)\beta}{\alpha}} & d+k<\alpha, \\ t^{-\beta}(|\log(\Omega)|+1) & d+k=\alpha, \\ t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{1-\frac{(d+k)}{\alpha}} & d+k>\alpha, \end{cases} \tag{4.1.15}$$

  for all $k \leq l$ and $i_1, \cdots, i_k$.

- For $\Omega \geq 1$,

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha}(t, x-y)\right| \leq Ct^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{(d+k)}{\alpha}}, \tag{4.1.16}$$

  for all $k \leq l$ and $i_1, \cdots, i_k$.

Proof. Using the asymptotic behaviour of w_β followed by (2.3.8) we have

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha}(t, x-y)\right| \leq I_1 + I_2,$$

where

$$I_1 := C\int_0^1 \min\left(t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{d+k}{\alpha}}z, t^{-\frac{(d+k)\beta}{\alpha}}z^{-\frac{d+k}{\alpha}}\right)\, \mathrm{d}z,$$

and

$$I_2 := C\int_1^\infty \min\left(t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{d+k}{\alpha}}z, t^{-\frac{(d+k)\beta}{\alpha}}z^{-\frac{d+k}{\alpha}}\right) z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z. \tag{4.1.17}$$

Again we are in the situation where these integrals are the same as those found in (4.1.14) after changing $d \mapsto d+k$. Thus we have for $\Omega \geq 1$,

$$I_1 = \frac{C}{2}t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{d+k}{\alpha}},$$

and

$$
\begin{aligned}
I_2 &= ct^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{d+k}{\alpha}}\int_1^{\Omega} z^{-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z + ct^{-\frac{(d+k)\beta}{\alpha}}\int_{\Omega}^{\infty} z^{-\frac{d+k}{\alpha}-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z \\
&\le Ct^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{d+k}{\alpha}} + ct^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-\frac{d+k}{\alpha}-\frac{1}{2(1-\beta)}}\exp\{-c\Omega^{\frac{1}{1-\beta}}\} \\
&\le C_{d,k,\alpha,\beta}t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{d+k}{\alpha}}.
\end{aligned}
$$

Combining the estimates for I_1 and I_2 gives us (4.1.16). For $\Omega \le 1$ we have

$$
I_2 = ct^{-\frac{(d+k)\beta}{\alpha}}\int_1^{\infty} z^{-\frac{d+k}{\alpha}-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z \le Ct^{-\frac{(d+k)\beta}{\alpha}}.
$$

It only remains to check the estimate for I_1 when $\Omega \le 1$,

$$
\begin{aligned}
I_1 &= ct^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{d+k}{\alpha}}\int_0^{\Omega} z\,\mathrm{d}z + ct^{-\frac{(d+k)\beta}{\alpha}}\int_{\Omega}^{1} z^{-\frac{d+k}{\alpha}}\,\mathrm{d}z \\
&= \frac{c}{2}t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{1-\frac{d+k}{\alpha}} + ct^{-\frac{(d+k)\beta}{\alpha}}\int_{\Omega}^{1} z^{-\frac{d+k}{\alpha}}\,\mathrm{d}z.
\end{aligned}
$$

Thus we have

$$
I_1 \le C\begin{cases} t^{-\frac{(d+k)\beta}{\alpha}}, & d+k<\alpha, \\ t^{-\beta}(|\log\Omega|+1), & d+k=\alpha, \\ t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{1-\frac{(d+k)}{\alpha}}, & d+k>\alpha. \end{cases}
$$

Combining the estimates for I_1 and I_2 for $\Omega \le 1$ gives us (4.1.15). □

4.2 Local Estimates

In the following two sections we look at two other families of spatial operators which extend the global estimates obtained in the previous sections. Firstly we consider a more general second order elliptic operator (not necessarily in divergence form), then we consider homogeneous pseudo-differential operators with variable coefficients. In both cases we provide local (i.e., small-time) two-sided estimates for the Green's functions of the associated fractional evolution equations. The key point here is that for these spatial operators, we no longer have global (in time) estimates for the associated Green's functions. Before going to the new estimates, we describe how one turns local estimates into global estimates. If for some Green's functions $G_0(t,x,y), G_1(t,x,y)$, one has the local two-sided estimate for some constant $c > 0$

$$\frac{1}{c}G_1(t,x,y) \leq G_0(t,x,y) \leq cG_1(t,x,y), \quad (t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d,$$

for some fixed $T > 0$, then by taking convolutions and using the Chapman-Kolmogorov equations,

$$\begin{aligned} G_0(2t,x,y) &= \int_{\mathbb{R}^d} G_0(t,x,z)G_0(t,z,y)\ dz \\ &\leq \int_{\mathbb{R}^d} cG_1(t,x,z)cG_1(t,z,y)\ dz \\ &= c^2 G_1(2t,x,y). \end{aligned}$$

Repeating this procedure n-times,

$$\begin{aligned} G_0(nt,x,y) &= \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} G_0(t,x,x_1)\cdots G_0(t,x_n,y)dx_1\cdots dx_n \\ &\leq \int_{\mathbb{R}^d} \cdots \int_{\mathbb{R}^d} cG_1(t,x,x_1)\cdots cG_1(t,x_n,y)dx_1\cdots dx_n \\ &= c^n G_1(nt,x,z). \end{aligned}$$

By fixing t and setting $\tau = nt$ (so that $\tau \approx n$ for large values of n and τ), we then get

$$\begin{aligned} G_0(\tau,x,y) &\leq c^{\tau/t} G_1(\tau,x,y) \\ &= e^{\frac{\tau}{t}\log c} G_1(\tau,x,y) \\ &\approx e^{\tau\tilde{c}} G_1(\tau,x,y), \quad \forall \tau > 0, x, y \in \mathbb{R}^d \end{aligned}$$

Applying the same procedure to the lower bound gives us the global two-sided estimate

$$e^{-c\tau}G_1(\tau,x,y) \leq G_0(\tau,x,y) \leq e^{c\tau}G_1(\tau,x,y) \tag{4.2.1}$$

for all $(\tau,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$.

4.2.1 Non-degenerate Diffusions

In Section 4.1.1 we derived global two-sided estimates for the Green's function of fractional evolution equations involving a fractional derivative in time and a second order elliptic operator *in divergence form* as the spatial operator. The key point in that case is that Aronsons estimates provides two-sided Gaussian estimates that hold globally for all time $t > 0$. In this section we consider the case that the spatial operator is any non-degenerate diffusion operator, which can generally be of the form

$$Lu(t,x) := a_{ij}(x)\partial_{x_i}\partial_{x_j}u(t,x) + b_i(x)\partial_{x_i}u(t,x) + c(x)u(t,x). \tag{4.2.2}$$

Assuming that $a(x)$ is uniformly elliptic and continuously differentiable, $b(x)$ and $c(x)$ are continuous, and the uniform bound holds:

$$\sup_x \max(|\nabla a(x)|, |b(x)|, |c(x)|) \leq M.$$

Then the Green's function associated with (4.2.2), satisfies the following local estimates:

$$\frac{t^{-\frac{d}{2}}}{C}\exp\left\{-C\frac{|x-y|^2}{t}\right\} \leq G(t,x,y) \leq Ct^{-\frac{d}{2}}\exp\left\{-C\frac{|x-y|^2}{t}\right\}, \tag{4.2.3}$$

for $(t,x,y) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$ for some fixed $T > 0$. We also have the following estimates for the spatial derivative of the Green's function G,

$$\left|\frac{\partial}{\partial x}G(t,x,y)\right| \leq Ct^{-\frac{d+1}{2}}\exp\left\{-C\frac{|x-y|^2}{t}\right\}, \tag{4.2.4}$$

for $(t,x,y) \in (0,T) \times \mathbb{R}^d \times \mathbb{R}^d$. The main obstacle now is that the estimates for the Green's function of (4.2.2) are only for small-time, thus a serious problem seems to arise when trying to insert the local estimate into the Pollard-Zolotarev formula, which involves integrating over all time $z \in (0,\infty)$. However we use the trick described in the previous section to make the local estimates global, in (4.2.1). To this end, the following two-sided estimate holds for $G(t,x,y)$ for all $(\tau,x,y) \in (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$e^{-c\tau}\tau^{-\frac{d}{2}}\exp\left\{-C\frac{|x-y|^2}{\tau}\right\} \leq G(\tau,x,y) \leq e^{c\tau}\tau^{-\frac{d}{2}}\exp\left\{-C\frac{|x-y|^2}{\tau}\right\}, \tag{4.2.5}$$

for some constant c. In addition, for all $(\tau, x, y) \in (0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$,

$$\left|\frac{\partial}{\partial x} G(\tau, x, y)\right| \leq e^{c\tau} \max(\tau^{-\frac{1}{2}}, 1)\tau^{-\frac{d}{2}} \exp\left\{-C\frac{|x-y|^2}{\tau}\right\}.$$

Alternatively we can split the estimates for the spatial derivative up into small-time and large-time - for $\tau \in (0, 1)$,

$$\left|\frac{\partial}{\partial x} G(\tau, x, y)\right| \leq C\tau^{-\frac{d+1}{2}} \exp\left\{-C\frac{|x-y|^2}{\tau}\right\}, \tag{4.2.6}$$

and for $\tau \in (1, \infty)$,

$$\left|\frac{\partial}{\partial x} G(\tau, x, y)\right| \leq e^{c\tau}\tau^{-\frac{d}{2}} \exp\left\{-C\frac{|x-y|^2}{\tau}\right\}. \tag{4.2.7}$$

Now we proceed to obtain estimates for the Green's function of the fractional evolution

$$D_0^\beta u(t, x) = Lu(t, x),$$

where L is defined as above in (4.2.2). The Green's function for this fractional evolution equation is given by

$$G^{(\beta)}(t, x, y) = \frac{1}{\beta}\int_0^\infty G(t^\beta z, x, y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z. \tag{4.2.8}$$

Again let $\Omega = |x-y|^2 t^{-\beta}$. We have the following local estimates for the Green's function $G^{(\beta)}$ above.

Theorem 4.2.1. *Assume that $a(\cdot) \in C^1(\mathbb{R}^d)$ is uniformly elliptic and $b(\cdot), c(\cdot) \in C(\mathbb{R}^d)$. Suppose also that,*

$$\sup_x \max(|\nabla a(x)|, |b(x)|, |c(x)|) \leq M.$$

Then for a fixed $T > 0$, there exists constants C_1, C_2, C_3 such that for $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$ the Green's function $G^{(\beta)}(t, x, y)$ defined by (4.2.8) satisfies the following estimates,

- *For* $\Omega \leq 1$,

$$G^{(\beta)}(t,x,y) \asymp C_1 \begin{cases} t^{-\frac{\beta}{2}}, & d=1, \\ t^{-\beta}(|\log\Omega|+1), & d=2, \\ t^{-\frac{d\beta}{2}}\Omega^{1-\frac{d}{2}}, & d\geq 3. \end{cases} \tag{4.2.9}$$

- *For* $\Omega \geq 1$,

$$G^{(\beta)}(t,x,y) \asymp C_2 t^{-\frac{d\beta}{2}}\Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)}\exp\{-C_3\Omega^{\frac{1}{2-\beta}}\}, \tag{4.2.10}$$

where C_1, C_2 *depends on* T, d, β *and* C_3 *depends on* T *and* β.

Proof. First splitting up to the stable density,

$$\begin{aligned} G^{(\beta)}(t,x,y) &\asymp C_\beta \int_0^1 G(t^\beta z,x,y)\ dz + C_\beta \int_1^\infty G(t^\beta z,x,y) z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z \\ &=: I_1 + I_2. \end{aligned}$$

Note that on using the estimate (4.2.3) in I_1, we have the same integral of the same name appearing in (4.1.5). Thus for $\Omega \leq 1$,

$$\begin{aligned} I_1 = C_\beta \int_0^1 G(t^\beta z,x,y)\ \mathrm{d}z &\asymp C_T t^{-\frac{d\beta}{2}} \int_0^1 z^{-\frac{d}{2}}\exp\{-\Omega z^{-1}\}\ \mathrm{d}z \\ &\asymp C \begin{cases} t^{-\frac{\beta}{2}}, & d=1, \\ t^{-\beta}(|\log\Omega|+1), & d=2, \\ t^{-\frac{d\beta}{2}}\Omega^{1-\frac{d}{2}}, & d\geq 3. \end{cases} \end{aligned} \tag{4.2.11}$$

In addition for $\Omega \geq 1$,

$$I_1 \asymp C_T t^{-\frac{d\beta}{2}}\Omega^{-1}\exp\{-\Omega\}. \tag{4.2.12}$$

Turning our attention to I_2, let us consider separately the upper and lower bound.

<u>**Upper bound for I_2**</u>

First applying the upper bound from (4.2.5) to G,

$$I_2 \leq C t^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1-\frac{1}{\beta}}\exp\{ct^\beta z - \Omega z^{-1}\} f_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z$$

$$= Ct^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - \Omega z^{-1} - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z. \tag{4.2.13}$$

For $\Omega \le 1$, we have

$$\begin{aligned} I_2 &\le Ct^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1-\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\ &\le Ct^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1-\frac{1}{2(1-\beta)}} \exp\{cT^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\ &= C_{T,d,\beta} t^{-\frac{d\beta}{2}}, \end{aligned}$$

for $t < T$ for some fixed $T > 0$. Combining this with (4.2.11) gives (4.2.9).

For $\Omega \ge 1$, we use again that the decay of $\exp\{-c_\beta z^{1/(1-\beta)}\}$ for large z is stronger than the growth of $\exp\{ct^\beta z\}$ for large z as long as $t < T$ for some fixed $T > 0$. That is,

$$\begin{aligned} I_2 &\le Ct^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\left\{-\Omega z^{-1} + ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\right\} \, \mathrm{d}z \\ &\le Ct^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\left\{-\Omega z^{-1} - C_{T,\beta} z^{\frac{1}{1-\beta}}\right\} \, \mathrm{d}z \\ &= Ct^{-\frac{d\beta}{2}} \int_0^1 w^{\frac{d}{2}-1-\frac{1}{2(1-\beta)}} \exp\left\{-\Omega w - C_{T,\beta} w^{-\frac{1}{1-\beta}}\right\} \, \mathrm{d}w, \end{aligned}$$

where we have made the substitution $w = z^{-1}$ in the last line. Now we apply Proposition 2.4.1 with $N = \frac{d}{2} - 1 - \frac{1}{2(1-\beta)}$ and $a = \frac{1}{1-\beta}$,

$$I_2 \le Ct^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\}.$$

Note the constants in the above estimate depend on T. Combining this with (4.2.12) gives us the required upper bound in (4.2.10).

Lower bound for I_2

Using the lower bound from (4.2.5) in I_2,

$$I_2 \ge ct^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{-ct^\beta z - \Omega z^{-1} - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z.$$

Firstly for $\Omega \le 1$,

$$I_2 \ge C_\beta t^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{-\Omega z^{-1} - ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z$$

$$
\begin{aligned}
&\geq C_\beta t^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{-ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\geq C_{\beta,T} t^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{-cT^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&= C_{T,\beta,d} t^{-\frac{d\beta}{2}}.
\end{aligned}
$$

Finally for $\Omega \geq 1$,

$$
\begin{aligned}
I_2 &= C_\beta t^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{-\Omega z^{-1} - ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\geq C_\beta t^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{-\Omega z^{-1} - (ct^\beta + c_\beta) z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\geq C_\beta t^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{-\Omega z^{-1} - C_{T,\beta} z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z,
\end{aligned}
$$

where we have used the fact that $\exp\{-ct^\beta z\} \geq \exp\{-ct^\beta z^{\frac{1}{1-\beta}}\}$ for $z > 1$. After making the substitution $z = w^{-1}$ we apply Proposition 2.4.1,

$$
\begin{aligned}
I_2 &\geq C_\beta t^{-\frac{d\beta}{2}} \int_0^1 w^{\frac{d}{2}-1-\frac{1}{2(1-\beta)}} \exp\{-\Omega w - C_{T,\beta} w^{-\frac{1}{1-\beta}}\} \, \mathrm{d}w \\
&\geq C_1 t^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C_2 \Omega^{\frac{1}{2-\beta}}\},
\end{aligned}
$$

where C_1 depends on T, β and d, and C_2 depends on T and β. Combining this with (4.2.12) gives us the lower bound in (4.2.10), as required. □

Next we look at estimating the spatial derivative of the Green's function $G^{(\beta)}$, firstly for large-time using (4.2.7) then for small-time using (4.2.6). As usual, let $\Omega := |x-y|^2 t^{-\beta}$. Firstly for large finite time,

Proposition 4.2.2. Under the same assumptions as Theorem 4.2.1, suppose further that $a(x)$ is twice continuously differentiable, and $b(x)$, $c(x)$ are continuously differentiable (with all derivatives bounded). Then for a fixed finite $T > 1$, the following estimates hold for the spatial derivative of the Green's function $G^{(\beta)}(t, x, y)$ for $(t, x, y) \in (1, T) \times \mathbb{R}^d \times \mathbb{R}^d$,

- For $\Omega \leq 1$,

$$
\left| \frac{\partial}{\partial x} G^{(\beta)}(t, x, y) \right| \leq C_{T,d,\beta} \begin{cases} t^{-\beta}(|\log \Omega| + 1), & d = 1, \\ |x-y|^{1-d}, & d \geq 2. \end{cases} \tag{4.2.14}
$$

- For $\Omega \geq 1$,

$$\left|\frac{\partial}{\partial x}G^{(\beta)}(t,x,y)\right| \leq C_{T,d,\beta}|x-y|^{-d\left(\frac{1-\beta}{2-\beta}\right)}\exp\{-C_{T,\beta}|x-y|^{\frac{2}{2-\beta}}\}. \quad (4.2.15)$$

Proof. We start as usual by first splitting up the integral into small and large z, and also use the triangle inequality,

$$\begin{aligned}\left|\frac{\partial}{\partial x}G^{(\beta)}(t,x,y)\right| \leq C_\beta \int_0^1 \left|\frac{\partial}{\partial x}G(t^\beta z,x,y)\right| \mathrm{d}z \\ +C_\beta \int_1^\infty \left|\frac{\partial}{\partial x}G(t^\beta z,x,y)\right| z^{-1-\frac{1}{\beta}}f_\beta(z^{-\frac{1}{\beta}}) \mathrm{d}z.\end{aligned}$$

Note that $t \in (1,T)$ means that $t^{-\beta} \in (T^{-\beta},1)$. Thus for $z \in (1,\infty)$ we have $z \geq t^{-\beta}$. Now we use the local estimate (4.2.6) for the first integral and (4.2.7) for the second,

$$\begin{aligned}\left|\frac{\partial}{\partial x}G^{(\beta)}(t,x,y)\right| &\leq Ct^{-\frac{(d+1)\beta}{2}}\int_0^1 z^{-\frac{d+1}{2}}\exp\{-\Omega z^{-1}\} \mathrm{d}z \\ &\quad + Ct^{-\frac{d\beta}{2}}\int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}}\exp\left\{-\Omega z^{-1}+ct^\beta z-c_\beta z^{\frac{1}{1-\beta}}\right\} \mathrm{d}z \\ &=: I_1+I_2.\end{aligned}$$

Note that the integral in I_1 is the same as (4.1.9), and thus for $\Omega \leq 1$

$$I_1 = Ct^{-\frac{(d+1)\beta}{2}}\int_0^1 z^{-\frac{d+1}{2}}\exp\{-\Omega z^{-1}\}\mathrm{d}z \leq \begin{cases} t^{-\beta}(|\log\Omega|+1), & d=1, \\ t^{-\frac{(d+1)\beta}{2}}\Omega^{1-\frac{d+1}{2}}, & d\geq 2. \end{cases}$$

Note however that $t \in (1,T)$, which means that $t^{-\beta} \in (T^{-\beta},1)$. Thus

$$I_1 \leq C_{T,\beta,d}\begin{cases} t^{-\beta}(|\log\Omega|+1), & d=1, \\ |x-y|^{1-d}, & d\geq 2. \end{cases} \quad (4.2.16)$$

For $\Omega \geq 1$,

$$I_1 \leq Ct^{-\frac{(d+1)\beta}{2}}\Omega^{-1}\exp\{-\Omega\} \leq C_{T,d,\beta}|x-y|^{-2}\exp\{-C_{T,\beta}|x-y|^2\}.$$

As for the integral I_2, this is the same one which appeared in the previous proof, (4.2.13), and thus for $\Omega \leq 1$,

$$I_2 \leq Ct^{-\frac{d\beta}{2}} \leq C_{T,d,\beta}.$$

Combining this with (4.2.16) which gives both (4.2.14). Finally an application of Proposition 2.4.1 gives for $\Omega \geq 1$,

$$\begin{aligned} I_2 &= Ct^{-\frac{d\beta}{2}} \int_1^{\infty} z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\left\{-\Omega z^{-1} + ct^{\beta} z - c_\beta z^{\frac{1}{1-\beta}}\right\} \, \mathrm{d}z \\ &\leq Ct^{-\frac{d\beta}{2}} \int_1^{\infty} z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\left\{-\Omega z^{-1} - C_{T,\beta} z^{\frac{1}{1-\beta}}\right\} \, \mathrm{d}z \\ &\leq Ct^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\} \\ &\leq C_{T,d,\beta} |x-y|^{-d\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C_{T,\beta}|x-y|^{\frac{2}{2-\beta}}\}. \end{aligned}$$

Combining this with the estimate for I_1, gives the estimate (4.2.15) for $\Omega \geq 1$, as required. $\square$

Next we have the estimates for small-time.

Proposition 4.2.3. Under the same assumptions as Theorem 4.2.1, suppose further that $a(x)$ is twice continuously differentiable, and $b(x)$, $c(x)$ are continuously differentiable (with all derivatives bounded). Then the following estimates hold for the spatial derivative of the Green's function $G^{(\beta)}(t,x,y)$ for $(t,x,y) \in (0,1) \times \mathbb{R}^d \times \mathbb{R}^d$,

- For $\Omega \leq 1$,

$$\left|\frac{\partial}{\partial x} G^{(\beta)}(t,x,y)\right| \leq C_{d,\beta} \begin{cases} t^{-\beta}(|\log \Omega| + 1), & d = 1, \\ t^{-\frac{(d+1)\beta}{2}} \Omega^{1-\frac{d+1}{2}}, & d \geq 2. \end{cases}$$

- For $1 \leq \Omega \leq t^{-\beta\left(\frac{2-\beta}{1-\beta}\right)}$,

$$\left|\frac{\partial}{\partial x} G^{(\beta)}(t,x,y)\right| \leq Ct^{-\frac{(d+1)\beta}{2}} \Omega^{-\left(\frac{d+1}{2}\right)\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\}.$$

- For $\Omega \geq t^{-\beta\left(\frac{2-\beta}{1-\beta}\right)}$,

$$\left|\frac{\partial}{\partial x} G^{(\beta)}(t,x,y)\right| \leq Ct^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\}.$$

Proof. Splitting the integral up using the stable density then using the estimates (4.2.6) and (4.2.7),

$$\begin{aligned}\left|\frac{\partial}{\partial x}G^{(\beta)}(t,x,y)\right| &\le C_\beta t^{-\frac{(d+1)\beta}{2}}\int_0^1 z^{-\frac{d+1}{2}}\exp\{-\Omega z^{-1}\}\,\mathrm{d}z\\ &\quad + C_\beta\int_1^\infty \max((t^\beta z)^{-\frac{1}{2}},1)z^{-\frac{d}{2}-1-\frac{1}{\beta}}\exp\left\{-\frac{\Omega}{z}+ct^\beta z\right\}f_\beta(z^{-\frac{1}{\beta}})\,\mathrm{d}z\\ &= C_\beta t^{-\frac{(d+1)\beta}{2}}\int_0^1 z^{-\frac{d+1}{2}}\exp\{-\Omega z^{-1}\}\,\mathrm{d}z\\ &\quad + C_\beta t^{-\frac{(d+1)\beta}{2}}\int_1^{t^{-\beta}} z^{-\frac{d+1}{2}-1+\frac{1}{2(1-\beta)}}\exp\left\{-\Omega z^{-1}+ct^\beta z-c_\beta z^{\frac{1}{1-\beta}}\right\}\,\mathrm{d}z\\ &\quad + C_\beta t^{-\frac{d\beta}{2}}\int_{t^{-\beta}}^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}}\exp\left\{-\Omega z^{-1}+ct^\beta z-c_\beta z^{\frac{1}{1-\beta}}\right\}\,\mathrm{d}z\\ &=: I_1+I_2+I_3.\end{aligned}$$

Now we investigate the usual cases.

<u>**Case 1:** $\Omega\le 1$</u> The integral in I_1, being the same as the one in (4.2.16), has the upper bound

$$I_1 = Ct^{-\frac{(d+1)\beta}{2}}\int_0^1 z^{-\frac{d+1}{2}}\exp\{-\Omega z^{-1}\}\,\mathrm{d}z \le C\begin{cases} t^{-\beta}(|\log\Omega|+1), & d=1,\\ t^{-\frac{(d+1)\beta}{2}}\Omega^{1-\frac{d+1}{2}}, & d\ge 2.\end{cases}$$

The other two integrals in I_2 and I_3 approach convergent integrals for bounded Ω, so

$$\begin{aligned} I_2 = &\ Ct^{-\frac{(d+1)\beta}{2}}\int_1^{t^{-\beta}} z^{-\frac{d+1}{2}-1+\frac{1}{2(1-\beta)}}\exp\{-\Omega z^{-1}+ct^\beta z-c_\beta z^{\frac{1}{1-\beta}}\}\,\mathrm{d}z\\ \le &\ Ct^{-\frac{(d+1)\beta}{2}}\int_1^\infty z^{-\frac{d+1}{2}-1+\frac{1}{2(1-\beta)}}\exp\{ct^\beta z-c_\beta z^{\frac{1}{1-\beta}}\}\,\mathrm{d}z\\ \le &\ C_{d,\beta}t^{-\frac{(d+1)\beta}{2}},\end{aligned}$$

and

$$\begin{aligned} I_3 &= Ct^{-\frac{d\beta}{2}}\int_{t^{-\beta}}^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}}\exp\left\{-\Omega z^{-1}+ct^\beta z-c_\beta z^{\frac{1}{1-\beta}}\right\}\,\mathrm{d}z\\ &\le Ct^{-\frac{d\beta}{2}}\int_{t^{-\beta}}^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}}\exp\left\{ct^\beta z-c_\beta z^{\frac{1}{1-\beta}}\right\}\,\mathrm{d}z\\ &\le t^{\beta-\frac{\beta}{2(1-\beta)}}\exp\{-C_\beta t^{-\frac{\beta}{1-\beta}}\}\end{aligned}$$

$$\leq C_{d,\beta} t^{-\frac{d\beta}{2}}.$$

Thus in this case,

$$\left|\frac{\partial}{\partial x} G^{(\beta)}(t,x,y)\right| \leq C \begin{cases} t^{-\beta}(|\log \Omega| + 1), & d = 1, \\ t^{-\frac{(d+1)\beta}{2}} \Omega^{1-\frac{d+1}{2}}, & d \geq 2. \end{cases}$$

<u>**Case 2:** $\Omega \geq 1$</u> A direct application of the Laplace method gives

$$I_1 \leq t^{-\frac{(d+1)\beta}{2}} \Omega^{-1} \exp\{-\Omega\}.$$

For the second integral we have,

$$\begin{aligned} I_2 &\leq C t^{-\frac{(d+1)\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{-\Omega z^{-1} - C_\beta z^{\frac{1}{1-\beta}}\}\, \mathrm{d}z \\ &\leq C t^{-\frac{(d+1)\beta}{2}} \Omega^{-\left(\frac{d+1}{2}\right)\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\} \end{aligned}$$

where we have used Proposition 2.4.1 in the last estimate. Finally since $t \in (0,1)$, another application of Proposition 2.4.1 gives

$$\begin{aligned} I_3 &\leq C t^{-\frac{d\beta}{2}} \int_1^\infty z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}} \exp\{-\Omega z^{-1} - C z^{\frac{1}{1-\beta}}\}\, \mathrm{d}z \\ &\leq C t^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\}, \end{aligned}$$

Note that

$$t^{-\frac{(d+1)\beta}{2}} \Omega^{-\left(\frac{d+1}{2}\right)\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\} \leq C t^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\},$$

when $t^{-\frac{\beta}{2}} \Omega^{-\frac{1}{2}\left(\frac{1-\beta}{2-\beta}\right)} \leq 1$. Thus for $\Omega \geq t^{-\beta\left(\frac{2-\beta}{1-\beta}\right)}$,

$$\left|\frac{\partial}{\partial x} G^{(\beta)}(t,x,y)\right| \leq C t^{-\frac{d\beta}{2}} \Omega^{-\frac{d}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\},$$

while for $1 \leq \Omega \leq t^{-\beta\left(\frac{2-\beta}{1-\beta}\right)}$,

$$\left|\frac{\partial}{\partial x} G^{(\beta)}(t,x,y)\right| \leq C t^{-\frac{(d+1)\beta}{2}} \Omega^{-\frac{d+1}{2}\left(\frac{1-\beta}{2-\beta}\right)} \exp\{-C\Omega^{\frac{1}{2-\beta}}\}.$$

□

4.2.2 Pseudo-differential Operators: Variable Coefficients

Finally we derive two-sided estimates for the Green's function of time-fractional stable-like equations. Stable-like operators are homogeneous pseudo-differential operators with variable coefficients (that depend on the spatial variable x, but not time). As noted earlier in (2.3.7) the fundamental solution G_ψ of the evolution equation

$$\partial_t u = -\Psi_\alpha(-i\nabla)u,$$

with $\psi_\alpha(p) = |p|^\alpha S_\mu(p/|p|)$, satisfies the following two-sided estimate for all $(t, x-y) \in (0,\infty) \times \mathbb{R}^d$,

$$G_{\psi_\alpha}(t, x-y) \asymp C \min\left(\frac{t}{|x-y|^{d+\alpha}}, t^{-d/\alpha}\right). \tag{4.2.17}$$

When the coefficients of the operator Ψ_α depends also on the spatial variable, the same kind of estimates hold for small-time. Using the same technique as the previous section to extend these small-time estimates to global estimates, we have the following two-sided estimates for $\tau > 0$, $x, y \in \mathbb{R}^d$,

$$e^{-C\tau} \min\left(\frac{\tau}{|x-y|^{d+\alpha}}, \tau^{-\frac{d}{\alpha}}\right) \le G_{\psi_\alpha,x}(\tau, x, y) \le e^{C\tau} \min\left(\frac{\tau}{|x-y|^{d+\alpha}}, \tau^{-\frac{d}{\alpha}}\right), \tag{4.2.18}$$

and

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G_{\psi_\alpha,x}(\tau, x, y)\right| \le e^{C\tau} \max\left(\tau^{-\frac{k}{\alpha}}, 1\right) \min\left(\frac{\tau}{|x-y|^{d+\alpha}}, \tau^{-\frac{d}{\alpha}}\right). \tag{4.2.19}$$

Now consider the following fractional evolution equation,

$$D^\beta_{0+*} u(t,x) = -\Psi_\alpha(x, -i\nabla)u(t,x), \quad u(0,x) = Y(x), \tag{4.2.20}$$

where the symbol of Ψ_α is of the form

$$\psi_\alpha(x,p) = |p|^\alpha S_\mu(x, p/|p|), \tag{4.2.21}$$

where S_μ satisfies the assumptions of Theorem 2.3.2. The solution of (4.2.20) is given by

$$u(t,x) = E_\beta(-\Psi_\alpha(x, -i\nabla)t^\beta)Y(x),$$

where E_β is the Mittag-Leffler function. The Green's function of Equation (4.2.20) is then

$$G^{(\beta)}_{\psi_\alpha,x}(t,x,y) = \frac{1}{\beta}\int_0^\infty G_{\psi_\alpha,x}(t^\beta z,x,y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z. \tag{4.2.22}$$

Let $\Omega = |x-y|^\alpha t^{-\beta}$.

Theorem 4.2.4. *Let $\alpha \in (0,2)$ and $\beta \in (0,1)$. Assume that the function S_μ in (4.2.21) is γ-Hölder continuous in the first variable and k-times continuously differentiable in the second variable. Assume further that the spectral measure μ has a strictly positive density. Then for a fixed $T>0$ there exists constants C such that for $(t,x,y) \in (0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ the following two-sided estimates for (4.2.22) hold,*

- *For $\Omega \leq 1$,*

$$G^{(\beta)}_{\psi_\alpha,x}(t,x,y) \asymp\ C \begin{cases} t^{-\frac{d\beta}{\alpha}}, & d<\alpha, \\ t^{-\beta}(|\log(\Omega)|+1), & d=\alpha, \\ t^{-\frac{d\beta}{\alpha}}\Omega^{1-\frac{d}{\alpha}}, & d>\alpha. \end{cases}$$

- *For $\Omega \geq 1$,*

$$G^{(\beta)}_{\psi_\alpha,x}(t,x,y) \asymp\ C t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}},$$

where the constants C depend on d, α, β and T.

Proof. We start by estimating the stable density with (2.3.12),

$$\begin{aligned} G^{(\beta)}_{\psi_\alpha,x}(t,x,y) \asymp C_\beta \int_0^1 G_{\psi_\alpha,x}(t^\beta z,x,y)\ dz \\ + C_\beta \int_1^\infty G_{\psi_\alpha,x}(t^\beta z,x,y) z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\ \mathrm{d}z. \end{aligned} \tag{4.2.23}$$

In the first integral, we use the estimate (2.3.9), and for the second term we use the global version (4.2.18) with $\tau = t^\beta z$. Starting with the upper bound, we have

$$G^{(\beta)}_{\psi_\alpha,x}(t,x,y) \leq C_T \int_0^1 \min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} z, t^{-\frac{d\beta}{\alpha}} z^{-\frac{d}{\alpha}}\right)\ \mathrm{d}z$$

$$+ c\int_1^{\infty} \min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}z, t^{-\frac{d\beta}{\alpha}}z^{-\frac{d}{\alpha}}\right) z^{-1-\frac{1}{\beta}}e^{ct^{\beta}z}f_{\beta}(z^{-\frac{1}{\beta}})\ \mathrm{d}z.$$

Recall that,

$$\min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}z, t^{-\frac{d\beta}{\alpha}}z^{-\frac{d}{\alpha}}\right) = \begin{cases} t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}z, & z \leq \Omega, \\ t^{-\frac{d\beta}{\alpha}}z^{-\frac{d}{\alpha}}, & z \geq \Omega. \end{cases}$$

Then we have

$$\begin{aligned} G^{(\beta)}_{\psi_\alpha,x}(t,x,y) \leq c&\int_0^1 \min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}z, t^{-\frac{d\beta}{\alpha}}z^{-\frac{d}{\alpha}}\right)\ \mathrm{d}z \\ &+ c\int_1^{\infty} \min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}z, t^{-\frac{d\beta}{\alpha}}z^{-\frac{d}{\alpha}}\right) z^{-1-\frac{1}{\beta}}e^{ct^{\beta}z}f_{\beta}(z^{-\frac{1}{\beta}})\ \mathrm{d}z \end{aligned} \tag{4.2.24}$$

$$:= I_1 + I_2^{up},$$

for the upper bound, and

$$\begin{aligned} G^{(\beta)}_{\psi_\alpha,x}(t,x,y) \geq \frac{1}{c}&\int_0^1 \min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}z, t^{-\frac{d\beta}{\alpha}}z^{-\frac{d}{\alpha}}\right)\ \mathrm{d}z \\ &+ C\int_1^{\infty} \min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}z, t^{-\frac{d\beta}{\alpha}}z^{-\frac{d}{\alpha}}\right) z^{-1-\frac{1}{\beta}}e^{-ct^{\beta}z}f_{\beta}(z^{-\frac{1}{\beta}})\ \mathrm{d}z \\ := I_1 + I_2^{lo},& \end{aligned}$$

for the lower bound. Note that the integral in I_1 is the as the one appearing in (4.1.14) and so, for $t < T$,

$$I_1 \asymp C_T \begin{cases} t^{-\frac{d\beta}{\alpha}}, & d < \alpha, \\ t^{-\beta}(|\log\Omega| + 1), & d = \alpha, \\ t^{-\frac{d\beta}{\alpha}}\Omega^{1-\frac{d}{\alpha}}, & d > \alpha, \end{cases}$$

for $\Omega \leq 1$, and

$$I_1 \asymp t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}},$$

for $\Omega \geq 1$. For the remaining integral I_2, we have the usual two cases.

Case 1: $\Omega \leq 1$. In this case we have

$$I_2^{up} = Ct^{-\frac{d\beta}{\alpha}} \int_1^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z.$$

This integral converges as long as $t < T$, since $\exp\{ct^\beta z\} \leq \exp\{Cz^{\frac{1}{1-\beta}}\}$ for sufficiently large z. Thus

$$I_2^{up} \leq C_{T,d,\beta,\alpha} t^{-\frac{d\beta}{\alpha}}.$$

On the other hand, we have

$$I_2^{lo} = Ct^{-\frac{d\beta}{\alpha}} \int_1^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{-ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z,$$

which is strictly positive for $t < T$, thus

$$I_2^{lo} \geq C_{T,d,\beta,\alpha} t^{-\frac{d\beta}{\alpha}}.$$

Combining these estimates with those for I_1 gives the estimates for $G^{(\beta)}_{\psi_\alpha,x}$ for $\Omega \leq 1$.

Case 2: $\Omega \geq 1$. In this case we have

$$I_2^{up} = Ct^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} \int_1^\Omega z^{\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\ + Ct^{-\frac{d\beta}{\alpha}} \int_\Omega^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z,$$

and

$$I_2^{lo} = Ct^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} \int_1^\Omega z^{\frac{1}{2(1-\beta)}} \exp\{-ct^\beta z - z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\ + Ct^{-\frac{d\beta}{\alpha}} \int_\Omega^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{-ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z.$$

Firstly we have,

$$t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} \int_1^\Omega z^{\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \leq C_{T,d,\beta,\alpha} t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}}$$

and

$$t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}\int_1^\Omega z^{\frac{1}{2(1-\beta)}}\exp\{-ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\}\,\mathrm{d}z \geq \tilde{C}_{T,d,\beta,\alpha}t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}.$$

Next note that $\exp\{t^\beta z\} \leq \exp\{T^\beta z\}$ and $\exp\{-t^\beta z\} \geq \exp\{-T^\beta z\}$ for $t < T$. Thus,

$$\begin{aligned}
I_2^{up} &\leq Ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} + Ct^{-\frac{d\beta}{\alpha}}\int_\Omega^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}}\exp\{t^\beta z - c_\beta z^{\frac{1}{1-\beta}}\}\,\mathrm{d}z \\
&\leq Ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} + Ct^{-\frac{d\beta}{\alpha}}\int_\Omega^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}}\exp\{-C_{T,\beta} z^{\frac{1}{1-\beta}}\}\,\mathrm{d}z \\
&\leq Ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} + Ct^{-\frac{d\beta}{\alpha}}\Omega^{-\frac{d}{\alpha}-\frac{1}{2(1-\beta)}}\exp\{-C_{T,\beta}\Omega^{\frac{1}{1-\beta}}\} \\
&\leq C_{T,\beta,\alpha,d}t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}, \qquad (4.2.25)
\end{aligned}$$

and

$$\begin{aligned}
I_2^{lo} &\geq Ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} + Ct^{-\frac{d\beta}{\alpha}}\int_\Omega^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}}\exp\{-t^\beta z - c_\beta z^{\frac{1}{1-\beta}}\}\,\mathrm{d}z \\
&\geq Ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} + Ct^{-\frac{d\beta}{\alpha}}\int_\Omega^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}}\exp\{-C_{T,\beta} z^{\frac{1}{1-\beta}}\}\,\mathrm{d}z \\
&\geq Ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} + Ct^{-\frac{d\beta}{\alpha}}\Omega^{-\frac{d}{\alpha}-\frac{1}{2(1-\beta)}}\exp\{-C_{T,\beta}\Omega^{\frac{1}{1-\beta}}\} \\
&\geq C_{T,\beta,\alpha,d}t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}}.
\end{aligned}$$

Thus for $\Omega \geq 1$, we have

$$G^{(\beta)}(t,x,y) \asymp C_{T,d,\beta,\alpha}t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}},$$

as claimed. □

Next we look at the spatial derivatives, where we consider separately small and large (but finite) time.

Proposition 4.2.5. Under the same assumptions as Theorem 4.2.4 and Theorem 2.3.2, the spatial derivatives of the Green's function $G^{(\beta)}_{\psi_\alpha,x}(t,x,y)$ for $(t,x,y) \in (0,1) \times \mathbb{R}^d \times \mathbb{R}^d$ satisfy,

- For $\Omega \le 1$,

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t,x,y)\right| \le \ C \begin{cases} t^{-\frac{(d+k)\beta}{\alpha}}, & d+k<\alpha, \\ t^{-\beta}(|\log\Omega|+1), & d+k=\alpha, \\ t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{1-\frac{d+k}{\alpha}}, & d+k>\alpha. \end{cases} \tag{4.2.26}$$

for all $k \le l$ and all indicies $i_i, \cdots, i_k$.

- For $1 \le \Omega \le t^{-\beta}$,

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t,x,y)\right| \le Ct^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{(d+k)}{\alpha}} \tag{4.2.27}$$

for all $k \le l$ and all indicies $i_i, \cdots, i_k$.

- For $\Omega \ge t^{-\beta}$,

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t,x,y)\right| \le Ct^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} \tag{4.2.28}$$

for all $k \le l$ and all indices $i_i, \cdots, i_k$.

Proof. Splitting up the stable density followed by using the estimates (2.3.10) and (4.2.19) we have

$$\begin{aligned}\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t,x,y)\right| &\le c_\beta \int_0^1 \left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G_{\psi_\alpha,x}(t^\beta z,x,y)\right| \, \mathrm{d}z \\ &\quad + c_\beta \int_1^\infty \left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t^\beta z,x,y)\right| z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z \\ &=: I_1 + I_2,\end{aligned}$$

where

$$I_1 := C\int_0^1 \min\left(t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{d+k}{\alpha}} z, t^{-\frac{(d+k)\beta}{\alpha}} z^{-\frac{d+k}{\alpha}}\right)\, \mathrm{d}z,$$

and

$$I_2 := C\int_1^\infty \max((t^\beta z)^{-\frac{k}{\alpha}},1)\min\left(t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} z, t^{-\frac{d\beta}{\alpha}} z^{-\frac{d}{\alpha}}\right) z^{-1-\frac{1}{\beta}} e^{ct^\beta z} f_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z.$$

Now note that since $t \in (0,1)$, the integral in I_1 is the same as that one appearing in (4.1.17), and thus for $\Omega \le 1$,

$$
\begin{aligned}
I_1 &= Ct^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}} \int_0^\Omega z \, \mathrm{d}z + Ct^{-\frac{(d+k)\beta}{\alpha}} \int_\Omega^1 z^{-\frac{d+k}{\alpha}} \, \mathrm{d}z \\
&\le C \begin{cases} t^{-\frac{(d+k)\beta}{\alpha}}, & d+k < \alpha, \\ t^{-\beta}(|\log \Omega| + 1), & d+k = \alpha, \\ t^{-\frac{(d+k)\beta}{\alpha}} \Omega^{1-\frac{d+k}{\alpha}}, & d+k > \alpha. \end{cases}
\end{aligned}
$$

For $\Omega \ge 1$,

$$
I_1 = Ct^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}} \int_0^1 z \, \mathrm{d}z = \frac{C}{2} t^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{(d+k)\beta}{\alpha}}. \tag{4.2.29}
$$

Turning to I_2, we need to consider some different cases.

Case 1: $\Omega \le 1$. In this case

$$
\begin{aligned}
I_2 &= Ct^{-\frac{(d+k)\beta}{\alpha}} \int_1^{t^{-\beta}} z^{-\frac{d+k}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\quad + Ct^{-\frac{d\beta}{\alpha}} \int_{t^{-\beta}}^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\le Ct^{-\frac{(d+k)\beta}{\alpha}} \int_1^\infty z^{-\frac{d+k}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{-C_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\quad + Ct^{\beta-\frac{\beta}{2(1-\beta)}} \exp\{-Ct^{-\frac{\beta}{1-\beta}}\} \\
&\le C_{\beta,d,\alpha,k} t^{-\frac{(d+k)\beta}{\alpha}}.
\end{aligned}
$$

Combining this with the estimate for I_1 shows (4.2.26).

Case 2: $1 \le \Omega \le t^{-\beta}$. In this case we have

$$
\begin{aligned}
I_2 &= Ct^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}} \int_1^\Omega z^{\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\quad + Ct^{-\frac{(d+k)\beta}{\alpha}} \int_\Omega^{t^{-\beta}} z^{-\frac{d+k}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\quad + Ct^{-\frac{d\beta}{\alpha}} \int_{t^{-\beta}}^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\le Ct^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}} \int_1^\infty z^{\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z \\
&\quad + Ct^{-\frac{(d+k)\beta}{\alpha}} \int_\Omega^\infty z^{-\frac{d+k}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\} \, \mathrm{d}z
\end{aligned}
$$

$$
\begin{aligned}
&+ Ct^{\beta-\frac{1}{2(1-\beta)}} \exp\{-Ct^{-\frac{\beta}{1-\beta}}\} \\
\le\ & C_\beta t^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}} \\
&+ C_{d,\alpha,\beta,l} t^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-\frac{d+k}{\alpha}-\frac{1}{2(1-\beta)}} \exp\{-C\Omega^{\frac{1}{1-\beta}}\} \\
&+ Ct^{\beta-\frac{1}{2(1-\beta)}} \exp\{-Ct^{-\frac{\beta}{1-\beta}}\} \\
\le\ & C_{\beta,d,\alpha,k} t^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}}.
\end{aligned}
$$

Combining this with (4.2.29) shows (4.2.27).

Case 3: $t^{-\beta} \le \Omega$.

$$
\begin{aligned}
I_2 =\ & Ct^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}} \int_1^{t^{-\beta}} z^{\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\}\, \mathrm{d}z \\
&+\ Ct^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} \int_{t^{-\beta}}^{\Omega} z^{\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\}\, \mathrm{d}z \\
&+\ Ct^{-\frac{d\beta}{\alpha}} \int_\Omega^\infty z^{-\frac{d}{\alpha}-1+\frac{1}{2(1-\beta)}} \exp\{ct^\beta z - c_\beta z^{\frac{1}{1-\beta}}\}\, \mathrm{d}z \\
\le\ & C_\beta t^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}} \\
&+\ C_\beta t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} \\
&+\ C_{d,\beta,\alpha,l} t^{-\frac{d\beta}{\alpha}} \Omega^{-\frac{d}{\alpha}-\frac{1}{2(1-\beta)}} \exp\{-C\Omega^{\frac{1}{1-\beta}}\} \\
\le\ & C_{d,\beta,\alpha,k} t^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}}.
\end{aligned}
$$

Finally combining this with (4.2.29) shows (4.2.28). □

Next, for large (finite) time.

Proposition 4.2.6. Under the same assumptions as Theorem 4.2.4 and Theorem 2.3.2, then for fixed $T > 0$, the following estimates hold for the spatial derivatives of the Green's function $G^{(\beta)}_{\psi_\alpha,x}(t,x,y)$ for $(t,x,y) \in (1,T) \times \mathbb{R}^d \times \mathbb{R}^d$,

- For $\Omega \le 1$,

$$
\left| \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t,x,y) \right| \le\ C_{T,d,\beta,\alpha,k} \begin{cases} 1, & d+k < \alpha, \\ t^{-\beta}(|\log \Omega| + 1), & d+k = \alpha, \\ |x-y|^{\alpha-d-k}, & d+k > \alpha, \end{cases} \tag{4.2.30}
$$

for all $k \le l$ and all indicies $i_i, \cdots, i_k$.

- For $\Omega \geq 1$,

$$\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t,x,y)\right| \leq C|x-y|^{-\alpha-d}, \tag{4.2.31}$$

for all $k \leq l$ and all indicies $i_i, \cdots, i_k$.

Proof. As usual we first use the asymptotic behaviour of the stable density,

$$\begin{aligned}\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t,x,y)\right| \leq & c_\beta \int_0^1 \left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G_{\psi_\alpha,x}(t^\beta z,x,y)\right| \,\mathrm{d}z \\ & + c_\beta \int_1^\infty \left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G_{\psi_\alpha,x}(t^\beta z,x,y)\right| z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}}) \,\mathrm{d}z\end{aligned}$$

Next, we use the estimate (2.3.10) for the first term and (4.2.19) for the second. Note that since $t \in (1,T)$, then

$$\begin{aligned}\left|\frac{\partial^k}{\partial x_{i_1}\cdots\partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t,x,y)\right| \leq & \; c\int_0^1 \min\left((t^\beta z)^{-\frac{d+k}{\alpha}}, t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{-1-\frac{d+k}{\alpha}} z\right) \,\mathrm{d}z \\ & + c\int_1^\infty \min\left((t^\beta z)^{-\frac{d}{\alpha}}, t^{-\frac{d\beta}{\alpha}}\Omega^{-1-\frac{d}{\alpha}} z\right) z^{-1-\frac{1}{\beta}} e^{ct^\beta z} f_\beta(z^{-\frac{1}{\beta}}) \,\mathrm{d}z \\ := & \; I_1 + I_2\end{aligned}$$

The integral in I_1 is the same as that one appearing in (4.1.17), and thus for $\Omega \leq 1$,

$$\begin{aligned} I_1 &= Ct^{-\frac{(d+1)\beta}{\alpha}}\Omega^{-1-\frac{d+1}{\alpha}}\int_0^\Omega z\,\mathrm{d}z + Ct^{-\frac{(d+1)\beta}{\alpha}}\int_\Omega^1 z^{-\frac{d+1}{\alpha}}\,\mathrm{d}z \\ &\leq C\begin{cases} t^{-\frac{(d+k)\beta}{\alpha}}, & d+k<\alpha, \\ t^{-\beta}(|\log\Omega|+1), & d+k=\alpha, \\ t^{-\frac{(d+k)\beta}{\alpha}}\Omega^{1-\frac{d+k}{\alpha}}, & d+k>\alpha. \end{cases}\end{aligned}$$

However in this situation $t \in (1,T)$, so t is away from both 0 and ∞. Thus, recalling that $\Omega = |x-y|^\alpha t^{-\beta}$,

$$I_1 \leq C_{T,d,\beta,k,\alpha}\begin{cases} 1, & d+k<\alpha, \\ t^{-\beta}(|\log\Omega|+1), & d+k=\alpha, \\ |x-y|^{\alpha-d-k}, & d+k>\alpha. \end{cases}$$

For $\Omega \geq 1$,

$$I_1 = Ct^{-\frac{(d+1)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}} \int_0^1 z \, \mathrm{d}z = \frac{C}{2} t^{-\frac{(d+k)\beta}{\alpha}} \Omega^{-1-\frac{d+k}{\alpha}} \leq C_{T,\beta,d,\alpha,k} |x-y|^{-\alpha-d-k}.$$

Furthermore, the integral I_2 is the same as the one defined as I_2^{up} in (4.2.24), thus for $\Omega \leq 1$,

$$I_2 \leq Ct^{-\frac{d\beta}{\alpha}} \leq C_{d,\beta,\alpha,T}$$

So for $\Omega \leq 1$,

$$\left| \frac{\partial^k}{\partial x_{i_1} \cdots \partial x_{i_k}} G^{(\beta)}_{\psi_\alpha,x}(t,x,y) \right| \leq I_1 + I_2 \leq C_{T,d,\beta,\alpha,k} \begin{cases} 1, & d+k < \alpha, \\ t^{-\beta}(|\log \Omega| + 1), & d+k = \alpha, \\ |x-y|^{\alpha-d-k}, & d+k > \alpha, \end{cases}$$

which thus gives (4.2.30). Finally for $\Omega \geq 1$, using (4.2.25),

$$I_2 \leq Ct^{-\frac{d\beta}{\alpha}} \Omega^{-1-\frac{d}{\alpha}} \leq C_{T,d,\beta,\alpha} |x-y|^{-\alpha-d},$$

thus combining the estimates for I_1 and I_2 for $\Omega \geq 1$ gives us (4.2.31). □

4.3 Generalised Evolution Equations

In this last section, we look at the following generalised evolution,

$$\begin{cases} D^{(\nu)}_{0+*} u(t,x) = Au(t,x), & (0,\infty) \times \mathbb{R}^d \\ u(0,x) = \phi(x), & \{0\} \times \mathbb{R}^d, \end{cases} \tag{4.3.1}$$

where $D^{(\nu)}_{0+*}$ is the Caputo-type operator

$$D^{(\nu)}_{0+*} u(t) = -\int_0^t (f(t-r) - f(t)) \nu(t, \mathrm{d}r) - (f(0) - f(t)) \int_t^\infty \nu(t, \mathrm{d}r).$$

Here $\nu(t,\cdot)$ is a Lévy transition kernel that satisfies

$$\sup_t \int \min(1,r) \nu(t, \mathrm{d}r) < \infty.$$

The solution to Equation (4.3.1) is given by

$$u(t,x) = E_{(\nu),t}(A)\phi(x),$$

where $E_{(\nu),t}(A)$ is the *operator-valued generalised Mittag-Leffler* function which is defined by the operator-valued integral

$$E_{(\nu),t}(A) = \int_0^\infty \mathrm{d}s e^{As}\left(-\frac{\partial}{\partial s}\left(\int_0^t G_{(\nu)}(s,t,\mathrm{d}r)\right)\right) = 1 + A\Pi_{(\nu)}^{-A}(t,[0,t]), \tag{4.3.2}$$

where $\Pi_{(\nu)}^{-A}$ is the operator-valued potential measure of the semigroup $T_s^{(\nu)}e^{sA}$ generated by $(-D_{0+*}^{(\nu)} + A)$,

$$\Pi_{(\nu)}^{-A}(t,\mathrm{d}r) = \int_0^\infty \mathrm{d}s e^{As}\int_0^t G_{(\nu)}(s,t,\mathrm{d}r).$$

Then we can rewrite this solution to get the Green's function,

$$\begin{aligned} E_{(\nu),t}(A)\phi(x) &= \int_{\mathbb{R}^d}\phi(y)\int_0^\infty G^A(s,x,y)\frac{\partial}{\partial s}\left(\int_{-\infty}^0 G_{(\nu)}(s,t,\mathrm{d}r)\right)\mathrm{d}s\mathrm{d}y \\ &= \int_{\mathbb{R}^d}\phi(y)G_A^{(\nu)}(t,x,y)\ \mathrm{d}y, \end{aligned}$$

where $G_A^{(\nu)}$ is the Green's function of the evolution Equation (4.3.1) given by

$$G_A^{(\nu)}(t,x,y) := \int_0^\infty G^A(s,x,y)\frac{\partial}{\partial s}\left(\int_{-\infty}^0 G^{(\nu)}(s,t,\mathrm{d}r)\right)\mathrm{d}s.$$

We will use the following comparison principle from Kolokoltsov (2019b).

Theorem 4.3.1. *Let ν and $\tilde{\nu}$ be two Lévy measures satisfying*

$$\nu(t,\mathrm{d}r) \geq \tilde{\nu}(\mathrm{d}r),$$

$$\sup_t \int_0^\infty \min(1,r)\nu(t,\mathrm{d}r) < \infty, \qquad \int_0^\infty \min(1,r)\tilde{\nu}(\mathrm{d}r) < \infty,$$

and $\nu(t,(0,\infty)) = \tilde{\nu}((0,\infty)) = \infty$. Then for any non-increasing function f we have the comparison principle for the semigroups:

$$T_t^\nu f \geq T_t^{\tilde{\nu}} f,$$

where $T_t^{(\nu)}$ and $T_t^{\tilde{\nu}}$ are the semigroups generated by $-D^{(\nu)}$ and $-D^{(\tilde{\nu})}$

$$D^{(\nu)}f(t) = -\int_0^\infty (f(t-r) - f(t))\nu(t, \mathrm{d}r)$$

and

$$D^{(\tilde{\nu})}f(t) = -\int_0^\infty (f(t-r) - f(t))\tilde{\nu}(\mathrm{d}r)$$

respectively. Moreover, the potential measures of the semigroups T_t^{ν} and $T_t^{\tilde{\nu}}$ satisfy the comparison principle,

$$U^{(\nu)}(t, [0,t]) \leq U^{(\tilde{\nu})}([0,t]).$$

A direct application of this comparison principle gives us the following result, which allows us to obtain estimates for the solutions of generalised evolution equations by using our estimates for $G^{(\beta)}$ from the previous sections.

Theorem 4.3.2. *Let A be one of the spatial operators from (4.1.1), Theorem 4.1.3, (4.2.2) or (4.2.20) along with their relevant assumptions. Let $\nu(t, \mathrm{d}s)$ be a Lévy transition kernel which has upper and lower bounds of β-fractional type,*

$$(-1/\Gamma(-\beta_1))C_\nu s^{-1-\beta_1}\mathrm{d}s \leq \nu(t, \mathrm{d}s) \leq (-1/\Gamma(-\beta_2))C_\nu s^{-1-\beta_2}\mathrm{d}s,$$

for some $\beta_1, \beta_2 \in (0,1)$ and $C_\nu > 0$. Then

$$c_2 E_{\beta_2}(At^{\beta_2})\phi(x) \leq E_{(\nu),t}(A)\phi(x) \leq c_1 E_{\beta_1}(At^{\beta_1})\phi(x),$$

for a non-increasing function ϕ, where

$$E_{(\nu),t}(A)\phi(x) = \int_{\mathbb{R}^d} \phi(y) G_A^{(\nu)}(t,x,y)\ \mathrm{d}y,$$

and

$$E_\beta(At^\beta)\phi(x) = \int_{\mathbb{R}^d} \phi(y) G_A^{(\beta)}(t,x,y)\ \mathrm{d}y.$$

Proof. This follows from the formula (4.3.2) and an application of the comparison principle for potential operators. □

Thus the estimates obtained in Theorem 4.1.1, Theorem 4.1.3, Theorem 4.2.1 and Theorem 4.2.4 can be used to estimate solutions of generalised evolutions (4.3.1).

Remark 7. *In order to see why this result is expected, let us give some intuition behind the comparison principle. The assumption that the Lévy kernel is bounded below by the Lévy kernel of a β-stable subordinator, means that the Lévy subordinator generated by the operator $-D_+^{(\nu)}$ where*

$$D_+^{(\nu)} f(t) = -\int_0^\infty (f(x-y) - f(x))\nu(\mathrm{d}y),$$

has on average jumps that are larger than those of the process generated by

$$D_+^\beta f(t) = \int_0^\infty (f(x-y) - f(x))[y^{1+\beta}\Gamma(-\beta)]^{-1}\, \mathrm{d}y.$$

So on the sample paths level, the jumps of $X^{(\nu)}$ will typically be larger than those of X^β, which means that the inverse process of $X^{(\nu)}$ will typically be constant for longer times than the inverse process of X^β. Thus when we subordinate the spatial process, $Y(t)$, generated by the operator A by the inverse subordinator given by

$$S_t^\nu := \inf\{s \geq 0 : X_s^\nu \geq t\},$$

and compare its paths to the spatial process subordinated by an inverse stable subordinator S_t^β, we will see that $Y(S_t^\nu)$ is dominated by $Y(S_t^\beta)$ in the sense that $Y(S_t^\nu)$ will have longer trapping times.

Conclusion

In this chapter, we have looked at two-sided estimates for the Green's function of fractional evolution equations of the form

$$D^\beta u(t,x) = Lu(t,x), \quad u(0,x) = Y(x).$$

The solution of such fractional evolution equations can be written with the help of operator-valued Mittag-Leffler functions,

$$\begin{aligned} u(t,x) = E_\beta(t^\beta L)Y(x) &= \int_0^\infty e^{zt^\beta L} Y(x) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z \\ &= \int_{\mathbb{R}^d} Y(y) \int_0^\infty G_L(t^\beta z, x, y) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\, \mathrm{d}z \end{aligned}$$

$$= \int_{\mathbb{R}^d} Y(y) G_L^{(\beta)}(t,x,y) \, \mathrm{d}z.$$

We have given two-sided estimates for the Green's function $G_L^{(\beta)}(t,x,y)$ (and its spatial derivatives) in several different situations. The situations can be split up into two broad cases: when the Green's function G_L associated with L does or does not have known global in time estimates. In those two cases, we consider generators of diffusion processes in Theorems 4.1.1 and 4.2.1; and we consider generators of stable and stable-like processes in Theorems 4.1.3 and 4.2.4. Finally, we looked at generalised evolution equations where the operator acting on the time variable is given by a Caputo-type operator

$$D_{0+*}^{(\nu,t)} u(t) = -\int_0^t (u(t-s) - u(t))\nu(t, \mathrm{d}s) - \int_t^\infty (u(0) - u(t))\nu(t, \mathrm{d}s).$$

We concluded that solutions to generalised evolution equations of the form

$$D_{0+*}^{(\nu,t)} u(t,x) = Lu(t,x), \quad u(0,x) = Y(x), \tag{4.3.3}$$

where $\nu(t, \mathrm{d}s)$ is a Lévy-type kernel which for fixed t is comparable to the Lévy measure of a β-stable subordinator, could be estimated using the estimates obtained for $G_L^{(\beta)}$. Then whenever one is looking at evolution equations of the form (4.3.3), or, from the probabilistic point of view, at stochastic processes generated by $-D^{(\nu)} + L$, then under the assumption that ν is comparable to β-stable, the estimates shown in this article can be used to gain a lot of information.

Note that in this article we have viewed $G_L^{(\beta)}(t,x,y)$ as the Green's function of the evolution equation

$$D^\beta u(t,x) = Lu(t,x).$$

Probabilistically speaking, $G_L^{(\beta)}$ are the transition densities of the process $X_t^{L,\beta}$ generated by $-D^\beta - L$. The process $X_t^{L,\beta}$ is the subordination of the process generated by L by the inverse of the process generated by D^β. In this view one could use the estimates in this article to obtain sample path properties of a subordinated process $X_t^{L,\beta}$.

Chapter 5

Mixed fractional evolution equations

This chapter is based on the article Johnston and Kolokoltsov (2019b), where we obtain two-sided estimates for the Green's function of the following boundary value problem,

$$
\begin{aligned}
{}_{t_1}D^{\beta}_{0+*}u(t_1,t_2,x) + {}_{t_2}D^{\gamma}_{0+*}u(t_1,t_2,x) &= L_x u(t_1,t_2,x), \qquad (5.0.1)\\
u(0,t_2,x) &= \phi_1(t_2,x),\\
u(t_1,0,x) &= \phi_2(t_1,x).
\end{aligned}
$$

In Section 5.4 we look at a higher dimensional version of (5.0.1) in the sense that we have k fractional derivatives on the left hand side (cf. Section 3.3), each acting on a different variable,

$$\sum_{i=1}^{k} {}_{t_i}D^{\beta_i}_{0+*}u(t,x) = L_x u(t,x), \tag{5.0.2}$$

where $(t,x) \in \mathbb{R}^k_+ \times \mathbb{R}^d$, with some specified boundary behaviour. The estimates obtained in this article can be used to study more general CD-type evolution equations (see Kolokoltsov (2019a, Section 8.5)) of the form

$$\sum_{i=1}^{k} {}_{t_i}D^{\nu_i(t_i,\cdot)}_{0+*}u(t,x) = L_x u(t,x), \tag{5.0.3}$$

where each $\nu_i(t_i, \cdot)$ is a Lévy-type kernel, under the assumption that each $\nu_i(t_i, \cdot)$ has a density which is comparable to the density of a β_i-stable process. This was done for the case $k = 1$ in Johnston and Kolokoltsov (2019a), so we do not repeat it here.

5.1 Transition density of spatial process

Let $Y_x(s)$ be a diffusion process with generator $L = \nabla \cdot (a(x)\nabla)$ for some symmetric measurable function a on $\mathbb{R}^d$. Recall that Aronsons estimates, (Aronson, 1967), say that the transition densities $G^Y(s, x, y)$ of $Y_x(s)$ satisfy the following two-sided Gaussian estimates for all $s > 0$,

$$G^Y(s, x, y) \asymp s^{-\frac{d}{2}} \exp\left\{-c\frac{|x-y|^2}{s}\right\}. \tag{5.1.1}$$

Let $X_r^\alpha(s)$ be the process (independent of $Y_x(s)$) generated by $-D_{0+*}^\alpha$, which is a decreasing β-stable process absorbed at 0 on an attempt to cross it. The transition density of the process $(Y_x(s), X_r^\alpha(s))$ is given by

$$G^{Y,\gamma}(s, r, x, y) = G^Y(s, x, y) s^{-\frac{1}{\gamma}} w_\gamma(r s^{-\frac{1}{\gamma}}).$$

The following result is obtained by applying Aronsons estimate for G^Y and (2.3.12) for w_γ.

Lemma 5.1.1. The transition density of $(X_r^\gamma(s), Y_x(s))$ satisfy the following estimates

- For $s \leq r^\gamma$,

$$G^{Y,\gamma}(s, r, x, y) \asymp C r^{-1-\gamma} s^{1-\frac{d}{2}} \exp\left\{-c\frac{|x-y|^2}{s}\right\}.$$

- For $s > r^\gamma$,

$$G^{Y,\gamma}(s, r, x, y) \asymp C r^{-\frac{2-\gamma}{2(1-\gamma)}} s^{\frac{1}{2(1-\gamma)}-\frac{d}{2}} \exp\left\{-c\frac{|x-y|^2}{s} - c s^{\frac{1}{1-\gamma}} r^{-\frac{\gamma}{1-\gamma}}\right\}.$$

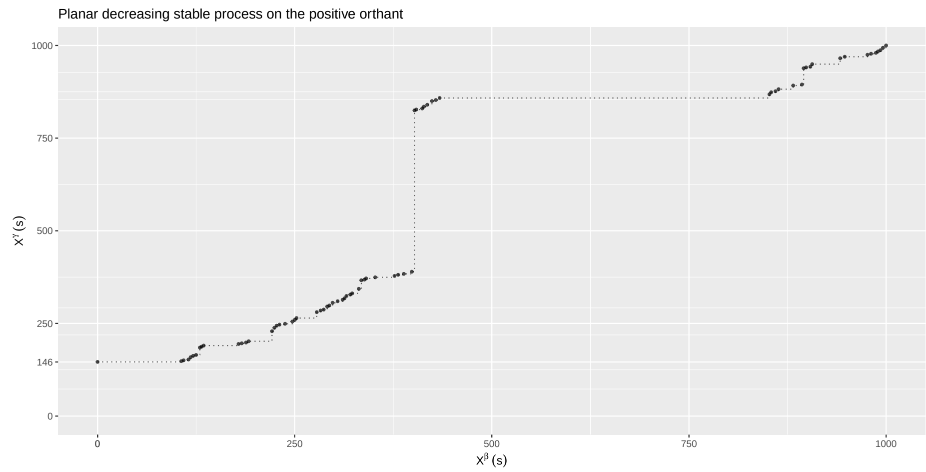

Figure 5.1: Sample path of $X^{\beta,\gamma}_{t_1,t_2}(s)$ until the time $s = \tau^{\beta,\gamma}_0$ when it hits the boundary and $X^{\beta,\gamma}_{t_1,t_2}(\tau^{\beta,\gamma}_0) = (149, 0)$ in this case. Here $\beta = \gamma = 0.8$ and $t_1 = t_2 = 1000$. Made using the R packages ggplot2 (Wickham, 2016) and stabledist (Wuertz et al., 2016).

5.2 Processes on the orthant

Consider the process living on $\mathbb{R}^2_+$ defined by $X^{\beta,\gamma}_{t_1,t_2}(s) := (X^\beta_{t_1}(s), X^\gamma_{t_2}(s))$, where each coordinate a one-dimensional stable subordinator (with inverted sign) which absorbed at 0, as described in the previous subsection. The process $X^{\beta,\gamma}_{t_1,t_2}(s)$ is generated by $-\,{}_{t_1}D^\beta_{0+*} - {}_{t_2}D^\gamma_{0+*}$, where $\beta, \gamma \in (0, 1)$, and it is started at $(t_1, t_2) \in \mathbb{R}_+ \times \mathbb{R}_+$. For clarity, see Figure 5.1 for a typical sample path of $X^{\beta,\gamma}_{t_1,t_2}(s)$. We assume that the processes X^β and X^γ are independent. This independence assumption implies that the first time the process $X^{\beta,\gamma}_{t_1,t_2}$ hits the boundary of $\mathbb{R}_+ \times \mathbb{R}_+$ is given by

$$\tau^{\beta,\gamma}_0 = \min\left(\tau^\beta_0, \tau^\gamma_0\right).$$

5.3 Mixed linear evolution

Consider the problem

$$\begin{aligned}({}_{t_1}D^{\beta}_{0+*} + {}_{t_2}D^{\gamma}_{0+*})f(t_1,t_2,x) &= Af(t_1,t_2,x),\\ f(0,t_2,x) &= \phi_1(t_2,x),\\ f(t_1,0,x) &= \phi_2(t_1,x).\end{aligned} \tag{5.3.1}$$

Here A is the generator of a Feller process $Y_x(s)$ started at $x \in \mathbb{R}^d$. For simplicity let us take $A = \nabla\cdot(a(x)\nabla)$, where $a(x)$ is a symmetric, uniformly elliptic and measurable function so that A generates a non-degenerate diffusion, with transition densities $G^Y(s,x,y)$ which satisfy Aronsons two-sided estimates (5.1.1).

Remark 8. *Note that we could also obtain estimates for the Green's function in the case when L is, say, a non-isotropic homogeneous pseudo-differential operator of order $\alpha \in (0,2)$ whose symbol is of the form*

$$\psi_\alpha(x,p) = |p|^\alpha w(x,p/|p|), \quad x \in \mathbb{R}^d,$$

where $w(x,\cdot)$ is some strictly positive function on $\mathbb{S}^{d-1}$. See Eidelman, Ivasyshen, et al. (2004) and Kolokoltsov (2000) for the relevant estimates for G^Y in that case.

5.3.1 Well-posedness of the mixed boundary value problem

Let us briefly discuss the well-posedness of problem (5.3.1). We only sketch the main steps, but see Kolokoltsov (2019a, Chapter 8), Hernández-Hernández, Kolokoltsov, and Toniazzi (2017, Theorem 4.20) or Kolokoltsov (2019b, Section 4) for a full account of well-posedness for these types of problems. For even more general operators A generating Feller semigroups (and even generalised versions of Caputo-derivatives), one can obtain both uniqueness and the stochastic representation (5.3.3) of the solution to (5.3.1) via Dynkin's formula (Dynkin, 1965, Theorem 5.1). To obtain existence of a classical solution, the main idea is to first transform the problem to an equivalent one involving zero boundary conditions and Riemann-Liouville fractional derivatives (by introducing a new unknown function $u(t_1,t_2,x) = f(t_1,t_2,x) - \mathbf{1}_{\{t_2>0\}}\phi_1(t_2,x) - \mathbf{1}_{\{t_1>0\}}\phi_2(t_1,x))$. This equivalent problem is then the following RL-type mixed boundary value problem,

$$\begin{aligned}({}_{t_1}D^{\beta}_{0+} + {}_{t_2}D^{\gamma}_{0+} - A)u(t_1,t_2,x) &= g_\phi(t_1,t_2,x), \\ u(0,t_2,x) = u(t_1,0,x) &= 0,\end{aligned} \tag{5.3.2}$$

where

$$g_\phi(t_1,t_2,x) = ({}_{t_2}D^{\gamma}_{0+*} - A)\phi_1(t_2,x) + ({}_{t_1}D^{\beta}_{0+*} - A)\phi_2(t_1,x).$$

Notice that here we require ϕ_1 and ϕ_2 to be in the domain of the generators $(-{}_{t_2}D^{\gamma}_{0+*} + A)$ and $(-{}_{t_1}D^{\beta}_{0+*} + A)$ respectively. The unique solution in the domain of the generator to (5.3.2) is then found by applying the potential operator (of the semigroup $T^{\beta}_s T^{\gamma}_s e^{sA}$ generated by $(-{}_{t_1}D^{\beta}_{0+} - {}_{t_2}D^{\gamma}_{0+} + A)$) to the forcing term $g_\phi(t_1,t_2,x)$. The solution to the Caputo problem (5.3.1) is then recovered by undoing the shift by ϕ_1 and ϕ_2,

$$\begin{aligned}f(t_1,t_2,x) = \mathbf{1}_{\{t_1=0\}}&\phi_1(t_2,x) + \mathbf{1}_{\{t_2=0\}}\phi_2(t_1,x) \\ &+ \mathbf{1}_{\{t_1=0\}}\int_0^{t_2}\int_0^{\infty} e^{rA}G_{\beta,\gamma}(r,s_2)\mathrm{d}r(-{}_{t_2}D^{\gamma}_{0+*} + A)\phi_1(t_2-s_2,x)\mathrm{d}s_2 \\ &+ \mathbf{1}_{\{t_2=0\}}\int_0^{t_1}\int_0^{\infty} e^{rA}G_{\beta,\gamma}(r,s_1)\mathrm{d}r(-{}_{t_1}D^{\beta}_{0+*} + A)\phi_2(t_1-s_1,x)\mathrm{d}s_1,\end{aligned}$$

where $G_{\beta,\gamma}(r,s)$ is the transition density of the process generated by $({}_{t_1}D^{\beta}_{0+*} + {}_{t_2}D^{\gamma}_{0+*})$. Rearranging and using Kolokoltsov (2019b, Equation 4.126) we have

$$f(t_1,t_2,x) = E^{(\beta,\gamma)}\left[t_1^{\beta}L^{\gamma}\right]\phi_1(t_2,x) + E^{(\beta,\gamma)}\left[t_2^{\gamma}L^{\beta}\right]\phi_2(t_1,x)$$

for $L^{\gamma} := (-{}_{t_2}D^{\gamma}_{0+*} + A)$ and $L^{\beta} := (-{}_{t_1}D^{\beta}_{0+*} + A)$. Here $E^{(\beta,\gamma)}[D]$ are *generalised operator-valued Mittag-Leffler functions*, which are introduced and extensively studied in the survey Kolokoltsov (2019b),

$$E^{(\beta,\gamma)}\left[t_1^{\beta}L^{\gamma}\right]\phi_1(t_2,x) = \int_0^{\infty} e^{sL^{\gamma}}\phi_1(t_2,x)\mu_0^{\beta}(s)\ \mathrm{d}s,$$

where $\mu_0^\beta(s)$ is the density of the exit time τ_0^β; we will justify this in the next subsection.

5.3.2 Estimates for Green's function

As mentioned in the previous section, an application of Dynkin's formula followed by Doobs optimal stopping theorem gives the following stochastic representation of the solution (whenever it exists) to (5.3.1),

$$f(t_1,t_2,x) = \mathbb{E}\left[\phi_1\left(X^\gamma_{t_2}(\tau_0^\beta), Y_x(\tau_0^\beta)\right)\mathbf{1}_{\{\tau_0^\beta<\tau_0^\gamma\}} + \phi_2\left(X^\beta_{t_1}(\tau_0^\gamma), Y_x(\tau_0^\gamma)\right)\mathbf{1}_{\{\tau_0^\gamma<\tau_0^\beta\}}\right] \tag{5.3.3}$$

A simple conditioning argument (see Appendix A.1), shows that this solution can be written as

$$\begin{aligned} f(t_1,t_2,x) &= \int_0^{t_2}\int_{\mathbb{R}^d}\phi_1(r,y)\left(\int_0^\infty G^Y(s,x,y)p_s^\gamma(t_2,r)\mu_0^\beta(s)\,\mathrm{d}s\right)\,\mathrm{d}y\mathrm{d}r \\ &\quad+\int_0^{t_1}\int_{\mathbb{R}^d}\phi_2(r,y)\left(\int_0^\infty G^Y(s,x,y)p_s^\beta(t_1,r)\mu_0^\gamma(s)\,\mathrm{d}s\right)\,\mathrm{d}y\mathrm{d}r, \qquad (5.3.4) \\ &=: \int_0^{t_2}\int_{\mathbb{R}^d}\phi_1(r,y)G_1^{(\beta,\gamma)}(t_1,r,x,y)\,\mathrm{d}y\mathrm{d}r \\ &\quad+\int_0^{t_1}\int_{\mathbb{R}^d}\phi_2(r,y)G_2^{(\beta,\gamma)}(t_2,r,x,y)\,\mathrm{d}y\mathrm{d}r, \end{aligned}$$

where

$$G_1^{(\beta,\gamma)}(t_1,r,x,y) := \frac{t_1}{\beta}\int_0^\infty G^Y(s,x,y)s^{-1-\frac{1}{\beta}-\frac{1}{\gamma}}w_\gamma(rs^{-\frac{1}{\gamma}})w_\beta(t_1 s^{-\frac{1}{\beta}})\,\mathrm{d}s,$$

and

$$G_2^{(\beta,\gamma)}(t_2,r,x,y) := \frac{t_2}{\gamma}\int_0^\infty G^Y(s,x,y)s^{-1-\frac{1}{\beta}-\frac{1}{\gamma}}w_\beta(rs^{-\frac{1}{\beta}})w_\gamma(t_2 s^{-\frac{1}{\gamma}})\,\mathrm{d}s.$$

Note in the above we have used (2.3.13) for densities μ_0^α and μ_0^β of the exit times τ_0^α and τ_0^β. On the other hand, rearranging (5.3.4) we find

$$f(t_1,t_2,x) = \int_0^\infty\left(\int_0^{t_2}\left(\int_{\mathbb{R}^d}\phi_1(r,y)G^Y(s,x,y)\mathrm{d}y\right)p_s^\gamma(t_2,r)\mathrm{d}r\right)\mu_0^\beta(s)\mathrm{d}s$$

$$
\begin{aligned}
&+ \int_0^\infty \left(\int_0^{t_1} \left(\int_{\mathbb{R}^d} \phi_2(r,y) G^Y(s,x,y) \mathrm{d}y \right) p_s^\beta(t_1,r)\mathrm{d}r \right) \mu_0^\gamma(s)\mathrm{d}s \\
&= \frac{t_1}{\beta} \int_0^\infty e^{s(A-D_{0+*}^\gamma)} \phi_1(t_2,x) s^{-1-\frac{1}{\beta}} w_\beta(t_1 s^{-\frac{1}{\beta}})\mathrm{d}s \\
&\quad + \frac{t_2}{\gamma} \int_0^\infty e^{s(A-D_{0+*}^\beta)} \phi_2(t_1,x) s^{-1-\frac{1}{\gamma}} w_\gamma(t_2 s^{-\frac{1}{\gamma}})\mathrm{d}s \\
&= \frac{1}{\beta} \int_0^\infty e^{zt_1^\beta(A-D_{0+*}^\gamma)} \phi_1(t_2,x) z^{-1-\frac{1}{\beta}} w_\beta(z^{-\frac{1}{\beta}})\mathrm{d}z \\
&\quad + \frac{1}{\gamma} \int_0^\infty e^{zt_2^\gamma(A-D_{0+*}^\beta)} \phi_2(t_1,x) z^{-1-\frac{1}{\gamma}} w_\beta(z^{-\frac{1}{\gamma}})\mathrm{d}z,
\end{aligned}
$$

where we have made in the last step the substitutions $z = st^\beta$ in the first integral and $z = st^\gamma$ in the second.

Remark 9. *Note that this means*

$$
f(t_1,t_2,x) = E_\beta[t_1^\beta L^\gamma]\phi_1(t_2,x) + E_\gamma[t_2^\gamma L^\beta]\phi_2(t_1,x)
$$

where $L^\gamma = A - D_{0+*}^\gamma$ *and* $L^\beta = A - D_{0+*}^\beta$*, where* E_α *is the (operator) valued Mittag-Leffler function,*

$$
E_\beta(L)\phi(t,x) = \int_0^\infty e^{sL}\phi(t,x) s^{-1-\frac{1}{\beta}} w_\beta(s^{-\frac{1}{\beta}})\ \mathrm{d}s.
$$

Thus, the *Green's function* associated to (5.3.1) are the coordinates of the integral kernel of the operator which acts on the boundary functions ϕ_1 and ϕ_2:

$$
\begin{aligned}
(\phi * G_{full})(t_1,t_2,x) &= \int_{\partial\mathbb{R}_+^2 \times \mathbb{R}^d} \phi(r_1,r_1,y) G_{full}^{\beta,\gamma}(t_1,t_2,r_1,r_2,x,y)\ \mathrm{d}y\mathrm{d}r_1\mathrm{d}r_2 \\
&= \int_{\partial\mathbb{R}_+ \times \mathbb{R}^d} \phi_1(r_2,y) G_1^{(\beta,\gamma)}(t_1,r_2,x,y)\mathrm{d}r_2\mathrm{d}y \\
&\quad + \int_{\partial\mathbb{R}_+ \times \mathbb{R}^d} \phi_2(r_1,y) G_2^{(\beta,\gamma)}(t_2,r_1,x,y)\mathrm{d}r_1\mathrm{d}y \\
&= (\phi_1 * G_1^{(\beta,\gamma)})(t_1,x) + (\phi_2 * G_2^{(\beta,\gamma)})(t_2,x).
\end{aligned}
$$

Remark 10. *More generally, the function*

$$
f(x) = (\phi * G_A)(x) = \int_{\partial X} \phi(z) G_A(x,z)\ \mathrm{d}z,
$$

solves the boundary value problem

$$\begin{aligned} Af(x) &= 0, \quad x \in X, \\ f(z) &= \phi(z), \quad z \in \partial X, \end{aligned}$$

where ϕ is a suitable function on the boundary of X.

For this reason, to obtain global two-sided estimates for the *full Green's function* $G_{full} = (G_1^{(\beta,\gamma)}, G_2^{(\beta,\gamma)})$, it suffices to obtain estimates for $G_1^{(\beta,\gamma)}$, since the estimates for $G_2^{(\beta,\gamma)}$ will be the same up to exchanging coordinates. For the sake of readability we drop the subscripts from $G_1^{(\beta,\gamma)}$ and t_1 and look only at the function

$$G^{(\beta,\gamma)}(t,x;r,y) := G_1^{(\beta,\gamma)}(t_1,x;r,y).$$

Making the substitution $s = t^{\beta} z$, we have

$$\begin{aligned} G^{(\beta,\gamma)}(t,x;r,y) &= t^{-\frac{\beta}{\gamma}} \int_0^{\infty} G^Y(t^{\beta}z,x,y) z^{-1-\frac{1}{\beta}-\frac{1}{\gamma}} w_{\gamma}(rt^{-\frac{\beta}{\gamma}} z^{-\frac{1}{\gamma}}) w_{\beta}(z^{-\frac{1}{\beta}})\ \mathrm{d}z \\ &= \int_0^{\infty} G^{Y,\gamma}(t^{\beta}z,r,x,y) t^{\beta} \mu_0^{\beta}(t^{\beta}z)\ \mathrm{d}z, \end{aligned} \tag{5.3.5}$$

where $G^{Y,\gamma}$ and μ_0^{β} are as in Lemma 2.3.3 and Lemma 5.1.1. Let $\Omega := |x-y|^2 t^{-\beta}$, $A = r^{\gamma} t^{-\beta}$.

Proposition 5.3.1. For $(t,r,x,y) \in (0,\infty) \times (0,t_2) \times \mathbb{R}^d \times \mathbb{R}^d$ and $t_2 \in (0,\infty)$, the following estimates hold,

- For $\Omega \leq 1$,

$$G^{(\beta,\gamma)}(t,r,x,y) \asymp C t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\gamma} \begin{cases} C, & d \leq 3 \\ (|\log(\Omega(A^{-1} \wedge 1))| + 1), & d = 4, \\ \Omega^{2-\frac{d}{2}}, & d \geq 5. \end{cases} \tag{5.3.6}$$

- For $\Omega \geq 1$,

$$G^{(\beta,\gamma)}(t,r,x,y) \asymp C t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} \Omega^{N_1} A^{N_2} \exp\left\{ - \left(\Omega(\max\{A^{-1},1\}\right)^{\frac{1}{2-\min(\beta,\gamma)}} \right\}, \tag{5.3.7}$$

where

$$N_1 = -\frac{d}{2}\left(\frac{1-\alpha}{2-\alpha}\right) + \frac{1-\alpha}{2(2-\alpha)(1-\tilde{\alpha})}$$
$$N_2 = -\frac{d}{2}\left(\frac{1}{2-\alpha}\right) + \frac{1}{2(2-\alpha)(1-\tilde{\alpha})} + \frac{1}{2(1-\alpha)} - \frac{2-\gamma}{2\gamma(1-\gamma)}$$
$$\alpha = \min(\beta,\gamma)$$
$$\tilde{\alpha} = \max(\beta,\gamma).$$

Proof. We sketch the main ideas of the proof here, see Appendix A.2 for the full details of the calculations. After applying Lemma 2.3.3 and Lemma 5.1.1 in (5.3.5), we end up with 4 integrals which contribute to the estimate for $G^{(\beta,\gamma)}$. For $\Omega \le 1$, the main contribution comes from the integral

$$I_1 = t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\frac{1}{\gamma}} \int_0^{A\wedge 1} z^{1-\frac{d}{2}} \exp\left\{-\Omega z^{-1}\right\} \,\mathrm{d}z.$$

After a substitution of $w = \Omega z^{-1}$, we immediately recognise the integral form of the incomplete gamma function, see Section 2.4,

$$I_1 = t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\frac{1}{\gamma}} \Omega^{2-\frac{d}{2}} \int_{(A^{-1}\vee 1)\Omega}^{\infty} w^{\frac{d}{2}-3} \exp\{-\Omega w\} \,\mathrm{d}w.$$

Thus we have the two-sided estimate for I_1 for $\Omega \le 1$,

$$I_1 \asymp C t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\gamma} \begin{cases} C, & d=1,2,3, \\ (|\log(\Omega(\max\{A^{-1},1\})| + 1), & d=4, \\ \Omega^{2-\frac{d}{2}}, & d \ge 5. \end{cases}$$

Since the integral I_1 is the main contributor to the estimate, this proves (5.3.6). For $\Omega \ge 1$, the main contribution to the estimate comes from the integral

$$I_4 = t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_{A\vee 1}^{\infty} z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}+\frac{1}{2(1-\gamma)}} \exp\left\{-\Omega z^{-1} - A^{-\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}} - z^{\frac{1}{1-\beta}}\right\} \,\mathrm{d}z.$$

To estimate this integral, let $\alpha = \min(\beta,\gamma)$ and $\tilde{\alpha} = \max(\beta,\gamma)$. Then as an upper (resp. lower) bound for I_4 we replace the powers in the exponential term with α (resp. $\tilde{\alpha}$). That is, the upper estimate

$$I_4 \leq C_1 t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_{A\vee 1}^{\infty} z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}+\frac{1}{2(1-\gamma)}} \exp\left\{-\Omega z^{-1} - A^{-\frac{1}{1-\alpha}} z^{\frac{1}{1-\alpha}}\right\} \mathrm{d}z,$$

and the lower estimate

$$I_4 \geq C_2 t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_{A\vee 1}^{\infty} z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}+\frac{1}{2(1-\gamma)}} \exp\left\{-\Omega z^{-1} - A^{-\frac{1}{1-\tilde{\alpha}}} z^{\frac{1}{1-\tilde{\alpha}}}\right\} \mathrm{d}z.$$

Then an application of Proposition 2.4.1 from the Preliminaries proves (5.3.7), and we are done. $\square$

5.4 Extension to higher dimension

Let us outline how to extend the previous sections to the case where we have more than two fractional derivatives. Let $\mathcal{O}$ be the orthant in $\mathbb{R}^k$ defined by

$$\mathcal{O} := \{(t_1, \cdots, t_k) \in \mathbb{R}^k, t_i \geq 0, i \in \{1, \cdots, k\}\}.$$

Let $\mathcal{O}_{i,0}$ denote the collection of vectors $t_{i,0}$ from $\mathcal{O}$ whose i-th coordinate is zero,

$$\mathcal{O}_{i,0} := \{t_{i,0} = (t_1, \cdots, t_{i-1}, 0, t_{i+1}, \cdots, t_k)\}.$$

Define $h_{i,0}(t)$ to be the projection of $\mathcal{O}_{i,0}$ onto the subspace $\mathcal{O}_i \subset \mathbb{R}^{k-1}$ by removing the coordinate which is zero, that is, $h_{i,0}(t) : \mathcal{O}_{i,0} \mapsto \mathcal{O}_i$

$$h_{i,0}(t_{i,0}) = (t_1, \cdots, t_{i-1}, t_{i+1}, \cdots, t_k).$$

We look at the equations on $\mathcal{O} \times \mathbb{R}^d$,

$$\begin{aligned} \left(\sum_{i=1}^{k} {}_{t_i}D_{0+*}^{\beta_i} - A\right) f(t,x) &= 0, && \text{on } \mathcal{O} \times \mathbb{R}^d, \qquad (5.4.1) \\ f(t_{i,0}, x) &= \phi_i(h_{i,0}(t_{i,0}), x), && \text{on } \mathcal{O}_{i,0} \times \mathbb{R}^d, \end{aligned}$$

where each ϕ_i is a function on $\mathcal{O}_i \times \mathbb{R}^d$.

Remark 11. *In order to have continuity of the solution to the above boundary value problem, we would need to also impose additional boundary conditions* *in order to ensure that the solution coincides at the points where the boundary meets - i.e, at the origin. Without this additional assumption we only have a* generalised *solution, which is enough for our purposes.*

As before, let $X^{\beta_i}_{t_i}(s)$ denote the process started at $t_i \in \mathbb{R}_+$ generated by $-D^{\beta_i}_{0+*}$ where $\beta_i \in (0,1)$, and let $\tau^{\beta_i}_0$ denote the exit time of this process from $(0,\infty)$,

$$\tau^{\beta_i}_0 := \inf\{s > 0 : X^{\beta_i}_{t_i}(s) \le 0\}.$$

Let $X^{\overline{\beta}}_t(s) = (X^{\beta_1}_{t_1}(s), \cdots, X^{\beta_k}_{t_k})$ be the process on $\mathcal{O}$ generated by

$$-\,{}_tD^{\overline{\beta}}_{0+*} := -\sum_{i=1}^{k} {}_{t_i}D^{\beta_i}_{0+*},$$

and due to the independence of each process $X^{\beta_i}_{t_i}$, the exit time of $X^{\overline{\beta}}_t(s)$ from the orthant $\mathcal{O}$ is given by

$$\tau^{\overline{\beta}}_0 = \min_{i\in\{1,\cdots,k\}} \tau^{\beta_i}_0.$$

For $t \in \mathbb{R}^k_+$, let $B_i(t)$ denote the subset of $\mathcal{O}_i$ defined by

$$B_i(t) := \{r \in \mathcal{O}_i,\ r_j \le t_j,\ j \ne i\},$$

i.e, B_i consists of elements of the form

$$[0,t_1] \times \cdots \times [0,t_{i-1}] \times [0,t_{i+1}] \times \cdots \times [0,t_k] \in \mathcal{O}_i.$$

The solution to (5.4.1) is given by

$$\begin{aligned} f(t,x) &= \mathbb{E}\left[\sum_{i=1}^{k} \phi_i(h_{i,0}(X^{\overline{\beta}}_t(\tau^{\overline{\beta}}_0), Y_x(\tau^{\overline{\beta}}_0))\mathbf{1}_{\{\tau^{\overline{\beta}}_0 = \tau^{\beta_i}_0\}}\right] \\ &= \sum_{i=1}^{k}\left(\int_{B_i(t)}\int_{\mathbb{R}^d} \phi_i(r,y))\left(\int_0^\infty p^Y(s,x,y)\prod_{j\ne i}^{k} p^{\beta_j}_s(t_j,r_j)\mu^{\beta_i}_0(s)\ \mathrm{d}s\right)\ \mathrm{d}y\mathrm{d}r\right). \end{aligned}$$

Remark 12. *The last equality above is a straightforward combination (or extension) of Proposition A.1.1 and the proof of (3.3.5).*

Thus the objects we are interested in is

$$G^{(\beta_i)}(t_i, x; r, y) = \int_0^\infty p^Y(s, x, y) \prod_{j \neq i}^{k} p_s^{\beta_j}(t_j, r_j) \mu_0^{\beta_i}(s) \, \mathrm{d}s,$$

where $(t_i, x) \in \mathbb{R}_+ \times \mathbb{R}^d$, and $(r, y) \in \mathcal{O}_i \times \mathbb{R}^d$. Note that

$$\prod_{j \neq i}^{k} p_s^{\beta_j}(t_j, r_j) = \prod_{j \neq i}^{k} s^{-\frac{1}{\beta_j}} w_{\beta_j}(r_j s^{-\frac{1}{\beta_j}})$$

and

$$\mu_0^{\beta_i}(s) = \frac{t_i}{\beta_i} s^{-1-\frac{1}{\beta_i}} w_{\beta_i}(t_i s^{-\frac{1}{\beta_i}}),$$

thus

$$\begin{aligned} G^{(\beta_i)}(t_i, x; r, y) &= \frac{t_i}{\beta_i} \int_0^\infty p^Y(s, x, y) s^{-1-\sum_{i=1}^k \frac{1}{\beta_i}} \prod_{j \neq i}^{k} w_{\beta_j}(r_j s^{-\frac{1}{\beta_j}}) w_{\beta_i}(t_i s^{-\frac{1}{\beta_i}}) \, \mathrm{d}s \\ &= \frac{t_i}{\beta_i} \int_0^\infty G^Y(s, r, x, y) \mu_0^\beta(s) \, \mathrm{d}s \end{aligned}$$

Focusing on the first coordinate, we have

$$\begin{aligned} G^{(\beta_1)}(t_1, x; r, y) &= \frac{t_1}{\beta_1} \int_0^\infty p^Y(s, x, y) s^{-1-\sum_{i=1}^k \frac{1}{\beta_i}} \prod_{j=2}^{k} w_{\beta_j}(r_j s^{-\frac{1}{\beta_j}}) w_{\beta_1}(t_1 s^{-\frac{1}{\beta_1}}) \, \mathrm{d}s \\ &= \int_0^\infty G^{Y,k}(s, r, x, y) \mu_0^{\beta_1}(s) \, \mathrm{d}s, \end{aligned}$$

where $\mu_0^\beta(s)$ is the density of the exit time $\tau_0^{\beta_1}$, and $G^{Y,k}(s, r, x, y)$ is the density of the process $(Y_x(s), X_{r_2}^{\beta_2}(s), X_{r_3}^{\beta_3}(s), \cdots, X_{r_k}^{\beta_k}(s))$,

$$\begin{aligned} G^{Y,k}(s, r, x, y) &= p^Y(s, x, y) s^{-\sum_{i=2}^k \frac{1}{\beta_i}} \prod_{j=2}^{k} w_{\beta_j}(r_j s^{-\frac{1}{\beta_j}}) \\ &\asymp s^{-\frac{d}{2}} \exp\left\{ -\frac{|x-y|^2}{s} \right\} s^{-\sum_{i=2}^k \frac{1}{\beta_i}} \prod_{j=2}^{k} w_{\beta_j}(r_j s^{-\frac{1}{\beta_j}}) \\ &\asymp s^{-\frac{d}{2}} \exp\left\{ -\frac{|x-y|^2}{s} \right\} \Bigg(s^{k-1} \prod_{j=2}^{k} r_j^{-1-\beta_j} \mathbf{1}_{\{s < r_j^{\beta_j}\}} \\ &\quad + \cdots \end{aligned}$$

$$
\begin{aligned}
&+ s^{n-1}\prod_{j=2}^{n} r_j^{-1-\beta_j}\mathbf{1}_{\{s<r_j^{\beta_j}\}}\prod_{i=n+1}^{k} s^{-\frac{1}{\beta_i}} f_{\beta_j}(r_i s^{-\frac{1}{\beta_i}})\mathbf{1}_{\{s>r_i^{\beta_i}\}}\\
&+\cdots\\
&+\prod_{i=2}^{k} s^{-\frac{1}{\beta_i}} f_{\beta_i}(r_i s^{-\frac{1}{\beta_i}})\mathbf{1}_{\{s>r_j^{\beta_j}\}}\Bigg)
\end{aligned}
$$

where the cross terms runs from $n = k-1$ down to $n = 1$ in the above above and are the mixtures of long and short tails. Note that we use the convention that $\prod_{i=2}^{1} = \prod_{i=k+1}^{k} = 1$. The RHS in the above can be written more compactly as

$$
s^{-\frac{d}{2}}\exp\left\{-\frac{|x-y|^2}{s}\right\}\sum_{n=1}^{k} s^{n-1}\prod_{j=2}^{n} r_j^{-\beta_j-1}\mathbf{1}_{\{s<r_j^{\beta_j}\}}\prod_{i=n+1}^{k} s^{-\frac{1}{\beta_i}} f_{\beta_i}(r_i s^{-\frac{1}{\beta_i}})\mathbf{1}_{\{s>r_i^{\beta_i}\}}.
$$

Note also that

$$
\prod_{i=n+1}^{k} s^{-\frac{1}{\beta_i}} f_{\beta_i}(r_i s^{-\frac{1}{\beta_i}}) = \prod_{i=n+1}^{k} s^{\frac{1}{2(1-\beta_i)}}\exp\left\{-c_{\beta_j}\left(r^{-\beta_j}s\right)^{\frac{1}{1-\beta_j}}\right\} r_j^{-\frac{2-\beta_j}{2(1-\beta_j)}}
$$

Let $A_1 = t_1^{-\beta_1}\prod_{i=2}^{k} r_i^{\beta_i}$ and $\Omega = |x-y|^2 t_1^{-\beta_1}$.

Conjecture 5.4.1. *For $(t_1, r, x, y) \in (0,\infty)\times\mathcal{O}_1\times\mathbb{R}^d\times\mathbb{R}^d$, we have the following two-sided estimates for the Green's function $G^{(\beta_1)}$,*

- *For $\Omega \leq 1$,*

$$
G^{(\beta_1)}(t_1, r, x, y) \asymp C t_1^{-\frac{d\beta_1}{2}}\Pi_1\begin{cases} C, & d \leq 2k-1,\\ \left|\log\left(\frac{\Omega}{\min\{A_1,1\}}\right)\right| + 1, & d = 2k,\\ \Omega^{2-\frac{d}{2}}, & d \geq 2k+1,\end{cases}
$$

where $\Pi_1 = \prod_{i=2}^{k} t_1^{-\frac{\beta_1}{\beta_i}} A_1^{-1-\frac{1}{\beta_i}}$.

- *For $\Omega \geq 1$,*

$$
G^{(\beta_1)}(t_1, r, x, y) \asymp \Pi_2 t_1^{-\frac{d\beta_1}{2}} A_1^{N_1}\Omega^{N_1}\exp\left\{-\left(\frac{\Omega}{\min\{A_1,1\}}\right)^{\frac{1}{2-\alpha}}\right\},
$$

where $\Pi_2 = \left(\prod_{i=2}^{k} t_1^{-\frac{\beta_1}{\beta_i}} A_1^{-\frac{2-\beta_i}{2\beta_i(1-\beta_i)}}\right)$, $\alpha = \min\{\beta_1, \cdots, \beta_k\}$, and the pow-

ers N_1 *and* N_2 *depend on* k, d *and* β_i *for* $1 \leq i \leq k$.

Chapter 6

Applications

In this final Chapter we discuss a possible application of our results to the area of finance. In the previous Chapter we looked at evolution equations on $\mathcal{O}^k \times \mathbb{R}^d$ (in particular the case $k = 2$), where $\mathcal{O}^k$ is the orthant in $\mathbb{R}^k$. A natural place where processes on the orthant $\mathcal{O}^2$ appear is in the modelling of *limit order books*. In particular we will consider the following boundary value problem,

$$\begin{aligned} {}_{t_1}D^{\beta}_{0+*}u(t_1,t_2) + {}_{t_2}D^{\gamma}_{0+*}u(t_1,t_2) &= 0, \quad \text{on } \mathcal{O}^2, \\ u(0,t_2) &= \phi_1(t_2), \quad \text{on } \{0\} \times \mathbb{R}_+, \\ u(t_1,0) &= \phi_2(t_1), \quad \text{on } \mathbb{R}_+ \times \{0\}. \end{aligned} \tag{6.0.1}$$

6.1 Limit order books: overview

A limit order book device used by many organized electronic markets to keep track of the interest of market participants. When an order arrives at an exchange, it waits in a *limit order book* to be executed. Market participants have two main options to post buy or sell orders, namely, *limit orders* and *market orders*. A limit order is an order to trade a certain volume of a stock[1] at a given specified price. The limit order book is the collection of all available limit orders. A market order is an order to buy or sell a certain volume of the stock at the best available price in the limit order book. Market orders are not added to the order book, instead the trade occurs immediately and the order

[1]or to trade another type of security, like equities, futures or derivatives.

book gets updated. Orders may also be cancelled at any time, but if one is concerned with just the evolution of the volume of orders, this has the same effect on the volume of trades available as a market order (or a limit order which is submitted at a price which can be immediately executed).

At any given time an order book holds the number of outstanding orders which are awaiting execution. Each order submitted to an order book consists of a collection of numbers indicating, among other things, (i) the price, (ii) the volume and (iii) the direction (buy or sell). From this point of view, the volume of trades available at a given price behaves like a queuing process, whose *arrivals* are the incoming orders while *jobs* are completed through one of two ways:

- A trade is executed via a market order (or limit buy/sell order that is less/more than the bid/ask price) of the opposite type.
- A cancellation occurs (either because the time limit was reached, or the traded cancelled the order).

The bid price p_b is defined as the highest price at which there is a limit buy order, while the ask price p_a is the lowest price at which there is a limit sell order. See Figure 6.1 for a schematic illustration of limit order book. There are many works that model the limit order books, see the recent book Abergel, Anane, et al. (2016) and references therein for a nice overview of the various methods used. Viewing the dynamics of orders and prices as a queuing system is a popular method of modelling limit order books. For example in Abergel and Jedidi (2013), Cont and De Larrard (2012), Kruk (2003), and Lipton et al. (2013), the authors consider various diffusive and fluid limits of bid/ask prices, essentially by considering the heavy traffic limits of order arrivals. In this view the volume process $V = (V_b, V_a)$ converges to a reflected Brownian motion in the positive orthant $\mathcal{O}^2$, which is restarted from within the orthant whenever it hits the boundary, according to some distribution $R = (R_1, R_2)$. The distribution R is the distribution of the sizes of the queues at prices 'behind' the prices p_b and p_a. Some works try to model each level of the order books, which requires knowing the distribution R, see Cont, Stoikov, et al. (2010) and Hambly et al. (2018), while it is simpler to consider only the highest level of the order book (i.e, only the prices and volumes at the bid and ask prices) see for example Avellaneda et al. (2011). Typically the arrival of orders

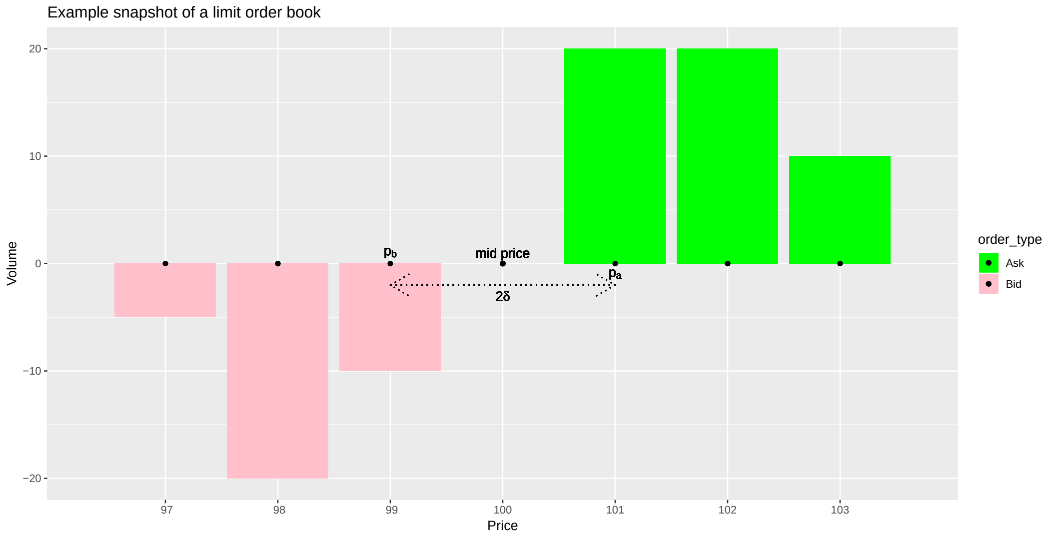

Figure 6.1: Example of a fictional limit order book of a stock on an exchange. Here the ask price is $p_a = 101$ and the bid price is $p_b = 99$. The spread is $S = 2\delta = 2$ and the mid-price is 100. The volume of orders available at the bid and ask are $V_b = 10$ and $V_a = 20$ respectively. If a market order came along to sell 10 units of this stock, after the trade is executed the volume V_b would be depleted and the new ask price would be $p_b = 98$ with a volume of $V_b = 20$.

are modelled by simple independent Poissonian arrival times, or by the more complex Hawkes processes which are state-dependent, see for example Blanc et al. (2017) and Lu and Abergel (2018).

Here we propose a simple toy model, where we consider only the top level of the order book and we assume that the dynamics of the volume process $V(s) = (V_b(s), V_a(s))$ is governed by the operator $-D^{\beta,\gamma}_{0+*}$ from (6.0.1) which we discussed in Chapter 5. Note in particular that this means that we are assuming the processes $V_b(s)$ and $V_a(s)$ are strictly decreasing. This could for example be reasonable if the *net flow* of orders is always negative, so that the number of orders executed per second is always larger than the number of incoming orders. We then assume that each time this volume process restarts at a uniform point somewhere inside $\mathcal{O}^2$ whenever it hits $\partial\mathcal{O}^2$. Finally we assume that each time the boundary is hit (which means one of the queues V_b or V_a has depleted), there occurs a price change in that same direction, while keeping the spread $S := p_a - p_b$ to always be equal to one tick δ. In Figure 6.2 we give an possible sample path of the dynamics of the process on the orthant *up until it first hits the boundary*. Figure 6.3 shows an example sample

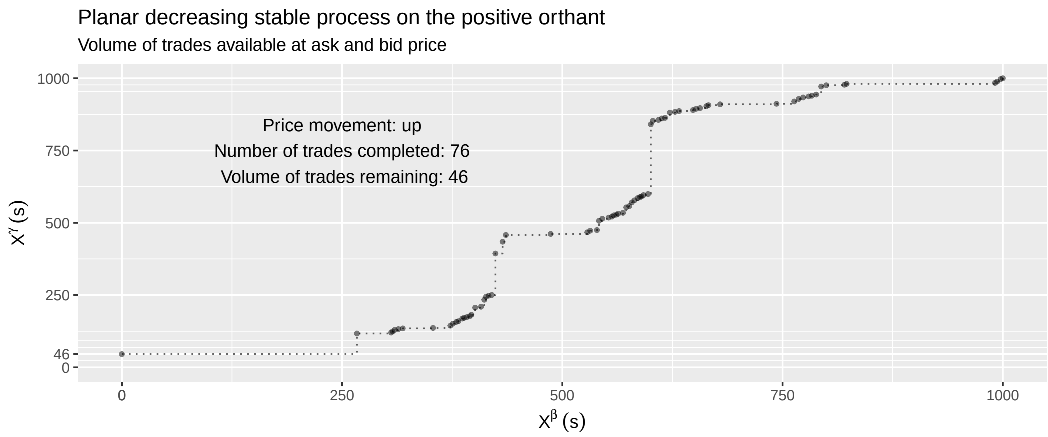

Figure 6.2: The volume process on the positive orthant. Both processes are decreasing α-stable subordinators with $\alpha = 0.8$, starting at $(1000, 1000)$.

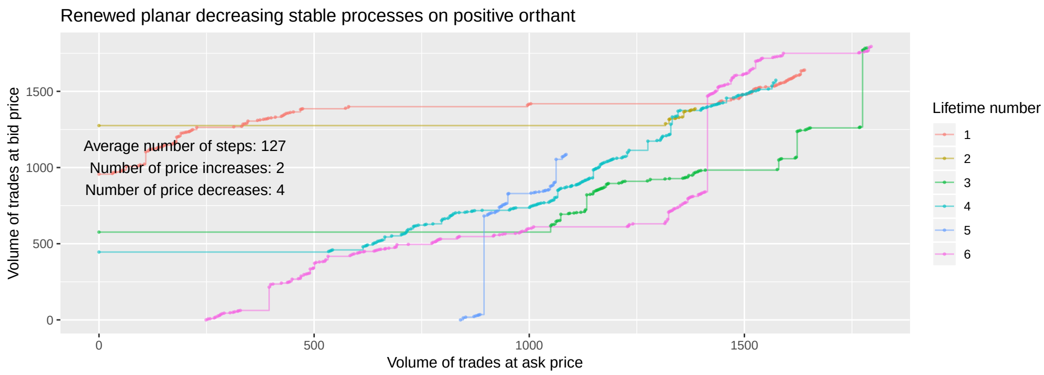

Figure 6.3: The volume process on the positive orthant, restarted at some uniformly distributed point (between 1000 and 2000) each time one coordinate hits the boundary of the orthant. As before, $\alpha = 0.8$. In this example, the net price change is -2, with the price changing on average every 127 time steps.

path of the process after being restarted several times, resulting in a net price change. Finally, Figure 6.4 shows a possible price process arising from letting the process $X^{(\beta,\gamma)}$ run for many lifetimes, restarting inside the orthant each time it reaches the boundary.

So far we have only considered a process on the orthant whose coordinates are independent decreasing β-stable subordinators. In Figure 6.5 we plot an α-stable process in $\mathbb{R}^2$, with $\alpha \in (0, 2)$, whose characteristic function is given by (2.2.2). Aesthetically, this process seems to exhibit behaviour that is more realistic for limit order books than the one already discussed, and may be an interesting avenue of future work.

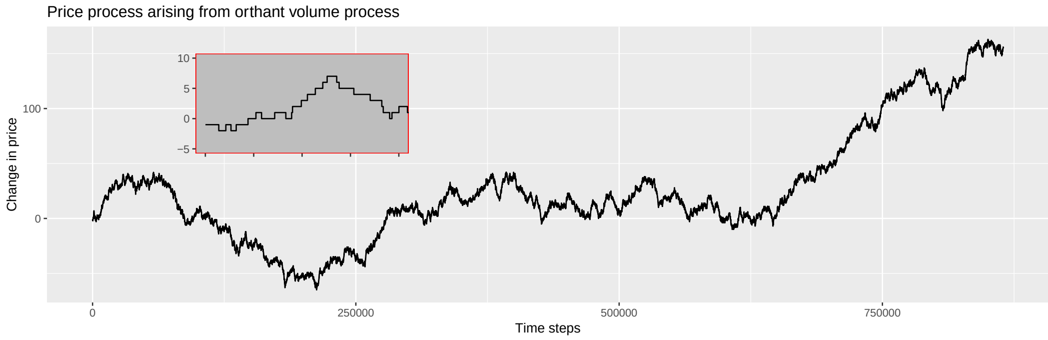

Figure 6.4: Example of price process which is driven by the process on the orthant: each time $X^{\beta,\gamma}$ hits the x-axis (respectively y-axis) the price moves down (respectively up). Here $\beta = \gamma = 0.8$, with 10000 price changes. Also shown is a zoomed in portion of the price process, highlighting the fact that the process remains constant for however long it takes for the volume process to hit the boundary of the orthant.

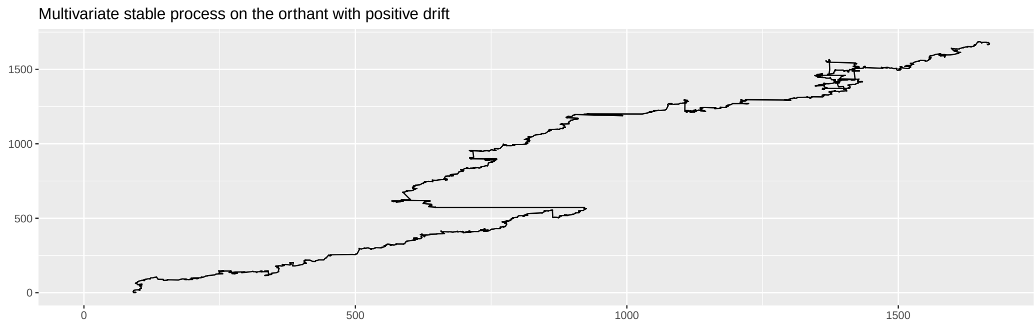

Figure 6.5: Example of bivariate α-stable on the orthant with two-sided jumps and positive drift. Here the order of stability is $\alpha = 1.5$, centred at $(1,1)$ and the spectral measure μ has 4 masses at $(1,0), (0,1), (-1,0), (0,-1)$. Plotted with the aid of the R packages ggplot2 and alphastable, see Wickham (2016) and Teimouri et al. (2019).

There are several interesting questions that one may think about when modelling limit order books;

i) How many times did the price change over a certain time period?
ii) How long does it take for a price to move?
iii) When the price *does* change, which direction does it move in?
iv) What is the long-time behaviour of the price process?

Items i) and ii) are settled by understanding the distribution of $\tau_0^{\beta,\gamma}$, in particular $\mathbb{E}[\tau_0^{\beta,\gamma}]$. Item iii) requires understanding the conditional distribution of the price process. In reality, the processes modelling the sizes of the queues at the ask and bid price should have a more complex structure. Firstly, the order of stability should depend on the current size of the queue. Secondly, the sizes of the queues should depend on each other - indeed, empirical studies have shown that the queue sizes of the bid and ask price are negatively correlated. Possible steps to take in this direction is to replace in (6.0.1) the classic fractional derivatives with $-D_{0+*}^{(\nu)}$ where ν is the Lévy measure of a stable-like process, for example. This will be the subject of future work.

Appendix A

Appendix

A.1 Conditioning argument

Recall that μ_0^α is the density of the random variable τ_0^α and $p_s^\alpha(t,r)$ are the transition densities of the monotone process $X_t^\alpha(s)$ started at $t \in (0,\infty)$.

Proposition A.1.1. For $\beta, \gamma \in (0,1)$,

$$\mathbb{E}\left[\phi_1(X_{t_2}^\gamma(\tau_0^\beta), Y_x(\tau_0^\beta))\mathbf{1}_{\{\tau_0^\beta < \tau_0^\gamma\}}\right] \\ = \int_0^{t_2}\int_{\mathbb{R}^d}\phi_1(r,y)\int_0^\infty p^Y(s,x,y)p_s^\gamma(t_2,r)\mu_0^\beta(s)\ \mathrm{d}s\mathrm{d}y\mathrm{d}r, \quad \text{(A.1.1)}$$

and similarly,

$$\mathbb{E}\left[\phi_2(X_{t_1}^\beta(\tau_0^\gamma), Y_x(\tau_0^\gamma))\mathbf{1}_{\{\tau_0^\gamma < \tau_0^\beta\}}\right] \\ = \int_0^{t_1}\int_{\mathbb{R}^d}\phi_2(r,y)\int_0^\infty p^Y(s,x,y)p_s^\beta(t_1,r)\mu_0^\gamma(s)\ \mathrm{d}s\mathrm{d}y\mathrm{d}r. \quad \text{(A.1.2)}$$

Proof. In the LHS of (A.1.1) condition first on $\{\tau_0^\beta = s\}$,

$$\mathbb{E}\left[\phi_1(X_{t_2}^\gamma(\tau_0^\beta), Y_x(\tau_0^\beta))\mathbf{1}_{\{\tau_0^\beta < \tau_0^\gamma\}}\right] = \int_0^\infty \mathbb{E}\left[\phi_1(X_{t_2}^\gamma(s), Y_x(s))\mathbf{1}_{\{s<\tau_0^\gamma\}}\right]\mu_0^\beta(s)\ \mathrm{d}s.$$

Due to the monotonicity of the process $X_{t_2}^\gamma$, the events $\{\tau_0^\gamma > s\}$ and $\{X_{t_2}^\gamma(s) > 0\}$ are equivalent. Thus we next condition on $\{X_{t_2}^\gamma(s) = r\}$,

$$= \int_0^\infty \mathbb{E}\left[\phi_1(X_{t_2}^\gamma(s), Y_x(s))\mathbf{1}_{\{X_{t_2}^\gamma(s)>0\}}\right]\mu_0^\beta(s)\ \mathrm{d}s$$

$$= \int_0^\infty \int_0^{t_2} \mathbb{E}\left[\phi_1(r, Y_x(s))\mathbf{1}_{\{r>0\}}\right] \mu_0^\beta(s) p_s^\gamma(t_2, r) \, \mathrm{d}r\mathrm{d}s.$$

Finally conditioning on $\{Y_x(s) = y\}$ and rearranging, we have

$$\int_0^{t_2} \int_{\mathbb{R}^d} \phi_1(r, y) \int_0^\infty p^Y(s, x, y) p_s^\gamma(t_2, r) \mu_0^\beta(s) \, \mathrm{d}s\mathrm{d}y\mathrm{d}r,$$

where $p^Y(s, x, y)$ are the transition densities of the process $(Y_x(s))_{s\geq 0}$ started at $x \in \mathbb{R}^d$. The proof of (A.1.2) is similar and is omitted. □

A.2 Proof of Proposition 5.3.1

Let $A := r^\gamma t^{-\beta}$, $\Omega := |x - y|^2 t^{-\beta}$. First we use Lemma 2.3.3 to estimate the density μ_0^β, then we use Lemma 5.1.1 to estimate the spatial density

$$\begin{aligned}
G^{(\beta,\gamma)}(t, x; r, y) &= \int_0^\infty G^{(Y,\gamma)}(t^\beta z, r, x, y) t^\beta \mu_0^\beta(t^\beta z) \, \mathrm{d}z \\
&\asymp \int_0^1 G^{(Y,\gamma)}(t^\beta z, r, x, y) \, \mathrm{d}z \\
&\quad + \int_1^\infty G^{(Y,\gamma)}(t^\beta z, r, x, y) z^{-1-\frac{1}{\beta}} f_\beta(z^{-\frac{1}{\beta}}) \, \mathrm{d}z \\
&\asymp t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} \int_0^1 z^{-\frac{1}{\gamma}-\frac{d}{2}} \exp\left\{-\Omega z^{-1}\right\} w_\gamma(A^{\frac{1}{\gamma}} z^{-\frac{1}{\gamma}}) \, \mathrm{d}z \\
&\quad + \ t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} \int_1^\infty z^{-1+\frac{1}{2(1-\beta)}-\frac{1}{\gamma}-\frac{d}{2}} \exp\left\{-\Omega z^{-1} - c_\beta z^{\frac{1}{1-\beta}}\right\} \\
&\qquad\qquad \cdot w_\gamma(A^{\frac{1}{\gamma}} z^{-\frac{1}{\gamma}}) \, \mathrm{d}z \\
&\asymp I_1 + I_2 + I_3 + I_4
\end{aligned}$$

where

$$\begin{aligned}
I_1 &= t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\frac{1}{\gamma}} \int_0^{A\wedge 1} z^{1-\frac{d}{2}} \exp\left\{-\Omega z^{-1}\right\} \, \mathrm{d}z \ \mathbf{1}_{\{A\in\mathbb{R}_+\}} \\
I_2 &= t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_A^1 z^{\frac{1}{2(1-\gamma)}-\frac{d}{2}} \exp\left\{-\Omega z^{-1} - A^{-\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}}\right\} \, \mathrm{d}z \ \mathbf{1}_{\{A<1\}} \\
I_3 &= t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\frac{1}{\gamma}} \int_1^A z^{\frac{1}{2(1-\beta)}-\frac{d}{2}} \exp\left\{-\Omega z^{-1} - c_\beta z^{\frac{1}{1-\beta}}\right\} \, \mathrm{d}z \ \mathbf{1}_{\{A>1\}}
\end{aligned}$$

$$I_4 = t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_{A\vee 1}^{\infty} z^{-\frac{d}{2}-1+\frac{1}{2(1-\beta)}+\frac{1}{2(1-\gamma)}} \exp\left\{-\Omega z^{-1} - A^{-\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}} - z^{\frac{1}{1-\beta}}\right\} \, \mathrm{d}z \, \mathbf{1}_{\{A\in\mathbb{R}_+\}}$$

Now we have 4 regimes to consider, which are

- **Case 1a):** $A \leq 1$ and $\Omega \leq 1$
- **Case 1b):** $A \geq 1$ and $\Omega \leq 1$
- **Case 2a):** $A \leq 1$ and $\Omega \geq 1$
- **Case 2b):** $A \geq 1$ and $\Omega \geq 1$.

By directly comparing the powers of z, Ω and A in the integrals above, we can reduce our attention to the integrals I_1 and I_4. Indeed for $\Omega \leq 1$ we have

$$0 = I_3 < I_4 \leq I_2 \leq I_1, \quad A \leq 1,$$

and

$$0 = I_2 < I_4 \leq I_3 \leq I_1, \quad A \geq 1.$$

For $\Omega \geq 1$ we have

$$0 = I_3 < I_1 \leq I_2 \leq I_4, \quad A \leq 1,$$

and

$$0 = I_2 < I_1 \leq I_3 \leq I_4, \quad A \geq 1.$$

Thus we have a preliminary two-sided estimate for $G^{(\beta,\gamma)}(t,r,x,y)$,

$$C_1 I_1 \leq G^{(\beta,\gamma)}(t,r,x,y) \leq C_2 I_1, \quad \text{for } \Omega \leq 1,$$

and

$$C_3 I_4 \leq G(\beta,\gamma)(t,r,x,y) \leq C_4 I_4, \quad \text{for } \Omega \geq 1,$$

for some constants C_1, C_2, C_3, C_4.

A.2.1 Estimates for I_1

For the first integral, we have for $A \leq 1$,

$$I_1 = t^{-\frac{\beta}{\gamma}} A^{-1-\frac{1}{\gamma}} \int_0^A z^{1-\frac{d}{2}} \exp\left\{-\Omega z^{-1}\right\} \, \mathrm{d}z$$

Then for $\Omega \to 0$ and $A \to 0$,

$$I_1 \sim C_{\beta,d,\gamma} t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\frac{1}{\gamma}} \begin{cases} 1, & d \leq 3, \\ |\log \Omega A^{-1}| + 1, & d = 4, \\ \Omega^{2-\frac{d}{2}}, & d \geq 5. \end{cases}$$

For $\Omega \to \infty$ and $A \to 0$,

$$I_1 \sim C_{\beta,d,\gamma} t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{1-\frac{d}{2}-\frac{1}{\gamma}} \Omega^{-1} \exp\left\{-\Omega A^{-1}\right\}.$$

For $A \geq 1$ we have

$$I_1 = t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\frac{1}{\gamma}} \int_0^1 z^{1-\frac{d}{2}} \exp\left\{-\Omega z^{-1}\right\} \, \mathrm{d}z,$$

so for $\Omega \to 0$ and $A \to \infty$,

$$I_1 \sim C_{\beta,d,\gamma} t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\frac{1}{\gamma}} \begin{cases} 1, & d \leq 3, \\ |\log \Omega| + 1, & d = 4, \\ \Omega^{2-\frac{d}{2}}, & d \geq 5. \end{cases}$$

For $\Omega \to \infty$ and $A \to \infty$,

$$I_1 \sim t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-1-\frac{1}{\gamma}} \Omega^{-1} \exp\left\{-\Omega\right\}.$$

A.2.2 Estimates for I_4

For $A \leq 1$,

$$I_4 = t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_1^\infty z^n \exp\left\{-\Omega z^{-1} - A^{-\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}} - c_\beta z^{\frac{1}{1-\beta}}\right\} \, \mathrm{d}z.$$

Let $\alpha := \min(\beta,\gamma)$ and $\tilde{\alpha} = \max(\beta,\gamma)$. For bounded $\Omega \le 1$, we have

$$\begin{aligned}
I_4 \le & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_1^\infty z^n \exp\left\{-c_\gamma A^{-\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}} - c_\beta z^{\frac{1}{1-\beta}}\right\} \,\mathrm{d}z \\
\le & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_1^\infty z^n \exp\left\{-c_\gamma A^{-\frac{1}{1-\gamma}} z^{\frac{1}{1-\alpha}}\right\} \,\mathrm{d}z \\
\le & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} A^{\frac{1}{1-\gamma}} \exp\left\{-C A^{-\frac{1}{1-\gamma}}\right\},
\end{aligned}$$

where we have used (2.4.2). Next we use (2.4.2) to get for $\Omega \ge 1$,

$$\begin{aligned}
I_4 \le\ & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_1^\infty z^n \exp\left\{-\Omega z^{-1} - A^{-\frac{1}{1-\alpha}} z^{\frac{1}{1-\alpha}}\right\} \,\mathrm{d}z \\
\sim & C t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \Omega^{\frac{2(n+1)-c}{2(c+1)}} A^{\frac{2c(n+1)+c}{2(c+1)}} \exp\left\{-C_2 \left(\Omega A^{-1}\right)^{\frac{c}{c+1}}\right\}, \quad \Omega A^{-1} \to \infty,
\end{aligned}$$

where $c = \frac{1}{1-\alpha}$ and $n = -\frac{d}{2} - 1 + \frac{1}{2(1-\beta)} + \frac{1}{2(1-\gamma)}$. Thus

$$I_4 \le C t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} \left(\Omega A^{\frac{1}{1-\alpha}}\right)^{-\frac{d}{2}\left(\frac{1-\alpha}{2-\alpha}\right)+\frac{1-\alpha}{2(2-\alpha)(1-\tilde{\alpha})}} A^{\frac{1}{2(1-\alpha)}-\frac{2-\gamma}{2\gamma(1-\gamma)}} \exp\left\{-C \left(\Omega A^{-1}\right)^{\frac{1}{2-\alpha}}\right\}.$$

Finally for $A \ge 1$, we have

$$I_4 = t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_A^\infty z^n \exp\left\{-\Omega z^{-1} - c_\gamma A^{-\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}} - c_\beta z^{\frac{1}{1-\beta}}\right\} \,\mathrm{d}z.$$

For bounded Ω, but unbounded A we have

$$\begin{aligned}
I_4 \le & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_A^\infty z^n \exp\left\{-c_\beta A^{-\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}} - c_\gamma z^{\frac{1}{1-\beta}}\right\} \,\mathrm{d}z \\
\le & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} A^n \exp\left\{-c_\beta A^{\frac{1}{1-\beta}}\right\}, \quad \text{for } A \to \infty.
\end{aligned}$$

For unbounded Ω and A, the term $A^{-\frac{1}{1-\gamma}}$ is negligible since A is large, then we apply the usual Laplace approximation Proposition 2.4.1 to get

$$\begin{aligned}
I_4 \le & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_A^\infty z^n \exp\left\{-\Omega z^{-1} - c_\beta A^{-\frac{1}{1-\gamma}} z^{\frac{1}{1-\gamma}} - c_\gamma z^{\frac{1}{1-\beta}}\right\} \,\mathrm{d}z \\
\le & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_1^\infty z^n \exp\left\{-\Omega z^{-1} - c_\alpha z^{\frac{1}{1-\alpha}}\right\} \,\mathrm{d}z \\
\le & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \int_0^1 w^{-n-2} \exp\left\{-\Omega w - c_\alpha w^{-\frac{1}{1-\alpha}}\right\} \,\mathrm{d}w \\
\le & t^{-\frac{\beta}{\gamma}-\frac{d\beta}{2}} A^{-\frac{2-\gamma}{2\gamma(1-\gamma)}} \Omega^{\frac{n+1}{2-\alpha}-\frac{1}{2(2-\alpha)}} \exp\left\{-\Omega^{\frac{1}{2-\alpha}}\right\}
\end{aligned}$$

for $\Omega \to \infty$ and $A \geq 1$. For the lower bound of I_4, simply reverse the role of α and $\tilde{\alpha}$ in each case - otherwise structure of the estimates are the same.

Bibliography

Abergel, F., M. Anane, A. Chakraborti, A. Jedidi, and I. M. Toke (2016). *Limit order books*. Cambridge University Press.

Abergel, F. and A. Jedidi (2013). "A mathematical approach to order book modeling". In: *Int. J. Theor. Appl. Finance* 16.5, pp. 1350025, 40. ISSN: 0219-0249. DOI: 10.1142/S0219024913500258.

Aronson, D. G. (1967). "Bounds for the fundamental solution of a parabolic equation". In: *Bulletin of the American Mathematical society* 73.6, pp. 890–897. ISSN: 0002-9904. DOI: 10.1090/s0002-9904-1967-11830-5.

Asmussen, S. and H. Albrecher (2010). *Ruin probabilities*. Second. Vol. 14. Advanced Series on Statistical Science & Applied Probability. World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, pp. xviii+602. ISBN: 978-981-4282-52-9; 981-4282-52-9. DOI: 10.1142/9789814282536.

Avellaneda, M., J. Reed, and S. Stoikov (2011). "Forecasting prices from level-I quotes in the presence of hidden liquidity". In: *Algorithmic Finance* 1.1, pp. 35–43. ISSN: 2158-5571.

Barlow, M. T. and J. Cerny (2011). "Convergence to fractional kinetics for random walks associated with unbounded conductances". In: *Probability theory and related fields* 149.3-4, pp. 675–677. ISSN: 0178-8051. DOI: 10.1007/s00440-011-0344-9.

Blanc, P., J. Donier, and J.-P. Bouchaud (2017). "Quadratic Hawkes processes for financial prices". In: *Quant. Finance* 17.2, pp. 171–188. ISSN: 1469-7688. DOI: 10.1080/14697688.2016.1193215.

Böttcher, B., R. Schilling, and J. Wang (2014). *Lévy matters III: Lévy-type processes: construction, approximation and sample path properties*. Vol. 2099. Springer.

Bruijn, N. G. de (1981). *Asymptotic methods in analysis*. third. Dover Publications, Inc., New York, pp. xii+200. ISBN: 0-486-64221-6.

Cabezas, M. et al. (2015). "Sub-Gaussian bound for the one-dimensional Bouchaud trap model". In: *Brazilian Journal of Probability and Statistics* 29.1, pp. 112–131.

Chen, Y., Y. Wang, and K. Wang (2013). "Asymptotic results for ruin probability of a two-dimensional renewal risk model". In: *Stoch. Anal. Appl.* 31.1, pp. 80–91. ISSN: 0736-2994. DOI: 10.1080/07362994.2013.741386.

Chen, Z.-Q., P. Kim, T. Kumagai, and J. Wang (2018). "Heat kernel estimates for time fractional equations". In: *Forum Mathematicum*. Vol. 30. 5. De Gruyter, pp. 1163–1192.

Cont, R. and A. De Larrard (2012). "Order book dynamics in liquid markets: limit theorems and diffusion approximations". In: *Available at SSRN 1757861*.

Cont, R., S. Stoikov, and R. Talreja (2010). "A stochastic model for order book dynamics". In: *Oper. Res.* 58.3, pp. 549–563. ISSN: 0030-364X. DOI: 10.1287/opre.1090.0780.

Courrege, P. (1965). "Sur la forme intégro-différentielle des opérateurs de C_k^∞ dans C satisfaisant au principe du maximum". In: *Séminaire Brelot-Choquet-Deny. Théorie du Potentiel* 10.1, pp. 1–38.

Deng, C.-S. and R. L. Schilling (2018). "Exact Asymptotic Formulas for the Heat Kernels of Space and Time-Fractional Equations". In: *arXiv preprint arXiv:1803.11435*.

Diethelm, K. (2010). *The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type*. Springer.

Djehiche, B. (1993). "A large deviation estimate for ruin probabilities". In: *Scand. Actuar. J.* 1993.1, pp. 42–59. ISSN: 0346-1238. DOI: 10.1080/03461238.1993.10413912.

Duffie, D., D. Filipović, W. Schachermayer, et al. (2003). "Affine processes and applications in finance". In: *The Annals of Applied Probability* 13.3, pp. 984–1053.

Dynkin, E. B. (1965). "Markov processes". In: *Markov Processes*. Springer, pp. 77–104.

Eidelman, S. D., S. D. Ivasyshen, and A. N. Kochubei (2004). *Analytic methods in the theory of differential and pseudo-differential equations of parabolic type*. Vol. 152. Springer Science & Business Media.

Eidelman, S. D. and A. N. Kochubei (2004). "Cauchy problem for fractional diffusion equations". In: *Journal of differential equations* 199.2, pp. 211–255.

Fedoryuk, M. V. (1987). *Asymptotics: integrals and series*. Mathematical Reference Library. "Nauka", Moscow, p. 544.

Feller, W. (2008). *An introduction to probability theory and its applications*. Vol. 2. John Wiley & Sons.

Grigoryan, A. and T. Kumagai (2008). “On the dichotomy in the heat kernel two sided estimates”. In: *Analysis on Graphs and its Applications (P. Exner et al.(eds.)), Proc. of Symposia in Pure Math.* Vol. 77, pp. 199–210.

Hairer, M., G. Iyer, L. Koralov, A. Novikov, Z. Pajor-Gyulai, et al. (2018). “A fractional kinetic process describing the intermediate time behaviour of cellular flows”. In: *The Annals of Probability* 46.2, pp. 897–955.

Hambly, B., J. Kalsi, and J. Newbury (2018). “Limit order books, diffusion approximations and reflected SPDEs: from microscopic to macroscopic models”. In: *arXiv preprint arXiv:1808.07107*.

Hernández-Hernández, M. E. and V. Kolokoltsov (2016). “On the solution of two-sided fractional ordinary differential equations of Caputo type”. In: *Fractional Calculus and Applied Analysis* 19.6, pp. 1393–1413.

Hernández-Hernández, M. E., V. Kolokoltsov, and L. Toniazzi (2017). “Generalised fractional evolution equations of Caputo type”. In: *Chaos, Solitons & Fractals* 102, pp. 184–196.

Herrmann, R. (2014). *Fractional calculus: an introduction for physicists*. World Scientific.

Johnston, I. and V. Kolokoltsov (2019a). “Green’s Function Estimates for Time-Fractional Evolution Equations”. In: *Fractal and Fractional* 3.2, p. 36.

Johnston, I. and V. Kolokoltsov (2019b). “Mixed linear fractional boundary value problems”. In: *arxiv preprint arXiv:1908.03158*.

Kaleta, K., M. Kwaśnicki, and J. Lörinczi (2018). “Contractivity and ground state domination properties for non-local Schrödinger operators”. In: *J. Spectr. Theory* 8.1, pp. 165–189. ISSN: 1664-039X. DOI: 10.4171/JST/193.

Kaleta, K. and J. Lörinczi (2019). “Zero-Energy Bound State Decay for Non-local Schrödinger Operators”. In: *Communications in Mathematical Physics*. ISSN: 1432-0916. DOI: 10.1007/s00220-019-03515-3. URL: https://doi.org/10.1007/s00220-019-03515-3.

Kelbert, M., V. Konakov, and S. Menozzi (2016). “Weak error for continuous time markov chains related to fractional in time P (I) DEs”. In: *Stochastic Processes and their Applications* 126.4, pp. 1145–1183.

Kochubei, A. N., Y. Kondratiev, and J. L. da Silva (2018). “From Random Times to Fractional Kinetics”. In: *arXiv preprint arXiv:1811.10531*.

Kochubei, A. N., Y. Kondratiev, and J. L. da Silva (2019). “Random Time Change and Related Evolution Equations”. In: *arXiv preprint arXiv:1901.10015*.

Kolokoltsov, V. (2000). “Symmetric stable laws and stable-like jump-diffusions”. In: *Proceedings of the London Mathematical Society* 80.3, pp. 725–768.

Kolokoltsov, V. (2011). *Markov Processes, Semigroups, and Generators*. Vol. 38. Walter de Gruyter.

Kolokoltsov, V. (2015). "On fully mixed and multidimensional extensions of the Caputo and Riemann-Liouville derivatives, related Markov processes and fractional differential equations". In: *Fractional Calculus and Applied Analysis* 18.4, pp. 1039–1073.

Kolokoltsov, V. (2017). "Chronological operator-valued Feynman-Kac formulae for generalized fractional evolutions". In: *arXiv preprint arXiv:1705.08157*.

Kolokoltsov, V. (2019a). *Differential equations on measures and functional spaces*. Springer.

Kolokoltsov, V. (2019b). "The probabilistic point of view on the generalized fractional PDEs". In: *Fractional Calculus and Applied Analysis*.

Kolokoltsov, V. and M. Veretennikova (2014). "Well-posedness and regularity of the Cauchy problem for nonlinear fractional in time and space equations". In: *Fractional Differential Calculus*.

Konstantinides, D. G. and J. Li (2016). "Asymptotic ruin probabilities for a multidimensional renewal risk model with multivariate regularly varying claims". In: *Insurance Math. Econom.* 69, pp. 38–44. ISSN: 0167-6687. DOI: 10.1016/j.insmatheco.2016.04.003.

Kruk, Ł. (2003). "Functional limit theorems for a simple auction". In: *Mathematics of Operations Research* 28.4, pp. 716–751.

Leccadito, A., T. Paletta, and R. Tunaru (2016). "Pricing and hedging basket options with exact moment matching". In: *Insurance Math. Econom.* 69, pp. 59–69. ISSN: 0167-6687. DOI: 10.1016/j.insmatheco.2016.03.013.

Leonenko, N. N., M. M. Meerschaert, and A. Sikorskii (2013). "Correlation structure of fractional Pearson diffusions". In: *Computers & Mathematics with Applications* 66.5, pp. 737–745.

Li, X., J. Wu, and J. Zhuang (2015). "Asymptotic multivariate finite-time ruin probability with statistically dependent heavy-tailed claims". In: *Methodol. Comput. Appl. Probab.* 17.2, pp. 463–477. ISSN: 1387-5841. DOI: 10.1007/s11009-013-9375-2.

Lipton, A., U. Pesavento, and M. G. Sotiropoulos (2013). "Trade arrival dynamics and quote imbalance in a limit order book". In: *arXiv preprint arXiv:1312.0514*.

Lu, X. and F. Abergel (2018). "High-dimensional Hawkes processes for limit order books: modelling, empirical analysis and numerical calibration". In: *Quant. Finance* 18.2, pp. 249–264. ISSN: 1469-7688. DOI: 10.1080/14697688.2017.1403142.

Mainardi, F. (2010). *Fractional calculus and waves in linear viscoelasticity: an introduction to mathematical models*. World Scientific.

Meerschaert, M. M. and H.-P. Scheffler (2004). "Limit theorems for continuous-time random walks with infinite mean waiting times". In: *J. Appl. Probab.* 41.3, pp. 623–638. ISSN: 0021-9002. DOI: 10.1239/jap/1091543414.

Meerschaert, M. M. and A. Sikorskii (2012). *Stochastic models for fractional calculus*. Vol. 43. Walter de Gruyter.

Mijatović, A. and M. Pistorius (2013). "Continuously monitored barrier options under Markov processes". In: *Math. Finance* 23.1, pp. 1–38. ISSN: 0960-1627.

Murray, J. D. (1984). *Asymptotic analysis*. Second. Vol. 48. Applied Mathematical Sciences. Springer-Verlag, New York, pp. vii+164. ISBN: 0-387-90937-0. DOI: 10.1007/978-1-4612-1122-8.

Pollard, H. (1948). "The completely monotonic character of the Mittag-Leffler function $E_\alpha(-x)$". In: *Bulletin of the American Mathematical Society* 54.12, pp. 1115–1116.

Porper, F. and S. D. Èidel'man (1984). "Two-sided estimates of fundamental solutions of second-order parabolic equations, and some applications". In: *Russian Mathematical Surveys* 39.3, pp. 119–178.

Ramasubramanian, S. (2016). "A multidimensional ruin problem and an associated notion of duality". In: *Stoch. Models* 32.4, pp. 539–574. ISSN: 1532-6349. DOI: 10.1080/15326349.2016.1175949.

Samorodnitsky, G. and M. S. Taqqu (1994). *Stable non-Gaussian random processes*. Stochastic Modeling. Stochastic models with infinite variance. Chapman & Hall, New York, pp. xxii+632. ISBN: 0-412-05171-0.

Sato, K.-i. (1999). *Levy Processes and Infinitely Divisible Distributions*. Vol. 85. Cambridge university press, p. 568.

Scalas, E., R. Gorenflo, and F. Mainardi (2000a). "Fractional calculus and continuous-time finance". In: *Phys. A* 284.1-4, pp. 376–384. ISSN: 0378-4371. DOI: 10.1016/S0378-4371(00)00255-7.

Scalas, E., R. Gorenflo, and F. Mainardi (2000b). "Fractional Calculus and continuous-time finance II: the waiting-time distribution". In: *Physica A. Stastical Mechanics and its Applications* 287.3-4, pp. 468–481. ISSN: 0378-4371.

Scalas, E., R. Gorenflo, F. Mainardi, and M. Raberto (2001). "Fractional calculus and continuous-time finance. III. The diffusion limit". In: *Mathematical finance (Konstanz, 2000)*. Trends Math. Birkhäuser, Basel, pp. 171–180.

Schilling, R. L., R. Song, and Z. Vondracek (2012). *Bernstein functions: theory and applications*. Vol. 37. Walter de Gruyter.

Tarasov, V. E. (2011). *Fractional dynamics: applications of fractional calculus to dynamics of particles, fields and media*. Springer Science & Business Media.

Teimouri, M., A. Mohammadpour, and S. Nadarajah (2019). *alphastable: Inference for Stable Distribution*. R package version 0.2.1. URL: https://CRAN.R-project.org/package=alphastable.

Uchaikin, V. V. and V. M. Zolotarev (1999). *Chance and stability*. Modern Probability and Statistics. Stable distributions and their applications, With a foreword by V. Yu. Korolev and Zolotarev. VSP, Utrecht, pp. xxii+570. ISBN: 90-6764-301-7. DOI: 10.1515/9783110935974.

Van Den Berg, C. and G. Forst (2012). *Potential theory on locally compact abelian groups*. Vol. 87. Springer Science & Business Media.

Wickham, H. (2016). *ggplot2: Elegant Graphics for Data Analysis*. Springer-Verlag New York. ISBN: 978-3-319-24277-4. URL: https://ggplot2.tidyverse.org.

Wuertz, D., M. Maechler, and R. core team members. (2016). *stabledist: Stable Distribution Functions*. R package version 0.7-1. URL: https://CRAN.R-project.org/package=stabledist.

Zhang, Y. (2000). "Sufficient and necessary conditions for stochastic comparability of jump processes". In: *Acta Math. Sin. (Engl. Ser.)* 16.1, pp. 99–102. ISSN: 1000-9574. DOI: 10.1007/s101149900029.

Zolotarev, V. M. (1957). "Mellin-Stieltjes transforms in probability theory". In: *Theory of Probability & Its Applications* 2.4, pp. 433–460.

Zolotarev, V. M. (1961). "On analytic properties of stable distribution laws". In: *Selected Translations in Mathematical Statistics and Probability* 1, pp. 202–211.

Zolotarev, V. M. (1986). *One-dimensional stable distributions*. Vol. 65. American Mathematical Soc.

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