

# Master of Science in Advanced Mathematics and Mathematical Engineering

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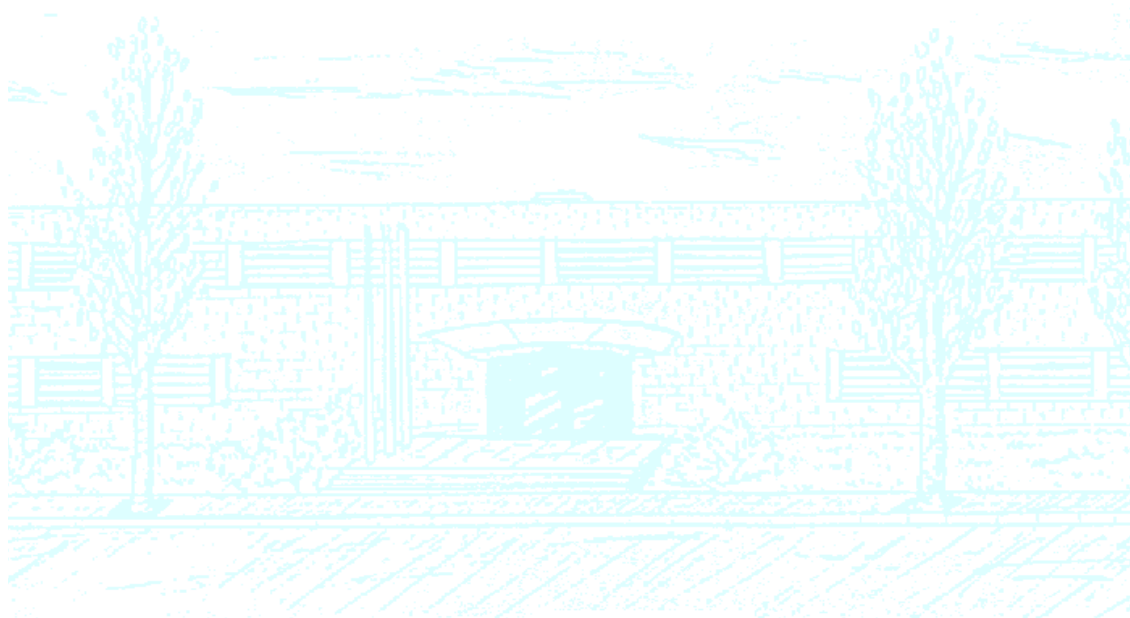
**Title: Semi-Analytical Implementation for the Name Concentration Measurement in a Credit Portfolio**

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Facultat de Matemàtiques  
i Estadística



Universitat Politècnica de Catalunya  
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Centre de Recerca Matemàtica

MASTER'S DEGREE THESIS  
MASTER OF SCIENCE IN ADVANCED  
MATHEMATICS AND MATHEMATICAL  
ENGINEERING

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**Semi-Analytical  
Implementation for the Name  
Concentration Measurement in  
a Credit Portfolio**

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# Chapter 1

## Introduction

Basel II is the second of the Basel Accords, which are recommendations on banking regulations issued by the Basel Committee on Banking Supervision. The purpose of Basel II, which was initially published in 2004, is to create an international standard that regulators can use, about how much capital banks need to cover the potential losses derived from its financial activities. We concentrate in credit risk, which is the most important risk a bank has to deal with.

Basel II is structured in a three pillar framework. Pillar one sets out details for adopting more risk sensitive minimal requirements, so called *regulatory capital*, for banking organizations. Pillar two lays out principle for the *supervisory review process* of capital adequacy and Pillar three seeks to establish *market discipline* by enhancing transparency in bank's financial reporting.

The Pillar one capital charge for credit risk is based on the Asymptotic Single-Risk Factor (ASRF) model, also called Vasicek model. The regulatory capital requirements are calculated evaluating the credit portfolio loss distribution at the 99% confidence level. One of its important assumptions is that a portfolio is well diversified, this is, there is no exposure (or name) concentration among obligors in the credit portfolio. In the real world, however, this main assumption is violated and then the measured risk can be underestimated.

Monte Carlo method is a standard method for measuring credit portfolio risk in order to deal with exposure concentration. However, this method is known to be very time consuming making the approximation impractical in many situations, over all when the size of the portfolio increases. For this

main reason, any analytical method is welcome.

This project focuses on the study and implementation of a semi-analytical technique to measure the name concentration of a credit portfolio recently published in the paper *Name Concentration* in the *Risk Magazine* journal by Jasper Hommels and Viktor Tchistiakov. We refer to [8].

Chapter 1 is devoted to the presentation of the regulatory framework and the different categories of financial risks.

In the second chapter we will give a brief introduction to credit risk modeling and measurement and we will explain some of the best-known models.

In Chapter 3 we focus on concentration risk and we introduce the most used measures and methods in order to quantify it. Moreover, we explain a Monte Carlo simulation which we will use in order to validate our results.

In Chapter 4 we will explain and try to reproduce the correction to the Vasicek model presented in the paper. Firstly, we will introduce the analytic solution for a single large counterparty. A procedure for determining capital reserves in the case of multiple large counterparties is given. Furthermore, we will run Monte Carlo simulations, which will serve us as a benchmark. After that, we will test the accuracy of the method by using a wide variety of credit portfolios with exposure concentration.

Finally, computation times and the implications from name concentration for economic capital and risk-adjusted return on capital are briefly discussed. The advantages and drawbacks of the analytical solution are described.

## 1.1 The regulatory framework

### 1.1.1 About the Basel Committee

The *Basel Committee on Banking Supervision* (BCBS) was founded by the Central Bank Governors of the Group of Ten (G10) at the end of 1974.

It provides a forum for regular cooperation on banking supervisory matters. Its objective is to enhance understanding of key supervisory issues and improve the quality of banking supervision worldwide. It seeks to do so by exchanging information on national supervisory issues, approaches and techniques, with a view to promoting common understanding. At times, the Committee uses this common understanding to develop guidelines and supervisory standards in areas where they are considered desirable.

The Committee's work is organised under four main sub-committees:

**The Standards Implementation Group** shares information and promotes consistency in implementation of the Basel II Framework.

**The Policy Development Group (PGD)** supports the Committee by identifying and reviewing emerging supervisory issues and proposing and developing policies that promote a sound banking system and high supervisory standards.

**The Accounting Task Force** works to help ensure that international accounting and auditing standards and practices promote sound risk management at banks, support market discipline through transparency, and reinforce the safety and soundness of the banking system.

**The Basel Consultative Group** provides a forum for deepening the Committee's engagement with supervisors around the world on banking supervisory issues and facilitates broad supervisory dialogue with non-member countries on new Committee initiatives.

### 1.1.2 Basel I

The First Basel Accord, known as Basel I, laid the basis for international minimum capital standard and banks became subject to *regulatory capital requirements*, coordinated by the BCBS.

The downfall of Herstatt-Bank underpinned the concern that the equity of the most important internationally active banks decreased to a worrisome

level. Equity is used to absorb losses and to assure liquidity. To decrease insolvency risk of banks and to minimize potential costs in the case of a bankruptcy, the target of Basel I was to assure a suitable amount of equity and to create consistent international competitive conditions.

The rules of the BCBS do not have any legal force. The supervisory rules are rather intended to provide guidelines for the supervisory authorities of the individual nations such that they can implement them in a suitable way for their banking system.

The main focus of Basel I was on credit risk, as the most important risk in the banking industry. Within Basel I banks are supposed to keep at least 8% equity in relation to their assets. The assets are weighted according to their degree of riskiness where the risk weights are determined for different borrower categories. The required equity can be computed as,

$$\text{Minimum Capital} = \text{Risk Weighted Assets} \cdot 8\%.$$

Hence the portfolio credit risk is measured as the sum of notional exposures weighted, *risk weighted assets*, by a coefficient reflecting the creditworthiness of the counterparty, *the risk weight*.

Since this approach did not take care of market risk, in 1996 an amendment to Basel I has been released which allows for both a standardized approach and a method based on internal VaR models for market risk in larger banks. However, the main criticism of Basel I remained. It does not account for methods for decrease risk as, for example, by means of portfolio diversification. Moreover, the approach measures risk in an insufficiently differentiated way since minimal capital requirements are computed independent of the borrower's creditworthiness. These drawbacks lead to the development of the Second Basel Accord. In June 2004 the BCBS released a *Revised Framework on International convergence of capital measurement and capital standards*, Basel II. These new rules officially came into force on January 1st, 2008.

### 1.1.3 Basel II

For a full revision see [5], [6] and [7].

The main targets of Basel II are the same as in Basel I as well. However, Basel II focuses not only on market and credit risk but also puts operational risks on the agenda, and its main purposes are:

- Ensuring that capital allocation is more risk sensitive.
- Separating operational risk from credit risk, and quantifying both.
- Attempting to align economic and regulatory capital more closely to reduce the scope for regulatory arbitrage.

Basel II is structured in a three pillar framework. The first (Minimum Capital) focuses on capital requirements to better reflect the true nature of risks for banking organizations. Pillar 2 (Supervisory Review) lays out principles for a more involved supervisory and regulatory system and Pillar 3 (Market Discipline) will potentially result in greater discipline imposed by the market by enhancing transparency in bank's financial reporting.

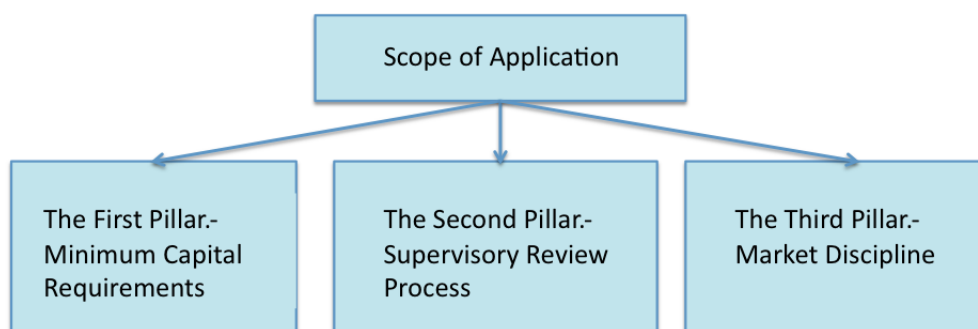


Figure 1.1: Structure of Basel II.

The former regulation lead banks to reject riskless positions, such as asset-backed transactions, since risk weighted assets for these positions were the same as for more risky and more profitable positions. The main goal of Pillar 1 is to take care of the specific risk of a bank when measuring minimal capital requirements. Pillar 1 therefore accounts for all three types of risk: credit risk, market risk and operational risk.

Concerning credit risk the new accord is more flexible and risk sensitive than the former Basel I accord. Within Basel II banks may opt for the standard approach, which is quite conservative with respect to capital charge and the more advanced, so-called *Internal Ratings Based* (IRB) approach when calculating regulatory capital for credit risk. In the standard approach, credit risk is measured by means of external ratings provided by certain rating agencies such as Standard&Poor's, Moody's or Fitch ratings.

In the IRB approach, the bank evaluates the risk itself. This approach, however, can only be applied when the supervisory authorities, which evaluate and audit the compliance of regulations with respect to the methods and transparency which are necessary for a bank to be allowed to use internal ratings, accept it. The bank, therefore, has to prove that certain conditions concerning the method and transparency are fulfilled.

The capital charge for market risk within Basel II is similar to the approach in Basel I. It is based mainly on Value at Risk approaches that statistically measure the total amount a bank can maximally lose. An innovation of Basel II is the creation of a new risk category, operational risk, which is explicitly taken into account in the new framework.

The main target of Pillar 3 is to improve market discipline by means of transparency of information concerning a bank's external accounting. Transparency can, for example, increase the probability of a decline in a bank's own stocks and therefore, motivate the bank to hold appropriate capital for potential losses.

### **Organizing the Risk Management function: Three-Pillar framework**

#### **1. Best-Practice Policies**

Risk tolerance must be expressed in terms that are consistent with the bank's business strategy. The business strategy should express the objectives of the financial institution in terms of risk/return targets. This should lead to setting risk limits, or tolerances, for the organization as a whole, and for its major activities.

**Market Risk Policy.** Business and risk managers should establish a policy that explicitly states their risk policy in terms of a statistically defined potential or 'worst case' loss: dealers and loan officers require a policy that states how much money can be put at risk. To this end, most major financial institutions are moving toward a value at risk framework, which calculates risk in terms of a probabilistic worst-case loss.

A best-practice market risk policy should state the statistically defined worst case loss in a way that considers the probability of both parallel and nonparallel shifts in the yield curve. The policy should also define the worst case loss in terms of a sufficiently low level of probability, say percent.

Management should decide how to allocate capital and risk units across activities and divisions in the institution in order to achieve their goals, while controlling exposure to market risk. The greater the market risk, the higher the expected rate of return that the bank can expect.

Management should also set the authorities for assuming market risks, and specify the nature of the market risks to which the institution should be exposed.

**Credit Risk Policy.** Every bank must determine a credit risk policy: how much credit to supply, for what duration, for which type of clients, and so on.

Profitability is only one consideration, the second being the risk of the loan. Therefore, bank policy should specify the extent of diversification, limits on size, and more. Banks need to tie their tolerance for risk and associated economic capital into their desired credit rating.

Some of the credit risks can be diversified away, and others should be priced. Management should specify its tolerance to credit risk, and limit the loan losses.

A reporting system to track exposures to credit risk is required, coupled with a routine for updating information about creditors.

**Operational Risk Policy.** Operational risks are the risks stemming from human errors, computer failures, employing large amounts of data for estimation purposes, and implementing pricing and valuation models. Management should decide which operational risks it should insure, and which it should manage. Assigning responsibilities is of utmost importance, but it cannot be effective without control procedures.

It is particularly important to set policies that establish how to review the introduction of all new products and evaluate all the pricing models that are used to value positions.

## 2. Best-Practice Methodologies

The best-practice methodologies refer to the application of appropriate analytic models to measure market risk, credit risk, operational risk and so on. The objective is not solely to measure risk, but also to ensure that the pricing and valuation methodologies are appropriate.

The G-30 recommends that dealers should value derivatives at market prices. Further, it recommends that risk managers should quantify market and credit risk using a value at risk (VaR) framework.

Finally, measurement tools should be developed to ensure that the bank's positions are on the efficient frontier of the trade-off between risk and reward.

**Risk Measurement Methodology.** An approach that permits a consistent measurement of market risk across all business units is needed.

Senior management should adopt a credit risk measurement policy which calls for measuring credit risk for the loan book and off-balance-sheet derivative products according to an analytic approach that is consistent with the approach implemented for market risk.

**Pricing and Valuation Methodologies.** It is really important that banks develop appropriate techniques to differentiate between transactions where prices are transparent, and those where price discovery is more limited.

Banks need to ask themselves whether their approach to estimating the expected credit loss is reasonable.

**Accounting for Portfolio Effects.** Pricing risk at the transaction level, without considering portfolio effects tends to 'price in' too much risk because it does not take into account portfolio effects. On the other hand, pricing risk at the portfolio level is complicated.

If portfolio effects are taken into account, then one can calculate the required economic capital for the entire organization.

A well designed portfolio risk measurement approach enables one to 'slice and dice' risk vertically and horizontally across an organization to facilitate the pricing of risk.

### 3. Best-Practice Infrastructure

The first and most important component of infrastructure in a financial services company is people. Given the right environment and support, it is people who make everything else happen. Best-practice risk measurement cannot be derived solely from complex analytical approaches: judgment will always be a significant input.



Likewise, ensuring the integrity of data provides an important competitive advantage, as data are translated into risk management information for both transaction makers and policy makers.

Finally, a key goal, critical to the successful management of risk, is to integrate risk management operations and technology.

The implementation of an integrated risk management system should enable a firm to maintain a competitive advantage by allowing the firm to monitor and manage all of its risk on a global basis.

### 1.1.4 Basel III

The third of the Basel Accords was developed in a response to the deficiencies in financial regulation revealed by the global financial crisis. Basel III is a comprehensive set of reform measures, developed by the BCBS, to strengthen the regulation, supervision and risk management of the banking sector. These measures aim to:

- Improve the banking sector's ability to absorb shocks arising from financial and economic stress, whatever the source.
- Improve risk management and governance.
- Strengthen banks' transparency and disclosures.

The performs target:

- Bank-level, or microprudential, regulation, which will help raise the resilience of individual banking institutions to periods of stress.
- Macroprudential, system wide risks that can build up across the banking sector as well as the procyclical amplification of these risks over time.

These two approaches to supervision are complementary as greater resilience at the individual bank level reduces the risk of system wide shocks.

Let provide an overview of the various measures taken by the Committee.

## Capital

### i) Pillar 1

**Quality and level of capital.** Greater focus on common equity. The minimum will be raised to 4.5% of riskweighted assets.

***Gone concern* contingent capital.** *Gone concern* contingent capital increases the contribution of the private sector to resolving future banking crises and thereby reduces moral hazard.

**Capital conservation buffer.** Comprising common equity of 2.5% of risk-weighted assets, bringing the total common equity standard to 7%. Constraint on a bank's discretionary distributions will be imposed when banks fall into the buffer range.

**Countercyclical buffer.** Imposed within a range of 0-2.5% comprising common equity, when authorities judge credit growth is resulting in an unacceptable build up of systematic risk.

**Securitisations.** Strengthens the capital treatment for certain complex securitisations. Requires banks to conduct more rigorous credit analyses of externally rated securitisation exposures.

**Trading book.** Significantly higher capital for trading and derivatives activities, as well as complex securitisations held in the trading book.

**Counterparty credit risk.** Substantial strengthening of the counterparty credit risk framework. Includes: more stringent requirements for measuring exposure; capital incentives for banks to use central counterparties for derivatives; and higher capital for inter-financial sector exposures.

**Leverage ratio.** A non-risk-based leverage ratio that includes off-balance sheet exposures will serve as a backstop to the risk-based capital requirement.

## ii) Pillar 2

**Supplemental Pillar 2 requirements.** Address firm-wide governance and risk management; capturing the risk of off-balance sheet exposures and securitisation activities; managing risk concentrations; providing incentives for banks to better manage risk and returns over the long term; sound compensation practices; valuation practices; stress testing; accounting standards for financial instruments; corporate governance; and supervisory colleges.

## iii) Pillar 3

**Revised Pillar 3 disclosures requirements.** The requirements introduced relate to securitisation exposures and sponsorship of off-balance sheet vehicles. Enhanced disclosures on the detail of the

components of regulatory capital and their reconciliation to the reported accounts will be required, including a comprehensive explanation of how a bank calculates its regulatory capital ratios.

In addition to meeting the Basel III requirements, global systemically important financial institutions (SIFIs) must have higher loss absorbency capacity to reflect the greater risks that they pose to the financial system. The Committee has developed a methodology that includes both quantitative indicators and qualitative elements to identify global SIFIs. The additional loss absorbency requirements are to be met with a progressive Common Equity Tier 1 (CET1) capital requirement. A consultative document was submitted to the Financial Stability Board (FSB), which is coordinating the overall set of measures to reduce the moral hazard posed by global systemically important financial institutions.

### Liquidity

**Liquidity coverage ratio.** The liquidity coverage ratio (LCR) will require banks to have sufficient high-quality liquid assets to withstand a 30-day stressed funding scenario that is specified by supervisors.

**Net stable funding ratio.** The net stable funding ratio (NSFR) is a longer-term structural ratio designed to address liquidity mismatches. It covers the entire balance sheet and provides incentives for banks to use stable sources of funding.

### Principles for Sound Liquidity Risk Management and Supervision.

The Committee's 2008 guidance entitled Principles takes account of lessons learned during the crisis and are based on a fundamental review of sound practices for managing liquidity risk in banking organisations.

**Supervisory monitoring.** The liquidity framework includes a common set of monitoring metrics to assist supervisors in identifying and analysing liquidity risk trends at both the bank and system-wide level.

## 1.2 Categories of Financial Risks

Risks are uncertainties resulting in adverse variations of profitability or losses. When dealing with banking issues there are a large number of risk, most of them are well known. There has been a significant extension of focus, from the traditional qualitative risk assessment towards the quantitative management of risk, due to both evolving risk practices and strong regulatory incentives.

**Banking Risks** are defined as adverse impacts on profitability of several distinct sources of uncertainty. Risk measurement requires capturing the source of the uncertainty and the magnitude of its potential adverse effect on profitability <sup>1</sup>.

**Financial Risks** are the risks related to the market movements or the economic changes of the environment.

### 1.2.1 Credit Risk

*Credit risk* or *credit worthiness* is the risk of loss due to a counterparty defaulting on a contract or, more generally, the risk of loss due to some credit event. In the European Central Bank (ECB) glossary, it is defined as 'the risk that a counterparty will not settle an obligation in full, either when due or at any time thereafter'. Is the first of all risks in terms of importance.

Traditionally this is applied to bonds where debt holders were concerned that the counterparty, to whom they've made a loan, might default on a payment. For that reason, credit risk is sometimes also called *default risk*.

We will study Credit Risk and the most important methods in order to measure it in a further section.

### 1.2.2 Market Risk

*Market risk* refers to the risk to an institution resulting from movements in market prices, in particular, changes in interest rates, foreign exchange rates, and equity and commodity prices. The associated market risks are:

**Equity Risk:** Refers to the risk that stock <sup>2</sup> prices or the implied volatility will change.

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<sup>1</sup>Profitability refers, in the following, to both accounting and mark-to-market measures.

<sup>2</sup>A stock is a part ownership interest in a company.

**Interest Rate Risk:** Is the risk that interest rates or the implied volatility will change. In general, as rates rise, the price of a fixed rate bond will fall, and viceversa.

**Currency Risk:** Risk arising from the change in exchange rates. Investors face an exchange rate risk when they have assets or operations across national borders or if they have loans or borrowings in a foreign currency.

**Commodity Risk:** Is the risk that commodity prices (metals, gas, . . .) will change. A commodity enterprise needs to deal with price risk, quantity risk, cost risk and political risk.

The market risk factors cited above are not exhaustive. Depending on the instruments traded by an institution, exposure to other factors may also arise. Moreover, market risk is often propagated by other forms of financial risk such as credit and market-liquidity risks. The institutions consideration of market risk should capture all risk factors that it is exposed to, and it must manage these risks soundly.

### 1.2.3 Operational Risk

Operational risk can be defined as the risk of loss resulting from inadequate or failed internal processes, people and systems, or from external events.

Basel II defined event types are internal fraud (misappropriation of assets, tax evasion), external fraud (theft of information, hacking damage), employment practices and workplace safety (workers compensation, employee health and safety), clients, products and business practice (market manipulation, antitrust, improper trade, product defects), damage to physical assets (natural disasters, terrorism, vandalism), business disruption and systems failures (software failures, hardware failures) and execution, delivery and process management (data entry errors, accounting errors, failed mandatory reporting).

However, the Basel Committee recognizes that operational risk is a term that has a variety of meanings and therefore, for internal purposes, banks are permitted to adopt their own definitions of operational risk, provided that the minimum elements in the Committee's definition are included.

A bank should develop a framework for managing operational risk and evaluate the adequacy of capital given this framework. The framework should cover the bank's appetite and tolerance for operational risk, as specified

through the policies for managing this risk, including the extent and manner in which operational risk is transferred outside the bank. It should also include policies outlining the bank's approach to identifying, assessing, monitoring and controlling the risk.

The failure to properly manage operational risk can result in a misstatement of an institution's risk/return profile and expose the institution to significant losses.

### 1.2.4 Liquidity Risk

Liquidity risk is financial risk due to uncertain liquidity. Is the risk that a given security or asset cannot be traded quickly enough in the market to prevent a loss.

Liquidity risk arises from situations in which a party interested in trading an asset cannot do it because nobody in the market wants to trade that asset. Liquidity risk becomes particularly important to parties who are about to hold or currently hold an asset, since it affects their ability to trade.

An institution might lose liquidity if its credit rating falls, it experiences sudden unexpected cash outflows, or some other event causes counterparties to avoid trading with or lending to the institution. A firm is also exposed to liquidity risk if markets on which it depends are subject to loss of liquidity. Liquidity risk tends to compound other risks.

Liquidity is fundamental to the ongoing viability of any banking organization. Bank's capital positions can have an effect on their ability to obtain liquidity, especially in a crisis. Each bank must have adequate systems for measuring, monitoring and controlling liquidity risk. Banks should evaluate the adequacy of capital given their own liquidity profile and the liquidity of the markets in which they operate.

The scope and frequency of a banks internal liquidity risk management reports will vary according to the complexity of the institutions operations and risk profile. Reportable items may include cash flow gaps, asset and funding concentrations, critical assumptions used in cash flow projections, key early warning or risk indicators, funding availability, status of contingent funding sources and collateral usage.

# Chapter 2

## Credit Risk Measurement and Modeling

There exists a wide range of literature on credit risk. We refer mainly to [1], [2], [3] and [4].

### 2.1 Risk Parameters

#### 2.1.1 Probability of Default (PD):

Is the likelihood that a loan will not be repaid and will fall into default. PD is calculated for each client who has a loan or for a portfolio of clients with similar attributes and it measures the uncertainty whether an obligor will default or not.

For comparison reasons it is usually specified with respect to a given time horizon, typically one year. Then the probability of default describes the probability that the default event occurs before the specified time horizon.

The credit history of the counterparty or portfolio and nature of the investment are also taken into account to calculate the PD.

Default probability cannot be measured directly. Many banks use external ratings agencies such as Standard&Poors, Fitch or Moody's. However, banks are also encouraged to use their own Internal Rating Methods as well.

#### 2.1.2 Exposure at Default (EAD):

The Exposure at Default of an obligor denotes the portion of the exposure to the obligor which is lost in case of default.

If an asset suffers from a lower valuation or a loan defaults, the EAD figure is how much the firm will lose as a result of the default. The loss is

contingent upon the amount to which the bank was exposed to the borrower at the time of default, commonly expressed as Exposure at Default.

EAD is a measure of potential exposure as calculated by a Basel Credit Risk Model for the period of one year or until maturity whichever is soonest.

A bank must provide an estimate of the exposure amount for each transaction. All these loss estimates should seek to fully capture the risks of an underlying exposure. These values are calculated taking account of the underlying asset, forward valuation, facility type and commitment details. However, they do not take account of guarantees, collateral or security.

In most cases EAD will equal the nominal amount of the facility, but for certain facilities it will include an estimate of future lending prior to default. Under the foundation methodology EAD is estimated through the use of standard supervisory rules. In the advanced methodology, the bank itself determines the appropriate EAD to be applied to each exposure.

### 2.1.3 Loss Given Default (LGD):

The Loss Given Default is the magnitude of a transaction describes the extent of the loss incurred in the event of default. It is expressed as a percentage of the exposure.

Loss Given Default is facility-specific because such losses are generally understood to be influenced by key transaction characteristics such as the presence of collateral and the degree of subordination.

LGD is usually modeled as a random variable describing the severity of losses in the default event and it is determined in one of two ways.

On the one hand, under the foundation methodology, LGD is estimated through the application of standard supervisory rules, which differentiate the level of Loss Given Default based upon the characteristics of the underlying transaction. The supervisory rules and treatments were chosen to be conservative. The starting point proposed by the BCBS is use a 50% LGD value for most unsecured transactions, with a higher LGD (75%) applied to subordinated exposures.

On the other hand, in the advanced methodology, the bank itself determines the appropriate LGD to be applied to each exposure, on the basis of robust data and analysis which is capable of being validated both internally and by supervisors. Thus, a bank using internal Loss Given Default estimates for capital purposes might be able to differentiate LGD values on the basis of a wider set of transaction characteristics as well as borrower characteristics. A bank wishing to use its own estimates of LGD will need to demonstrate to its supervisor that it can meet additional minimum requirements pertinent to the integrity and reliability of these estimates.



## 2.2 Risk Measures

In order to assess concentrations in credit risk, it is important to first measure credit risk. Many measures have been used and developed to communicate credit risk levels. In this section we follow mainly the work by [17].

### 2.2.1 Risk as the random variable

Risk is related to the variability of the future value of a position, due to market changes or more generally to uncertain events.

The random variables on the set of states of nature at a future date can be interpreted as possible future values of positions of portfolios currently held.

A first crude, but crucial, measurement of the risk of a position will be whether its future value belongs or does not belong to the subset of acceptable risks, as decided by a supervisor such as:

- i) A regulator who takes into account the unfavorable states when allowing a risky position that may draw on the resources of the government, for example as a guarantor of last resort.
- ii) An exchanges clearing firm, which has to make good on the promises to all parties of transactions being securely completed.
- iii) An investment manager who knows that his firm has basically given to its traders an exit option in which the strike price consists in being fired in the event of big trading losses on ones position.

In each of these cases, there is a trade-off between the severity of the risk measurement and the level of activities in the supervised domain.

Let consider one period of uncertainty  $(0, T)$  between two dates 0 and  $T$ . The various currencies are numbered by  $i$ ,  $1 \leq i \leq I$ , and for each of them one reference instrument is given, which carries one unit of date 0 currency  $i$  into  $r_i$  units of date  $T$  currency  $i$ .

The period  $(0, T)$  can be the period between hedging and rehedging, a fixed interval, the period required to liquidate a position, or the length of coverage provided by an insurance contract.

Date 0 exchange rates are supposed to be one, and  $e_i$  denotes the random number of units of currency 1 which one unit of currency  $i$  buys at date  $T$ .

An investors initial portfolio consists of positions  $A_i$ ,  $1 \leq i \leq I$ . The position  $A_i$  provides  $A_i(T)$  units of currency  $i$  at date  $T$ .

**Definition 1.** We call risk the investor's future net worth

$$\sum_{1 \leq i \leq I} e_i \cdot A_i(T).$$

### 2.2.2 Axioms on acceptance sets

We suppose that the set of all possible states at the end of the period is known, but the probabilities of the various states occurring may be unknown or not subject to common agreement. This assumes that markets at date  $T$  are liquid; if they are not, more complicated models are required in which we can distinguish the risks of a position a the mapping from the former to the latter may not be linear.

#### Notation

- Let  $\Omega$  the set of states of nature, and assume it is finite. Considering  $\Omega$  as the set of outcomes of an experiment, we compute the final net worth or a position for each element of  $\Omega$ . It is a random variable denoted by  $Y$ .
- Let  $\mathcal{G}$  be the set of all risks, that is the set of all real-valued functions on  $\Omega$ . Since  $\Omega$  is assumed to be finite,  $\mathcal{G}$  can be identified with  $\mathbb{R}^n$ , where  $n = \text{card}(\Omega)$ . The cone of nonnegative elements in  $\mathcal{G}$  shall be denoted by  $L_+$ , its negative by  $L_-$ .
- We call  $A_{i,j}$  a set of final net worths, expressed in currency  $i$ , which are accepted by regulator  $j$ .
- Let  $A_i = \bigcap_j A_{i,j}$  and use the generic notation  $\mathcal{A}$ .

#### Axioms for acceptance sets

Let  $\mathcal{A}$  be the acceptance set.

1.  $\mathcal{A}$  contains  $L_+$ .
2.  $\mathcal{A} \cap L_- = \{0\}$ .
3.  $\mathcal{A}$  is convex.
4.  $\mathcal{A}$  is a positively homogeneous cone.

### 2.2.3 Correspondence between Acceptance Sets and Measures of Risk

Sets of acceptable future net worths are the primitive objects to be considered in order to describe acceptance or rejection of a risk. Given some reference instrument, there is a natural way to define a measure of risk by describing how close or how far from acceptance a position is.

**Definition 2.** *A measure of risk is a mapping from  $\mathcal{G}$  into  $\mathbb{R}$ .*

**Definition 3.** *Risk measure associated with an acceptance set. Given the total rate of return  $r$  on a reference instrument, the risk measure associated with the acceptance set  $\mathcal{A}$  is defined by*

$$\rho_{\mathcal{A},r} = \inf\{m \mid m \cdot r + Y \in \mathcal{A}\}.$$

**Definition 4.** *Acceptance set associated with a risk measure  $\rho$ ,*

$$\mathcal{A}_\rho = \{Y \in \mathcal{G} \mid \rho(Y) \leq 0\}.$$

We will consider now several possible properties for a risk measure  $\rho$  defined on  $\mathcal{G}$ .

### 2.2.4 Properties of the risk measures

A risk measure that is used for specifying capital requirements can be thought of as the amount of cash (or capital) that must be added to a position to make its risk acceptable to regulators. A number of properties that such a risk measure should have have been proposed. These are:

- **Monotonicity:** if a portfolio has lower returns than another portfolio for every state of the world, its risk measure should be greater.

$$\forall Y_1, Y_2 \in \mathcal{G} \text{ with } Y_1 \leq Y_2, \text{ we have } \rho(Y_2) \leq \rho(Y_1).$$

- **Translation invariance:** if we add an amount of cash  $\alpha$  to a portfolio, its risk measure should go down by  $\alpha$ .

$$\forall Y \in \mathcal{G} \text{ and all real numbers } \alpha, \text{ we have } \rho(Y + \alpha \cdot r) = \rho(Y) - \alpha.$$

- **Positive homogeneity:** changing the size of a portfolio by a factor  $\lambda$  while keeping the relative amounts of different items in the portfolio the same should result in the risk measure being multiplied by  $\lambda$ .

$$\forall \lambda \geq 0 \text{ and } \forall Y \in \mathcal{G}, \rho(\lambda Y) = \lambda \rho(Y).$$

- **Subadditivity:** the risk measure for two portfolios after they have been merged should be no greater than the sum of their risk measures before they were merged.

$$\forall Y_1, Y_2 \in \mathcal{G}, \rho(Y_1 + Y_2) \leq \rho(Y_1) + \rho(Y_2).$$

The first three conditions are straightforward given that the risk measure is the amount of cash needed to be added to the portfolio to make its risk acceptable. The fourth condition states that diversification helps reduce risks. When two risks are aggregated, the total of the risk measures corresponding to the risks should either decrease or stay the same.

**Definition 5.** *Coherence.*

*A risk measure satisfying the four properties of translation invariance, subadditivity, positive homogeneity and monotonicity is called coherent.*

On the one hand we shall speak of a *model-dependent* measure of risk when an explicit probability on  $\Omega$  is used to construct it. We will study in a next section the most important models in order to measure credit risk.

On the other hand, we will have a *model-free* measure otherwise. Let see some currently used model-free measures in order to quantify risk. Some of them are coherent risk measures, attending to the four axioms that they must satisfy, and others are not.

### 2.2.5 Value at Risk (VaR)

Value at Risk is not a risk measure, since it does not verify the subadditivity axiom. Nevertheless, it is probably the most widely used measure in financial institutions (see [3]). It is used to measure potential losses.

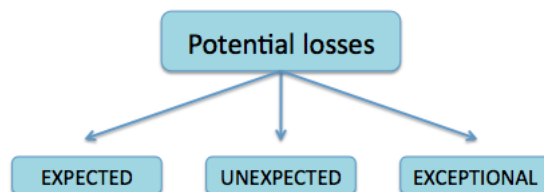


Figure 2.1: There are several types of potential losses.

It can be defined as the worst loss that might be expected from holding a security of portfolio over a given period of time, given a specified level of probability, the so-called *confidence level*,

$$VaR_q(L) = \inf\{x \in \mathbb{R} : \mathbb{P}(L > x) \leq 1 - q\} = \inf\{x \in \mathbb{R} : F_L(x) \geq q\},$$

where  $F_L(x)$  is the distribution function of the loss variable. Thus, VaR is simply a quantile of the loss distribution.

In general, in credit risk management high values of  $q$  are used:  $q = 99.9\%$  in the Second Basel Accord and  $q = 99.98\%$  is the value used in practice for many banks. A higher confidence level leads to a higher VaR. The reason for these high values for  $q$  is that banks want to demonstrate external rating agencies a solvency level that corresponds at least to the achieved rating class.

VaR has many benefits when compared to traditional measures of risk:

- It is a value which has a very simple meaning of unexpected losses against which some hedge should be provided.
- VaR is fungible. All risks are expressed with the same units of unexpected losses at a given tolerance level.
- It is a synthetic measure of risk which includes multiple dimensions (volatility, sensitivity to market movements and gaps) into a single number.

- VaR applies to all level of risk management and can capture diversification effects.
- Its methodology is used to define economic capital, which applies directly to the issue of capital adequacy.

It can be used in many applications:

- It measures the risk at the levels of business units, customers and product classes.
- It is also used in order to set limits, at those same levels of aggregation, as a maximum allowed VaR which should constrain the VaR resulting from a transaction.
- It used as the measure of risk-adjusted performance, using VaR as the measure of risks.

### Expected Loss (EL)

For a default model, the expected loss is the expectation of all book value losses at the horizon.

The EL is frequently used for assessing credit risk. It represents a statistical average over a portfolio of a large number of loans or transactions. The law of large numbers says that losses will be sometimes high or low, but the average is the expected loss. Statistical losses are more a portfolio concept than an individual transaction concept.

**Definition 6.** *The portfolio loss  $L_N$  is defined as the random variable*

$$L_N = \sum_{n=1}^N EAD_n \cdot LGD_n \cdot D_n,$$

where  $D_n$  denotes the default indicator of obligor  $n$  and it can be represented as a Bernoulli random variable taking the values

$$D_n = \begin{cases} 1 & \text{if obligor } n \text{ defaults} \\ 0 & \text{otherwise} \end{cases}$$

with probabilities  $\mathbb{P}(D_n = 1) = PD_n$  and  $\mathbb{P}(D_n = 0) = 1 - PD_n$ .

**Definition 7.** *The expected loss of a certain obligor  $n$ ,  $EL_n$ , represents a kind of risk premium which a bank can charge for taking the risk that obligor  $n$  might default. It is defined as <sup>1</sup>*

$$EL_n = \mathbb{E}[L_n] = EAD_n \cdot ELGD_n \cdot PD_n.$$

By additivity of the expectation operator, the EL for a given portfolio can be defined as it follows:

**Definition 8.** *The expected loss for a given portfolio containing  $N$  loans can be written as*

$$EL = \sum_{n=1}^N EAD_n \cdot ELGD_n \cdot PD_n.$$

Provisions should hedge EL. They differ from capital that provides a protection against deviations from this average. Capital should provide protection against unexpected losses only, expected losses being netted out of revenues.

### Unexpected Loss (UL)

Whereas expected losses can be described as the usual or average losses that an institution incurs in its natural course of business, unexpected losses are deviations from the average that may put an institutions stability at risk. Peak losses, although occurring quite seldom, can be very large when they occur, therefore a bank should also reserve money for them.

The deviation of losses from the EL is measured by using the standard deviation of the loss variable.

**Definition 9.** *The unexpected loss of obligor  $n$  is defined as <sup>2</sup>*

$$UL_n = \sqrt{\mathbb{V}[L_n]} = \sqrt{\mathbb{V}[EAD_n \cdot LGD_n \cdot D_n]}.$$

---

<sup>1</sup> $ELGD_n$ : expectation of LGD variable of obligor  $n$ .

<sup>2</sup> $VLGD_n$ : volatility of LGD variable of obligor  $n$ .

- a) If  $D_n$  and  $LGD_n$  are uncorrelated, the unexpected loss of obligor  $n$ ,  $UL_n$ , is given by

$$UL_n = EAD_n \cdot \sqrt{VLGD_n^2 \cdot PD_n + ELGD_n^2 \cdot PD_n(1 - PD_n)},$$

and the UL of the total portfolio

$$UL = \sum_{n=1}^N EAD_n \cdot \sqrt{VLGD_n^2 \cdot PD_n + ELGD_n^2 \cdot PD_n(1 - PD_n)}.$$

- b) In the correlated case, additivity is lost, and the UL of the total portfolio is given by

$$UL = \sqrt{\mathbb{V}[L]} = \sqrt{\sum_{n=1}^N \sum_{k=1}^N EAD_n \cdot EAD_k \cdot Cov[LGD_n \cdot D_n; LGD_n \cdot D_k]}.$$

### Exceptional Loss

Exceptional events are not included in UL. To include them would drastically increase the VaR measures. At the limit, the total assets of a bank could be lost, although this theoretical possibility is extremely unlikely to occur, owing to diversification effects. This raises the issue of drawing a border between unexpected and truly exceptional losses. The border is defined by the tolerance level, plus the practicality of measuring unexpected losses at very low tolerance levels.

Truly exceptional losses cannot be evaluated with statistical laws. Stress scenarios address this issue. They provide some idea of what the losses could be under extreme conditions, but the probability of such scenarios is judgmental rather than subject to statistical benchmarks.



### 2.2.6 Expected Shortfall (ES)

In the literature on risk management, VaR has shown itself to be a very useful risk indicator. However, it is also now widely accepted that VaR is not the finest measure available. VaR is often criticized for its failure to reflect the severity of losses in the worst scenarios in which the loss exceeds VaR. In other words, VaR is not a measure of the heaviness of the tail of the distribution. A novel theoretical development in recent years is the Expected Shortfall. We review work by [18].

*Expected Shortfall (ES)* is also a risk measure used in financial risk measurement to evaluate the market risk or credit risk of a portfolio. This statistic arises in a natural way from the estimation of the average of the  $100 \cdot q\%$  worst losses in a sample of returns to a portfolio. Here  $q$  is some fixed confidence level. It is closely related to VaR. Instead of using a fixed confidence level, as in the concept of VaR, one averages VaR over all confidence levels  $q \in (0, 1)$  for some  $q \in (0, 1)$ . Thus, the tail behavior of the loss distribution is taken into account. This measure is also sometimes referred to as conditional VAR, or tail loss. Formally,

**Definition 10.** For a loss  $L$  with  $\mathbb{E}(|L|) < \infty$  and distribution function  $F_L$ , the *Expected Shortfall* at a confidence level  $q \in (0, 1)$  is defined as

$$ES_q = \frac{1}{1-q} \int_q^1 VaR_u(L) du.$$

If the loss variable is integrable with continuous distribution function, the following Lemma holds:

**Lemma 1.** For integrable loss variable  $L$  with continuous distribution function  $F_L$  and any  $q \in (0, 1)$  we have

$$ES_q = \frac{\mathbb{E}(L; L \geq VaR_q(L))}{1-q} = \mathbb{E}(L | L \geq VaR_q(L)).$$

Hence, in this situation expected shortfall can be interpreted as the expected loss that is incurred in the event that VaR is exceeded. In the discontinuous case, a more complicated formula holds:

$$ES_q = \frac{1}{1-q} (\mathbb{E}(L; L \geq VaR_q(L))) + VaR_q(L) \cdot (1 - q - \mathbb{P}(L \geq VaR_q(L))).$$

### Properties of ES

- $ES_q$  increases as  $q$  increases.
- For the 100% quantile,  $ES_{1.0}$  equals the expected value of the portfolio.
- For a given portfolio,  $ES_q \geq VaR_q$ .

### Example

Let  $Z_1, Z_2$  be two random variables such that  $Z_1 \sim N(0, 1)$  and  $Z_2 \sim \Gamma(3, 1)$ .

Figure 2.2 shows the pdf of  $Z_1$ . The green vertical line shows the Value at Risk while the blue one indicates the Expected Shortfall, both at level 95%.

$$VaR_{Z_1,0.95} = 1.6 \text{ and } ES_{Z_1,0.95} = 2.1.$$

In particular, for heavy-tailed distributions, the difference between VaR and ES is more pronounced than for normal distributions. Figure 2.3 shows the probability density function of the variable  $Z_2$ .

$$VaR_{Z_2,0.95} = 6.3 \text{ and } ES_{Z_2,0.95} = 7.6.$$

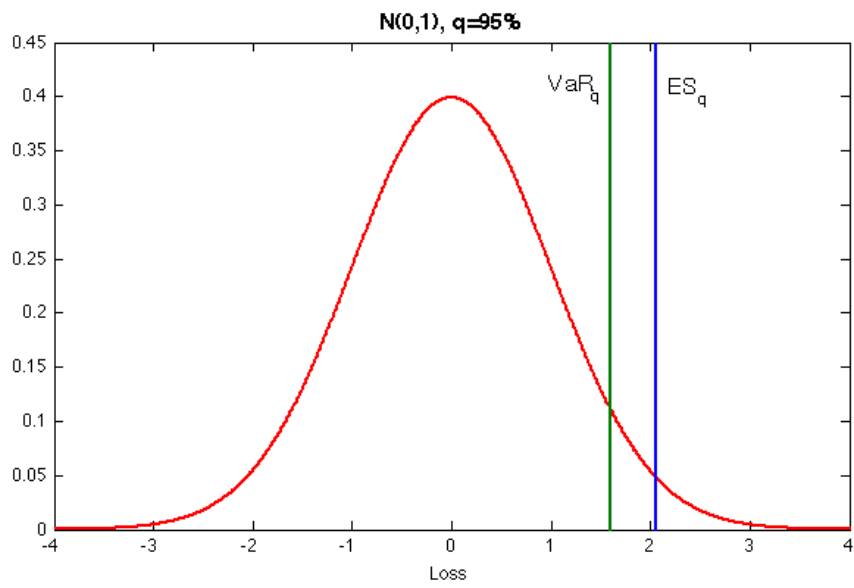


Figure 2.2: VaR and ES for standard normal distribution.

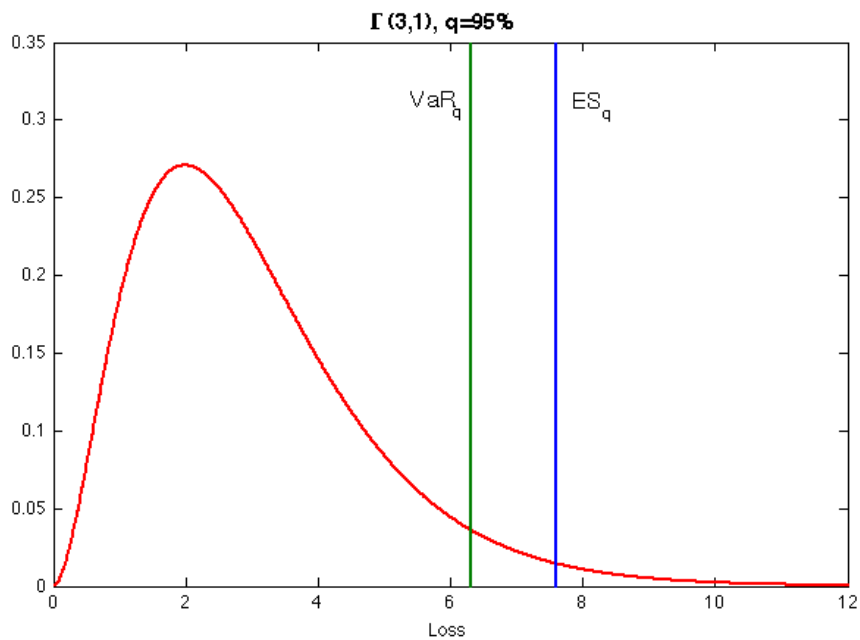


Figure 2.3: VaR and ES for Gamma distribution.

### Advantages of ES

Regulators make extensive use of VaR and its importance as a risk measure is therefore unlikely to diminish. However, expected shortfall has a number of advantages over VaR. This has led many financial institutions to use it as a risk measure internally.

- ES is universal, it can be applied to any instrument and to any underlying source of risk.
- ES is complete, it produces a unique global assessment for portfolios exposed to different sources of risk.
- ES is a simple concept since it is the answer to a natural and legitimate question on the risks run by a portfolio.
- Any bank that has a VaR based risk management system could switch to ES with virtually no additional computational effort.

### 2.2.7 Economic Capital (EC)

EC is also a measure of potential losses for the entire portfolio of the bank. It is the capital necessary to absorb unexpected losses. The EC at a given tolerance level is identical to the value of potential loss at the same tolerance level. Since EC is the capital required to cover the risk of potential losses, it is also called *risk-based capital* or *Capital At Risk (CAR)*

**Definition 11.** *The Economic Capital for a given confidence level  $q$  is defined as the Value at Risk at level  $q$  of the portfolio loss  $L$  minus the expected loss of the portfolio,*

$$EC_q(L) = VaR_q(L) - EL.$$

For a confidence level  $q = 99.98\%$  it can be interpreted as the appropriate capital to cover unexpected losses in 9.998 out of 10 years, where a time horizon of one year is assumed. Hence it represents the capital a bank should reserve to limit the probability of default to a given confidence level. The VaR is reduced by the EL due to the common decomposition of total risk capital, that is VaR, into a part covering EL and a part reserved for UL.

Suppose a bank wants to include a new loan in its portfolio and, thus, has to adopt its risk measurement. While the EL is independent from the composition of the reference portfolio, the EC strongly depends on the composition of the portfolio in which the new loan will be included. The EC charge for a new loan of an already well diversified portfolio, for example, might be much lower than the EC charge of the same loan when included in a portfolio where the new loan induces some concentration risk. For this reason the EL charges are said to be *portfolio independent*, while the EC charges are *portfolio dependent* which makes the calculation of the contributory EC much more complicated, since the EC always has to be computed based on the decomposition of the complete reference portfolio.

In the worst case, a bank could lose its entire credit portfolio in a given year. Holding capital against such an unlikely event is economically inefficient. As banks want to spend most of their capital for profitable investments, there is a strong incentive to minimize the capital a bank holds. Hence the problem of risk management in a financial institution is to find the balance between holding enough capital to be able to meet all debt obligations also

in times of financial distress, on the one hand, and minimizing EC to make profits, on the other hand.

### 2.2.8 Risk adjusted return on capital (RAROC)

The risk adjusted return on capital (RAROC) calculation is based on the trade-off between risk and return. It is a risk-adjusted profitability measurement and management framework for measuring and forecasting risk-adjusted financial performance, for maintaining financial integrity and boost confidence among stakeholders and for providing a consistent view of profitability across businesses.

**Definition 12.** *RAROC is defined as the ratio of risk-adjusted return to economic capital.*

$$RAROC = \frac{\text{Earnings} - \text{Expected losses}}{\text{Economic Capital}}.$$

A number of large banks have developed RAROC with the aim, in most cases, of quantifying the amount of equity capital necessary to support all their operating activities.

RAROC systems allocate capital for two basic reasons: risk management and performance evaluation. For risk management purposes, the main goal of allocating capital to individual business units is to determine the bank's optimal capital structure, that is economic capital allocation is closely correlated with individual business risk. As a performance evaluation tool, it allows banks to assign capital to business units based on the economic value added of each unit.

## 2.3 Asset Value Models and Reduced Form Models

### 2.3.1 Asset value models

These models trace back to the Merton model, which we will introduce in the next section. The default of a firm is modeled in terms of the relationship between its assets and the liabilities that it faces at the end of a given period time. The value of the firm's debt at maturity then equals the nominal value of the liabilities minus the pay-off of a European put option on the firm's value. The asset value process is modeled as a geometric Brownian motion and default occurs when the asset value at a maturity is lower than the liabilities.

In models of this type default risk depends mainly on the stochastic evolution of the asset value and default occurs when the random variable describing the asset value falls below a certain threshold which represent the liabilities. Therefore these structural models are also known as *latent variable models*.

**Definition 13.** *Latent variable model.*

Let  $V = (V_1, \dots, V_N)$  be a  $N$ -dimensional vector with continuous marginal distributions functions  $F_n(v) = \mathbb{P}(V_n \leq v)$ . Given a sequence of deterministic thresholds

$$-\infty = b_0^n < b_1^n < \dots < b_R^n < b_{R+1}^n = \infty,$$

we say that the obligor  $n$  is in state  $S_n = r$  if and only if

$$V_n \in (b_r^n, b_{r+1}^n] \text{ for some } r \in \{0, 1, \dots, R\} \text{ and } n \in \{1, 2, \dots, N\}.$$

Then  $(V_n, (b_r^n)_{0 \leq r \leq R+1})_{1 \leq n \leq N}$  defines a latent variable model for the state vector  $S = (S_1, \dots, S_N)$ . The individual default probability of firm  $n$  is given by  $F_n(b_1^n) = \mathbb{P}(V_n \leq b_1^n)$ .

Examples for this class of credit risk models are the theoretical Merton model and, based on this, important industry models like KMV's Portfolio Manager or the CreditMetrics Model.

### 2.3.2 Default rate models

In reduced form models one directly models the process of credit default instead of constructing a stochastic process of the firm's asset value which indirectly leads to a model of the firm's default.

In this class of models, defaults can happen at any discrete time interval and only the probability of default is specified. The default probability of a firm is usually modeled as a non-negative random variable, whose distribution typically depends on economic covariables. This class of models is sometimes called *mixture models*.

These models can be treated as two stage models. Conditional on the realization of economic factors the individual default probabilities are assumed to be independent whereas they can be unconditionally dependent. The conditional default probabilities are modeled as random variables with some mixing distribution which is specified in a second step. A prominent example of such a model is the CreditRisk<sup>+</sup> which we will introduce later.



## 2.4 The Merton Model

The Merton model can be understood as a multi-factor model as will be explained .

The Merton model is of Bernoulli type where the decision about default or survival of a firm at the end of a time period is made by comparing the firm's asset value to certain threshold value. If the firm value is below this threshold, the firm defaults and otherwise it survives. This model is frequently used in the different approaches for measuring concentration which we will discuss later.

### 2.4.1 The General Framework

The Merton model assumes the asset value of a firm to follow some stochastic process  $(V_t)_{t \geq 0}$ . There are only two cases of securities: equity and debt. It is assumed that equity receives no dividends and that the firm cannot issue new debt. The model assumes that the company's debt is given by a zero coupon bond with face value  $B$  that will become due at a future time  $T$ . The firm defaults if the value of its assets is less than the promised debt repayment at time  $T$ . In the Merton model default can occur only at the maturity  $T$  of the bond. Denote the value at time  $t$  of equity and debt by  $S_t$  and  $B_t$ . In a frictionless market (there are no taxes or transaction costs), the value of the firm's assets is given by the sum of debt and equity, i.e.,

$$V_t = S_t + B_t, \quad 0 \leq t \leq T.$$

At maturity there are two possible scenarios:

1.  $V_T > B$ : the value of the firm's assets exceeds the debt. In this case the debtholders receive  $B_T = B$ , the shareholders receive the residual value  $S_T = V_T - B$ , and there is not default.
2.  $V_T \leq B$ : the value of the firm's assets is less than its debt. Thus the firm cannot meet its financial obligations and defaults. In this case, the debtholders take ownership of the firm, and the shareholders are left with nothing, so we have  $B_T = V_T$ ,  $S_T = 0$ .

Hence, combining the above two results, the payment to the shareholders at time  $T$  is given by

$$S_T = \max(V_T - B, 0) = (V_T - B)^+,$$

and debtholders receive

$$B_T = \min(V_T, B) = B - (B - V_T)^+.$$

This shows that the value of the firm's equity is the payoff of an European call option on the assets of the firm with strike price equal to the promised debt payment. The Merton model treats the asset value  $V_t$  as any underlying. It assumes that under the real world probability measure  $\mathbb{P}$  the asset value process follows a geometric Brownian motion of the form

$$dV_t = \mu_V V_t dt + \sigma_V dW_t, \quad 0 \leq t \leq T, \quad (2.1)$$

for constants  $\mu_V \in \mathbb{R}$ ,  $\sigma_V > 0$  and a standard Brownian motion  $(W_t)_{t \geq 0}$ . Further, it makes all simplifying assumptions of the Black-Scholes option pricing formula. The solution at time  $T$  of the stochastic differential equation (2.1) with initial value  $V_0$  is given by

$$V_T = V_0 \cdot \exp \left( \left( \mu_V - \frac{1}{2} \sigma_V^2 \right) T + \sigma_V W_T \right),$$

which in particular implies that

$$\ln(V_T) \sim \Phi \left( \ln(V_0) + \left( \mu_V - \frac{1}{2} \sigma_V^2 \right) T, \sigma_V^2 T \right).$$

Hence the market value of the firm's equity at maturity  $T$  can be determined as the price of a European call option on the asset value  $V_t$  with exercise price  $B$  and maturity  $T$ . The risk neutral pricing theory then yields that the market value of equity at time  $t < T$  can be computed as the discounted expectation of the payoff function  $S_T$ ,

$$S_t = \mathbb{E} \left[ e^{-r(T-t)} \cdot (V_T - B)^+ | F_t \right],$$

and it is given by

$$S_t = V_t \cdot \Phi(d_{t,1}) - B \cdot e^{-r(T-t)} \cdot \Phi(d_{t,2}),$$

where

$$d_{t,1} = \frac{\ln\left(\frac{V_t}{B}\right) + \left(r + \frac{1}{2}\sigma_V^2\right)(T-t)}{\sigma_V \cdot \sqrt{T-t}} \quad \text{and} \quad d_{t,2} = d_{t,1} - \sigma_V \cdot \sqrt{T-t}.$$

Here  $r$  denotes the risk-free interest rate, assumed to be constant.

According to the equation for  $B_T$ , we are able to value the firm's debt at time  $t \leq T$  as

$$\begin{aligned} B_t &= \mathbb{E} \left[ e^{-r(T-t)} (B - (B - V_T)^+) | F_t \right] \\ &= B \cdot e^{-r(T-t)} - \left( B \cdot e^{-r(T-t)} \Phi(-d_{t,2}) - V_t \cdot \Phi(-d_{t,1}) \right). \end{aligned}$$

The default probability of the firm by time  $T$  is the probability that the shareholders will not exercise their call option to buy the assets of the company for  $B$  at time  $T$ , and it can be computed as

$$\mathbb{P}(V_T \leq B) = \mathbb{P}(\ln(V_T) \leq \ln(B)) = \Phi\left(\frac{\ln(B/V_0) - (\mu_V - \frac{1}{2}\sigma_V^2) \cdot T}{\sigma_V \sqrt{T}}\right). \quad (2.2)$$

The last equation shows that the default probability is increasing in  $B$ , decreasing in  $V_0$  and  $\mu_V$  and, for  $V_0 > B$ , increasing in  $\sigma_V$ , which is all perfectly in line with economic intuition. Under the risk-neutral measure  $\mathbb{Q}$  we have

$$\mathbb{Q}(V_T \leq B) = \mathbb{Q}\left(\frac{\ln(B/V_0) - (r - \frac{1}{2}\sigma_V^2) \cdot T}{\sigma_V \sqrt{T}} \leq -d_{0,2}\right) = 1 - \phi(d_{0,2}).$$

Hence the risk-neutral default probability, given information up to time  $t$ , is given by  $1 - \phi(d_{t,2})$ .

**Remark 1.**

The Merton model can also incorporate credit migrations and, thus, is not limited to the default-only mode as presented above. Therefore, we consider a firm which has been assigned to some rating category at time  $t_0 = 0$ . The time horizon is fixed  $T > 0$ . Assume that the transition probabilities  $p(r)$  for a firm are available for all rating grades  $0 \leq r \leq R$ . The transition probability thus denotes the probability that the firm belongs to rating class  $r$  at time horizon  $T$ . In particular,  $p(0)$  denotes the default probability of the firm.

Suppose that the asset-value process  $V_t$  of the firm follows the model given in 2.1. Let define thresholds

$$-\infty = b_0 < b_1 < \dots < b_R < b_{R+1} = \infty,$$

such that  $\mathbb{P}(b_r < V_T \leq b_{r+1}) = p(r)$  for  $r \in \{0, \dots, R\}$ , this is, the probability that the firms belongs to rating  $r$  at the time horizon  $T$  equals the probability that the firm's value at time  $T$  is between  $b_r$  and  $b_{r+1}$ . Hence we have translated the transition probabilities into a series of thresholds for an assumed asset-value process. We recall that  $b_1$  denotes the default threshold, i.e. the value of the firm's liabilities  $B$ . The higher thresholds are the asset-value levels marking the boundaries of higher rating categories.

Although the Merton model provides a useful context for modeling credit risk, and practical implementations of the model are used by many financial institutions, it also has some drawbacks.

It assumes the firm's debt financing consists of a one-year coupon bond. For most firms, however, this is an oversimplification. Moreover, the simplifying assumptions of the Black-Scholes model are questionable in the context of corporate debt. In particular, the assumption of normally distributed losses can lead to an underestimation of the potential risk in a loan portfolio.

Finally, and this might be the most important shortcoming of the Merton model, the firm's value is not observable which makes assigning values to it and its volatility problematic.

## 2.4.2 The Multi-Factor Merton Model

We consider a portfolio of  $N$  borrowers. Each of the obligors has one loan with exposure  $EAD_n$ . We express the loan as a share of total portfolio exposure, i.e., the exposure share of obligor  $n$  is given by

$$s_n = \frac{EAD_n}{\sum_{n=1}^N EAD_n}.$$

Fix a time horizon  $T > 0$ . We define  $V_t^{(n)}$  to be the asset value of counterparty  $n$  at time  $t < T$ .

For every counterparty there exists a threshold  $C_n$  such that counterparty  $n$  defaults in  $[0, T]$  if  $V_t^{(n)} < C_n$ . Hence,  $V_T^{(n)}$  can be seen as a latent variable driving the event of default. Thus, we define

$$D_n = \mathbf{1}_{\{V_T^{(n)} < C_n\}} \sim \text{Bern}(1; \mathbb{P}(V_T^{(n)} < C_n)). \quad (2.3)$$

Let  $r_n$  be borrower  $n$ 's asset-value log-returns,  $\log(V_T^{(n)}/V_0^{(n)})$ . The main assumption in the factor model is the following one:

**Assumption 1.** *The asset returns  $r_n$  depend linearly on  $K$  standard normally distributed risk factors  $X = (X_1, \dots, X_K)$  affecting borrower's defaults in a systematic way as well as on a standard normally distributed idiosyncratic term  $\epsilon_n$ . Moreover,  $\epsilon_n$  is independent of the systematic factors  $X_k$ ,  $\forall k \in \{1, \dots, K\}$  and the  $\epsilon_n$  are uncorrelated.*

Then, borrower  $n$ 's asset value log-returns, after standardization, admit a representation of the form

$$r_n = \beta_n \cdot Y_n + \sqrt{1 - \beta_n^2} \cdot \epsilon_n, \quad (2.4)$$

where  $Y_n$  represents the firm's composite factor,  $\epsilon_n$  denotes the idiosyncratic shock and  $\beta_n$  captures the correlation between  $r_n$  and  $Y_n$ .

$Y_n$  can be decomposed into  $K$  independent factors  $X = (X_1, \dots, X_K)$  by  $Y_n = \sum_{k=1}^K \alpha_{n,k} \cdot X_k$  with  $\alpha_{n,k}$  describing the dependence of obligor  $n$  on an industrial or regional sector  $k$  represented by factor  $X_k$ .

The correlation of the asset returns depends only on the correlation of the composite factors  $Y_n$  since the risk factors and the idiosyncratic shocks are assumed to be independent. Computing variances in (2.4) we get

$$\mathbb{V}(r_n) = \beta_n^2 \cdot \mathbb{V}(Y_n) + (1 - \beta_n^2) \cdot \mathbb{V}(\epsilon_n).$$

- i)  $\beta_n^2 \cdot \mathbb{V}(Y_n)$  quantifies the systematic risk of the counterparty  $n$ .
- ii)  $(1 - \beta_n^2) \cdot \mathbb{V}(\epsilon_n)$  captures the idiosyncratic risk which cannot be explained with the factors  $X_k$ .

In order to ensure that  $\mathbb{V}(Y_n) = 1$ , considering that  $r_n$ ,  $X_k$  and  $\epsilon_n$  are assumed to be standard normally distributed, the coefficients  $\alpha_{n,k}$  must verify  $\sum_{k=1}^K \alpha_{n,k}^2 = 1$ . Thus, we can rewrite (2.3) as

$$D_n = \mathbf{1}_{\{r_n < c_n\}} \sim \text{Bern}(1; \mathbb{P}(r_n < c_n)),$$

where  $c_n$  is the threshold corresponding to  $C_n$  after exchanging  $V_T^{(n)}$  by  $r_n$ .

Assume  $T = 1$ . Let  $PD_n$  the one year default probability of obligor  $n$ . We have  $PD_n = \mathbb{P}(r_n < c_n)$  and, since  $r_n \sim N(0, 1)$ ,

$$c_n = \Phi^{-1}(PD_n).$$

Hence, the condition  $r_n < c_n$  in the factor representation, can be written as

$$\epsilon_n < \frac{\Phi^{-1}(PD_n) - \beta_n \cdot Y_n}{\sqrt{1 - \beta_n^2}}.$$

Thus, the one year default probability of obligor  $n$ , conditional on  $Y_n$ , is given by

$$PD_n(Y_n) = \Phi \left( \frac{\Phi^{-1}(PD_n) - \beta_n \cdot Y_n}{\sqrt{1 - \beta_n^2}} \right). \quad (2.5)$$

The only remaining random part is the factor  $Y_n$ .

Representing  $Y_n$  by the independent systematic factors  $X = (X_1, \dots, X_K)$ , the default probability of obligor  $n$ , conditional on a specification  $x = (x_1, \dots, x_K)$  of  $X$ , can be written

$$PD_n(x) = \Phi \left( \frac{\Phi^{-1}(PD_n) - \beta_n \cdot \sum_{k=1}^K \alpha_{n,k} \cdot X_k}{\sqrt{1 - \beta_n^2}} \right). \quad (2.6)$$

The further step is to find an expression for the portfolio loss variable  $L$ .

If borrower  $n$  defaults, its rate of loss is determined by the variable loss given default,  $LGD_n$ , with mean  $ELGD_n$  and variance  $VLGD_n$ . LGD are assumed to be independent for different borrowers as well from all the other variables in the model. Then, the portfolio loss rate can be written as it follows

$$L = \sum_{n=1}^N s_n \cdot LGD_n \cdot \mathbf{1}_{\{r_n < \Phi^{-1}(PD_n)\}}.$$

Thus, the expected loss rate of borrower  $n$  is the probability of default times the expected rate of loss in case of default. The expected portfolio loss consists on the exposure weighted sum of all expected individual losses. Taking into account that  $LGD_n$  is independent from  $D_n$  and that the conditional expectation of default indicator equals the probability that  $r_n$  lies below  $c_n$  conditional on the risk factors, we obtain

$$\mathbb{E}(L|(X_1, \dots, X_K)) = \sum_{n=1}^N s_n \cdot ELGD_n \cdot \Phi \left( \frac{\Phi^{-1}(PD_n) - \beta_n \cdot \sum_{k=1}^K \alpha_{n,k} \cdot X_k}{\sqrt{1 - \beta_n^2}} \right). \quad (2.7)$$

The determination of the portfolio loss distribution requires a Monte Carlo simulation of the systematic risk factors. In a next section we will present an analytical approximation to compute the  $q^{th}$  percentile of the loss distribution in this multi-factor framework under the assumption that portfolios are infinitely fine grained such that the idiosyncratic risk is completely diversified away.

## 2.5 Industry Models based on the Merton Approach

### 2.5.1 The KMV Model

It was founded by KMV in 1989 and is now maintained by Moody's KMV.

It uses the Merton approach in a slightly varied way to determine the risk of a credit portfolio. The main contribution of KMV, however, is not the theoretical model but its calibration to achieve that the default probabilities correspond to a large extent to the empirically achieved ones. This calibration is based on a huge proprietary database.

Within the KMV model one computes the so-called *Expected Default Frequency* (EDF) based on the firm's capital structure, the volatility of the asset returns and the current asset value in three stages.

First, KMV uses an iterative procedure to estimate the asset value and the volatility of asset returns. Their method is based on the Merton approach of modeling equity as a Call option on the underlying assets of the firm with the firm's liabilities as the strike price. Using this property of equity, one can derive the underlying asset value and asset volatility from the implied market value, the volatility of equity and the book value of liabilities.

Recall that in the classical Merton model the default probability of a given firm is determined by the probability that the asset value  $V_1$  in one year lies below the threshold value  $B$  representing the firm's debt. Hence, the default probability,  $PD_{Merton}$ , in the Merton model is a function of the current asset value  $V_0$ , the asset value's annualized mean  $\mu_V$  and volatility  $\sigma_V$ , and the threshold  $B$ . With lognormally distributed asset values, this leads to a default probability (assuming a one year time horizon) of the form

$$PD_{Merton} = 1 - \Phi\left(\frac{\ln(V_0/B) + (\mu_V - \frac{1}{2}\sigma_V^2)}{\sigma_V}\right). \quad (2.8)$$

Since asset values are not necessarily lognormal, the above relationship between asset value and default probability may not be an accurate description of empirically observed default probabilities.

The EDF represents an estimated probability that a given firm will default within one year. In the KMV model, the EDF is slightly different but has a similar structure as the default probability of the Merton model. The

function  $1 - \phi$  in the previous formula is replaced by some decreasing function which is estimated empirically in the KMV model. Moreover, the threshold value  $B$  is replaced by a new default threshold  $\hat{B}$  representing the structure of the firm's liabilities more closely, and the argument of the normal distribution function in the above equation is replaced by a simpler expression.

Therefore, KMV computes, in a second step, the *distance to default* (DD) as

$$DD := \frac{V_0 - \hat{B}}{\sigma_V V_0}. \quad (2.9)$$

It represents an approximation of the argument of 2.8, since  $\mu_V$  and  $\sigma_V^2$  are typically small and since  $\ln(V_0) - \ln(\hat{B}) \approx (V_0 - \hat{B})/V_0$ .

Finally, in the last step, the DD is mapped to historical default events to estimate the EDF. In the KMV model, it is assumed that firms are homogeneous in default probabilities for equals DDs. The mapping between DD and EDF is determined empirically based on a database of historical default events. The estimated average EDF is then used as a proxy for the probability of default.

### 2.5.2 The CreditMetrics Model

It was developed by JPMorgan and the RiskMetrics Group.

It also descends from the Merton model. It deviates from the KMV model mainly through the determination of the default probability of a given firm by means of rating classes. Changes in portfolio value are only related to credit migrations of the single obligors, including both up and downgrades as well as defaults. While in the KMV model, borrower specific default probabilities are computed, CreditMetrics assumes default and migration probabilities to be constant within the rating classes. Each firm is assigned to a certain credit rating category at a given time period. The number of rating classes is finite and rating classes are ordered by credit quality including also the default class. One then determines the credit migration probabilities, that is, the probability of moving from one rating class to another in a given time (typically one year). These probabilities are usually presented in form of a rating transition probability matrix. Having assigned every borrower to a certain rating class and having determined the rating transition matrix as well as the expectation and volatility of the recovery rate, the distribution of the portfolio loss can be simulated. When embedding the CreditMetrics model in an asset-value model of the Merton type, this can be achieved as sketched in Remark 1.



## 2.6 The Asymptotic Single Risk Factor Model

As already mentioned, Basell II risk weight formulas are intended to ensure that unexpected losses can be covered up to a certain confidence level prescribed by the supervisors.

They are based on the **Asymptotic Single Risk Factor model**, also known as the **Vasicek model**. It is constructed in a way that the capital required for any risky loan should not depend on the particular portfolio decomposition it is added to, this so-called *portfolio invariance*.

### 2.6.1 Assumptions of the model

**Assumption 2.** *Let assume that*

1. *Portfolios are infinitely fine-grained, i.e. no exposure accounts for more than an arbitrarily small share of total portfolio exposure.*
2. *Dependence across exposures is driven by a single systematic risk factor.*

Let denote by  $N$  the number of risky loans and by  $X$  the systematic risk factor.  $PD_n$  represents the unconditional default probability and  $PD_n(X)$  the conditional default probability.  $EAD_n$  denotes the exposure at default of obligor  $n$ ,  $LGD_n$  the obligor  $n$ 's percentage loss in default and  $D_n$  is the default indicator variable of obligor  $n$ .

**Assumption 3.** *Let*

$$U_n = LGD_n \cdot D_n, n = 1 \dots N.$$

*Assume that these variables are bounded in  $[-1,1]$  and mutually independent conditional on the factor  $X$ . Therefore, denote the exposure share of obligor  $n$  by*

$$s_n = \frac{EAD_n}{\sum_{n=1}^N EAD_n}. \quad (2.10)$$

*Then the portfolio loss ratio is given by*

$$L = \sum_{n=1}^N D_n \cdot LGD_n \cdot s_n.$$

**Assumption 4.** *The first condition in Assumption 1 is satisfied when the sequence of  $EAD_n$  satisfies the following conditions:*

1.  $\sum_{n=1}^N EAD_n \rightarrow \infty$ .
2. *It exists  $\xi > 0$  such that the largest  $s_n \in O(N^{-(\frac{1}{2}+\xi)})$ , i.e. it shrinks to zero as  $N \rightarrow \infty$ .*

Thus, by the Strong Law of Large Numbers,

**Theorem 1.** *Under assumptions 2 and 3, the portfolio loss ratio conditional on a realization  $x$  of the systematic risk factor  $X$  satisfies*

$$L_N - \mathbb{E}(L_N|X) \rightarrow 0 \text{ almost surely as } N \rightarrow \infty.$$

**Conclusion:** The larger the portfolio is, the more idiosyncratic risk is diversified away. In the limit the portfolio is only driven by a systemic risk. This limiting portfolio is the so-called *asymptotic portfolio* or *infinitely fine grained*.

We also need some assumptions to guarantee that the neighborhood of the  $q^{\text{th}}$  quantile of  $\mathbb{E}(L_N|X)$  is associated with the neighborhood of the  $q^{\text{th}}$  quantile of the systematic factor. Otherwise, the tail quantiles of the loss distribution would depend in a complex way on the behaviour of the conditional expected loss for each borrower.

**Assumption 5.** *There is an open interval  $B$  containing the  $q^{\text{th}}$  percentile of the systematic risk factor  $X$  and there is a real number  $N_0 < \infty$  such that*

1.  $\mathbb{E}(U_n|x)$  is continuous in  $x$  on  $B$ ,  $n = \{1, \dots, N\}$ .
2.  $\mathbb{E}(L_N|x)$  is nondecreasing in  $x$  on  $B$ ,  $\forall N > N_0$ .
3.  $\inf_{x \in B} \mathbb{E}(L_N|x) \geq \sup_{x \leq \inf B} \mathbb{E}(L_N|x)$  and  $\sup_{x \in B} \mathbb{E}(L_N|x) \leq \inf_{x \geq \sup B} \mathbb{E}(L_N|x)$ ,  $\forall N > N_0$ .

**Theorem 2.** *Suppose previous assumptions hold. Then for  $N > N_0$  we have*

$$\alpha_q(\mathbb{E}(L_N|X)) = \mathbb{E}(L_N|\alpha_q(X)),$$

where  $\alpha_q(\mathbb{E}(L_N|X))$  denotes the  $q^{\text{th}}$  quantile of the random variable  $\mathbb{E}(L_N|X)$ .

### 2.6.2 The risk weight functions

The ASRF model can be described as a factor model such that the return on the firm's assets is of the form

$$r_n = \sqrt{\rho_n} \cdot X + \sqrt{1 - \rho_n} \cdot \epsilon_n, \quad n = 1 \dots N,$$

where

$X$  : Systematic risk factor, normally distributed.

$\epsilon_n$  : Idiosyncratic shocks.

$r_n$  : Log-asset return of obligor  $n$ .

$\rho_n$  : Captures the correlation between  $r_n$  and the single-factor  $X$ . It is determined by the borrower's asset class .

#### Conditional default probabilities

They are given by

$$PD_n(X) = \Phi \left( \frac{\Phi^{-1}(PD_n) - \sqrt{\rho_n} \cdot X}{\sqrt{1 - \rho_n}} \right). \quad (2.11)$$

Choosing a realization of the systematic risk to be equal to the  $q^{th}$  quantile,  $\alpha_q(x)$ ,

$$PD_n(\alpha_q(x)) = \Phi \left( \frac{\Phi^{-1}(PD_n) + \sqrt{\rho_n} \cdot \Phi^{-1}(q)}{\sqrt{1 - \rho_n}} \right). \quad (2.12)$$

Having assigned an unconditional default probability to each obligor,  $PD_n$ , and having computed the conditional default probabilities via (2.12),  $PD_n(X)$ , we can return to the computation of the capital in the ASRF model.

#### Economic capital

As we have

$$EC_\alpha = \alpha_q(L) - EL,$$

once the unconditional default probability has been assigned to each obligor and the conditional default probabilities have been also computed, we will be able to compute the regulatory capital.

The EL of loan  $n$  is given by

$$EL_n = PD_n \cdot EAD_n \cdot LGD_n. \quad (2.13)$$

The LGD must reflect financial distress. During these times or economic downturn losses on defaulted loans are higher than under normal conditions. Basel II uses a so-called *downturn* LGD. A method to estimate it is to construct a mapping function similar to the derivation of the conditional default probability, to transform average  $LGD_s$  into downturn  $LGD_s$ , but banks are usually allowed to use their own methods to derive downturn  $LGD_s$ .

The expected loss conditional on the  $q^{th}$  quantile can be estimated by

$$\mathbb{E}(L_n | \alpha_q(X)) = PD_n(\alpha_q(X)) \cdot LGD_n \cdot EAD_n.$$

### 2.6.3 The loss distribution of a granular portfolio

Our aim is to compute the loss distribution in an infinitely granular portfolio. For a portfolio of  $N$  loans, the percentage portfolio loss,  $L_N$  is given by

$$L_N = \sum_{n=1}^N s_n \cdot LGD_n \cdot D_n,$$

with

$s_n = \frac{EAD_n}{\sum_{n=1}^N EAD_n}$  : Exposure share of obligor  $n$ .

$D_n$  : Default indicator variable of obligor  $n$ . It is Bernoulli distributed.

$X$  : Systematic factor. Is  $N(0, 1)$  distributed.

$PD_n(X)$  : Conditional default probability of obligor  $n$ .

We assume an homogeneous portfolio, i.e.  $PD_n = PD$  and  $LGD_n = 100\%$ ,  $n = \{1, \dots, N\}$ . Then the conditional default probability becomes

$$PD(X) = \Phi \left( \frac{\Phi^{-1}(p) - \sqrt{\rho} \cdot X}{\sqrt{1 - \rho}} \right),$$

where  $\rho$  is the correlation coefficient between obligor  $n$  and the risk factor  $X$ . They are constant (since  $PD_s$  are constant).

(a) By Theorem 1,

$$P\left(\lim_{N \rightarrow \infty} (L_N - \mathbb{E}(L_N|X)) = 0\right) = 1.$$

(b) As the sum over the exposure shares equals to 1,

$$\mathbb{E}(L_N|X) = \sum_{n=1}^N s_n \cdot \mathbb{E}(D_n|X) = PD(X).$$

Hence, in the limit as  $N \rightarrow \infty$ , the percentage portfolio loss tends to the conditional default probability

$$\Rightarrow L_N \rightarrow PD(X) \text{ almost surely as } N \rightarrow \infty.$$

We can conclude that in an infinitely granular portfolio, the conditional PD describes the fraction of defaulted obligors.

Let  $L$  be the percentage number of defaults in the portfolio. Thus,  $\forall x \in [0, 1]$ ,

$$F_L(x) = P(L \leq x) = P(PD(X) \leq x) = P\left(-X \leq \frac{\sqrt{1-\rho} \cdot \Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right).$$

Therefore we will have that the loss distribution function is given by

$$F_L(x) = F_{Vasicek}(x) = \Phi\left(\frac{\sqrt{1-\rho} \cdot \Phi^{-1}(x) - \Phi^{-1}(PD)}{\sqrt{\rho}}\right). \quad (2.14)$$

We can also obtain the probability density function by deriving the above equation

$$f_L(x) = \sqrt{\frac{1-\rho}{\rho}} \exp\left(\frac{-1}{2\rho} \left(\sqrt{1-\rho} \cdot \Phi^{-1}(x) - \Phi^{-1}(PD)\right)^2\right) \cdot \exp\left(\frac{1}{2} \left(\Phi^{-1}(x)\right)^2\right). \quad (2.15)$$

The asymptotic Vasicek approximations work well for portfolios consisting of an infinite number of small obligors. These formulas are less suitable for and tend to underestimate risks for portfolios with few obligors or portfolios dominated by a few large exposures.

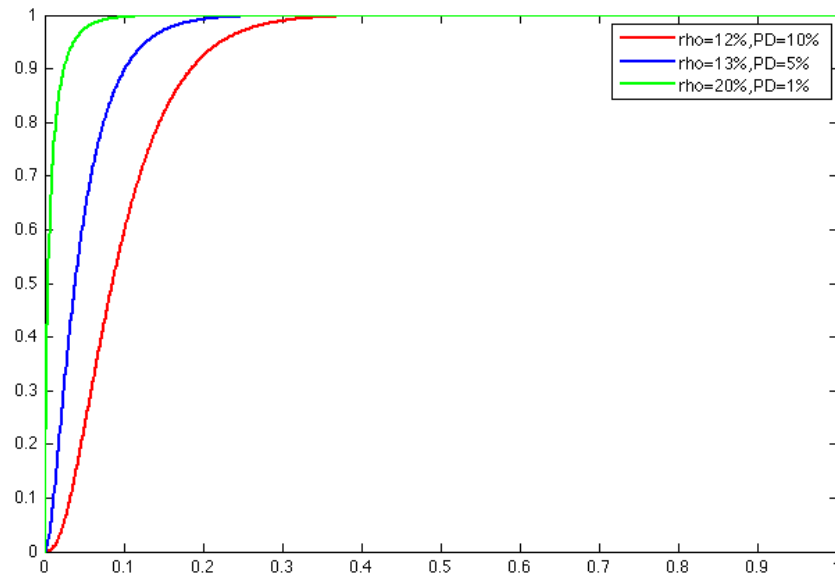


Figure 2.4: Portfolio loss distribution for different values of  $\rho$  and  $PD$ .

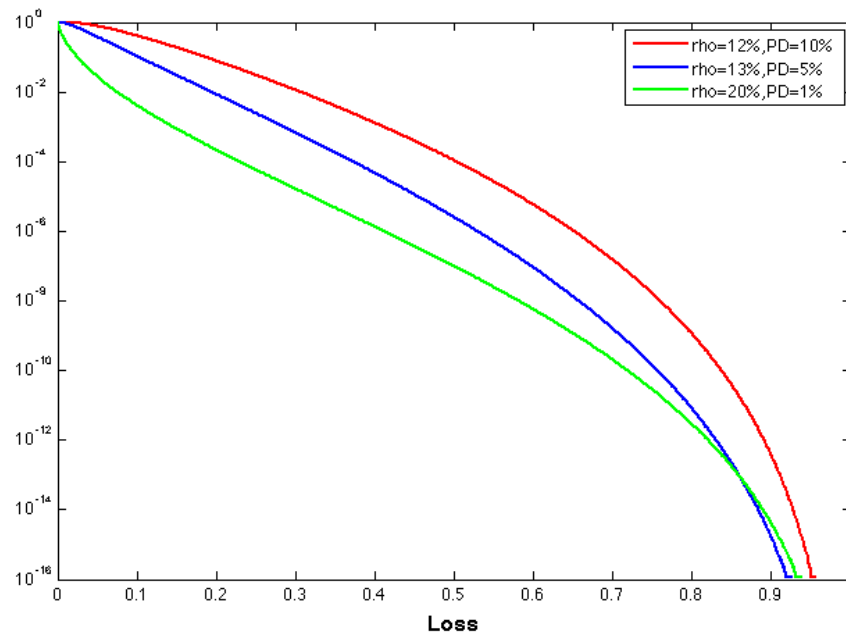


Figure 2.5: Tail probability distribution for different values of  $\rho$  and  $PD$ .

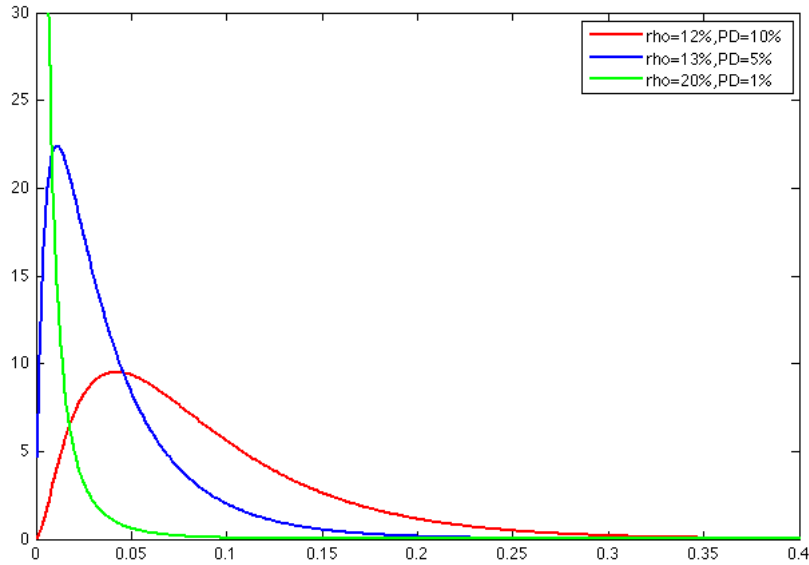


Figure 2.6: Densities of the distribution of the percentage number of defaults for different values of  $\rho$  and  $PD$ .

### Limiting cases

- If  $\rho = 0$ ,  $F_L$  follows a binomial distribution.
- If  $\rho = 1$ ,  $F_L$  is Bernoulli distributed.
- If  $PD = 0$ , all obligors survive almost surely.
- If  $PD = 1$ , all obligors default almost surely.

We can now easily compute the economic capital  $EC_q$  at level  $q$  for a granular portfolio with respect to  $PD$  and  $\rho$ . By Theorem 2,

$$\underbrace{\alpha_q(\mathbb{E}(L_N|X))}_{\alpha_q(PD(X))} = \mathbb{E}(L_N|\alpha_q(X)).$$

To compute the correspondent  $q^{th}$  quantile, we apply the formula

$$VaR_q = \Phi \left( \frac{\Phi^{-1}(PD) + \sqrt{\rho} \cdot \Phi^{-1}(q)}{\sqrt{1-\rho}} \right).$$

Let see some examples for the calculation of regulatory capital.

The following figures show the 95% and 99% quantiles of the percentage portfolio loss. In practice, the 99.9% or the 99.98% quantiles are used more

frequently, however, these values would be hard to recognize in the figures since they are far in the tail of the distributions.

(i)  $PD = 10\%$ ,  $\rho = 12\%$

(ii)  $PD = 5\%$ ,  $\rho = 13\%$

$q$	$VaR_q$	$EC_q$
95%	0.22	0.124
99%	0.306	0.206
99.9%	0.41	0.311

$q$	$VaR_q$	$EC_q$
95%	0.129	0.079
4 99%	0.193	0.144
99.9%	0.284	0.235

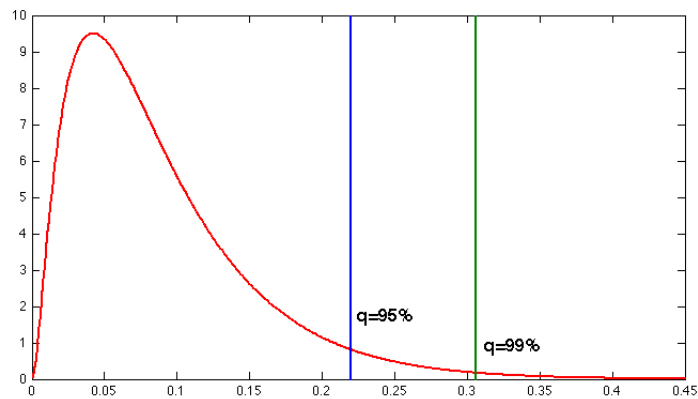


Figure 2.7:  $PD = 10\%$ ,  $\rho = 12\%$

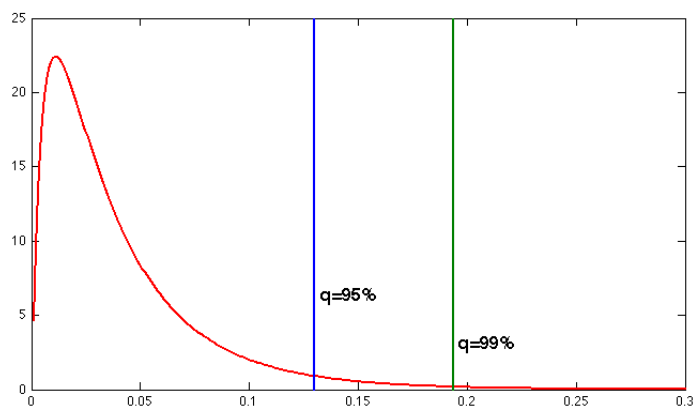


Figure 2.8:  $PD = 5\%$ ,  $\rho = 13\%$



## 2.7 The CreditRisk<sup>+</sup> model

It is an actuarial model. It is now one of the financial industry's benchmark models in the area of Credit Risk management. For an extended overview we refer to [16]. It is also widely used in the supervisory community since it uses as basic input the same data as also required for the Basel II IRB approach. Moreover, this model has some nice properties:

- One can derive an analytic solution for the loss distribution of a given credit portfolio instead of simulating it.
- It seems easier to calibrate data to the model.
- This model reveals one of the most essential Credit Risk drivers: Concentration.

### 2.7.1 General Model Setting

We consider a portfolio of  $N$  obligors. Obligor  $n$  constitutes a loss exposure  $E_n$  and has a probability of default  $PD_n$  over the time period  $[0, T]$ . The *loss exposure* is given by its exposure at default  $EAD_n$  times its expected percentage loss given default,  $ELGD_n$ , i.e.,  $E_n = EAD_n \cdot ELGD_n$ .

As in the previous models, the state of obligor  $n$  at the time horizon  $T$  can be represented as a Bernoulli random variable  $D_n$ , where

$$D_i = \begin{cases} 1 & \text{if obligor } n \text{ defaults at time } T \\ 0 & \text{otherwise} \end{cases}$$

Hence the default probability is  $\mathbb{P}(D_n = 1) = PD_n$  while the survival probability is given by  $\mathbb{P}(D_n = 0)$ .

**Definition 14.** *The probability generating function (PGF) of a non-negative integer valued random variable  $X$  is defined as*

$$G_X(z) = \mathbb{E}(z^X) = \sum_{i=0}^{\infty} z^i \cdot \mathbb{P}(X = i).$$

From this definition it immediately follows that

$$\mathbb{P}(X = i) = \frac{1}{i!} G_X^{(i)}(0), \quad i \in \mathbb{N}.$$

Thus the distribution of a random variable can easily be computed as soon as one knows the PGF. The CreditRisk<sup>+</sup> model makes use of this property as well. We will briefly state some properties of the PGF which will be also used in the following.

**Proposition 1.** *Let  $X, Y$  be two random variables.*

1. *Let  $X, Y$  be independent. Then*

$$G_{X+Y}(z) = G_X(z) \cdot G_Y(z).$$

2. *Let  $G_{X|Y}(z)$  be the PGF of  $X$  conditional on the random variable  $Y$  and denote the distribution function of  $Y$  by  $F$ . Then*

$$G_X(z) = \int G_{X|Y}(z)F(dy).$$

In the following we will denote:

$$L_n = D_n \cdot E_n : \text{loss of obligor } n.$$

$$L = \sum L_n : \text{total portfolio loss.}$$

$$\nu_n = \frac{EAD_n \cdot ELGD_n}{E} = \frac{E_n}{E} : \text{normalized exposure of obligor } n.$$

$$\lambda_n = D_n \cdot \nu_n : \text{normalized loss of obligor } n.$$

$$\lambda = \sum \lambda_n : \text{total normalized portfolio loss.}$$

In order to compute the VaR of the portfolio we need to derive the probability distribution of the portfolio loss  $L$ . We can also derived the distribution of  $\lambda$ .

### 2.7.2 The Poisson Approximation

When the default probabilities are random and default events are no longer independent, an analytical solution for the loss distribution can be derived by using an approximation for the distribution of the default events. Thus, consider the individual default probabilities to be sufficiently small for the compound Bernoulli distribution of the default events to be well approximated by a Poisson distribution. Under this assumption it is still possible to

derive an analytical solution of the loss distribution function.

Since default events are assumed to follow a Bernoulli distribution, the probability generating function of  $D_n$  can be computed to

$$G_{D_n}(z) = \sum_{x=0}^{\infty} \mathbb{P}(D_n = x) \cdot z^x = (1 - PD_n) + z \cdot PD_n \quad (2.16)$$

$$= \exp(\ln(1 + PD_n \cdot (z - 1))). \quad (2.17)$$

$$(2.18)$$

$PD_n$  small  $\rightarrow PD_n \cdot (z - 1)$  small whenever  $|z| \leq 1$ . And applying a Taylor series expansion around  $\omega = 0$  we obtain,

$$\ln(1 + \omega) = \omega - \frac{\omega^2}{2} + \frac{\omega^3}{3} + \dots$$

Thus,

$$\ln(1 + PD_n \cdot (z - 1)) = PD_n \cdot (z - 1) - \frac{(PD_n \cdot (z - 1))^2}{2} + \frac{(PD_n \cdot (z - 1))^3}{3} + \dots$$

$$\Rightarrow \ln(1 + PD_n \cdot (z - 1)) \simeq PD_n \cdot (z - 1).$$

And (2.18) becomes

$$G_{D_n}(z) \approx \exp(PD_n \cdot (z - 1)).$$

Using again a Taylor series expansion around  $z = 0$ , we can rewrite the latest formula

$$G_{D_n}(z) \approx \exp(PD_n \cdot (z - 1)) = \exp(-PD_n) \cdot \sum_{x=0}^{\infty} \frac{PD_n^x}{x!} \cdot z^x.$$

which is the PGF of the Poisson distribution with intensity  $PD_n$ . Hence for small values of  $PD_n$ , the Bernoulli distribution of  $PD_n$  can be approximated by a Poisson distribution with intensity  $PD_n$ , that is,

$$\mathbb{P}(D_n = x) = \exp(-PD_n) \cdot \frac{PD_n^x}{x!}.$$

Furthermore, the PGF of  $\lambda_n$  is defined as it follows

$$\begin{aligned} G_{\lambda_n}(z) &= \mathbb{E}(Z^{\lambda_n}) = \mathbb{E}(z^{D_n \nu_n}) = \sum_{x=0}^{\infty} \mathbb{P}(D_n = x) \cdot z^{D_n \nu_n} \\ &= \sum_{x=0}^{\infty} \exp(-PD_n) \cdot \frac{PD_n^x}{x!} \cdot (z^{\nu_n})^x \\ &= (1 + PD_n \cdot z^{\nu_n}) \cdot \exp(-PD_n), \end{aligned}$$

since  $D_n = 0, 1$ . Due to Taylor expansion we have

$$\exp(PD_n \cdot z^{\nu_n}) \approx 1 + PD_n \cdot z^{\nu_n}.$$

Thus,

$$G_{\lambda_n}(z) = \exp(PD_n \cdot (z^{\nu_n-1})). \quad (2.19)$$

### 2.7.3 Model with Random Default Probabilities

To obtain an analytic solution for the loss distribution we have to impose some assumptions on the model.

**Assumption 6.** *Assume that the default probabilities are random and that they are influenced by a common set of Gamma-distributed systematic risk factors. Thus, the default events are assumed to be mutually independent only conditional on the realizations of the risk factors.*

In the CreditRisk<sup>+</sup> model, correlation among default events is induced by the dependence of the default probabilities on a common set of risk factors. Assume that there exist  $K$  systematic risk factors  $X_1, \dots, X_K$  which describe the variability of the default probabilities  $PD_n$ . Each factor is associated with a certain sector (industry, country, region, etc). All risk factors are taken to be independent and Gamma distributed with shape parameter  $\alpha_k = \frac{1}{\xi_k}$  and scale parameter  $\beta_k = \xi_k$ . Recall that the Gamma distribution is defined by the probability density

$$\Gamma_{\alpha,\beta}(x) = \frac{1}{\beta^\alpha \Gamma(\alpha) \cdot e^{-x/\beta} \cdot x^{\alpha-1}}.$$

**Definition 15.** *The moment generating function (MGF) of a random variable  $Y$  with density  $f_Y$  is defined as the analytic function*

$$M_Y(z) = \mathbb{E}(e^{zY}) = \int e^{zt} \cdot f_Y(t) dt.$$

Thus, the MGF of  $X_k$  can be computed as

$$M_{X_k}(z) = \mathbb{E}(\exp(X_k \cdot z)) = (1 - \beta_k \cdot z)^{-\alpha_k} = (1 - \xi_k \cdot z)^{-1/\xi_k}.$$

We denote the idiosyncratic risk by  $X_0 = 1$  and let  $X = (X_0, X_1, \dots, X_K)$ . The link between the default probabilities and the risk factors  $X_k$  is given by the following factor model

$$PD_n(X) = PD_n \left( \sum_{k=0}^K \omega_{k,n} X_k \right), \quad n = 1 \dots N, \quad (2.20)$$

where  $PD_n$  is the average default probability of obligor  $n$  and the factor loading  $\omega_{k,n}$  measures the sensitivity of obligor  $n$  to the risk factor  $X_k$  where  $0 \leq \omega_{n,k} \leq 1$ , and,  $\forall n = 1 \dots N$ , we have also that  $\sum_{k=1}^K \omega_{k,n} \leq 1$ .

Define  $\omega_{0,n} = 1 - \sum_{k=1}^K \omega_{k,n}$  as the share of idiosyncratic risk of obligor  $n$ .

The PGF of the individual normalized loss, given by (2.19), can be rewritten by the above factor model specification as

$$\begin{aligned} G_{\lambda_n}(z) &= \exp \left[ PD_n \cdot \left( \sum_{k=0}^K \omega_{k,n} X_k \right) \cdot (z^{\nu_n} - 1) \right] \\ &= \prod_{k=0}^K \exp(PD_n \cdot \omega_{k,n} \cdot X_k \cdot (z^{\nu_n} - 1)). \end{aligned}$$

Let denote  $G_\lambda(z|X)$  the PGF of the total normalized loss, conditional on  $X$ . Since individual losses are mutually independent conditional on  $X$ , we have

$$\begin{aligned}
G_\lambda(z|X) &= \prod_{n=1}^N G_{\lambda_n}(z|X) \\
&= \prod_{n=1}^N \prod_{k=0}^K \exp(PD_n \cdot \omega_{k,n} \cdot X_k \cdot (z^{\nu_n} - 1)) \\
&= \exp \left[ \sum_{n=1}^N \sum_{k=0}^K PD_n \cdot \omega_{k,n} \cdot X_k \cdot (z^{\nu_n} - 1) \right].
\end{aligned}$$

Let define

$$\begin{aligned}
P_k(z) &:= \sum_{n=1}^N PD_n \cdot \omega_{k,n} \cdot X_k \cdot (z^{\nu_n} - 1). \text{ Then,} \\
G_\lambda(z|X) &= \exp \left[ \sum_{k=0}^K X_k \cdot P_k(z) \right].
\end{aligned}$$

The unconditional PGF of the total normalized loss, denoted by  $G_\lambda(z)$  is the expectation of  $G_\lambda(z|X)$  under  $X$ 's probability distribution, this is,

$$G_\lambda(z) = \mathbb{E}_X(G_\lambda(z|X)) = \mathbb{E}_X \left[ \exp \left( \sum_{k=0}^K X_k \cdot P_k(z) \right) \right].$$

Which equals, by definition of the joint MGF  $M_X$ , to:

$$G_\lambda(z) = \mathbb{E}_X(\exp(P(z) \cdot X)) = M_X(P(z)),$$

where  $P(z) = (P_0(z), \dots, P_K(z))$  and  $X = (X_0, \dots, X_K)$ . Due to the independence of the variables, the above expression can be rewritten as

$$\begin{aligned}
G_\lambda(z) &= M_X(P(z)) = \prod_{k=0}^K M_{X_k}(P_k(z)) \\
&= \prod_{k=0}^K (1 - \xi_k \cdot P_k(z))^{-1/x_{i_k}} = \prod_{k=0}^K \exp(\ln((1 - \xi_k \cdot P_k(z))^{-1/x_{i_k}})).
\end{aligned}$$

To sum up, we have that

$$G_{\lambda,CR^+}(z) = \exp \left( - \sum_{k=0}^K \frac{1}{\xi_k} \cdot \ln(1 - \xi_k \cdot P_k(z)) \right) \quad (2.21)$$

is the PGF of the total normalized portfolio loss for the CreditRisk<sup>+</sup> model.

# Chapter 3

## Concentration Risk in Credit Portfolios

When measuring credit risk, we are particularly interested in dependencies between certain extreme credit events. The *Business Week* stated in September 1998:

*Extreme, synchronized rises and falls in financial markets occur infrequently but they occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which many things go wrong at the same time- the perfect 'storm' scenario.*

This perfect storm scenario is what we mean by concentration of risk. The quote underlines the importance of a sufficient measurement of concentration risk since losses due to concentration risk can be extreme. In this chapter we follow [1], [11], [12] and [13].

### 3.1 Concentration Risk

*Concentration Risk* is the risk arising from an uneven distribution of counterparties in credit or any business relationships which are capable of generating losses large enough to jeopardise an institution's solvency.

Concentration risks, particularly concentrations in credit risk, have played a key role in the financial instability of the banking sector last years. The BCBS already recognized the high importance of credit risk concentrations in the Basel framework: 'Risk concentrations are arguably the single most important cause of major problems in banks'.

Concentration risk can be considered from either a macro (systemic) or a micro (idiosyncratic) perspective.

- *Systematic risk* represents the effect of unexpected changes in macroeconomic and financial market conditions on the performance of borrowers. Borrowers may differ in their degree of sensitivity to systematic risk, but few firms are completely indifferent to the wider economic conditions in which they operate. Therefore, the systematic component of portfolio risk is unavoidable and only partly diversifiable.
- *Idiosyncratic risk* represents the effects of risks that are particular to individual borrowers. As a portfolio becomes more fine-grained, in the sense that the largest individual exposures account for a smaller share of total portfolio exposure, idiosyncratic risk is diversified away at the portfolio level.

From the point of view of financial stability (macro perspective), the focus is on risks for groups of banks which, for example, emerge from a joint concentration in certain business lines. By contrast, the primary focus in internal risk management and from a supervisory point of view is on concentration risk at the level of individual institutions (micro perspective).

From the perspective of credit risk, concentration risks in credit portfolios (most significant source of risk to the solvency of banks) arise from:

- Unequal distribution of loans to single borrowers: *Name Concentration*.

On the one hand, the term *single name concentration risk* denotes the firm specific (idiosyncratic) risk in a credit portfolio which arises from the credit risk of large borrowers. Firm specific risk comprises the risks resulting from the potential default of a single borrower or a legally connected group of borrowers. The term *single name concentration risk* is used if the exposures to large individual borrowers account for the bulk of all loans in a portfolio.

On the other hand, systematic risk comprises all of the risks affecting several legally independent borrowers or the entire portfolio, for example, the state of the economy or industry sector dependent risks.

- Different industry or regional sectors: *Sector Concentration*.

Sectoral concentration in credit portfolios can be broken down into concentration in certain sectors of industry and concentration in individual countries or regions. While commercial credit risk models widely used in the financial sector usually measure both kinds of sectoral concentration using a similar methodology, there are many differences from a theoretical point of view. Credit concentration in industry sectors is a typical risk driver of corporate loans, while public and private borrowers can also play a key role in the case of country risk. Moreover,



country risk is a generic term for different, partly interdependent risk categories, for example political risk and transfer risk. By contrast, concentration risk from exposures to industry sectors arises from credit dependencies between enterprises, resulting from a common sector affiliation and the prevailing economic environment in that sector.

- Certain dependencies between business and different borrowers can increase the credit risk in a portfolio. This effect is called *Default Contagion*. The availability of suitable data on bilateral business relations and the resultant interdependencies represent a key problem. Compared with the measurement of granularity and sectoral concentration, there is still a long way to go before generally accepted models for micro contagion risk are available.

This risk is not limited to credit portfolios and may stem from various sources.

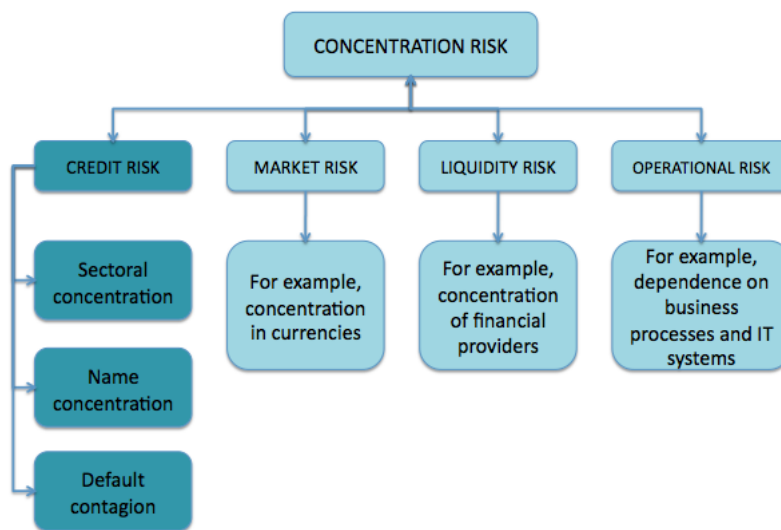


Figure 3.1: Overview of Concentration Risk.

In recent years there have been significant improvements in understanding and measuring concentration risk in credit portfolios. The measurement of these risks is important against the background of regulatory capital needs as well as for computing the economic capital. Unfortunately, the existing approaches are mostly not fully consistent with the new capital adequacy framework (Basel II), sometimes within the derivation and sometimes within

the implementation, so that the benefit of these approaches is restricted.

On this project, we focus on measuring name concentration, mainly by using a single-factor Merton model. In order to measure sector concentration, a multi-factor model would be required.

### 3.1.1 Basel II statements on Credit Concentration Risk

Understanding and analytically measuring concentration risk, such as undiversified idiosyncratic risk and industry or country risk, in credit portfolios is one of the major challenges in recent research. The measurement is necessary for the determination of regulatory capital under pillar 2 of Basel II as well as for managing the portfolio and allocating economic capital.

Some of the recommendations made on the Basel framework to deal with concentration risk are (for a whole description see [5]):

- Banks should have in place effective internal policies, systems and controls to identify, measure, monitor, and control their credit risk concentrations. Banks should explicitly consider the extent of their credit risk concentrations in their assessment of capital adequacy under Pillar 2. These policies should cover the different forms of credit risk concentrations to which a bank may be exposed.
- A framework for managing credit risk concentrations in banks should be clearly documented and should include a definition of the credit risk concentrations relevant to the bank and how these concentrations and their corresponding limits are calculated. Limits should be defined in relation to a bank's capital, total assets or, where adequate measures exist, its overall risk level.
- Management should conduct periodic stress tests of its major credit risk concentrations and review the results of those tests to identify and respond to potential changes in market conditions that could adversely impact the bank's performance.
- In the course of their activities, supervisors should assess the extent of a bank's credit risk concentrations, how they are managed, and the extent to which the bank considers them in its internal assessment of capital adequacy under Pillar 2. Such assessments should include reviews of the results of a bank's stress tests. Supervisors should take appropriate actions where the risks arising from a bank's credit risk concentrations are not adequately addressed by the bank.

## 3.2 Ad-Hoc measures of Concentration. Model-free methods

Various indexing techniques have been examined in the credit risk literature. All have a common approach: to identify the extent of the concentration in a portfolio through a single measure. Thus, ratios provide a simple approximation for measuring exposure or borrower concentrations.

A concentration index for a portfolio of  $N$  loans should satisfy the following properties:

1. *Transfer principle*: The reduction of a loan exposure and an equal increase of a bigger loan must not decrease the concentration measure.
2. *Uniform distribution principle*: The measure of concentration attains its minimum value, when all loans are equal of size.
3. *Lorenz-criterion*: If two portfolios, which are composed of the same number of loans, satisfy that the aggregate size of the  $k$  biggest loans of the first portfolio is greater or equal to the size of the  $k$  biggest loans in the second portfolio for  $1 \leq k \leq N$ , then the same inequality must hold between the measures of concentration in the two portfolios.
4. *Superadditivity*: If two or more loans are merged, the measure of concentration must not decrease.
5. *Independence of loan quantity*: Consider a portfolio consisting of loans of equal size. The measure of concentration must not increase with an increase in the number of loans.
6. *Irrelevance of small exposures*: Granting an additional loan of a relatively low amount does not increase the concentration measure. More formally, if  $\hat{s}$  denotes a certain percentage of the total exposure and a new loan with a relative share of  $s_n \leq \hat{s}$  of the total exposure is granted, then the concentration measure does not increase.

All these properties are essential for an index to qualify as a measure of concentration.

### 3.2.1 Concentration Ratio

Consider a portfolio with exposure shares  $s_1 \geq s_2 \geq \dots \geq s_N$ , given by (2.10), and such that  $\sum_{n=1}^N s_n = 1$ .

**Definition 16.** *The Concentration Ratio ( $CR_k$ ) of the portfolio is defined as the ratio of the sum of the  $k$  biggest exposures to the total sum of exposures in the portfolio, this is,*

$$CR_k = \sum_{i=1}^k s_i, \quad 1 \leq k \leq N.$$

The concentration ratio satisfies all six properties as can easily be seen.

However, the concentration ratio has quite a number of drawbacks as a measure of concentration:

- The number  $k$  is chosen by the investigator or the risk manager, so its choice is arbitrary although it has a strong impact on the outcome.
- The ratio considers only the size distribution of the  $k$  largest loans and does not take into account the full information about the loan distribution.
- Shifts in the portfolio structure can stay unrecognized by this measure depending on the choice of  $k$ .

Closely related to the concentration ratio is the *Lorenz curve*.

### 3.2.2 Lorenz Curve

The Lorenz curve is not exactly an index in the sense that it returns a single number for each credit portfolio. It is a mapping that assigns to every percentage  $q$  of the total loan number the cumulative percentages  $L(q)$  of loan sizes.

**Definition 17.** *Given an ordered data set  $A_1 \leq A_2 \leq \dots \leq A_N$ , the (empirical) Lorenz curve generated by the data set is defined for all  $q \in (0, 1)$  as the piecewise linear interpolation with breakpoints  $L(0) = 0$  and*

$$L(n/N) = \frac{\sum_{i=1}^n A_i}{\sum_{j=1}^N A_j}, \quad n = 1 \dots N.$$

Consider a distribution function  $F$  with density function  $f$  such that  $F$  increases on its support and that the mean of  $F$  exists. Then the  $q^{th}$  quantile of  $F$  is well defined and the (theoretical) *Lorenz curve*  $L(q)$  for  $q \in (0, 1)$  is defined by

$$L(q) = \frac{\int_0^q F^{-1}(t) dt}{\int_{-\infty}^{\infty} tf(t) dt}.$$

Reformulating, we obtain by substitution

$$L(F(x)) = \frac{\int_{-\infty}^{F(x)} F^{-1}(t) dt}{\int_{-\infty}^{\infty} tf(t) dt} = \frac{\int_{-\infty}^x tf(t) dt}{\int_{-\infty}^{\infty} tf(t) dt}.$$

Hence the Lorenz curve is a graph showing for each level  $q$  the proportion of the distribution assumed by the first  $q$  percentage of values.

Obviously, a Lorenz curve always starts at  $(0,0)$  and ends at  $(1,1)$  and is by definition a continuous function. If the variable being measured cannot take negative values, the Lorenz curve is an increasing and convex function.

The line of perfect equality in the Lorenz curve is  $L(F(x)) = F(x)$  and represents a uniform distribution while perfect inequality in the Lorenz curve is characterized by  $L(F(x)) = \delta(x)$ , the Dirac function with weight 1 at  $x = 1$ , representing a Dirac distribution.

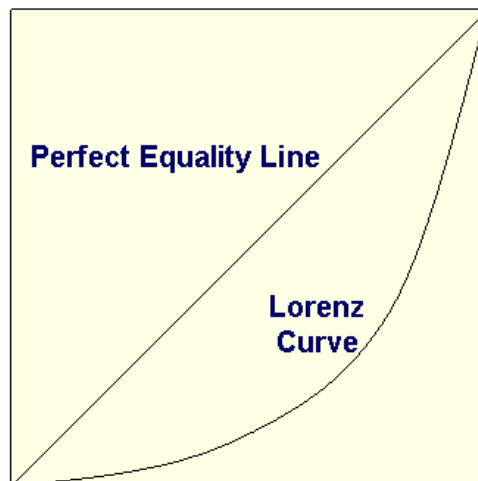


Figure 3.2: Lorenz curve.

The main drawback of the Lorenz curve as a measure of concentration is that it does not allow a unique ranking of two portfolios in terms of their concentration. The latter is only possible when the curves of the two portfolios do not intersect. In that case the portfolio with the lower Lorenz curve is said to be higher concentrated.

In general, the Lorenz curve is not optimal as a measure for concentration risk since it does not deliver a unique ranking of loans and rather measures a deviation from the uniform distribution where the number of loans in the portfolio is not taken into account. We want to point out here that inequality and concentration are, of course, not the same since concentration also depends on the number of loans in the portfolio. A portfolio of two loans of the same size will be considered as well diversified when applying the Lorenz curve as a concentration measure. However, a portfolio consisting of one hundred loans of different sizes might be considered as much more concentrated as its Lorenz curve will differ from the line of perfect equality. Thus, it might be questionable to measure the degree of concentration on the basis of deviation from equality.

### 3.2.3 Gini index (G)

Another ad-hoc measure which is closely linked to the Lorenz curve is the *Gini coefficient* which measures the size of the area between the Lorenz curve and the main diagonal and thus also represents a measure for the deviation from equal distribution.

**Definition 18.** For a portfolio of  $N$  loans with exposure shares  $s_1, \dots, s_N$ , the (empirical) Gini coefficient is defined as

$$G = \frac{\sum_{n=1}^N (2n-1) \cdot s_n}{N} - 1.$$

For a given distribution function the (theoretical) Gini coefficient is defined as the ratio of the area between the Lorenz curve  $L(q)$  of the distribution and the curve of the uniform distribution, to the area under the uniform distribution.

If the area between the line of perfect equality and the Lorenz curve is denoted by single name, and the area under the Lorenz curve is denoted by  $B$ , then the Gini coefficient is

$$G = \frac{A}{A+B}.$$

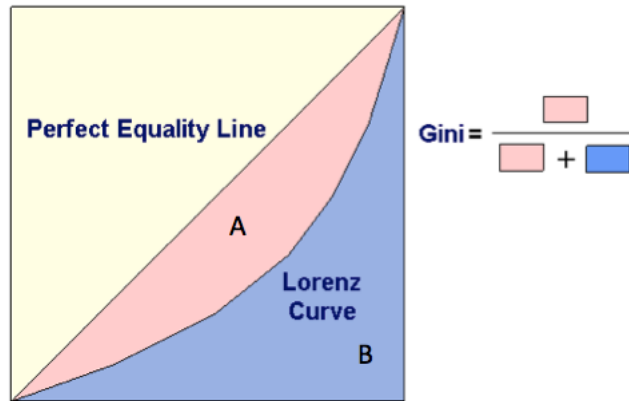


Figure 3.3: Gini coefficient.

The above figure illustrates the relation between the Gini coefficient and the Lorenz curve.

If the Lorenz curve is represented by the function  $L(q)$ , the value of B can be found with integration and

$$G = 1 - 2 \int_0^1 L(q) dq.$$

- i) A coefficient close to zero signifies a homogeneous portfolio in which all of the exposure amounts are distributed equally.
- ii) A coefficient close to one means a highly concentrated portfolio.

One can easily see that properties (1) and (5) are satisfied by the Gini coefficient. However, properties (4) and (6) are violated.

A potential disadvantage of using G is that the size of the portfolio is not taken into account. For example, a portfolio with a few equal sized loans has a lower coefficient than a better diversified, larger credit portfolio containing loans of different amounts.

Furthermore, this index may rise if a relatively small loan to another borrower is added to the portfolio despite the fact that this diminishes the concentration.

Therefore, we consider the Gini coefficient suitable to only a limited extent for the measurement of concentration risks.

### 3.2.4 Herfindahl-Hirschman index (HHI)

It is probably the most commonly used model-free measure, particularly in the empirical literature.

Originally used in the context of quantifying diversification within an industry to assess the level of competition in the marketplace, the HHI can be also used to calculate portfolio concentration risk.

**Definition 19.** *The Herfindahl-Hirschman index is defined as the sum of the squares of the relative portfolio shares of all borrowers.*

$$HHI = \sum_{n=1}^N s_n^2,$$

where  $s_n$  is the exposure share of borrower  $n$  and  $N$  is the number of borrowers under observation.

The HHI ranges from  $\frac{1}{N}$  to 1, so the normalized HHI index can be written as

$$HHI^* = \frac{H - \frac{1}{N}}{1 - \frac{1}{N}}.$$

- i) Well diversified portfolios with a large number of small credits have an HHI value close to zero.
- ii) Heavily concentrated portfolios can have a considerably higher HHI value.
- iii) In the extreme case where we observe only one credit,  $HHI=1$ .

This statistical measure has some drawbacks to be used for measuring concentration risk.

Firstly, it does not consider distribution of exposures across credit ratings, portfolios with the same HHI values can have different sizes of concentration risks.

Moreover, it does not allow concentration risk to be expressed directly as economic capital, so we will need additional functions in order to compute economic capital for concentration risk.



### 3.3 Model-based methods

Neither the HHI nor the Gini coefficient or any other model-free method for measuring concentration risk can incorporate the effects of obligor specific credit qualities, which are, for example, represented by obligor specific default probabilities. Regulators and other stakeholders are demanding more accurate and precise answers which can only be obtained by using more sophisticated models and providing more detailed analysis. For these reasons certain model based methods for measuring concentration risks have been developed which can deal more explicitly with exposure distribution, credit quality and default dependencies. Moreover, model-based methods allow the single name concentration risk to be expressed directly as economic capital. For an extended overview of methods that take into account concentration risk in credit portfolios we refer to [1].

#### 3.3.1 Granularity adjustment (GA) for the ASRF model

The GA for the ASRF model constitutes an approximation formula for calculating the appropriate economic capital needed to cover the risk arising from the potential default of large borrowers. We follow the revised methodology developed in [15].

It is an extension of the ASRF model which forms the theoretical basis of the Internal Ratings-Based (IRB) approaches. Through this adjustment, single name concentration is integrated into the ASRF model.

The ASRF model presumes that portfolios are fully diversified with respect to individual borrowers, so that economic capital depends only on systematic risk. Hence, the IRB formula omits the contribution of the residual idiosyncratic risk to the required EC.

We discuss an approach how to assess a potential add-on to capital for the effect of lack of granularity in the ASRF model.

#### Example as Motivation for GA Methodology

Assume that the systematic risk factor  $X \sim N(0, \nu^2)$ , and the loss rate on instrument  $n$ , conditional on the systematic risk factor  $U_n|X \sim N(X, \sigma^2)$ , where  $\nu, \sigma$  are known. Thus,  $L_N$  is also normally distributed. We can compute the  $q^{th}$  quantile,

$$\alpha_q = \sqrt{\frac{\nu^2 + \sigma^2}{N}} \cdot \Phi^{-1}(q).$$

Moreover, when  $N \rightarrow \infty$  the distribution of  $L_N$  converges to the distribution of  $X$ , ( $F_{L_N} \rightarrow F_X$ ). Thus, the quantile of the asymptotic distribution is  $\nu \cdot \Phi^{-1}(q)$ . This is the systemic component of VaR. Therefore, we can derive the idiosyncratic component.

$$VaR_q = \sqrt{\frac{\nu^2 + \sigma^2}{N}} \cdot \Phi^{-1}(q) = \nu \cdot \Phi^{-1}(q) + \text{idiosyncratic}. \quad (3.1)$$

By applying a Taylor's expansion around  $\frac{\nu^2}{N} = 0$ ,

$$\sqrt{\frac{\nu^2 + \sigma^2}{N}} \simeq \nu + \frac{1}{N} \cdot \frac{\nu^2}{2\nu} + O\left(\frac{1}{N^2}\right). \quad (3.2)$$

Thus, replacing (3.2) into (3.1), the idiosyncratic component can be expressed as it follows:

$$\begin{aligned} \text{Idiosyncratic component} &= \left(\nu + \frac{1}{N} \cdot \frac{\nu^2}{2\nu}\right) \cdot \Phi^{-1}(q) + \Phi^{-1}(q) + O\left(\frac{1}{N^2}\right) \\ &= \frac{1}{N} \cdot \frac{\nu^2}{2\nu} \cdot \Phi^{-1}(q) + O\left(\frac{1}{N^2}\right). \end{aligned}$$

The Granularity Adjustment is an application of the same logic to a proper credit risk model.

Let  $\nu = 1$ . The following figure shows the systematic and idiosyncratic components of VaR of the portfolio loss ratio  $L_N$  as the number  $N$  of obligors in the portfolio increases.

As the number of borrowers in the portfolio increases, the idiosyncratic component vanishes. This is also the main intuition of the *GA*. For large, portfolios, which are typically better diversified, the *GA* is lower than for small concentrated portfolios, this is illustrated in the next figure.

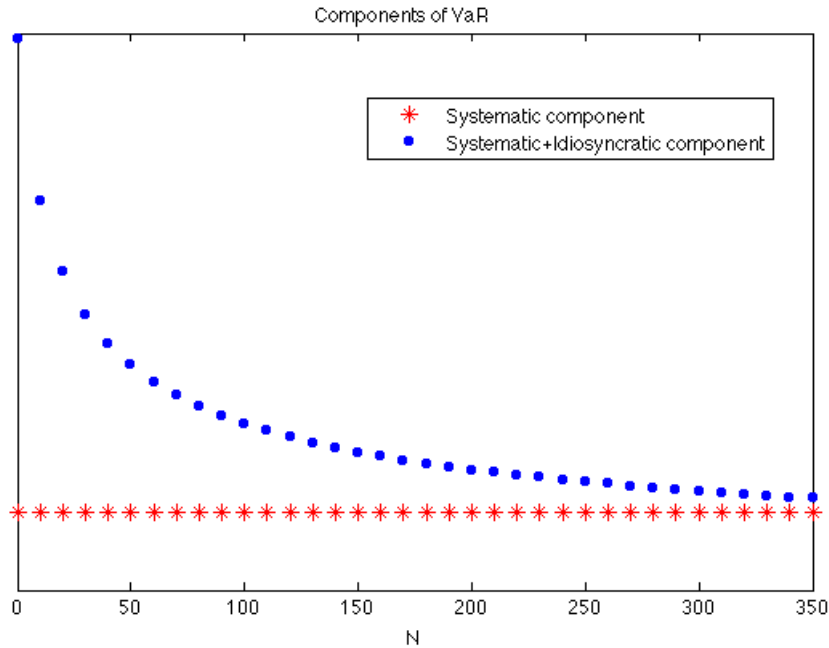


Figure 3.4: Systematic and idiosyncratic component of VaR of portfolio loss  $L_N$ .

### The General Framework

Let  $X$  denote the systematic risk factor. Let assume that it is unidimensional, that is, there is only a single systematic risk factor. For our portfolio of  $N$  risky loans, let  $U_n$  denote the loss rate on position  $n$ . Let  $L_N$  be the loss rate on the portfolio.

When economic capital is measured as VaR at the  $q^{th}$  percentile our aim is to estimate  $\alpha_q(L_N)$ . The IRB formula, however, delivers the  $q^{th}$  percentile of the expected loss conditional on the systematic factor  $\alpha_q(\mathbb{E}(L_N|X))$ .

The exact adjustment for the effect of undiversified idiosyncratic risk in the portfolio is the difference

$$\alpha_q(L_N) - \alpha_q(\mathbb{E}(L_N|X)).$$

As this adjustment cannot be derived in an analytical form, we will construct a Taylor series approximation in orders of  $\frac{1}{N}$ .

Let  $\mu(X) = \mathbb{E}(L_N|X)$  be the conditional mean of the portfolio loss and  $\sigma^2(X) = \mathbb{V}(L_N|X)$  the conditional variance of the portfolio loss.

If  $\epsilon = 1$ , the portfolio loss is given by:

$$L_N = \mathbb{E}(L_N|X) + \epsilon(L_N - \mathbb{E}(L_N|X)).$$

Thus, applying a second order Taylor expansion in powers of  $\epsilon$  around the conditional mean and evaluating the resulting formula at  $\epsilon = 0$ ,

$$\begin{aligned} \alpha_q(L_N) &= \alpha_q(\mathbb{E}(L_N|X) + \epsilon(L_N - \mathbb{E}(L_N|X))) \\ &= \alpha_q\mathbb{E}(L_N|X) + \epsilon\alpha_q(L_N - \mathbb{E}(L_N|X)) \\ &= \alpha_q\mathbb{E}(L_N|X) + \frac{\partial}{\partial\epsilon}\alpha_q(\mathbb{E}(L_N|X) + \epsilon(L_N - \mathbb{E}(L_N|X)))|_{\epsilon=0} \\ &\quad + \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\alpha_q(\mathbb{E}(L_N|X) + \epsilon(L_N - \mathbb{E}(L_N|X)))|_{\epsilon=0} + O(\epsilon^3). \end{aligned}$$

Hence the granularity adjustment of the portfolio is given by:

$$GA_N = \frac{\partial}{\partial\epsilon}\alpha_q(\nu(X) + \epsilon(L_N - \nu(X)))|_{\epsilon=0} \quad (3.3)$$

$$+ \frac{1}{2}\frac{\partial^2}{\partial\epsilon^2}\alpha_q(\nu(X) + \epsilon(L_N - \nu(X)))|_{\epsilon=0}. \quad (3.4)$$

$$(3.5)$$

The first derivative in the Taylor expansion of the quantile vanishes, so the GA (3.5) can be expressed as

$$GA_N = \frac{-1}{2f(\alpha_q(X))} \cdot \frac{d}{dx} \left( \frac{\sigma^2(x)f(x)}{\nu'(x)} \right) \Big|_{x=\alpha_q(X)}, \quad (3.6)$$

where  $f$  is the density function of the systematic risk factor  $X$ .

This general framework can be accommodate any definition of 'loss'.

Let  $UL_N$  denote the true  $UL$  for the portfolio and let  $UL_N^{asympt}$  be the asymptotic approximation of the portfolio.

When we split regulatory capital in its  $UL$  and  $EL$  components, we have

$$\alpha_q(\mathbb{E}(L_N|X)) = UL_N^{asympt} + \underbrace{EL_N^{asympt}}_{\mathbb{E}(\mathbb{E}(L_N|X))}.$$

Therefore, we obtain

$$GA_N = \alpha_q(L_N) - \alpha_q(\mathbb{E}(L_N|X)) = (UL_N + EL_N) - (UL_N^{asympt} + \mathbb{E}(\mathbb{E}(L_N|X))),$$

$$GA_N = UL_N - UL_N^{asympt}.$$

Expected loss vanishes out of the GA.

The granularity adjustment method can also be used in a more general context where a small perturbation is made to a distribution.

### 3.3.2 Normal Approximation

**Theorem 3.** *Central Limit Theorem (CLT).*

Let  $\{X_1, X_2, \dots, X_n\}$  be independent and identically distributed random variables with expected values  $\mu$  and variances  $\sigma^2$ . Suppose we are interested in the behavior of the sample average of these random variables  $S_n = \frac{1}{n}(X_1 + X_2 + \dots + X_n)$ . The CLT asserts that for large  $n$ , the distribution of  $S_n$  is approximately normal with mean  $\mu$  and variance  $\frac{1}{n}\sigma^2$ .

The normal approximation (NA) is a direct application of this theorem and can be found in [9]. We then need to take into account the variability of portfolio loss  $L$  conditional on the common factor  $Y$ . This can easily be approximated due to the CLT. Conditional on the common factor  $Y$ , the portfolio loss  $L$  follows a normal distribution,  $Y \sim N(\mu, \sigma^2)$ , such that

$$\begin{aligned}\mu(Y) &= \sum_{i=1}^n \omega_i P_i(Y), \\ \sigma^2(Y) &= \sum_{i=1}^n \omega_i P_i(Y)(1 - P_i(Y)),\end{aligned}$$

where  $\omega_i = EAD_i \cdot LGD_i$  and  $P_i = P(D_i = 1|Y)$ .

It follows that the conditional tail probability reads

$$P(L > x|Y) = \Phi\left(\frac{\mu(Y) - x}{\sigma(Y)}\right).$$

The unconditional tail probability can then be obtained by integrating over  $Y$ , i.e.,

$$P(L > x) = \mathbb{E}\left[\Phi\left(\frac{\mu(Y) - x}{\sigma(Y)}\right)\right] = \int_{\mathbb{R}} \Phi\left(\frac{\mu(Y) - x}{\sigma(Y)}\right) \phi(y) dy.$$

### 3.3.3 Saddlepoint Approximation

In credit risk management one is particularly interested in the portfolio loss distribution. As the portfolio loss is usually modeled as the sum of random variables, the main task is to evaluate the probability density function (pdf) of such a sum of random variables. When these random variables are independent, the pdf is just the convolution of the pdfs of the individual obligor loss distributions. The evaluation of this convolution, however, is computationally intensive, on the one hand, and, on the other hand, analytically in most cases quite complicated since the loss distribution usually does not possess a closed form solution. However, in some cases, moments can be computed which allows to approximate the loss distribution based on the moment generating function. In such situations a technique, which is frequently used, is the *Edgeworth expansion* method which works quite well in the center of a distribution. In the tails, however, this technique performs poorly. The *saddlepoint expansion* can be regarded as an improvement of the Edgeworth expansion. It is a technique for approximating integrals used in physical sciences, engineering and statistics. [10] were the first to apply the saddlepoint approximation method to credit risk and in particular to the computation of credit VaR. They derived a formula for calculating the marginal impact of a new exposure in a given portfolio. Therefore, the method can be used to quantify the concentration risk of a portfolio.

Since the saddlepoint approximation (SA) provides accurate estimates to very small tail probabilities it is a very suitable technique in the context of portfolio credit loss.

The saddlepoint approximation to a random variable of finite sum  $X = \sum_{i=1}^n X_i$  relies on the existence of the moment generating function (MGF)  $M_X(t) = \mathbb{E}(e^{tX})$ . For  $X_i$ , with known analytic MGF's,  $M_{X_i}$ , the MGF of the sum  $X$  is given by

$$M_X(t) = \prod_{i=1}^n M_{X_i}(t).$$

The inverse MGF of  $X$ , known as the Bromwich integral or inverse Laplace transform, can be written as

$$f_X(x) = \frac{1}{2\pi j} \int_{-j\infty}^{+j\infty} \exp(K_X(t) - tx) dt,$$

with  $j = \sqrt{-1}$  and  $K_X(t) = \log M_X(t)$  the Cumulant Generating Function of  $X$ .

Let  $t = t_0$  a point such that  $K'_X(t_0) = x$ . Then,  $K_X(t_0) - t_0 \cdot x$  is stationary, i.e.,  $t_0$  is a saddlepoint.

The density  $f_X(x)$  and the tail probability  $P(X > x)$  can be approximated by  $K_X(t)$  and its derivatives at  $t_0$ . There are several variants of saddlepoint approximations available, but the most used one for density is the Daniels formula:

$$f_X(x) \approx \frac{\phi(z_l)}{\sqrt{K''(t_0)}} \left[ 1 + \left( -\frac{5K'''(t_0)^2}{24K''(t_0)^3} + \frac{K^{(4)}(t_0)}{8K''(t_0)^2} \right) \right], \quad (3.7)$$

and the correspondent most used one for probability when applying saddlepoint approximation is the Lugannani-Rice formula

$$F_X(x) \approx 1 - \Phi(z_l) + \phi(z_l) \left( \frac{1}{z_\omega} - \frac{1}{z_l} \right), \quad (3.8)$$

with

$$z_\omega = t_0 \sqrt{K''(t_0)} \quad \text{and} \quad z_l = \text{sgn}(t_0) \cdot \sqrt{2(xt_0 - K(t_0))}.$$

The saddlepoint approximation is usually highly accurate in the tail of a distribution.

### 3.4 Monte Carlo Simulations

The value at risk and economic capital including name concentration can be calculated using a full Monte Carlo simulation.

Monte Carlo simulations are employed to approximate the loss distribution and estimate various risk measures.

Asset value simulations use a single-factor model of asset values, with a random specific error term, uniform correlation between assets, uniform sizes of exposures, under default mode only. The asset values are a linear function of a common factor, representing the state of the economy, and of an error term, independent of the state of the economy, representing specific risk. Monte Carlo simulations generate random sets of future values of the economic factor and of the specific risk, from which the asset values derive directly. This is a much more restrictive framework than actual portfolio models. However, it illustrates the essential mechanisms of the process. Moreover, the same technique allows us to vary the default probabilities and the sizes of exposures across exposures.

A large number of trials generate a portfolio value distribution, later converted into a portfolio distribution. Since asset values trigger default, all we need is to generate correlated asset values for all obligors. When running a default model, the target portfolio variable modelled is the portfolio loss.

To proceed, the sequential steps are:

- Generate a standardized random normal distribution of the common factor  $Z$ .
- Generate as many standardized random distributions of the specific residuals  $\epsilon_i$  as there are obligors.
- Calculate the resulting standardized random variable  $Z_i$ , which is the asset value of each obligor.
- Transform this asset value, for each obligor, into a default 0-1 variable.
- Calculate the loss for each obligor, given the value of the default variable and the exposure.
- Sum all losses to get one value for portfolio loss in each simulation.
- Repeat the simulation as many times as desired.



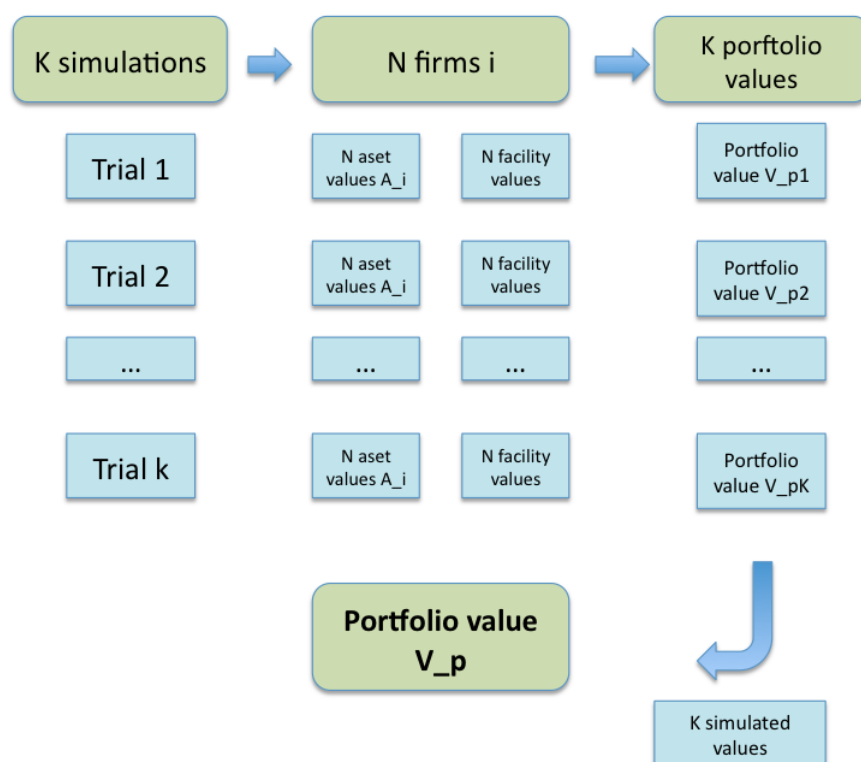


Figure 3.5: Monte Carlo simulation of asset values to generate correlated loss distribution.

Monte Carlo provides a number of advantages over deterministic analysis:

- Probabilistic results: Results show not only what could happen but how likely each outcome is.
- Graphical results: It is easy to create graphs of different outcomes.
- Sensitivity analysis: With just a few cases, deterministic analysis makes it difficult to see which variables impact the outcome the most. In Monte Carlo simulation, its easy to see which inputs had the biggest effect on bottom-line results.
- Scenario Analysis: In deterministic models, it is very difficult to model different combinations of values for different inputs to see the effects of truly different scenarios. Using Monte Carlo simulation, analysts can see exactly which inputs had which values together when certain outcomes occurred.

The employed code is included in Appendix A.

### 3.4.1 A first example

Let consider a stylized portfolio of 10000 counterparties with  $EAD = 1$ ,  $LGD = 100\%$ ,  $\rho = 20\%$  and  $PD = 1\%$ . This is the so-called portfolio A. Let number of trials equals to 1000000. We performed a Monte Carlo simulation of this granular portfolio and plot the obtained cumulative distribution function.

Define portfolio B as portfolio A with 10 large counterparties added, all of them with  $EAD = 20$ . The total portfolio is not granular anymore. Name concentration appears. If we perform a MC simulation we can compute the EC taking into account name concentration. We can notice the difference between the computed  $EC$  for both portfolios. As we have seen, we will be able to quantify this 'probability' correction by applying the analytical solution. This will be done in the next chapter.

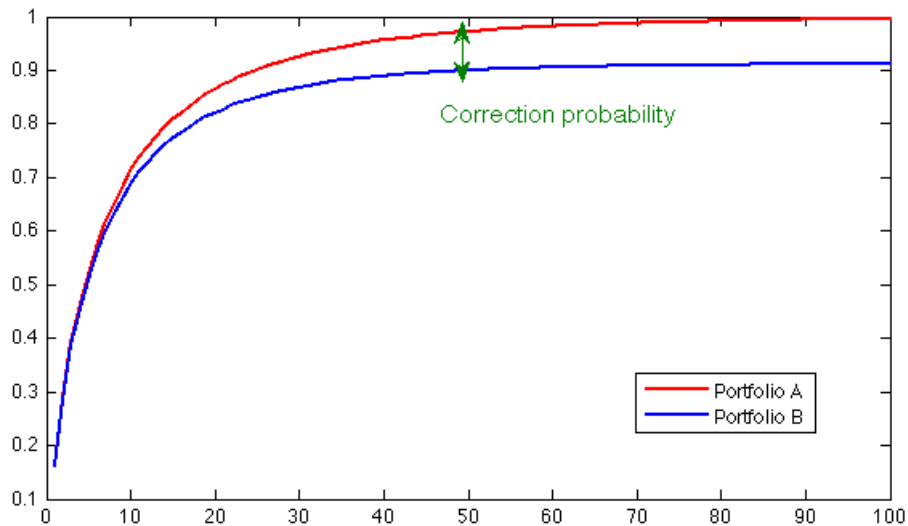


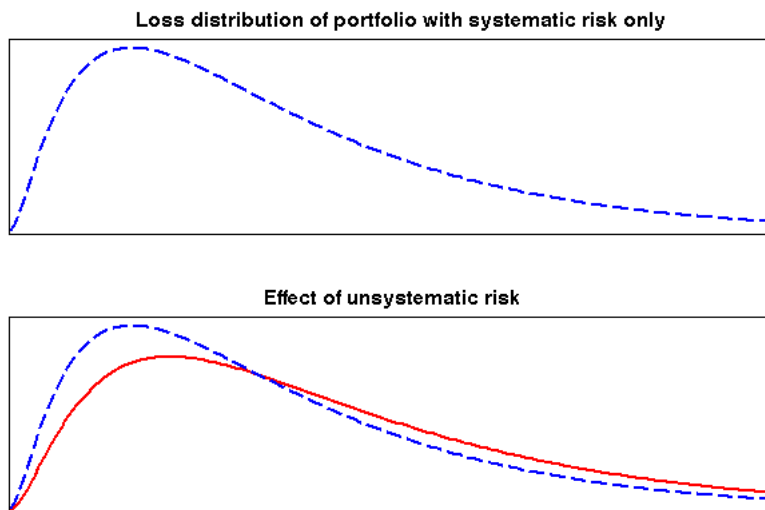
Figure 3.6: Correction probability when Name Concentration appears.

# Chapter 4

## Semi-Analytical Implementation

No real-world credit portfolio is perfectly diversified. In calculating Value at Risk for a credit portfolio, a correction has therefore to be made in order to account for the remaining unsystematic risk.

In the next two pictures we see the situation for increasing amounts of unsystematic risk. Now there is some uncertainty in portfolio loss even when we know the state of the world.

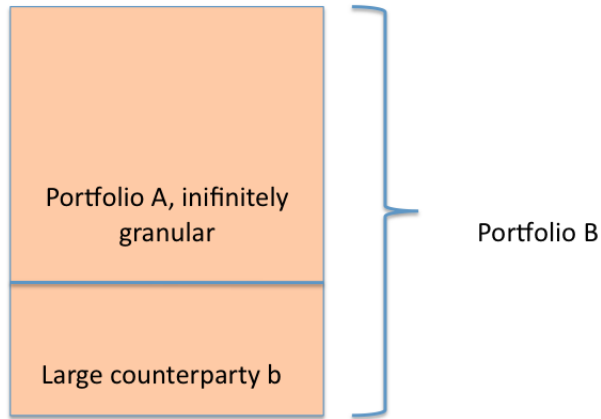


In the top picture we have the distribution of the infinitely fine-grained portfolio. In the lower picture we have replotted the infinitely granular dis-

tribution and also shown the distribution of a real portfolio that does have unsystematic risk. The tail of the distribution is fattened, there is a greater chance of big losses.

## 4.1 Single large name in the portfolio

Let consider a perfectly granular portfolio with one large exposure added. We consider portfolio A defined as a Vasicek portfolio containing  $N$  loans, and portfolio B is like portfolio A with a large counterparty b added to it. The total portfolio is not perfectly granular anymore.



Our goal is to calculate the Value at Risk at a confidence level  $q$  of portfolio B analytically .

We will show that the loss distribution function of the portfolio B containing the large exposure can be calculated by subtracting a certain probability  $\Delta P$ , from the loss distribution function of the original portfolio.

**Theorem 4.**

$$F_B(l) = F_A(l) - \Delta P(l),$$

where

$F_B(l)$  : Cumulative loss distribution function of portfolio B.

$F_A(l)$  : Cumulative loss distribution function of portfolio A.

**Proof:**

We prove the formula by deriving an analytical expression for  $\Delta P(l)$ . We need to calculate  $F_B(l)$ .

We conditionate this probability on two disjoint states, depending on the default, or not, of the large added counterparty  $b$ ,

$$F_B(l) = F_B(l|b \text{ is not in default}) + F_B(l|b \text{ is in default}).$$

(a) Counterparty  $b$  is not in default.

We assume that:

- There is a mark-to-market loss<sup>1</sup> that is a deterministic function,  $l_b(y)$ , of the market factor  $y$ .
- The loss is independent of  $y$ , and it is given by  $LGD_b, EAD_b$ .

$$\begin{aligned} F_B(l|b \text{ not in default}) &= P(L_B(y) < l | b \text{ not in default}) \\ &= P(L_A(y) + l_b(y) < l | b \text{ not in default}) \\ &= P(L_A(y) < l - l_b(y) | b \text{ not in default}). \end{aligned}$$

Since  $L_A(y)$ ,  $l_b(y)$  are strictly decreasing with  $y$ , the equation

$$L_A(y) = l - l_b(y)$$

has a solution and it is unique. Let  $y_1(l)$  be this solution. Thus, we have that

$$F_B(l|b \text{ not in default}) = P(L_A(y) < y_1(l) | b \text{ not in default}). \quad (4.1)$$

(b) Counterparty  $b$  is in default.

$$\begin{aligned} F_B(l|b \text{ in default}) &= P(L_B(y) < l) = P(L_A(y) + l_b(y) < l) \\ &= P(L_A(y) + LGD_b \cdot EAD_b < l) \\ &= P(L_A(y) < l - LGD_b \cdot EAD_b). \end{aligned}$$

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<sup>1</sup>A *mark-to-market loss* is a loss generated through an accounting entry rather than the actual sale of a security. Mark-to-market losses can occur when financial instruments held are valued at the current market value. If a security was purchased at a certain price and the market price later fell, the holder would have an unrealized loss, and marking the security down to the new market price results in the mark-to-market loss.

Let  $y_2(l)$  be the solution of the equation

$$L_A(y) = l - LGD_b \cdot EAD_b.$$

In this case, we find that

$$F_B(l|\text{b in default}) = P(L_A(y) < y_2(l)). \quad (4.2)$$

**Notes:**

- (i) As we assumed that the market-to-market loss is always smaller than the default loss,  $y_2(l) > y_1(l)$ ,  $\forall l$ .
- (ii) According to the Vasicek model, the conditional default probability is given by

$$PD_b(Y = y) = \Phi \left( \frac{\Phi^{-1}(PD_b) - \sqrt{\rho_b} \cdot y}{\sqrt{1 - \rho_b}} \right).$$

Thus, and taking into account (4.1) and (4.2), we compute the unconditional loss by integrating with respect to the variable  $y$ .

$$\begin{aligned} F_B(l) &= F_B(l|\text{b not in default}) + F_B(l|\text{b in default}) \\ &= \int_{y_1(l)}^{\infty} \left[ 1 - \Phi \left( \frac{\Phi^{-1}(PD_b) - \sqrt{\rho_b} \cdot y}{\sqrt{1 - \rho_b}} \right) \right] \phi(y) dy \\ &\quad + \int_{y_2(l)}^{\infty} \Phi \left( \frac{\Phi^{-1}(PD_b) - \sqrt{\rho_b} \cdot y}{\sqrt{1 - \rho_b}} \right) \phi(y) dy \\ &= \int_{y_1(l)}^{\infty} \phi(y) dy - \int_{y_1(l)}^{y_2(l)} \Phi \left( \frac{\Phi^{-1}(PD_b) - \sqrt{\rho_b} \cdot y}{\sqrt{1 - \rho_b}} \right) \phi(y) dy, \end{aligned}$$

where  $\phi(y)$  denotes the probability density function of a standard normal variable.

Then,

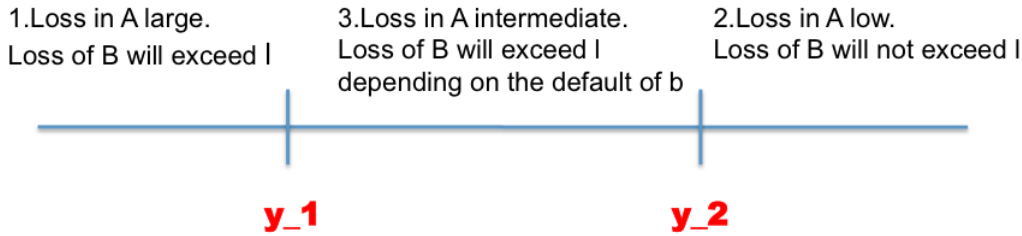
$$F_B(l) = F_A(l) - \underbrace{\int_{y_1(l)}^{y_2(l)} \Phi \left( \frac{\Phi^{-1}(PD_b) - \sqrt{\rho_b} \cdot y}{\sqrt{1 - \rho_b}} \right) \phi(y) dy}_{\Delta P(l)}.$$

In conclusion, we have found that the probability correction is given by the analytical expression

$$\Delta P(l) = \int_{y_1(l)}^{y_2(l)} \Phi \left( \frac{\Phi^{-1}(PD_b) - \sqrt{\rho_b} \cdot y}{\sqrt{1 - \rho_b}} \right) \cdot \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy. \quad (4.3)$$

### Determination of border values $y_1$ and $y_2$

Let  $F_B(l)$  be the cumulative loss distribution function of the portfolio B. Given a value of  $l$ , there are three possible options:



1. The loss of portfolio A is larger than  $l$ . Then, the loss in the portfolio B will be also larger than  $l$ .

$$l_B = l_A + l_b,$$

$$l_b > 0, l_A > l \Rightarrow l_B = l_A + l_b > l.$$

2. The loss of portfolio A is lower than  $l$  minus the potential loss of counterparty b. Then, a default of counterparty b will not make  $l_B$  larger than  $l$ .

$$l_A = l_B - l_b < l - LGD_b \cdot EAD_b \Rightarrow l_B < l.$$

3. The intermediate situation.

$$l - LGD_b \cdot EAD_b < l_A < l.$$

- If counterparty b defaults. The loss is independent of  $y$  and is given by  $LGD_b, EAD_b$ .

$$l - LGD_b \cdot EAD_b < l_A = l_B - l_b = l_B - LGD_b \cdot EAD_b, \\ \Rightarrow l_B > l.$$

- If counterparty b does not default

$$l_B < l.$$

As the loss in portfolio A is driven by the global factor  $y$ , one can determine the limits between the different situations.

According to the Vasicek model, the loss is given by the realisation of the global factor. Thus,

$$l = \sum_n N\left(\frac{\Phi^{-1}(PD_n) - \sqrt{\rho_n} \cdot y_1}{\sqrt{1 - \rho_n}}\right) LGD_n \cdot EAD_n \text{ and}$$

$$l - LGD_b EAD_b = \sum_n N\left(\frac{\Phi^{-1}(PD_n) - \sqrt{\rho_n} \cdot y_2}{\sqrt{1 - \rho_n}}\right) LGD_n \cdot EAD_n.$$

- Let A be homogeneous <sup>2</sup>,

In our particular case, we have that

$$PD_n = PD, \quad n = 1 \dots N + 1$$

$$\rho_n = \rho, \quad n = 1 \dots N + 1$$

$$LGD_n = LGD, \quad n = 1 \dots N + 1$$

$$EAD_n = \begin{cases} EAD_A, & n = 1 \dots N \\ EAD_b, & \text{large added counterparty} \end{cases}$$

Thus,

$$l = \Phi\left(\frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot y_1}{\sqrt{1 - \rho}}\right) LGD(N \cdot EAD_A + EAD_b),$$

$$\Phi^{-1}\left(\frac{l}{LGD(N \cdot EAD_A + EAD_b)}\right) = \frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot y_1}{\sqrt{1 - \rho}},$$

$$\Rightarrow y_1 = -\frac{\sqrt{1 - \rho} \cdot \Phi^{-1}\left(\frac{l}{LGD(N \cdot EAD_A + EAD_b)}\right) - \Phi^{-1}(PD)}{\sqrt{\rho}}.$$

<sup>2</sup>In the homogeneous portfolio the risk variables are independent and identically distributed. We assume an homogeneous portfolio in the sense that all obligors have the same default probability,  $PD_n = PD$  and  $LGD_n = 100\%$ ,  $\forall n \in [1, N]$ .



$$\begin{aligned}
l - LGD \cdot EAD_b &= \Phi \left( \frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot y_2}{\sqrt{1 - \rho}} \right) LGD(N \cdot EAD_A + EAD_b), \\
\Phi^{-1} \left( \frac{l - LGD \cdot EAD_b}{LGD(N \cdot EAD_A + EAD_b)} \right) &= \frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot y_2}{\sqrt{1 - \rho}}, \\
\sqrt{1 - \rho} \cdot \Phi^{-1} \left( \frac{l - LGD \cdot EAD_b}{LGD(N \cdot EAD_A + EAD_b)} \right) - \Phi^{-1}(PD) &= -\sqrt{\rho} y_2, \\
\Rightarrow y_2 &= -\frac{\sqrt{1 - \rho} \cdot \Phi^{-1} \left( \frac{l - LGD \cdot EAD_b}{LGD(N \cdot EAD_A + EAD_b)} \right) - \Phi^{-1}(PD)}{\sqrt{\rho}}.
\end{aligned}$$

- If A is not homogeneous, we could find the values of  $y_1$  and  $y_2$  by a root-finding algorithm.

Moreover,  $LGD = 1$ . Thus, we have obtained that the border values are given by

$$y_1(l) = -\frac{\sqrt{1 - \rho} \cdot \Phi^{-1} \left( \frac{l}{N \cdot EAD_A + EAD_b} \right) - \Phi^{-1}(PD)}{\sqrt{\rho}}. \quad (4.4)$$

$$y_2(l) = -\frac{\sqrt{1 - \rho} \cdot \Phi^{-1} \left( \frac{l - LGD \cdot EAD_b}{N \cdot EAD_A + EAD_b} \right) - \Phi^{-1}(PD)}{\sqrt{\rho}}. \quad (4.5)$$

## Summary

Finding the Value at Risk of portfolio B at a confidence level of  $q$  becomes a matter of finding the correct loss  $l$  that satisfies

$$F_{Vasicek} \left( \frac{l}{EAD_B} \right) - \Delta P(l) = q, \quad (4.6)$$

where

$F_{Vasicek}$  refers to the loss distribution function of the portfolio under the ASRF framework, and it is given, as we already saw by (2.14).

$EAD_B$  is the exposure of the total portfolio, i.e.,

$$EAD_B = N \cdot EAD_A + EAD_b.$$

Thus, we must find  $l$  that verifies

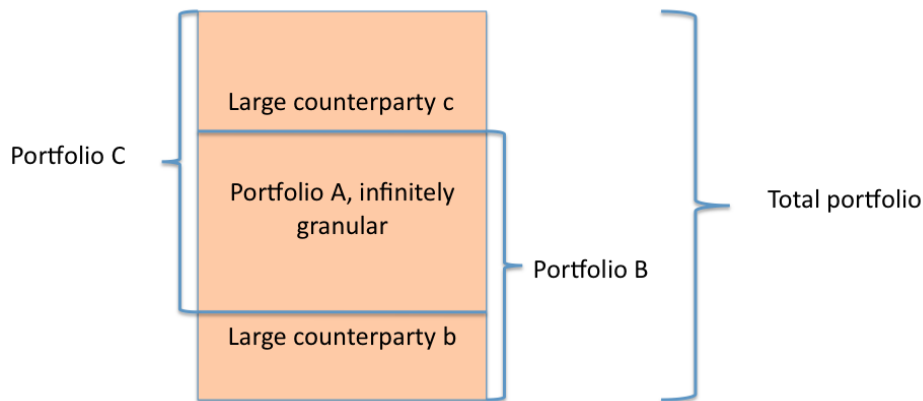
$$\Phi \left( \frac{\sqrt{1-\rho} \cdot \Phi^{-1} \left( \frac{l}{EAD_B} \right) - \Phi^{-1}(PD)}{\sqrt{\rho}} \right) - \int_{y_1(l)}^{y_2(l)} \Phi \left( \frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot y}{\sqrt{1-\rho}} \right) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy = q.$$

We will denote  $l = VaR_{T,q,Analytic}$  the Value at Risk of the total portfolio at a confidence level  $q$  computed with the analytical method.

## 4.2 Multiple large counterparties in the portfolio

In general, the portfolio of a bank includes several large counterparties, not only one. The analytic solution derived for the single large counterparty can be used to estimate the VaR of a portfolio containing multiple large counterparties.

Let A be a Vasicek portfolio containing  $N$  counterparties and portfolio B is defined as portfolio A with a large counterparty b added to it. Moreover, define C as portfolio A with a large counterparty c. The total portfolio contains portfolio A and both large counterparties b and c.



If B is considered to be infinitely granular and counterparty c to be the large counterparty, we can calculate the VaR of the total portfolio at a some confidence level  $q$  by using the analytic solution, denoted  $VaR_{T,q,Analytic}$  as we have already mentioned, and we will have,

$$VaR_{T,q,Analytic} = VaR_{B,q,Vasicek} + \Delta VaR_{c,q},$$

with

$VaR_{B,q,Vasicek}$  : Value at Risk of portfolio C at a confidence level  $q$  computed under the ASRF framework.

$\Delta VaR_{c,q}$  : Change in VaR at a confidence level  $q$  when the large counterparty  $c$  is added to portfolio B.

Subtracting the VaR of the portfolio B, results in:

$$\Delta VaR_{c,q} = VaR_{T,q,Analytic} - VaR_{B,q,Vasicek}.$$

This is, of course, an approximation because portfolio B contains a large counterparty  $b$ , then it is not perfectly granular. However, this effect can be considered a second-order effect, especially when counterparty  $c$  is not too large.

The same can be done considering counterparty  $b$  to be the large counterparty in the total portfolio, and  $C$  defined as a Vasicek portfolio.

$$VaR_{T,q,Analytic} = VaR_{C,q,Vasicek} + \Delta VaR_{b,q},$$

where

$VaR_{C,q,Vasicek}$  : Value at Risk of portfolio B at a confidence level  $q$  computed under the ASRF framework.

$\Delta VaR_{b,q}$  : Change in VaR at a confidence level  $q$  when the large counterparty  $b$  is added to portfolio C.

Subtracting the VaR of the portfolio C,

$$\Delta VaR_{b,q} = VaR_{T,q,Analytic} - VaR_{C,q,Vasicek}.$$

The Value at Risk of the total portfolio,  $VaR_{Total,q}$  can be calculated according to the approximation formula

$$VaR_{Total,q} = VaR_{A,q,Vasicek} + \Delta VaR_{b,q} + \Delta VaR_{c,q}, \quad (4.7)$$

this is,

$$\begin{aligned} VaR_{Total,q} = & VaR_{A,q,Vasicek} + VaR_{T,q,Analytic} - VaR_{C,q,Vasicek} \\ & + VaR_{T,q,Analytic} - VaR_{B,q,Vasicek}. \end{aligned}$$

Generalising (4.7) for  $M$  large counterparties, we will have

$$VaR_{Total,q} = VaR_{A,q,Vasicek} + \sum_{i=1}^M \Delta VaR_{i,q}. \quad (4.8)$$

If all large added counterparties are yield by the same risk factors we will be able to simplify the last equation since the contribution of each large added counterparty will be the same. If  $M$  large counterparties are added, we will consider  $M$  times the contribution of a large counterparty, which will be computed using the proposed analytical solution. Thus,

$$VaR_{Total,q} = VaR_{A,q,Vasicek} + M \cdot \Delta VaR_{i,q}.$$

Let C defined again as portfolio A with a large added counterparty b. The above equation becomes,

$$VaR_{Total,q} = VaR_{A,q,Vasicek} + M \cdot (VaR_{T,q,Analytic} - VaR_{C,q,Vasicek}) \quad (4.9)$$

$$= \underbrace{VaR_{A,q,Vasicek}}_{(i)} - M \cdot \underbrace{VaR_{C,q,Vasicek}}_{(ii)} + M \cdot \underbrace{VaR_{T,q,Analytic}}_{(iii)}. \quad (4.10)$$

(i) **Value at Risk of portfolio A**

As the granular portfolio A contains  $N$  loans the Value at Risk according to the Vasicek model will be computed according to the formula:

$$VaR_{A,q,Vasicek} = \sum_{n=1}^N EAD_n \cdot LGD_n \cdot \Phi \left( \frac{\Phi^{-1}(PD_n) + \sqrt{\rho_n} \cdot \Phi^{-1}(q)}{\sqrt{1 - \rho_n}} \right).$$

Since portfolio A is homogeneous ( $PD_n = PD$  and  $LGD_n = LGD$ ,  $n = 1 \dots N$ ) and we will assume  $LGD = 1$ ,  $EAD_n = EAD_A$ , and  $\rho_n = \rho$ ,  $n = 1 \dots N$ , we can simplify the above equation, it becomes

$$VaR_{A,q,Vasicek} = N \cdot EAD_A \cdot \Phi \left( \frac{\Phi^{-1}(PD) + \sqrt{\rho} \cdot \Phi^{-1}(q)}{\sqrt{1 - \rho}} \right).$$

(ii) **Value at Risk of portfolio C**

We remind that portfolio C is portfolio A with one large counterparty, b, added. Thus, it contains N+1 counterparties. All of them have the same risk drivers except the exposure at default, which in case of counterparty b,  $EAD_b$ , is larger.

$$\begin{aligned} VaR_{C,q,Vasicek} &= \sum_{n=1}^{N+1} EAD_n \cdot LGD_n \cdot \Phi \left( \frac{\Phi^{-1}(PD_n) + \sqrt{\rho_n} \cdot \Phi^{-1}(q)}{\sqrt{1 - \rho_n}} \right) \\ &= (N \cdot EAD_A + EAD_b) \cdot \Phi \left( \frac{\Phi^{-1}(PD) + \sqrt{\rho} \cdot \Phi^{-1}(q)}{\sqrt{1 - \rho}} \right). \end{aligned}$$

(iii) **Value at Risk of the total portfolio**

With the analytic solution we will compute  $VaR_{T,q,Analytic}$ .

- We define

$$G(y) = \Phi \left( \frac{\Phi^{-1}(PD) - \sqrt{\rho} \cdot y}{\sqrt{1 - \rho}} \right) \cdot \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}.$$

- We evaluate numerically  $G$  between the border values  $y_1$  and  $y_2$ ,

$$y_1 = -\frac{\sqrt{1 - \rho} \cdot \Phi^{-1} \left( \frac{l}{N \cdot EAD_A + EAD_b} \right) - \Phi^{-1}(PD)}{\sqrt{\rho}},$$

$$y_2 = -\frac{\sqrt{1 - \rho} \cdot \Phi^{-1} \left( \frac{l - LGD \cdot EAD_b}{N \cdot EAD_A + EAD_b} \right) - \Phi^{-1}(PD)}{\sqrt{\rho}}.$$

Let

$$I(l) = \int_{y_1(l)}^{y_2(l)} G(y) dy.$$

The above integral is numerically approximated within an error of  $10^{-6}$  using recursive Simpson quadrature.

- We look for  $l$  such that

$$F_{Vasicek} \left( \frac{l}{N \cdot EAD_A + EAD_b} \right) - I(l) = q.$$

Let

$$\begin{aligned} H(l) &= F_{Vasicek} \left( \frac{l}{N \cdot EAD_A + EAD_b} \right) - I(l) - q \\ &= \Phi \left( \frac{\sqrt{1-\rho} \cdot \Phi^{-1} \left( \frac{l}{N \cdot EAD_A + EAD_b} \right) - \Phi^{-1}(PD)}{\sqrt{\rho}} \right) - I(l) - q. \end{aligned}$$

$H$  is defined as a handle function.

Finding  $l$  is equivalent to search a zero of the function  $H$ . Thus, we compute numerically a root of this function.

In order to do that, we will employ the Matlab function,

```
l=fzero(H, VaR_{C,q,Vasicek})
```

which tries to find a zero of  $H$  near  $VaR_{C,q,Vasicek}$ .

It uses a combination of bisection, secant, and inverse quadratic interpolation (IQI) methods.

The idea behind this algorithm is to combine the reliability of bisection with the convergence speed of secant and inverse quadratic interpolation methods. Here is the outline:

- Start with  $a$  and  $b$  so that  $f(a)$  and  $f(b)$  have opposite signs.
- Use a secant step to give  $c$  between  $a$  and  $b$ .
- Repeat the following steps until

$$|b - a| < \epsilon|b| \text{ or } f(b) = 0.$$

- Arrange  $a$ ,  $b$  and  $c$  so that
  - \*  $f(a)$  and  $f(b)$  have opposite signs.
  - \*  $|f(b)| \leq |f(a)|$ .
  - \*  $c$  is the previous value of  $b$ .
- If  $c \neq a$ , consider an IQI step.
- If  $c = a$ , consider a secant step.

- If the IQI or secant step is in the interval  $[a, b]$ , take it.
- If the step is not in the interval, use bisection.

This algorithm is foolproof. It never loses track of the zero trapped in a shrinking interval. It uses rapidly convergent methods when they are reliable. It uses a slow, but sure, method when it is necessary.

As we have already explained, this is equivalent to compute the Value at Risk of the total portfolio at a given confidence level  $q$ , so we will have  $l = VaR_{T,q,Analytic}$ .

Now we are able to compute all the elements in (4.10) in order to obtain the Value at Risk of the total portfolio at a confidence level  $q$ ,

$$VaR_{Total,q} = VaR_{A,q,Vasicek} + M \cdot (VaR_{T,q,Analytic} - VaR_{C,q,Vasicek}).$$

We will consider this framework in most of the examples. Nevertheless, we will also take into account the case of adding large counterparties with different exposures at default. Moreover, we will also perform an example considering not an infinitely granular portfolio. Those cases will let us see how the analytical solution behaves in other scenarios.



### 4.3 Numerical Examples

Here, we investigate the accuracy of the analytical method relative to some representative bank portfolios. First, the Vasicek model has been applied, resulting in VaR ASRF. Then, a full Monte Carlo has been performed, obtaining VaR MC. We will compare these results with the obtained approximation by applying the analytical solution, VaR analytic.

From now on, let consider the following notation:

$N$ : Number of loans in the granular portfolio.

$EAD$ : Exposure at default of each counterparty in the test portfolio.

$M$ : Number of large added counterparties.

$EAD_b$ : Exposure of each large added counterparty.

$q$ : Confidence level.

$HHI$ : Herfindahl-Hirschman index of the total portfolio.

$G$ : Gini index of the total portfolio.

$r_e$ : Ratio exposure of large counterparties to total portfolio.

$$r_e = \frac{M \cdot EAD_b}{EAD_{Total}} = \frac{M \cdot EAD_b}{N \cdot EAD + M \cdot EAD_b}.$$

$\eta$ : Relative error between the analytic approximation and Monte Carlo,

$$\eta = \frac{\text{VaR analytic} - \text{VaR MC}}{\text{VaR MC}}.$$

The correspondent codes are included in Appendix A.

## 1. The portfolio remains homogeneous

First, let consider the effect on Value at Risk of adding counterparties such that the portfolio is still homogeneous. This is the simplest case to consider. We will add counterparties with the same exposure as counterparties in the granular portfolio.

### Test portfolio used for simulations

N	EAD	PD	LGD	$\rho$
10000	1	1.00 %	100 %	20 %

With very small  $EAD_b \cdot LGD$ , the limiting value  $y_1$  will be very close to  $y_2$ . This causes the correction probability term to tend to zero in the limit. The loss distribution of portfolio B becomes almost the same as the loss distribution of portfolio A.

In this case applying the analytical solution makes no sense since it reaches the same result as the ASRF method.

#### A) $EAD_b=1$ , $M=20$

$q$	VaR ASRF	VaR analytic	VaR MC	$\eta$
99%	754.01	754.02	756	-0.262%
99.9%	1418.16	1458.17	1467	-0.602%

#### B) $EAD_b=1$ , $M=1000$

$q$	VaR ASRF	VaR analytic	VaR MC	$\eta$
99%	827.76	827.87	830	-0.257%
99.9%	1600.78	1600.94	1584	1.069%

## 2. Effect on Value at Risk of increasing EAD of two large counterparties

Test portfolio used for simulations

N	10000	$EAD_b$	$r_e$	HHI	G
EAD	1	50	0.990 %	$1.4704 \cdot 10^{-4}$	0.0097
PD	1.00 %	100	1.961%	$2.8835 \cdot 10^{-4}$	0.0194
LGD	100 %	200	3.846%	$8.3210 \cdot 10^{-4}$	0.0383
$\rho$	20%	500	9.091%	0.0042	0.0907

As we will consider counterparties which have the same risk drivers we can simplify (4.7). Thus,

$$\begin{aligned}
 VaR_{Total,q} &= VaR_{A,q,Vasicek} + 2 \cdot \Delta VaR_{b,q} \\
 &= VaR_{A,q,Vasicek} + 2 \cdot (VaR_{T,q,Analytic} - VaR_{C,q,Vasicek}) \\
 &= VaR_{A,q,Vasicek} - 2 \cdot VaR_{C,q,Vasicek} + 2 \cdot VaR_{T,q,Analytic}.
 \end{aligned}$$

A)  $q=99\%$

$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
50	760.03	760.59	762	-0.19%
100	767.56	769.90	770	-0.01%
200	782.61	792.91	795	-0.26%
500	827.76	913.34	902	1.26%

B)  $q=99.9\%$

$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
50	1469.81	1470.64	1470	0.05%
100	1484.36	1487.79	1489	-0.08%
200	1513.46	1527.94	1528	- 0.004%
500	1600.78	1705.89	1705	0.05%

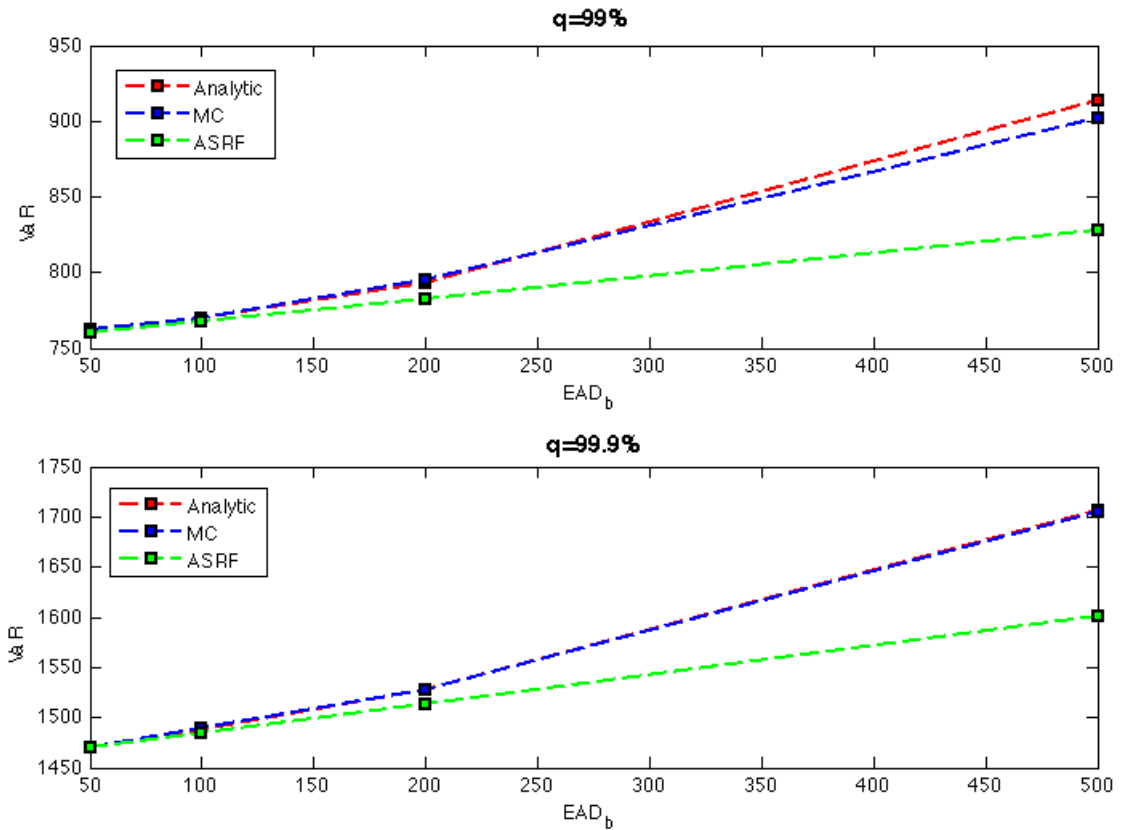


Figure 4.1: Effect on VaR of increasing EAD of two large counterparties.

Analytic and MC VaR figures show good agreement when the EAD of the large counterparties do not exceed about 4% of the total portfolio EAD.

For small values of the EAD of the large added counterparties the approximation is within the error margins of the MC result.

With large values of the EAD of the large counterparties, the analytic solution overestimates the VaR, and consequently, the EC.

### 3. Effect on Value at Risk of increasing EAD of ten large counterparties

Test portfolio used for simulations

N	10000	$EAD_b$	$r_e$	HHI	G
EAD	1	20	1.961%	$1.3456 \cdot 10^{-4}$	0.0186
PD	1.00 %	40	3.846%	$2.4038 \cdot 10^{-4}$	0.0375
LGD	100 %	100	9.091%	$9.0909 \cdot 10^{-4}$	0.0899
$\rho$	20%	200	16.667%	0.0028	0.1657
		400	28.571%	0.0082	0.2847

Taking into account that we are considering counterparties which have the same risk drivers we will obtain by simplyfing (4.7)

$$\begin{aligned}
 VaR_{Total,q} &= VaR_{A,q,Vasicek} + 10 \cdot \Delta VaR_{b,q} \\
 &= VaR_{A,q,Vasicek} + 10 \cdot (VaR_{T,q,Analytic} - VaR_{C,q,Vasicek}) \\
 &= VaR_{A,q,Vasicek} - 10 \cdot VaR_{C,q,Vasicek} + 10 \cdot VaR_{T,q,Analytic}.
 \end{aligned}$$

A)  $q=99\%$

$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
20	767.56	767.99	770	-0.26%
40	782.61	784.38	785	-0.08%
100	827.76	839.47	841	-0.18%
200	903.01	954.49	948	0.67%
400	1053.51	1302.94	1204	8.2%

B)  $q=99.9\%$

$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
20	1484.36	1485.01	1494	-0.6%
40	1513.46	1516.11	1518	-0.13%
100	1600.78	1617.89	1617	0.055%
200	1746.30	1818.69	1818	0.038%
400	2037.35	2538.61	2254	12.63%

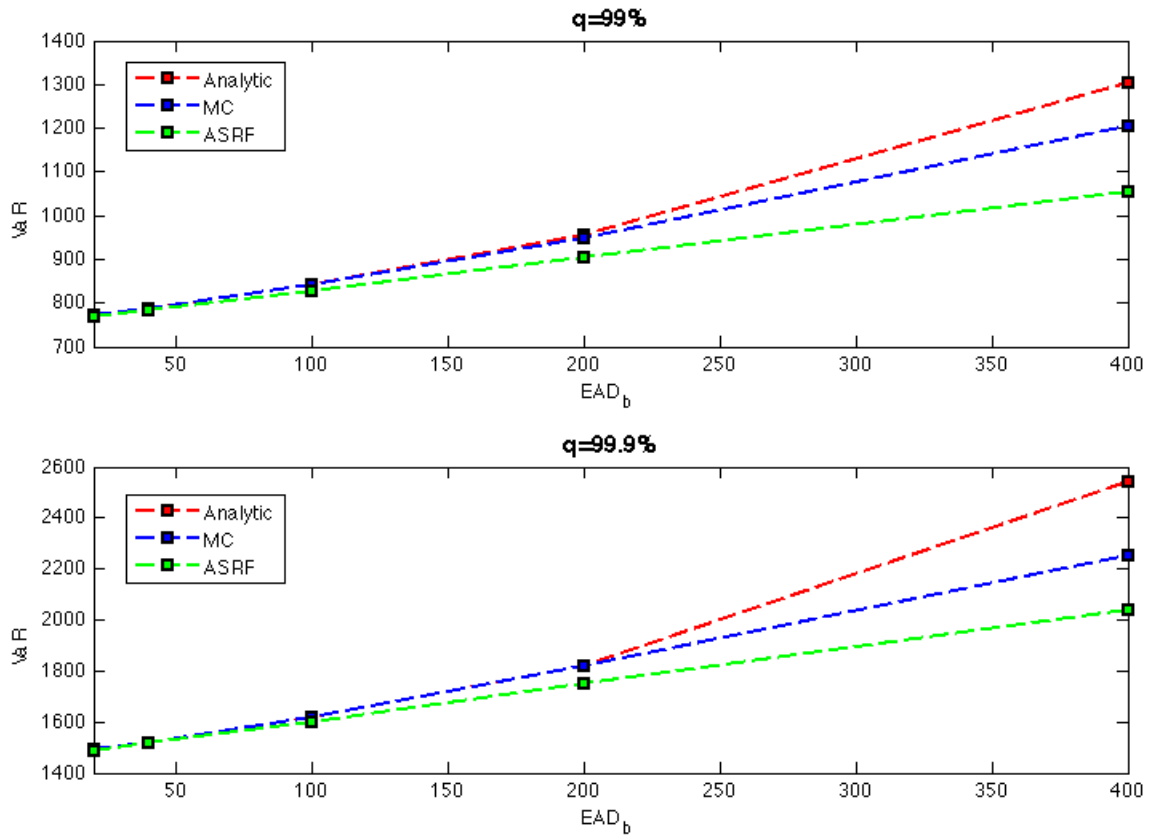


Figure 4.2: Effect on VaR of increasing EAD of 10 large counterparties.

There is a good agreement between both approaches when the total ratio exposure of large counterparties to the total portfolio does not exceed about 9%.

The more diversified the portfolio becomes, the worse results our analytical approximation provides. To have an idea about portfolio's concentration we can have a look at the concentration indices. As the exposure of the added counterparties grows up, these indices become higher, which means that the portfolio is more concentrated.

#### 4. Effect on Value at Risk of increasing EAD of 15 large counterparties

Test portfolio used for simulations

N	10000	$EAD_b$	$r_e$	HHI	G
EAD	1	20	2.913%	$1.5081 \cdot 10^{-4}$	0.0276
PD	1.00 %	40	5.669%	$3.0259 \cdot 10^{-4}$	0.0551
LGD	100 %	100	13.043%	0.0012098	0.1289
$\rho$	20%	200	23.080%	0.0036095	0.2293
		400	37.500%	0.0094	0.3735

Taking into account that we are considering counterparties which have the same risk drivers we will obtain by simplifying (4.7)

$$\begin{aligned}
 VaR_{Total,q} &= VaR_{A,q,Vasicek} + 15 \cdot \Delta VaR_{b,q} \\
 &= VaR_{A,q,Vasicek} + 15 \cdot (VaR_{T,q,Analytic} - VaR_{C,q,Vasicek}) \\
 &= VaR_{A,q,Vasicek} - 15 \cdot VaR_{C,q,Vasicek} + 15 \cdot VaR_{T,q,Analytic}.
 \end{aligned}$$

A)  $q=99\%$

$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
20	775.08	775.74	777	-0.16%
40	797.66	800.32	800	0.04%
100	865.38	882.95	880	0.34%
200	978.26	1055.48	1040	1.49%
400	1204.01	1578.16	1392	13.37%

B)  $q=99.9\%$

$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
20	1498.91	1499.89	1488	0.79%
40	1542.57	1546.53	1537	0.62 %
100	1673.54	1699.20	1674	1.51%
200	1891.83	2000.40	1937	3.3%
400	2328.40	2810.28	2560	9.8%

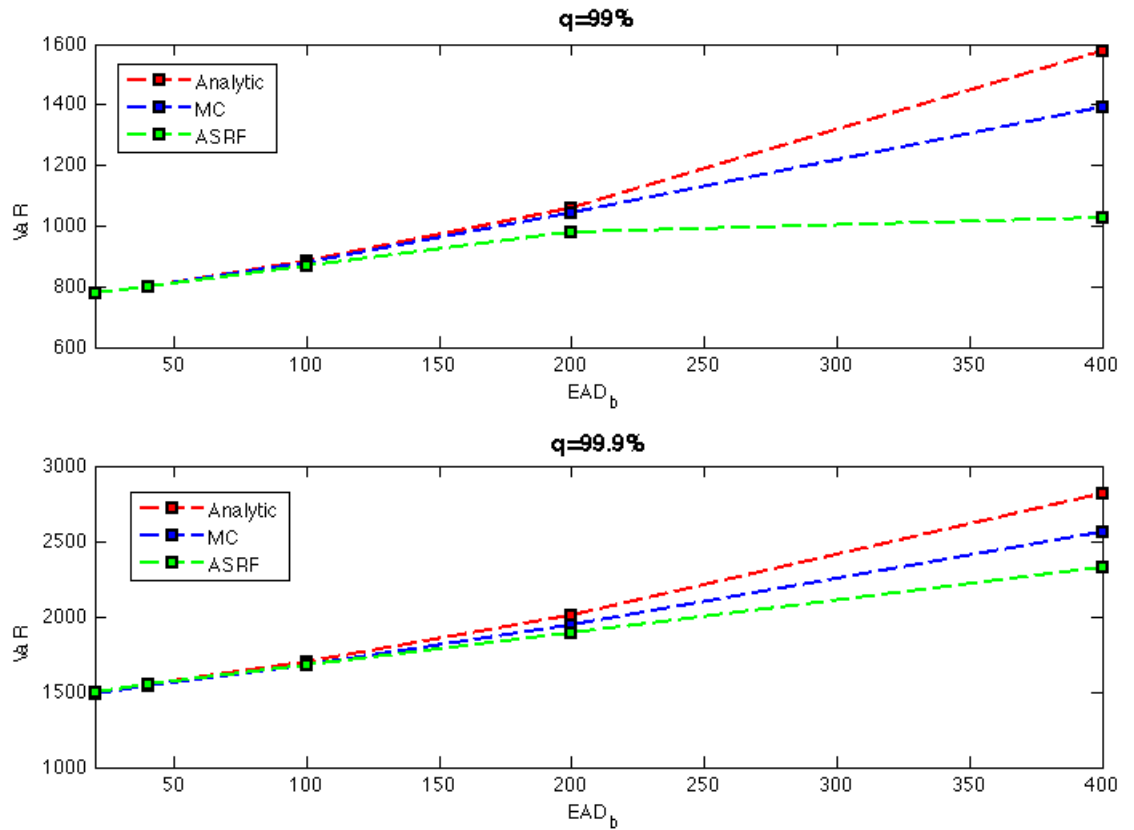


Figure 4.3: Effect on VaR of increasing EAD of 15 large counterparties.

The analytic solution matches MC results even for a ratio exposure of large counterparties about 13%.

The analytic solution performs better with a larger number of large counterparties in the portfolio.

This is due to a portfolio with many large counterparties is closer to a perfectly granular portfolio than the portfolio with one large counterparty. For example when considering  $M = 2, EAD_b = 100 \Rightarrow EAD_{total} = 10200$  and  $M = 10, EAD_b = 20 \Rightarrow EAD_{total} = 10200$  we have two portfolios built up from the test base portfolio and with the same exposure, but the first one is more concentrated (HHI=0.00029, G=0.0194) than the second one (HHI=0.00013, G=0.0186).

With the last three examples we can appreciate that as  $M$  increases the  $EC$  of the analytic solution increases. For this representative portfolio the analytic solution is sufficiently accurate.



## 5. Test portfolio with N 'small'

Portfolio 2 used for simulations

N	100	M	$EAD_b$	$r_e$	HHI	G
EAD	1	2	5	9.091%	0.0124	0.07130
PD	1.00 %	2	7	12.281%	0.0152	0.10320
LGD	100 %	2	10	16.667%	0.0208	0.14706
$\rho$	20%	2	12	19.335%	0.0252	0.17394
		10	5	33.333%	0.0156	0.24242
		10	7	41.176%	0.0204	0.32086
		10	10	50%	0.0275	0.40910

A)  $q=99\%$

$M$	$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
2	5	8.28	9.13	9	1.44%
2	7	8.58	10.56	10	5.30%
2	10	9.03	13.85	12	15.42%
2	12	9.33	16.66	14	19%
10	5	11.29	15.57	14	11.21%
10	7	12.79	22.72	17	33.65%
10	10	15.05	39.13	22	77.86%

In general we do not get accurate results by applying the analytical solution when adding large counterparties to a test portfolio composed by a small number of loans.

The method is specially suitable for a portfolio with a relatively large number of obligors.

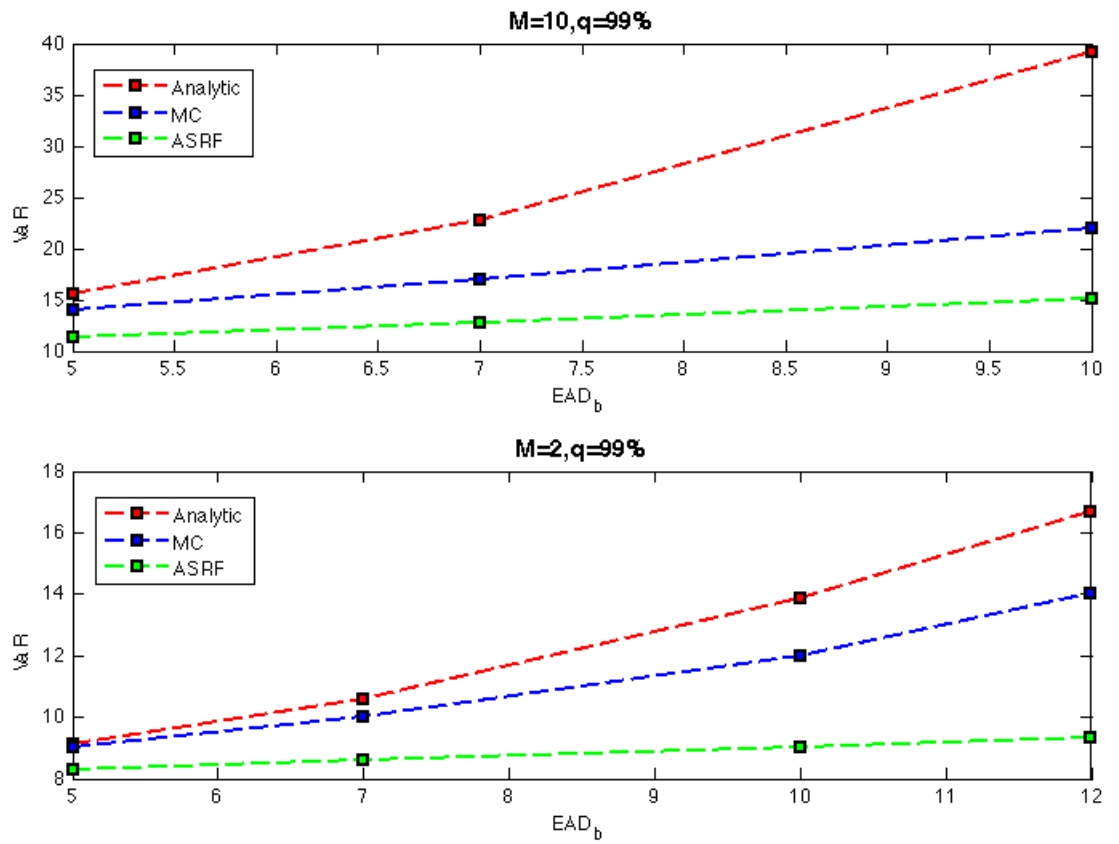


Figure 4.4: Effect of adding 2 and 10 large counterparties to portfolio 2.

## 6. Test portfolio with not 'small' exposures

Portfolio 3 used for simulations

N	1000
EAD	10
PD	1.00 %
LGD	100 %
$\rho$	20%

M	$EAD_b$	$r_e$	HHI	G
2	20	0.398%	0.00099	0.00199
2	50	0.990%	0.00103	0.00791
2	100	1.961%	0.00115	0.01761
2	150	2.913%	0.00137	0.02713
10	20	1.961%	0.00099	0.00971
10	50	4.762%	0.00113	0.03772
10	100	9.091%	0.00165	0.08101
10	150	13.043%	0.00246	0.12053

A)  $q=99\%$

M	$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
2	20	755.52	755.60	758	-0.317%
2	50	760.03	760.59	762	-0.185%
2	100	767.56	769.90	770	-0.013%
2	150	775.08	780.61	778	0.335%
10	20	767.56	767.99	770	-0.261%
10	50	790.13	792.93	792	0.117%
10	100	827.76	839.47	841	-0.182%
10	150	865.38	893.00	885	0.901%

We get a quite accurate result except when the ratio exposure of large counterparties to total portfolio is approximately bigger than 9%, the analytic solution starts to overestimate the VaR.

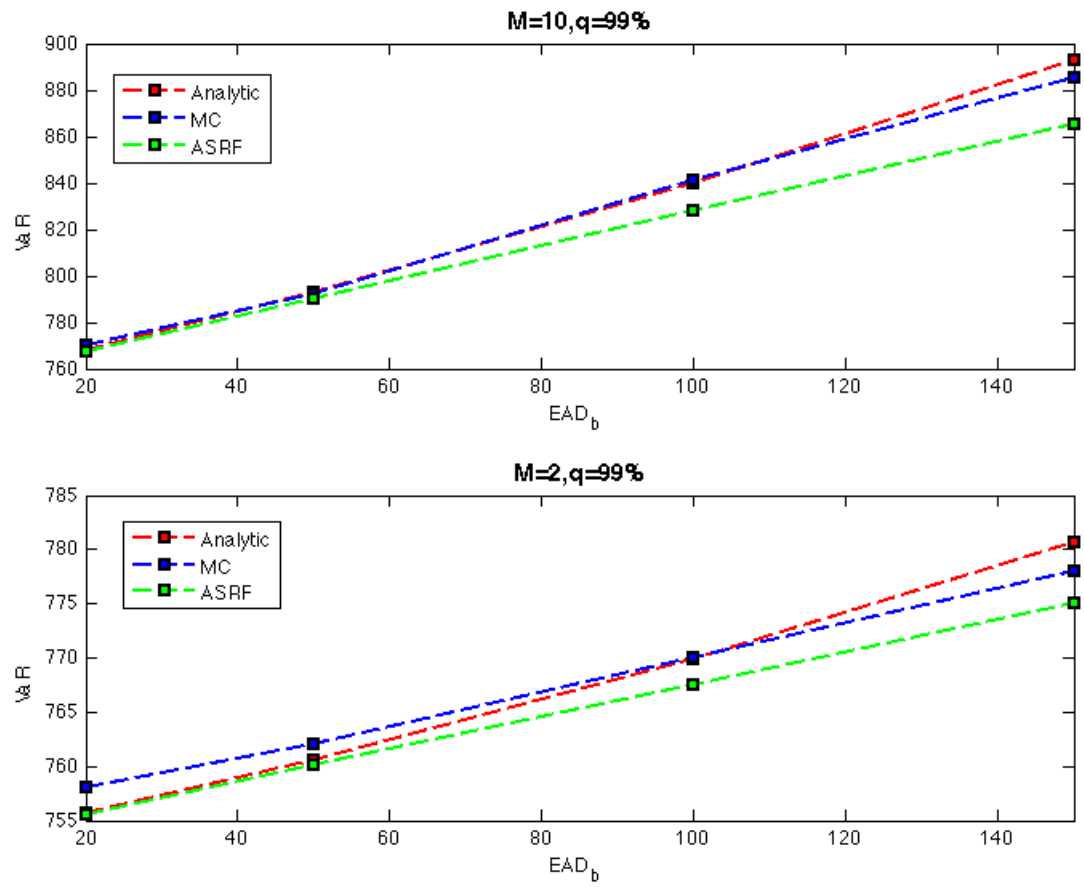
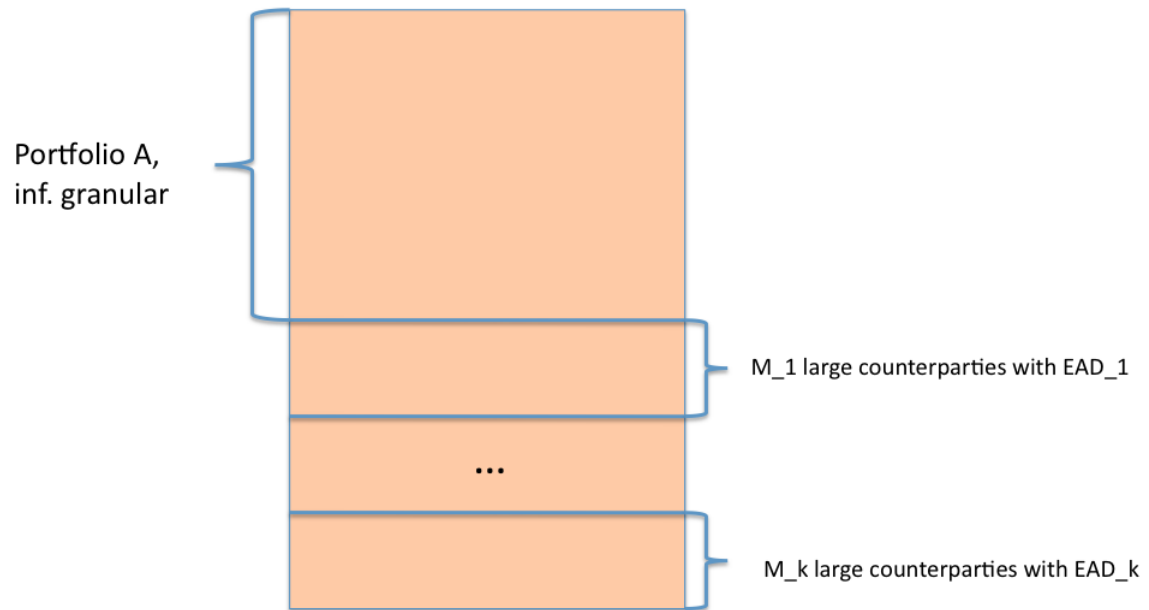


Figure 4.5: Effect of adding 2 and 10 large counterparties to portfolio 3.

## 7. Effect of adding large counterparties with different exposures



### Portfolio 4 used for simulations

In this case we will add many large counterparties with different exposures.

Exposure	1	10	50	100	500
# of obligors	10000	1000	200	100	20

It is a portfolio of so-called lower granularity since the largest obligor has an exposure 500 times larger than the smallest obligor.

To apply the analytical solution, we will quantify  $M_1$  times the contribution of a large counterparty with  $EAD_1$ ,  $M_2$  times the contribution of a large counterparty with  $EAD_2$ , etc.

Let the portfolio  $C_i, i \in \{1, \dots, k\}$  be the granular portfolio with a large added counterparty with exposure  $EAD_i$ .

$$\begin{aligned}
VaR_{Total,q} &= VaR_{A,q,Vasicek} + M_1 \cdot \Delta VaR_{1,q} + \dots + M_k \cdot \Delta VaR_{k,q} \\
&= VaR_{A,q,Vasicek} + M_1 \cdot (VaR_{T,q,Analytic} - VaR_{C_1,q,Vasicek}) \\
&\quad + \dots + M_k \cdot (VaR_{T,q,Analytic} - VaR_{C_k,q,Vasicek}).
\end{aligned}$$

$q$	VaR ASRF	VaR analytic	VaR MC	$\eta$
99%	3762.54	4802.19	3900	23.13%
99.9%	7276.26	8597.86	7500	14.64%

In this example the ratio exposure of large counterparties to total portfolio equals to 80%.

A first guess tells us that the approximation will not be very accurate since the analytical solution is expected to be an accurate approximation as long as the large counterparties are not too large compared with the total portfolio. In fact, computing it, we can check out that the analytical method is not a good choice to perform the VaR computation in this case.

The Vasicek model provides a much better approximation, even if it is not a really good one, than applying the probability correction.

## 8. The main portfolio is not a Vasicek one

One of the main assumptions when we applying the analytical method is that the base portfolio is a granular one.

Nevertheless, let consider a portfolio which is not infinitely granular. We will add some large counterparties to it and we will apply the analytical method.

N	10000	M	$EAD_b$	$r_e$	HHI	G
EAD	$i/N \forall i$	–	–	–	0.00013	0.33330
PD	1.00 %	10	50	4.762%	0.00094	0.33330
LGD	100 %	10	100	9.091%	0.00287	0.44313
$\rho$	20%	10	150	13.043%	0.00540	0.48589
		10	200	16.667%	0.00823	0.52254

Firstly, if we compute the VaR of this portfolio by applying the ASRF method and a Monte Carlo simulation we obtain:

$q$	VaR ASRF	VaR MC
99%	376.29	382
99.9%	727.70	733

Even if the test portfolio is not a granular one, we are not considering a highly concentrated portfolio and it is a case in which the Vasicek model provides a quite acceptable approach. It is logical to wonder about the performance of the analytical method in this scenario. Let add some large counterparties as in previous examples.

For small values of the EAD the analytical solution is a good approach to MC results.

When the total ratio of the large added counterparties is bigger approximately than a 4% the analytical method does not give us an accurate approximation.

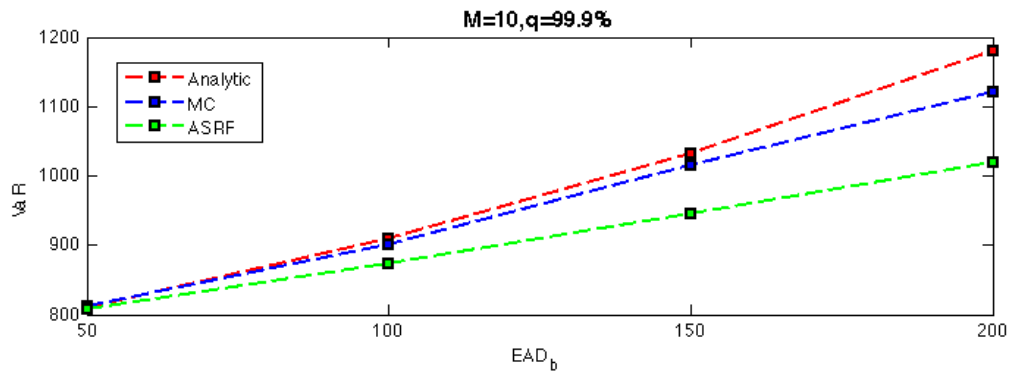
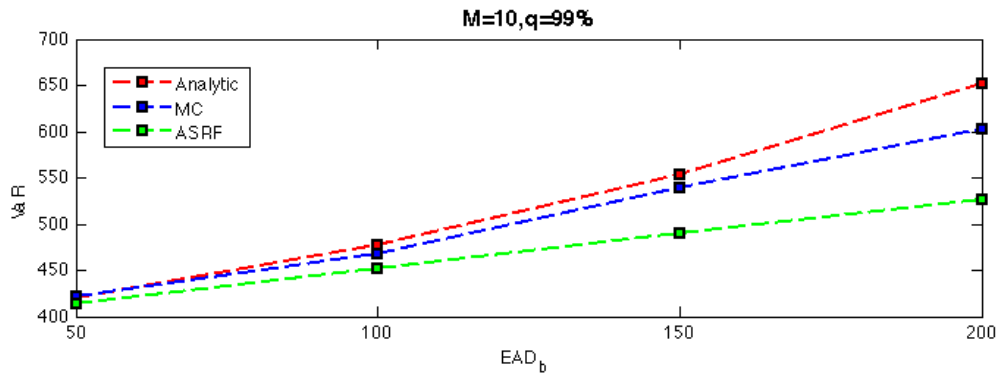
While the EAD gets bigger, the Vasicek method always matches much lower results than MC whereas the analytical solution overestimates the VaR.

Effect of increasing EAD of ten large counterparties ( $M=10$ )A)  $q=99\%$ 

$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
50	413.92	419.77	421	-0.292 %
100	451.54	477.28	468	1.983 %
150	489.17	552.90	539	2.579%
200	526.79	651.49	602	8.221%

B)  $q=99.9\%$ 

$EAD_b$	VaR ASRF	VaR analytic	VaR MC	$\eta$
50	808.46	809.01	811	-0.245 %
100	873.22	909.41	900	1.046 %
150	945.99	1031.90	1014	1.765%
200	1018.75	1179.36	1120	5.300%





## Computation times

Here we show the employed time by the three methods to obtain the VaR that we have performed in each example: the Asymptotic Single Risk Factor model, the analytical solution and Monte Carlo.

Each computation time is the average of all the trials with different number of counterparties and different EAD added to the test portfolio in each section.

We remind that while Monte Carlo simulations have been coded in **C**, both the ASRF and the analytical method have been run in **Matlab**.

Computation times are shown in seconds.

Example	ASRF	Analytic	MC time1	MC time2
2	0.00049	0.01488	471	4752
3	0.00035	0.01539	474	4773
4	0.00036	0.01572	477	4747
5	0.00034	0.02789	1	51
6	0.00037	0.01410	47	473
7	0.00049	0.08167	533	5722
8	0.08841	0.11506	463	4749

where

1. MC time1 is the time consuming while performing 100000 Monte Carlo simulations.
2. MC time2 is the time consuming while performing 1000000 Monte Carlo simulations.

While Monte Carlo simulation allows for detailed modeling at the individual exposure level, it comes with a substantial computational cost. This is particularly true as we are often interested in the far tail of the loss distribution, as well as sensitivity to portfolio parameters such as individual exposure size.

When computing VaR, for portfolios of the order of 10000 loans, we have used a Monte Carlo with 1000000 simulations to get an accurate result. In cases when we had  $N \sim 100$  and  $N \sim 10000$  we ran 100000 simulations to obtain the result. Anyway, we show all the computation times in all cases, and it becomes clear that depending on the number of simulations, time grows substantially.

**Note: Computer Hardware Specifications**

Mac OS X, Version 10.6.4

- Processor: 2.4 GHz Intel Core 2 Duo
- Memory: 4 GB 1067 MHz DDR3
- Startup Disk: Macintosh HD

**4.4 Implications for the EC and RAROC****EC with and without name concentration**

Here we present the effect of name concentration on the contribution of a counterparty to the portfolio EC.

The stylized portfolio from the previous section is again used. Its risk characteristics are summarized in the following table:

N	EAD	PD	LGD	$\rho$
10000	1	1.00 %	100 %	20 %

**Definition 20.** *VaR Contribution (VaRC).*

*Marginal VaR contribution of the EAD of a counterparty which measures how much each obligor contributes to the total VaR or EC of a portfolio.*

Ordinary Monte Carlo estimation is impractical for this problem because the conditional expectations defining the marginal risk contributions are conditioned on rare events. We will compute VaRC under the ASRF framework, and we will also be able to use the analytical method to measure it.

**Asymptotic Vasicek (ASRF) approximation**

One can calculate VaRC at the  $q$ -percentile due to an added counterparty  $b$  for an infinitely large portfolio without concentration as follows

$$VaRC_{b,q} = EAD_b \cdot LGD_b \cdot \Phi \left( \frac{\Phi^{-1}(PD_b) + \sqrt{\rho_b} \cdot \Phi^{-1}(q)}{\sqrt{1 - \rho_b}} \right).$$

**Analytic approximation**

The VaRC can be approximated by the incremental VaR given by

$$\Delta VaR_b = VaR_{Total,b} - VaR_{C,Vasicek},$$

for portfolios without large name concentrations since VaRC is the linear approximation. Thus,

$$VaRC_b \approx \Delta VaR_b.$$

This means that the analytic solution can be used to calculate the VaRC of large counterparties.

From now on, let consider **q=99%**.

Firstly, VaRC and EC are calculated using both Vasicek and the analytical framework for different values of a large added counterparty.

<i>EAD</i> large c.	VaRC ASRF	VaRC analytic	EC ASRF	EC analytic
20	1.51	1.55	1.31	1.35
40	3.01	3.19	2.61	2.79
60	4.52	4.92	3.92	4.32
80	6.02	6.76	5.22	5.96
100	7.53	8.70	6.53	7.69
150	11.29	14.05	9.79	12.55
200	15.05	20.19	13.05	18.19

The EC calculated using the Vasicek framework grows linearly with EAD. EC with name concentration taken into account has accelerating growth with respect to EAD.

$\Delta EAD$ large c.	$\Delta EC$ ASRF	$\Delta EC$ analytic
20	1.31	1.44
20	1.31	1.53
20	1.31	1.64
20	1.31	1.73
50	3.26	4.86
50	3.26	5.64

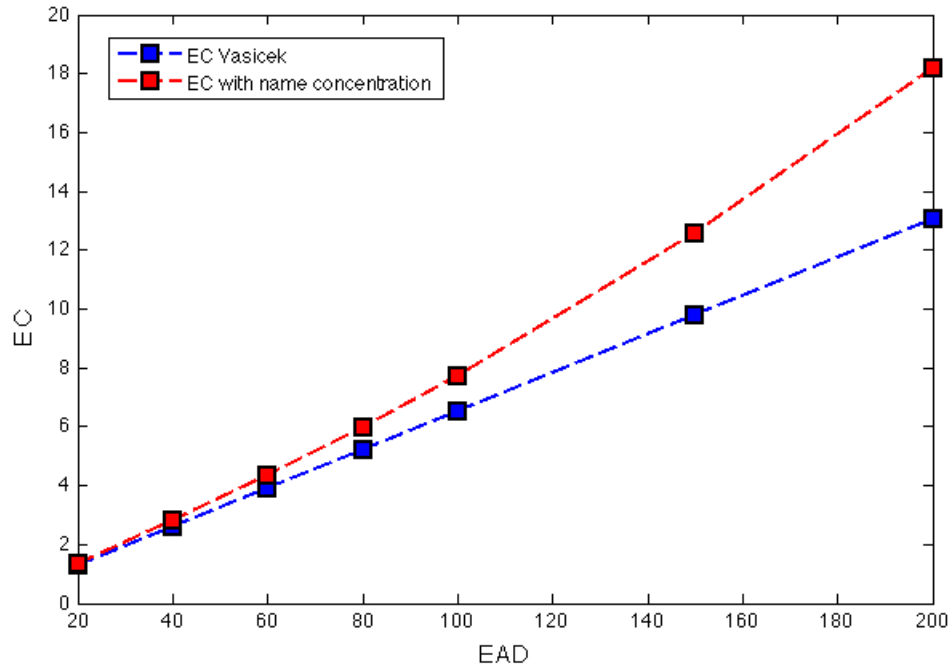


Figure 4.6: Effect of name concentration on EC of a large counterparty in the stylized portfolio.

## RAROC with and without name concentration

$$RAROC = \frac{\text{Earnings} - \text{Expected losses}}{\text{EC}}.$$

### Earnings. The hurdle rate

The earnings can be defined at various levels. Earnings can be limited to the net interest margin, excluding fees, or they can be defined as the net interest margin plus fees. They can be calculated both before and after operating costs. The minimum value of the required ratio obviously depends upon the perimeter of the earnings calculation.

**Definition 21.** *The hurdle rate is the minimum required return.*

The benchmark is the price of risk in the capital market. The hurdle rate should be set equal to the return required by shareholders, given the risk of

the stock. For example, if in the previous example the RAROC hurdle is set to 20%, this 20% benchmark applies to earnings net of all operating costs.

$$r(\%) = \text{Earnings}/\text{VaR} \geq 20\% \text{ or } \text{Earnings} \geq 20\% \cdot \text{VaR}.$$

### Expected losses

We quantify the *EL* due to a large counterparty according to

$$EL = EAD \cdot LGD \cdot PD.$$

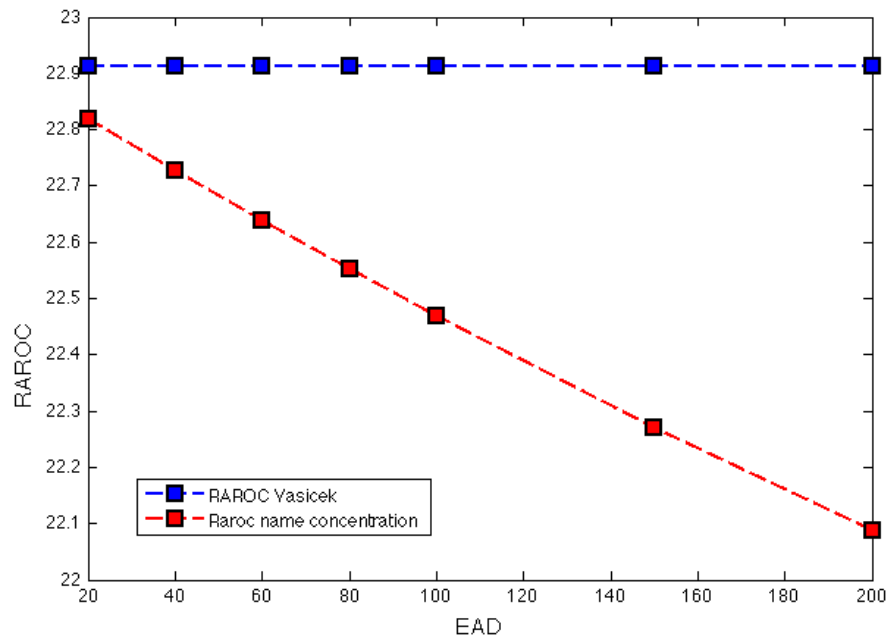
### Economic Capital

We will use the EC computed previously, both with name concentration (analytic) and without name concentration (Vasicek).

We apply it to the previous example. The RAROC is calculated for a large counterparty without name concentration taken into account, with the Vasicek framework, and with name concentration taken into account. This has been done for different values of EAD. The hurdle rate has been set to 20%.

<i>EAD</i> large c.	RAROC ASRF	RAROC analytic
20	22.9119	22.8179
40	22.9119	22.7267
60	22.9119	22.6381
80	22.9119	22.5521
100	22.9119	22.4688
150	22.9119	22.2710
200	22.9119	22.0881

The fact that VaRC with name concentration taken into account grows faster than the growth of the Vasicek approach means that RAROC decreases with EAD when name concentration is taken into account. This is illustrated in the next figure, which shows the effect of name concentration on RAROC of a large counterparty in the stylized portfolio.



# Conclusions

- The analytic solution allows the calculation of the VaR and EC contribution of large counterparties using deterministic formulas rather than a Monte Carlo approximation. The MC simulation contains a certain amount of stochastic noise, leading to an error interval around the calculated VaR. For certain applications, for example risk-based pricing, a stochastic VaR contribution is not desirable.
- We can obtain the name concentration formulas intuitively.
- The analytical approach is less technically and computationally involved than a full Monte Carlo approximation. Using a MC simulation involves separate evaluation of each individual counterparty, it is very computationally intensive, even with advanced Monte Carlo techniques. An interpreted programming language will get fast the analytical result.
- A potential disadvantage of the analytical solution is that it is only suitable under certain assumptions. It especially works better when we have an infinitely granular portfolio with relatively large number of obligors and relatively small exposures. However, when we are in this scenario the analytic solution matches the Monte Carlo results because the analytic solution delivers an exact number for a single large counterparty added to an infinitely granular portfolio.
- The brute force Monte Carlo simulation is mathematically less complicated than the analytical method.





# Appendix A

## Codes

### A.1 Monte Carlo Simulations in C

```
#include <stdio.h>
#include <stdlib.h>
#include <math.h>
#include <time.h>

#define IM1 2147483563
#define IM2 2147483399
#define AM (1.0/IM1)
#define IMM1 (IM1-1)
#define IA1 40014
#define IA2 40692
#define IQ1 53668
#define IQ2 52774
#define IR1 12211
#define IR2 3791
#define NTAB 32
#define NDIV (1+IMM1/NTAB)
#define EPS 1.2e-10
#define RNMX (1.0-EPS)
#define A1 (-3.969683028665376e+01)
#define A2 2.209460984245205e+02
#define A3 (-2.759285104469687e+02)
#define A4 1.383577518672690e+02
#define A5 (-3.066479806614716e+01)
#define A6 2.506628277459239e+00
```

```

#define B1 (-5.447609879822406e+01)
#define B2 1.615858368580409e+02
#define B3 (-1.556989798598866e+02)
#define B4 6.680131188771972e+01
#define B5 (-1.328068155288572e+01)
#define C1 (-7.784894002430293e-03)
#define C2 (-3.223964580411365e-01)
#define C3 (-2.400758277161838e+00)
#define C4 (-2.549732539343734e+00)
#define C5 4.374664141464968e+00
#define C6 2.938163982698783e+00
#define D1 7.784695709041462e-03
#define D2 3.224671290700398e-01
#define D3 2.445134137142996e+00
#define D4 3.754408661907416e+00
#define P_LOW 0.02425 /* P_high = 1 - p_low*/
#define P_HIGH 0.97575
#define SIM 1000000 /* Number of simulations */
#define N 10010 /* Number of loans in the portfolio */
#define m 1000 /* Number of intervals at the frequency graph */

/* Inverse cumulative normal distribution */
long double icdf(p)
long double p;
{
long double x;
long double q, r, u, e;
if ((0 < p ) && (p < P_LOW)){
q = sqrt(-2*log(p));
x = (((((C1*q+C2)*q+C3)*q+C4)*q+C5)*q+C6)/
(((D1*q+D2)*q+D3)*q+D4)*q+1);
}
else{
if ((P_LOW <= p) && (p <= P_HIGH)){
q = p - 0.5;
r = q*q;
x = (((((A1*r+A2)*r+A3)*r+A4)*r+A5)*r+A6)*q /
(((B1*r+B2)*r+B3)*r+B4)*r+B5)*r+1);
}
else{
if ((P_HIGH < p)&&(p < 1)){

```

```

        q = sqrt(-2*log(1-p));
        x = -((((C1*q+C2)*q+C3)*q+C4)*q+C5)*q+C6)/
        (((D1*q+D2)*q+D3)*q+D4)*q+1);
    }
}
return x;
}

/* L'Ecuyer's uniform random number generator with Bays-Durham shuffle */
double lecuyer(long *idum)
{  int j;
   long k;
   static long idum2 = 123456789;
   static long iy = 0;
   static long iv[NTAB];
   double temp;

   if (*idum <= 0){
       if (-(*idum) < 1) *idum = 1;
       else *idum = -(*idum);
       idum2 = *idum;
       for (j=NTAB+7;j>=0;j--){
k = *idum/IQ1;
*idum = IA1*( *idum-k*IQ1)-k*IR1;
if (*idum < 0) *idum +=IM1;
if (j < NTAB) iv[j] = *idum;
        }
        iy = iv[0];
}

k = *idum/IQ1;
*idum = IA1*( *idum-k*IQ1)-k*IR1;
if (*idum < 0) *idum +=IM1;
k = idum2/IQ2;
idum2 = IA2*(idum2-k*IQ2)-k*IR2;
if (idum2 < 0) idum2 += IM2;
j = iy/NDIV;
iy = iv[j]-idum2;
iv[j] = *idum;
if (iy < 1) iy += IMM1;

```

```

if ((temp=AM*iy) > RNMX) return RNMX;
else return temp;
}

double normdev(long *idum)
{ static int iset = 0;
  static double gset;
  double fac,rsq,v1,v2;

  if (iset == 0){
    do {
      v1 = 2.0*lecuyer(idum)-1.0;
      v2 = 2.0*lecuyer(idum)-1.0;
      rsq = v1*v1+v2*v2;
    } while (rsq >= 1.0 || rsq == 0);
    fac = sqrt(-2.0*log(rsq)/rsq);
    gset = v1*fac;
    iset = 1;
    return v2*fac;
  }
  else {
    iset = 0;
    return gset;
  }
}

/*MAIN ROUTINE*/
main()
{
  double rho,y,ep[N],T[N],r[N],loss;
  int i,j,D[N];
  long seed=0,seed2=1;
  double E[N],PD[N],Li[N];
  FILE *salida;
  clock_t timeMC_ini,timeMC_fin;
  double F[m][2];

  timeMC_ini=clock();

  for(i=0;i<N-10;i++) /* Elements in the granular portfolio */

```

```
{
  E[i]=1./10500;
  PD[i]=0.01;
  T[i]=icdf(PD[i]);
}

for(i=N-10;i<N;i++)
{
  E[i]=50./10500;
  PD[i]=0.01;
  T[i]=icdf(PD[i]);
}

rho=0.2; /* Correlation coefficient */

for(i=0;i<m;i++) F[i][0]=0;
for(j=0;j<SIM;j++)
{
  loss=0;
  y=normdev(&seed);
  for(i=0;i<N;i++)
  {
    ep[i]=normdev(&seed2);
    r[i]=sqrt(rho)*y+sqrt(1-rho)*ep[i];
    if(r[i]<T[i]) D[i]=1;
    else D[i]=0;
    Li[i]=E[i]*D[i];
    loss=loss+Li[i];
  }
  if(loss==0) F[0][0]+=1;
  else if(loss==1) F[m-1][0]+=1;
  else
  {
    for(i=0;i<m;i++)
      if(loss>=(1./m)*i && loss<(1./m)*(i+1)) F[i][0]+=1;
  }
}

for(i=0;i<m;i++) F[i][0]=F[i][0]/SIM;

F[0][1]=F[0][0];
```

```
for(i=1;i<m;i++)
    F[i][1]=F[i][0]+F[i-1][1];

timeMC_fin=clock();

salida = fopen("portfolioA_M10_EAD50.dat","w");
for(i=0;i<m;i++)
    fprintf(salida,"%lf;%lf;%lf;%.15lf\n",
        (1./m)*i,(1./m)*(i+1),F[i][0],F[i][1]);
fprintf(salida,"That took %d seconds .\n",
    (timeMC_fin-timeMC_ini)/CLOCKS_PER_SEC);
fclose(salida);
}
```

## Notes

- Depending on the number of simulations, the Monte Carlo approach involves large computations. As we will run many simulations, instead of using Matlab, since it is an interpreted language it can be unacceptably slow in cases like this, we code a routine in C in order to improve computational efficiency.
- We have predicted the loss distribution function of the portfolio, and also the cumulative distribution function.
- The results are normalized. Thus, to obtain the Value at Risk at a confidence level  $q$ , we will have to consider the correspondent value of loss, i.e. the value of the loss distribution which corresponds with a value of  $q$  of the CDF, times the exposure of the total portfolio.

## A.2 Analytic solution. Matlab code

```
function [VaR_Total,t,VaRC_analytic,RAROC_analytic,HHI,GI]
=Analytic_Solution(rho,EAD_A,EAD_b,LGD,PD,q,N,M)

%INPUT
    %rho: Correlation coefficient
    %EAD_A: Exposure of the counterparties in the base portfolio
    %EAD_b: Exposure of the added counterparties
    %LGD: Loss Given Default
    %PD: Probability at Default
    %q: Confidence level
    %N: Number of loans in the granular portfolio
    %M: Number of added counterparties
%OUTPUT
    %VaR_Total: VaR of the total portfolio applying the analytical method
    %t: Time to compute the total VaR
    %VaRC_analytic: VaR contribution of a large counterparty
    %RAROC_analytic: RAROC considering name concentration
    %HHI: Herfindahl-Hirschman index
    %GI: Gini index

tic

VaR_A=N*EAD_A*normcdf((norminv(PD)+sqrt(rho)*norminv(q))/sqrt(1-rho));
VaR_C=(N*EAD_A+EAD_b)*normcdf((norminv(PD)+sqrt(rho)*norminv(q))/sqrt(1-rho));

G=@(y)normcdf((norminv(0.01)-sqrt(rho).*y)/sqrt(1-rho)).*\exp(-(y.^2)/2)/sqrt(2*pi);

H=@(l)normcdf((sqrt(1-rho)*norminv(1/(N*EAD_A+EAD_b))-norminv(PD))/sqrt(rho))
-quad(G,-(sqrt(1-rho)*norminv(1/(N*EAD_A+EAD_b))
-norminv(PD))/sqrt(rho),-(sqrt(1-rho)*norminv((1-LGD*EAD_b)/(N*EAD_A+EAD_b))
-norminv(PD))/sqrt(rho))-q;

l=fzero(H,VaR_C);

VaR_analytic=l;
VaR_Total=VaR_A+ M*(VaR_analytic-VaR_C);

t=toc;
```

```

VaRC_analytic=1-VaR_C;
EC=VaRC_analytic-EAD_b*PD*LGD;
RAROC_analytic=(20*VaRC_analytic-EAD_b*PD*LGD)/EC;

EAD_t=N*EAD_A+M*EAD_b;
HHI=(N*EAD_A^2+M*EAD_b^2)/(EAD_t^2);
for i=1:N
    Gini(i)=(2*i-1)*EAD_A/EAD_t;
end
if M>0
    for j=N+1:N+M
        Gini(j)=(2*j-1)*EAD_b/EAD_t;
    end
end
GI=sum(Gini)/(N+M)-1;

```

### Notes

- The user has to introduce the risk parameters and also the number of large added counterparties and their exposure. The function computes the Value at Risk at a some confidence level  $q$  according to the analytical approach explained previously. VaR Contribution and RAROC are also computed, we will compare them with the results obtained in case of not considering name concentration. In order to have an initial guess about how concentrated the portfolio is, HHI and Gini indices are computed as well.
- We have used specifically inbuilt Matlab functions which allow us to perform numerical calculations without the need for complicated and time consuming programming.
- To test the accuracy of this solution, obtained results will be compared with a full Monte Carlo simulation. The Vasicek model <sup>1</sup> will be applied too since it is currently one of the most widely used tools to measure VaR and EC.

---

<sup>1</sup>A Matlab code, which will be shown later, is used.



## A.3 ASRF model in Matlab

```

function [VaR,t,VaRC_vasicek,RAROC_vasicek,HHI,GI]
=ASRF_Solution(rho,EAD_A,EAD_b,LGD,PD,q,N,M)
%INPUT
    %rho: Correlation coefficient
    %EAD_A: Exposure of the counterparties in the base portfolio
    %EAD_b: Exposure of the added counterparties
    %LGD: Loss Given Default
    %PD: Probability at Default
    %q: Confidence level
    %N: Number of loans in the granular portfolio
    %M: Number of added counterparties
%OUTPUT
    %VaR: VaR of the total portfolio applying the analytical method
    %t: Time to compute the total VaR
    %VaRC_vasicek: VaR contribution of a large counterparty
    %RAROC_vasicek: RAROC without considering name concentration
    %HHI: Herfindahl-Hirschman index
    %GI: Gini index
tic
VaR=(N*EAD_A+M*EAD_b)*normcdf((norminv(PD)+sqrt(rho)*norminv(q))/sqrt(1-rho));
t=toc;

%VaRC and RAROC of a large counterparty without name concentration
VaRC_vasicek=EAD_b*normcdf((norminv(PD)+sqrt(rho)*norminv(q))/sqrt(1-rho));
EC=VaRC_vasicek-EAD_b*PD*LGD;
RAROC_vasicek=(20*VaRC_vasicek-EAD_b*PD*LGD)/EC;

EAD_t=N*EAD_A+M*EAD_b;
HHI=(N*EAD_A^2+M*EAD_b^2)/(EAD_t^2);
for i=1:N
    Gini(i)=(2*i-1)*EAD_A/EAD_t;
end
if M>0
    for j=N+1:N+M
        Gini(j)=(2*j-1)*EAD_b/EAD_t;
    end
end
GI=sum(Gini)/(N+M)-1;

```



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