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Entanglement and non-locality of pure quantum states

Ana Belén Sainz

Supervised by Dr. Antonio Acín, (ICFO)

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Ana Belén Sainz

ICFO - The Institute of Photonic Sciences, Mediterranean Technology Park, Av. del Canal Olimpic, 08860 Castelldefels (Barcelona), SPAIN

E-mail: `belen.sainz@icfo.es`

Abstract. We study the process of majorisation and interconversion of bipartite states, and apply the formalism for analysing the local and non-local content of pure bipartite entangled states. We proved that the states which are majorised by the singlet are fully non-local. For that we introduce a particular chained Bell inequality and the corresponding set of measuremets for its violation.

1. Introduction

One of the most remarkable aspects of quantum mechanics is the strength of its predicted correlations, for they have no classical analogue. Indeed, since the 1930s, entanglement has been recognised as one of the main features of the quantum description of nature [1, 2]. The interest in quantum correlations has increased during the last two decades due to the emerging field of quantum information science, in which entanglement is the key ingredient for many applications (see [3]). This is a consequence of a "conceptual revolution": *exploit "quantum strangeness" to perform tasks that are classically impossible.*

On the other hand, it is well known that correlations between bipartite pure entangled states may not be ascribed to "shared randomness" [4] (local correlations), a phenomenon called *quantum nonlocality.* A natural question is: what happens in a experiment with several pairs of entangled subsystems? Can one consider that only some pairs behave non-locally, while the others would give rise to purely local correlations? Such a quantification of the non-local resources of a state should provide a detailed account of which tasks can be accomplished with it.

Elitzur, Popescu and Rohrlich were the first to address this question, proving that each pair in the ensemble behaves non-locally when the particles are spins coupled in a singlet state [5]. Later, Barrett, Kent and Pironio (BKP) generalised this result to arbitrary dimension and proved that the maximally entangled state of two d-dimensional quantum systems has no local component [6].

In the last decade, the emerging connection between majorisation theory and entanglement manipulation led to well known results in bipartite entanglement manipulations [7]. In this work we will use these results and those of Barrett, Kent and Pironio to prove that those bipartite entangled states which can be deterministically transformed by local operations and classical communication into a singlet (2-maximally entangled state) have only non-local component, thus generalising the result of BKP.

This work is divided in four main parts. In Section 2 the fundamental concepts of the field are presented: entanglement and non-locality. Section 3 first shows the main points of BKP work, and then presents the basics of majorisation theory with focus on bipartite pure state transformations and protocols.

In Section 4 we apply majorisation theory to our particular quantum states, and present the chained Bell Inequality with which the local and non-local parts of the state are tested. Finally, in Section 5 we sumarise the results and present future directions of work.

2. Concepts and tools.

2.1. Entanglement

Entanglement is an important feature of Quantum Mechanics and Quantum Information Theory. It was first noticed by Einstein, Podolsky and Rosen, in their famous EPR paradox [1], and was considered one of the characteristic things that appear in the quantum world without classical analog.

How to define entanglement is a complex task, which strongly depends on its different uses [8]. From "mathematical" basis, one can define a pure entangled state of a composite system as "a pure state that cannot be written as a product state" [9]. A bipartite pure state $|\psi\rangle_{AB} \in \mathcal{H}_A \otimes \mathcal{H}_B$ can always be expressed as

$$|\psi\rangle_{AB} = \sum_i \alpha_i |i\rangle_A |i\rangle_B \,, \tag{1}$$

where $\{|i\rangle_A\}$ and $\{|i\rangle_B\}$ are particular orthonormal basis of each Hilbert space $\mathcal{H}_{A,B}$. That is called the *Schmidt decomposition* of the state, and α_i are the *Schmidt coeficients*. For practical purposes, we will label them in decreasing order, such that $\alpha_j \geq \alpha_{j+1}$. If the state has only one non-zero Schmidt coeficient, it is a product state, otherwise it is entangled. [3].

From an operational point of view, entanglement can also be defined as follows. A bipartite state shared by two parties (Alice and Bob) is product if it can be created by both parties using only LOCC: "Local Operations and Classical Communication". Thus, if a bipartite state can not be created using only LOCC, it is called entangled [8]. This recent approach to "entanglement" highlights its importance as a resource, that can be useful for achieving tasks such as dense coding [10] and teleportation [11].

As a final remark, in this work we will constrain the set of operations to LOCC, based on technological and fundamental motivations, for this restriction arises naturally in many physical settings involving spatially-separated quantum communication [8].

2.2. Non Locality

As entanglement and non-locality are different resources [12], I will, in this part, describe general aspects of the latter. When correlations at distance are considered, the following generic setup is used [9]. A source emits a particle to each of two distant locations. In each one, a user Alice (Bob) chooses and performs a measurement A (B) and registers the outcome a (b). After repeating this procedure many times, they communicate, compare their results, and derive the probability distribution $P(a, b|A, B)$. The question that arises is: can these correlations be ascribed to a classical mechanism, such as a pre-established stategy? John Bell, in 1964 [4] proved that the whole family of probabilities predicted by quantum physics cannot. This interesting feature has some valuable applications, in fields such as communication complexity [13], secret key distribution [14] and randomness generation [15].

In a Local Hidden Variables model (LHV, also called local realism, *i.e.* classical correlations), there is some classical information λ shared by the particles, and thus the probability distribution takes the form

$$P(a, b|A, B) = \int d\lambda\, \rho(\lambda)\, P(a|A, \lambda)\, P(b|B, \lambda)\,, \tag{2}$$

where $\rho(\lambda)$ is the distribution from where λ is drawn. When this is not the case, $P(a, b|A, B)$ is said to form a "non-local distribution". Therefore, local variable theories cannot exhibit arbitrary correlations. The constraints these must obey can always be written in the form of inequalities (generically referred to as Bell inequalities, due to his pioneer work [4]) which must be satisfied by linear combination of elements of this distribution (see [16]). Examples of such inequalities are by Bell [4], Clauser, Horne, Shimony and Holt [17], and Collins *et al.* [18].

Throughout this work, I will study probability distributions compatible with the non-signaling principle (NS). A distribution is said to be non-signaling if the marginal distributions are well defined, *i.e.* if the probability that Alice obtains a given she measures A does not depend on Bob's masurement. Mathematically, a distribution is non-signalling *if and only if*

$$P(a|A) = \sum_b P(a, b|A, B) = \sum_b P(a, b|A, B')\,. \tag{3}$$

This is a case study of many background works, and its mathematical description is well established [9].

As a final remark, I will review the EPR2 approach [5], which consist on decomposing a non-signaling probability distribution as the convex sum of a local (P_L) and a non-local (P_{NL}) distributions, as follows:

$$P_{NS}(r_A, r_B|A, B) = p_L\, P_L(r_A, r_B|A, B) + [1 - p_L]\, P_{NL}(r_A, r_B|A, B)\,. \tag{4}$$

The equality must hold for all outcomes r_a, r_b and possible measurements A, B, and the weight $p_L \in [0, 1]$ of the local component is required to be independent of the measurements and the outcomes. In the particular case of a quantum state ρ, the probability distribution of all possible measurements may be written as (4) with $p_L = p_L(\rho)$. Finally, in order to make this decomposition unique, P_L and P_{NL} are require to maximise the weight of the local component. Thus, the state ρ is local if and only if $p_L(\rho) = 1$.

3. Previous works

3.1. Non-Local States

In 2006, Barrett, Kent and Pironio proved, by introducing a chained Bell inequality, that the maximally entangled state of two d-dimensional quantum systems has no local component [6]. In this section I will review their work.

In this setting, Alice and Bob each have a choice among measurements: $A_1, \cdots, A_N$ and $B_1, \cdots, B_N$. The chained Bell inequality is:

$$I_N = \langle [A_1 - B_1] \rangle + \langle [B_1 - A_2] \rangle + \cdots + \langle [A_N - B_N] \rangle + \langle [B_N - A_1 - 1] \rangle \geq d - 1 \quad (5)$$

where $\langle X \rangle = \sum_{i=1}^{d-1} i\, P(X = i)$, and $[X]$ denotes X modulo d.

Barrett *et al* proved that, if Alice and Bob share the maximally entangled state $|\psi_d\rangle = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} |q\rangle_A |q\rangle_B$, there exist measurement settings such that $I_N(QM) \to 0$ for large N. Here $I_N(QM)$ indicates that the probabilities computed for I_N have their origin in quantum mechanics.

The set of measurements is defined as:

- Alice: eigenvectors characterising A_k:

$$|r\rangle_{A_k} = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} \exp\left[\frac{2\pi i}{d}\, q\,(r - \alpha_k)\right] |q\rangle_A \,. \quad (6)$$

- Bob: eigenvectors characterising B_k:

$$|r\rangle_{B_k} = \frac{1}{\sqrt{d}} \sum_{q=0}^{d-1} \exp\left[-\frac{2\pi i}{d}\, q\,(r - \beta_k)\right] |q\rangle_B \,. \quad (7)$$

where $\alpha_k = (k - \frac{1}{2})/N$, $\beta_k = k/N$, and the eigenvectors $|r\rangle$ have eigenvalue $r = 0, \cdots, d-1$. With these settings and $|\psi_d\rangle$, it follows that $I_N(QM) \propto N^{-1} + O(N^{-2})$, which can be made arbitrarily small for sufficiently large N.

Finally, from eq. (4) I_N may be written as $I_N(QM) = p\, I_N(L) + (1-p)\, I_N(NL)$, where $I_N(L)$ is obtained from P_L, and $I_N(NL)$ from P_{NL}. The local component of the distribution cannot violate the Bell inequality (5), thus $I_N(L) \geq d - 1$. However, the non-local part must only satisfy $I_N(NL) \geq 0$, which implies $p \leq \frac{I_N(QM)}{d-1}$. It follows that $p = 0$ and the probability distribution that arises from $|\psi_d\rangle$ is fully non-local.

3.2. Majorisation and bipartite states transformation

What does it mean to say that a given probability distribution is more disordered than another? This question also arises in quantum theory when talking abouth quantum states: when is a state more disordered than another? Majorisation is a mathematical tool that was developed to answer these questions, and gives a means for comparing in an elegant way two probability distributions or two density matrices [7]. Another interesting topic is interconversion of bipartite states. Suppose Alice and Bob share a quantum state ψ. A natural question is: into what class of states ϕ may ψ be transformed, assuming that both parties can only use local operations on their respective systems, and unlimited classical communication? Majorisation also answers this.

There exist two equivalent definitions for "majorisation":

1. The d-dimensional real vector r is *majorised* by the d-dimensional real vector s, written $r \prec s$, if there exists a set of d-dimensional permutation matrices P_j and a probability distribution $\{p_j\}$ such that $r = \sum_j p_j\, P_j\, s$.
2. $r \prec s$ if and only if:

$$\begin{aligned} r_1^{\downarrow} &\leq s_1^{\downarrow} \\ r_1^{\downarrow} + r_2^{\downarrow} &\leq s_1^{\downarrow} + s_2^{\downarrow} \\ &\vdots \\ r_1^{\downarrow} + \cdots + r_{d-1}^{\downarrow} &\leq s_1^{\downarrow} + \cdots + s_{d-1}^{\downarrow} \\ r_1^{\downarrow} + \cdots + r_d^{\downarrow} &= s_1^{\downarrow} + \cdots + s_d^{\downarrow} \end{aligned} \tag{8}$$

   The vector $r^{\downarrow} = (r_1^{\downarrow}, \cdots, r_d^{\downarrow})$ has the same components as r but rewritten in decreasing order, *i.e.* $r_1^{\downarrow} \geq r_2^{\downarrow} \geq \cdots \geq r_d^{\downarrow}$. The same holds for $s^{\downarrow}$ and s.

Definition (1) is intuitive and useful when proving theoretical results, and (2) is more convenient for actual calculations.

In relation to the interconversion of bipartite states under LOCC, majorisation theory has some results about necessary and sufficient conditions for such transformations to be feasible. In what follows we will focus on pure quantum states of a composite system consisting of two spatially separated parts A and B.

Let $\psi = \sum_{i=1}^{d} \sqrt{\lambda_i}\, |i\rangle_A\, |i\rangle_B$ be the state of the system, and let us define $\lambda(\psi)$ as the vector whose components $\lambda(\psi)_j = \lambda_j$ are the square of the Schmidt coeficients (1). The following theorem, due to Nielsen [19], refers to a local deterministic conversion:

Theorem 1: *State ψ can be converted into state ϕ by means of LOCC if, and only if,*

$$\lambda(\psi) \prec \lambda(\phi)$$

Now we are only left with the problem of finding a "protocol" to perform the transformation. There are many studies tackling this question, leading to several procedures which are helpful in different scenarios.

Lo and Popescu [20] explained a detailed operational protocol in which, in a proven finite number of steps, Alice and Bob may transform state $|\psi\rangle$ of dimension d into a maximally entangled state of dimension $m < d$. They also proved that, as far as entanglement manipulations are concerned, there is always a way to reduce a sequencial two-way communication protocol into a strategy involving only a single (generalised) measurement by Alice, followed by the one way communication of the result from Alice to Bob, and finally a local unitary transformation by Bob. Moreover, Nielsen and Vidal [7] found a set of measurement operators and a corresponding protocol given $|\psi\rangle$ and $|\phi\rangle$.

As a final remark, when talking about entanglement transformations, the measurements performed are "generalised measurements", *i.e.* projective measurements in a larger Hilbert space of "system $\times$ ancilla" (see [3, 21]).

4. Testing non-local states

In their work, Barrett, Kent and Pironio [6] studied the local and non-local components of a maximally entangled state of arbitrary dimension. The aim of this work is to study the case for more general states: those entangled states which can be transformed deterministically by LOCC to a 2-dimension maximally entangled state. First we will discuss the general process of conversion, and then we will focus on the violation of a Bell inequality in two different scenarios.

4.1. Transformation.

Let us define as our initial state $|\psi\rangle = \sum_{i=0}^{d-1} \sqrt{\lambda_i}\,|ii\rangle$, where $|ij\rangle = |i\rangle_A\,|j\rangle_B$. If we want to transform $|\psi\rangle$ into $|\phi_2\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$, the majorisation condition states:

$$\begin{array}{rcl} \lambda_0 & \leq & \frac{1}{2} \\ \lambda_0+\lambda_1 & \leq & \frac{1}{2}+\frac{1}{2}=1 \\ & \vdots & \\ \lambda_0+\cdots+\lambda_k & \leq & 1 \\ \lambda_0+\cdots+\lambda_{d-1} & = & 1 \end{array} \tag{9}$$

As normalisation requires $\sum_{i=0}^{d-1} \lambda_i = 1$ and $\lambda_i \geq 0 \quad \forall i$, then the only condition for the transformation to be feasible is $\lambda_0 \leq 1/2$.

Now let us focus on the measurements involved in the transformation. Based on Lo and Popescu's work, we know that there exist a finite set of measurements for Alice $\{M_j\}$ and a corresponding set of unitaries for Bob $\{U_j\}$, such that $M_j\,U_j\,\psi\,U_j^\dagger\,M_j^\dagger = p_j\,\phi_2$. The factor p_j states the probability for Alice's majorisation outcome $r = j$ to occur. Regarding notation, $\psi = |\psi\rangle\langle\psi|$ and $\phi_2 = |\phi_2\rangle\langle\phi_2|$.

For an example of a set of this operators in arbitrary dimension, see the work by Nielsen and Vidal, section (4.3) [7]. Here I will comment on the qutrit case.

Example: qutrits

In this particular problem, the initial state is $|\psi_3\rangle = \sqrt{\lambda_0}|00\rangle + \sqrt{\lambda_1}|11\rangle + \sqrt{\lambda_2}|22\rangle$, where $\lambda_0 \leq 1/2$. The set of measurements for Alice is:

$$\begin{aligned} M_1 &= k_1\left(\sqrt{\lambda_1}|00\rangle\langle 00| + \sqrt{\lambda_0}|11\rangle\langle 11|\right)\,, \\ M_2 &= k_2\left(\sqrt{\lambda_2}|11\rangle\langle 11| + \sqrt{\lambda_1}|22\rangle\langle 22|\right)\,, \\ M_3 &= k_3\left(\sqrt{\lambda_2}|00\rangle\langle 00| + \sqrt{\lambda_0}|22\rangle\langle 22|\right)\,, \end{aligned} \tag{10}$$

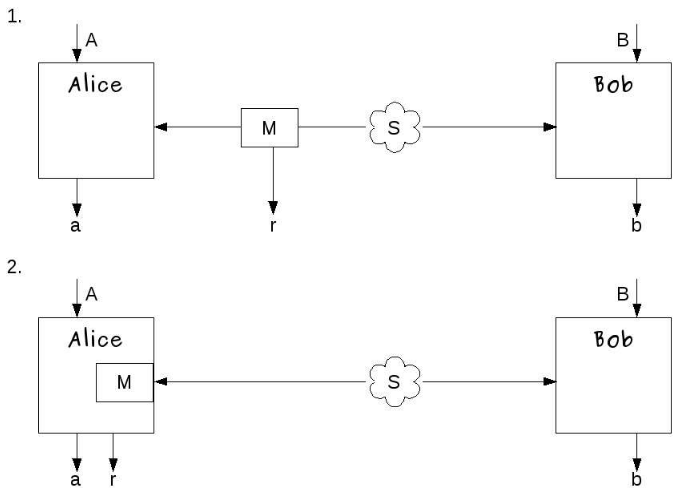

Figure 1. Two scenarios for Alice and Bob: Alice's majorisation (M) and BKP (A) measurements are performed separated (1) or very close (2).

and the corresponding unitaries for Bob are:

$$\begin{aligned} U_1 &= I\,, \\ U_2 &= |00\rangle\langle 22| + |11\rangle\langle 11| + |22\rangle\langle 00|\,, \\ U_3 &= |00\rangle\langle 00| + |22\rangle\langle 11| + |11\rangle\langle 22|\,. \end{aligned} \tag{11}$$

As $\{M_j\}$ define a quantum measurement, they must satisfy the condition $\sum_{j=1}^{3} M_j\, M_j^\dagger = I$. Thus, the following values for the constants k_j are found:

$$k_1^2 = \frac{\lambda_0 + \lambda_1 - \lambda_2}{2\,\lambda_0\,\lambda_1},\; k_2^2 = \frac{\lambda_1 + \lambda_2 - \lambda_0}{2\,\lambda_1\,\lambda_2},\; k_3^2 = \frac{\lambda_0 + \lambda_2 - \lambda_1}{2\,\lambda_0\,\lambda_2}. \tag{12}$$

Majorisation condition ensures that k_j are real.

4.2. Bell Inequality.

In this section the local and non-local components of $|\psi\rangle$ will be studied by means of a Bell inequality, with a scope similar to that of Barrett, Kent and Pironio [6].

The procedure is the following: a source sends one quantum particle to Alice and another to Bob, in the entangled state $|\psi\rangle$ which majorises the singlet. Then, Alice performs the majorisation measurement on her particle. After that, Alice and Bob choose each a local measurement to perform on their respective systems. Bob lists his inputs and outputs, and Alice lists her inputs, outputs and majorisation results. This last item correspond to the information she should classically send to Bob, in order for him to perform the unitary operation and thus obtain the singlet $|\phi_2\rangle$. However, in this setting, Bob is not capable of knowing the majorisation result before performing his measurements. Finally, they compare their lists, and compute some probabilities which I will coment later.

There are two possible scenarios (see fig. 1):

1. The majorisation measurements are performed before Alice decides which Bell observable to measure, and thus the former (and their results) do not depend on the latter.
2. Majorisation and Alice measurement are performed together, and thus their outcomes may be related.

In this work, we will study the first case. Let us suppose the case of n majorisation measurement M_j, and recall the N operators A_k and B_k defined in section (3.1). In this setup we thus define Alice's and Bob's sets as:

- Alice may choose among N measurements $\{A_k\}$.
- Bob may choose among $N \times n$ measurements defined as $\tilde{B}_{kj} = U_j^\dagger \, B_k \, U_j$, where U_j are the "majorisation unitaries".

Both Alice and Bob measurements have thus two outputs.

The inequality that will be used is:

$$\begin{aligned}\tilde{I}(N,n) = {} & P(a_1 \neq b_{11}, r=1) + P(a_2 \neq b_{11}, r=1) + P(a_2 \neq b_{21}, r=1) + \\ & \cdots + P(a_N \neq b_{N1}, r=1) + P(a_1 = b_{N1}, r=1) + \\ & P(a_1 \neq b_{12}, r=2) + P(a_2 \neq b_{12}, r=2) + P(a_2 \neq b_{22}, r=2) + \\ & \cdots + P(a_N \neq b_{N2}, r=2) + P(a_1 = b_{N2}, r=2) + \\ & \vdots \\ & P(a_1 \neq b_{1n}, r=n) + P(a_2 \neq b_{1n}, r=n) + P(a_2 \neq b_{2n}, r=n) + \\ & \cdots + P(a_N \neq b_{Nn}, r=n) + P(a_1 = b_{Nn}, r=n). \end{aligned} \tag{13}$$

Here the notation means: $P(a_k \neq b_{li}, r=i)$ is the probability that Alice and Bob obtain different outcomes and that the result r of the majorisation is $r=i$, given they have measured A_k, $\tilde{B}_{li}$, .

Following the ideas of section (3.1), let us check that $\tilde{I}^{QM}(N,n) \to 0$ when $N \to \infty$. For that purpose let us consider only one block of $\tilde{I}$, *i.e.* a given fixed r:

$$\begin{aligned}\tilde{I}_{N,j}(QM) = {} & P(a_1 \neq b_{1j}, r=j) + P(a_2 \neq b_{1j}, r=j) + P(a_2 \neq b_{2j}, r=j) + \\ & \cdots + P(a_N \neq b_{Nj}, r=j) + P(a_1 = b_{Nj}, r=j) \end{aligned} \tag{14}$$

Let us analyse each term on the right hand side of (14).

$$\begin{aligned} P(a_k \neq b_{lj}, r=j) &= P(a_k = 0, b_{lj} = 1, r=j) + P(a_k = 1, b_{lj} = 0, r=j) \\ &= Tr\left\{ |0\rangle\langle 0|_{A_k} \, U_j^\dagger \, |1\rangle\langle 1|_{B_l} \, U_j \, M_j \psi M_j^\dagger \right\} + \\ &\quad Tr\left\{ |1\rangle\langle 1|_{A_k} \, U_j^\dagger \, |0\rangle\langle 0|_{B_l} \, U_j \, M_j \psi M_j^\dagger \right\} \\ &= Tr\left\{ |0\rangle\langle 0|_{A_k} \, |1\rangle\langle 1|_{B_l} \, U_j M_j \psi M_j^\dagger U_j^\dagger \right\} + \\ &\quad Tr\left\{ |1\rangle\langle 1|_{A_k} \, |0\rangle\langle 0|_{B_l} \, U_j M_j \psi M_j^\dagger U_j^\dagger \right\} \\ &= Tr\left\{ |0\rangle\langle 0|_{A_k} \, |1\rangle\langle 1|_{B_l} \, p_j \, \phi_2 \right\} + \\ &\quad Tr\left\{ |1\rangle\langle 1|_{A_k} \, |0\rangle\langle 0|_{B_l} \, p_j \, \phi_2 \right\} \\ &= p_j \, \langle [A_k - B_l] \rangle \end{aligned} \tag{15}$$

where we've used the explicit form of the measurement eigenvectors and fact that, for the BKP two outcome case, $\langle [A_k - B_l] \rangle = P([A_k - B_l] = 1) = P(A_k \neq B_l)$.

Thus:

$$\frac{\tilde{I}_{N,j}(QM)}{p_j} = \langle [A_1 - B_1] \rangle + \langle [A_2 - B_1] \rangle + \langle [A_2 - B_2] \rangle + \cdots + \langle [A_N - B_N] \rangle + \langle [A_1 - B_N + 1] \rangle . \quad (16)$$

The right hand side equals the Bell inequality explained in section (3.1), which goes to zero as N goes to infinity. It follows that $\tilde{I}^{QM}(N,n) \to 0$ as $N \to \infty$, independently of n.

As was previously discussed in section (2.2), the local and non-local components of ψ result in a local and a non-local probability distributions, leading to a expression for $\tilde{I}^{QM}(N,n)$ of the form:

$$\tilde{I}^{QM}(N,n) = p\,\tilde{I}^{L}(N,n) + (1-p)\,\tilde{I}^{NL}(N,n). \quad (17)$$

The non-local constribution is bounded from below by the algebraic limit, thus $\tilde{I}^{NL}(N,n) \geq 0$. Therefore, $\tilde{I}^{QM}(N,n) \geq p\,\tilde{I}^{L}(N,n)$. The problem that arises now is to obtain a lower bound for $\tilde{I}^{L}(N,n)$. In order to do so, let us change the notation: instead of $P(a_j \neq b_{li}, r = i)$ we write $P(a \neq b, r = i | A_j, B_{li})$. For the local component:

$$\tilde{I}_{N,j}(L) = P(a \neq b, r = j | A_1, B_{1j}) + P(a \neq b, r = j | A_2, B_{1j}) + \cdots + P(a \neq b, r = j | A_N, B_{Nj}) + P(a = b, r = j | A_1, B_{Nj}) . \quad (18)$$

We will now show that $\tilde{I}_{N,j}(L) \geq p(r = j)$. If $p(r = j) = 0$ we should prove that $\tilde{I}_{N,j}(L) \geq 0$, which is trivial since the probabilities of outcomes are non-negative. If $p(r = j) \neq 0$, let us recall the identity: $P(a \neq b, r = j | A_i, B_{lj}) = P(a \neq b | A_i, B_{lj}, r = j)\, p(r = j | A_i, B_{lj})$. The non-signalling principle assures that $p(r = j | A_i, B_{lj})$ is independent of Bob's measurement. Besides, as we are in scenario 1 (fig. 1), it is also independent of Alice's measurement. Thus, $p(r = j | A_i, B_{lj}) = p(r = j)$. It follows that:

$$\frac{\tilde{I}_{N,j}(L)}{p(r=j)} = P(a \neq b | A_1, B_{1j}, r = j) + P(a \neq b | A_2, B_{1j}, r = j) + \cdots + P(a \neq b | A_N, B_{Nj}, r = j) + P(a = b | A_1, B_{Nj}, r = j) . \quad (19)$$

The right-hand side is just the Bell inequality presented in BKP, with an extra constrain $r = j$. As the inputs A_i, B_{lj} do not depend on r and are two-outcome measurements, it follows that $\frac{\tilde{I}_{N,j}(L)}{p(r=j)} \geq 1$. $\square$

The lower bound for the local probability distribution is $\tilde{I}^{L}(N,n) = \sum_{j=1}^{n} \tilde{I}_{N,j}(L) \geq \sum_{j=1}^{n} p(r = j) = 1$. As it is non-zero, and $\tilde{I}^{QM}(N,n) \to 0$ as $N \to \infty$, it follows that $p = 0$, ans thus the initial state $|\psi\rangle$ is fully non-local.

5. Summary and Conclusions

In this work we have used the results from majorisation theory in order to generalise the study for maximally entangled states done by Barrett, Kent and Pironio. We have presented an extension of their Bell inequality and a set of measurements for which it is violated by the bipartite entangled states that can be deterministically transformed into singlets. This was studied for a scenario in which Alice's actions (majorisation and Bell measurements) are space-like separated events. In this case, the correlations observed between Alice and Bob's outcomes proved to have no local component, thus characterising the set of states as fully nonlocal.

However, this scenario in which Alice's two operations are independent is not the most standard, since for a Bell experiment, a "box" with one input and two (one) outputs would be expected for Alice (Bob). Therefore, if her actions are not space-like separated, majorisation results may in principle be influenced by Alice's second measurements. This requires a different study for the local probability's bound, which is an interesting open question for further research.

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