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Title: Control of Robotic Systems Using Differential Flatness

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Per la Nieves i en Joaquim
També per les persones en qui m'he recolçat al llarg d'aquest any
Moltes gràcies!

Abstract

Key words: Chained form, control theory, drift-less system, exterior differential systems, Engel, form, Goursat, nonlinear control, N-Trailer, Pfaff, robot.

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In this work, a coordinate change of state variables is performed for drift-less systems of dimension $m+2$ with 2 inputs using Goursat Normal Form. Then, we define a feedback law that will allow us to convert the original system into chained form. Later on, we find the flat outputs and define a new feedback law. Finally, numerical simulations are presented for a planar space robot, a mobile robot with a trailer and a N-trailer.

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1 Introduction

Feedback linearization of control systems allows us to apply the theory of linear systems to the nonlinear ones and to design inputs in order to move the system along a trajectory given initial and final points.

A particular case of dynamic feedback linearization is to linearize using the Goursat normal form. Once the Goursat normal form is found, the flat outputs are derived easily. This procedure requires several computations to determine if a system can be linearizable by feedback linearization. However, for nonholonomic systems, it becomes an easier task.

The compilation of results involving feedback linearization and the computation of flat outputs using Pfaffian systems are presented in this work work. We will focus on applying feedback linearization to robotic systems.

This project is divided into 3 different topics. First of all, we give the algebraic notions and several results involving exterior differential systems that will be used through the different chapters as well as the theory about Goursat normal forms and how to obtain them. All this is contained in Chapters 2 to 5.

Then, Chapter 6 contains a simplified model of a planar space robot that is feedback linearized using Pfaff's Theorem. In Chapter 7, a feedback linearization of a mobile robot with a trailer is presented using Engel's Theorem. Numerical simulations are presented in Chapters 6 and 7.

Finally, in Chapter 8 the Goursat normal form for the N-Trailer problem is realized. We will prove that the N-Trailer can be transform into Goursat Normal Form and therefore, into chained form. Later on, we will proceed to transformed the N-Trailer taking coordinates from the last trailer. Finally, numerical results are presented for a 2-trailer and a 3-trailer.

2
Algebra

2.1 Multilinear Algebra and Ideals

Definition 2.1.1 (Algebra)**.** An *algebra* $(V, \odot)$, is a vectorial space V over a field (we will normally use the real field), with a multiplicative operation $\odot : V \times V \longrightarrow V$ that satisfies:

- Given a scalar $\alpha \in \mathbb{R}$, $\alpha(a \odot b) = (\alpha a) \odot b = a \odot (\alpha b)$.
- If there exists an element $e \in V$ such that $x \odot e = e \odot x = x$, $\forall x \in V$, then it is unique and we call it *neutral* or *identity element.*

Definition 2.1.2 (Algebraic Ideal)**.** Let $(V, \odot)$ be an algebra, we say that a subspace $W \subset V$ is an *algebraic ideal* if

$$x \in W, \quad y \in V \Longrightarrow x \odot y, y \odot x \in W.$$

We recall that the intersection of ideals is also an ideal.

Definition 2.1.3 (Minimal Ideal)**.** Let $(V, \odot)$ be an algebra and let $A := \{a_i \in V, \quad 1 \leq i \leq K\}$ be any finite collection of linearly independent elements in V. Let S be the set of all ideals containing A, i.e.

$$S = \{I \subset V, I \text{ ideal}, A \subset I\}.$$

The ideal I_A generated by A is defined as:

$$I_A = \bigcap_{I \in S} I$$

and it is the *minimal ideal* in S containing A.

Theorem 2.1.1. *Let $(V, \odot)$ be an algebra with an identity element. Let $A := \{a_i \in V, \quad 1 \leq i \leq K\}$ be a finite collection of elements in V and I_A the ideal generated by A. Then for each $x \in I_A$ there exist vectors $v_1, \ldots, v_K \in V$ such that*

$$x = v_1 \odot a_1 + v_2 \odot a_2 + \ldots + v_K \odot a_K.$$

Definition 2.1.4. Let $(V, \odot)$ be an algebra and $I \subset V$ an ideal. Two vectors $x, y \in V$ are said to be *equivalent modulus* I if and only if $x - y \in I$. This equivalence is denoted by

$$x \equiv y \mod I.$$

If the space $(V, \odot)$ has an identity element, the above definition implies that there exists equivalence between vectors if and only if

$$x - y = \sum_{i=1}^{K} \theta_i \odot \alpha_i$$

for any $\theta_1, \ldots, \theta_K \in V$. We will denote it as

$$x \equiv y \mod \alpha_1, \alpha_2, \ldots, \alpha_K$$

due to the fact that the modulus operation is performed over the ideal generated by $\alpha_1, \alpha_2, \ldots, \alpha_K$.

2.2 Exterior Algebra

We consider V a vectorial space, V^* its dual space and $\Lambda^k(V^*)$ the vectorial space of the alternating k-tensors with a multiplicative operation. The wedge product is the usual operation but this is not closed in the space $\Lambda^k(V^*)$. Therefore, $\Lambda^k(V^*)$ is not an algebra with this operation.

We define the direct sum operation on the all alternating tensors space as

$$\Lambda(V^*) = \Lambda^0(V^*) \oplus \Lambda^1(V^*) \oplus \cdots \oplus \Lambda^m(V^*).$$

Then, given $\xi \in \Lambda(V^*)$, this tensor can be writen as $\xi = \xi^0 + \xi^1 + \cdots + \xi^m$ where each $\xi^p \in \Lambda^p(V^*)$. Notice that $\Lambda(V^*)$ is closed under the exterior multiplication. It is therefore an algebra.

Definition 2.2.1 (Exterior Algebra)**.** The space of all the alternating tensors with the exterior product, $(\Lambda(V^*), \wedge)$, is an algebra called the *exterior algebra* over V^*.

We note that the algebra $(\Lambda(V^*), \wedge)$ has the identity element since $1 \in \Lambda^0(V^*)$. The Theorem 2.1.1 implies that the ideal generated by a finite set

$$\Sigma = \left\{\alpha^i \in \Lambda(V^*), \quad 1 \le i \le K\right\}.$$

can be written as

$$I_\Sigma = \left\{ \pi \in \Lambda(V^*) \, : \, \pi = \sum_{i=1}^{K} \theta^i \wedge \alpha^i, \quad \theta^i \in \Lambda(V^*) \right\}.$$

Given an arbitrary set Σ of linearly independent generators, it may also be possible to generate I_Σ with a smaller set of generators Σ'.

2.3 Systems of Exterior Equations

The goal of this section is to solve the following system of equations

$$\alpha^1 = 0, \ldots, \alpha^K = 0$$

where $\alpha^i \in \Lambda(V^*)$.

Definition 2.3.1 (System of Exterior Equations)**.** A *system of exterior equations* over V is a finite set of linearly independent equations

$$\alpha^1 = 0, \ldots, \alpha^K = 0$$

where each $\alpha^i \in \Lambda^k(V^*)$ for some $1 \leq k \leq m$. A solution to a system of exterior equations is any subspace $W \subset V$ such that

$$\alpha^1|_W \equiv 0, \ldots, \alpha^K|_W \equiv 0$$

where $\alpha|_W$ stands for $\alpha(v_1, \ldots, v_k)$ for all $v_1, \ldots, v_k \in W$.

We have to keep in mind that there is not uniqueness of the solutions of this system since any subspace $W_1 \subset W$ satisfies $\alpha|_{W_1} \equiv 0$ if $\alpha|_W \equiv 0$.

Theorem 2.3.1. *Given a system of exterior equations $\alpha^1 = 0, \ldots, \alpha^K = 0$, and the corresponding I_Σ generated by the collection of alternating tensors $\Sigma = \left\{\alpha^1, \ldots, \alpha^K\right\}$ where $\alpha^i \in \Lambda(V^*)$. A subspace W solves the system of exterior equations if and only if also satisfies $\pi|_W \equiv 0$ for all $\pi \in I_\Sigma$.*

Proof. If $\pi|_W \equiv 0$ for all $\pi \in I_\Sigma$ then, since the ideal is generated by $\Sigma = \left\{\alpha^1, \ldots \alpha^K\right\}$, each α^i belong in I_Σ and consequently $\alpha^i|W \equiv 0$, $\forall \alpha^i \in I_\Sigma$.

Reciprocally, if $\pi \in I_\Sigma$, it can be written as

$$\pi = \sum_{i=1}^{K} \theta^i \wedge \alpha^i, \quad \theta^i \in \Lambda(V^*).$$

Hence, if $\alpha^i|_W \equiv 0$ for $1 \leq i \leq K$ implies that $\pi|_W \equiv 0$.

□

This result allows us to treat the system of exterior equations, the set of generators for the ideal, and the algebraic ideal as essentially equivalent objects. From here, we may abuse notations and denote the system of equations as its corresponding generator and the generator set as its corresponding ideal.

Definition 2.3.2 (Generators Algebraically Equivalents)**.** Let Σ_1 and Σ_2 be two sets of generators. If $I_{\Sigma_1} = I_{\Sigma_2}$, i.e., they generate the same ideal, we will say that the *generators* are *algebraically equivalents.*

We will use this definition to represent the system of exterior equations in a simplified way.

Definition 2.3.3 (Associated Space)**.** Let Σ be a system of exterior equations and I_Σ the ideal which it generates. The *associated space* of the ideal I_Σ is defined by

$$A\left(I_\Sigma\right) = \left\{v \in V \,:\, v \lrcorner \alpha \in I_\Sigma, \forall \alpha \in I_\Sigma\right\}.$$

Definition 2.3.4 (Retracting Space)**.** The dual associated space, or *retracting space* of the ideal is defined by $C(I_\Sigma) = A\left(I_\Sigma\right)^\perp \subset V^*$.

Once the retracting space is determined, one can find an algebraic equivalent system Σ' that is a subset of $\Lambda(C(I_\Sigma))$, the exterior algebra over the retracting space.

Theorem 2.3.2. *Let $a_1, \ldots, a_m$ be a basis for V. Then the value of an alternating k-tensor $\omega \in \Lambda^k(V^*)$ is independent of a basis element a_i if and only if $a_i \lrcorner \omega \equiv 0$.*

Proof. Let $\phi^1, \ldots, \phi^m$ be a dual basis of $a_1, \ldots, a_m$. Then ω can be written with respect to the dual basis as

$$\omega = \sum_J d_J \phi^{j_1} \wedge \phi^{j_2} \wedge \ldots \wedge \phi^{j_k} = \sum_J d_J \psi^J$$

where the sum is taken over all ascending k-tuples J. If a basis element ψ^J does not contain ϕ^i, then clearly $a_i \lrcorner \psi^J \equiv 0$.

If a basis element contains ϕ^i, then $a_i \lrcorner \wedge \phi^{j_1} \wedge \phi^{j_2} \wedge \ldots \wedge \phi^{j_k} \not\equiv 0$ because a_i can always be matched with ϕ^i through a permutation that affects only the sign. Consequently, $(a_i \lrcorner \omega) \equiv 0$ if and only if the coefficients d_J of all the terms containing ϕ^j are zero.

□

Theorem 2.3.3 (Characterization of Retracting Space)**.** *Let Σ be a system of exterior equations and I_Σ its corresponding algebraic ideal. Then there exists an algebraically equivalent system Σ' such that $\Sigma' \subset \Lambda(C(I_\Sigma))$.*

Proof. Let $v_1, \ldots, v_m$ be a basis of V and $\phi^1, \ldots, \phi^m$ be the dual basis, selected such that $v_{r+1}, \ldots, v_m$ span $A(I_\Sigma)$. Consequently $\phi^1, \ldots, \phi^r$ must span $C(I_\Sigma)$. By induction:

Consider α be any 1-tensor in I_Σ. With respect to the chosen basis, α can be written as

$$\alpha = \sum_{i=1}^{m} a_i \phi^i.$$

Taking into account that $v \lrcorner \alpha \equiv 0 \mod I_\Sigma$ for all $v \in A(I_\Sigma)$, then $a_i = 0$ for $i = r+1, \ldots, m$. Hence,

$$\alpha = \sum_{i=1}^{r} a_i \phi^i.$$

Therefore, all the 1-tensors in Σ are contained in $\Lambda^1(C(I_\Sigma))$. Now, suppose that all the tensors of degree less or equal than k in I_Σ are contained in $\Lambda(C(I_\Sigma))$. Let α be any $(k+1)$-tensor in I_Σ. We consider the tensor

$$\alpha' = \alpha - \phi^{r+1} \wedge (v_{r+1} \lrcorner \alpha).$$

The term $v_{r+1} \lrcorner \alpha$ is a k-tensor in I_Σ by the definition of associated space, and thus, by the induction hypothesis, it must be in $C(I_\Sigma)$. The wedge product of this term with ϕ_{r+1} belongs in $\Lambda(C(I_\Sigma))$. Furthermore,

$$v_{r+1} \lrcorner \alpha' = v_{r+1} \lrcorner \alpha - (v_{r+1} \lrcorner \phi^{r+1}) \wedge (v_{r+1} \lrcorner \alpha) + \phi^{r+1} \wedge (v_{r+1} \lrcorner (v_{r+1} \lrcorner \alpha)) \equiv 0.$$

By the Theorem 2.3.2, α' has no terms involving ϕ^{r+1}.

If we now replace α by α', the ideal generated will be unchanged since

$$\theta \wedge \alpha = \theta \wedge \alpha' + \theta \wedge \phi^{r+1} \wedge (v_{r+1} \lrcorner \alpha)$$

and $v_{r+1} \lrcorner \alpha \in I_\Sigma$.
We can repeat this process for $v_{r+2}, \ldots, v_m$ to produce an $\hat{\alpha}$ that it is a generator of I_Σ and an element of $\Lambda(C(I_\Sigma))$.

□

Definition 2.3.5 (Space of Linear Divisors)**.** Given α a p-form, we define the *space of linear divisors* of α as

$$L_\alpha = \{\omega \in V^* \,:\, \omega \wedge \alpha = 0\}.$$

Theorem 2.3.4. *Let I_Σ be an ideal generated by the set:*

$$\Sigma = \left\{\omega^1, \ldots, \omega^s, \Omega\right\}$$

where $\omega^i \in V^*$ and $\Omega \in \Lambda^2(V^*)$. Let r be the smallest integer such that

$$(\Omega)^{r+1} \wedge \omega^1 \wedge \ldots \wedge \omega^s = 0.$$

Then, the retracting space $C(I_\Sigma)$ has dimension $2r + s$.

Proof. We consider the first case $s = 0$. Then,

$$\Sigma = \{\Omega\} \text{ and } (\Omega)^{r+1} = 0.$$

Since the ideal generated by Σ is defined as

$$I_\Sigma = \left\{\pi \in \Lambda(V^*) \,:\, \pi = \sum_{i=1}^{m} \theta^i \wedge \Omega, \quad \theta^i \in \Lambda(V^*)\right\}.$$

any element of I_Σ will be a linear combination of $\Omega, \Omega^2, \ldots, \Omega^r$.
Since $\Omega \in \Lambda(C(I_\Sigma))$ and $\Omega^r \in \Lambda^{2r}(C(I_\Sigma))$ then

$$\dim(C(I_\Sigma)) \geq 2r.$$

Let's consider $f : V \longrightarrow V^*$ a linear map defined as

$$f(x) = x \lrcorner \Omega, \quad x \in V.$$

Note that the ideal generated by Σ does not contain any 1-form, hence,

$$x \lrcorner \Omega = 0 \Longleftrightarrow x \in A(I_\Sigma).$$

Which proves that

$$\ker f = A(I_\Sigma).$$

Therefore, $\dim(\ker f) = \dim(A(I_\Sigma))$. Since $A(I_\Sigma) = C(I_\Sigma)^\perp$, then

$$\dim(\ker f) \leq m - 2r.$$

On the other hand, for $s=0$

$$x \lrcorner \Omega^{r+1} = (r+1)(x \lrcorner \Omega) \wedge \Omega^r = 0,$$

the last equality is true since $\Omega^{r+1}=0$.
An element of the image of f belong in L_{Ω^r} since

$$\operatorname{Im} f = \{\omega \in V^* \,:\, \omega = x \lrcorner \Omega, \quad x \in V\}.$$

The definition of $\Im f$ implies that $\omega \wedge \Omega^p = (x \lrcorner \Omega) \wedge \Omega^r = 0$, then, $\omega \in L_{\Omega^r}$. Therefore, $\operatorname{Im} f \subset L_{\Omega^r}$.

Since Ω^r it has degree $2r$ and has at most $2r$ linear divisors,

$$\dim(\operatorname{Im} f) \leq 2r.$$

An elemental linear algebra result states that

$$\dim(\ker f) + \dim(\operatorname{Im} f) = m.$$

Hence, $\dim(\operatorname{Im} f) = 2r$, $\dim(\ker f) = m - 2r$ and, consequently, $\dim(C(I_\Sigma)) = 2r$.

In the general case, we consider $W^* = \{\omega^1, \ldots, \omega^s\}$ that has dimension s.
Then $W = (W^*)^\perp \subset V$ and the quotient space V^*/W^* has a relation induced by the relation of V with V^*, and they are dual vectorial spaces. By hypothesis

$$\Omega^r \wedge \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^s \neq 0$$

and $\Omega^r \wedge \omega^1 \wedge \omega^2 \wedge \ldots \wedge \omega^s \in \Lambda^{2r+s}(C(I_\Sigma))$, so that

$$\dim(C(I_\Sigma)) \geq 2r + s.$$

The following linear map is considered

$$f' : W \xrightarrow{\ f\ } V^* \xrightarrow{\ \pi\ } V^*/W^*$$

where π is the projection to the quotient space and f is the map defined before.

As in the trivial case, we wish to find upper bounds for the dimensions of the kernel and the image of f'. Using the algebra result, we know

$$\dim(\ker f') + \dim(\operatorname{Im} f') = \dim(W) = m - s.$$

Reasoning similarly to the previous case, we find

$$\dim(\ker f) \leq m - 2r - s$$
$$\dim(\operatorname{Im} f) \leq 2r.$$

Consequently, $\dim(C(I_\Sigma)) = 2r + s$.

□

2.4 Codistributions

Definition 2.4.1 (Distribution)**.** A smooth *distribution* associates a subspace of the tangent space with each point $p \in M$. It is represented as the span of d smooth vector fields with

$$\Delta = \{X_1, \ldots, X_d\}$$

The dimension of the codistribution at a point is defined to be the dimension of the subspace $\Delta(p)$. A distribution is said to be regular if its dimension does not vary with p.

Definition 2.4.2 (Codistribution)**.** A *codistribution* is defined as the map that associates each point of the variety with a set of 1-forms. This linear combination of 1-forms will be a subspace of the cotangent space T_p^*M. We denote the codistribution as

$$\Theta(p) = \{\omega^1(p), \ldots, \omega^d(p)\}.$$

There is notion of duality between distributions and codistributions which allows us to construct codistributions from distributions and vice versa.

Given a distribution Δ, for each p in a neighborhood U, consider all the 1-forms which pointwise annihilate all vectors in $\Delta(p)$,

$$\Delta^{\perp}(p) = \{\omega(p) \in T_p^*M : \omega(p)(X) = 0, \quad \forall X \in \Delta(p)\}.$$

Clearly, $\Delta^{\perp}(p)$ is a subspace of T_p^*M and it is, therefore, a codistribution. We call $\Delta^{\perp}$ the annihilator or dual of Δ. Conversely, given a codistribution Θ, we construct the annihilating or dual distribution pointwise as

$$\Theta^{\perp}(p) = \{v \in T_pM : \omega(p)(v) = 0, \quad \forall \omega(p) \in \Omega(p)\}.$$

3 Exterior Differential Systems

3.1 Exterior algebra on a manifold

The space of all forms on a manifold M,

$$\Omega(M) = \Omega^0(M) \oplus \cdots \oplus \Omega^n(M),$$

together with the wedge product is called *exterior algebra in M*. An algebraic ideal of this algebra is defined as a subspace I such that if $\alpha \in I$ then $\alpha \wedge \beta \in I$ for any $\beta \in \Omega(M)$.

Definition 3.1.1 (Closed Ideal)**.** An ideal $I \subset \Omega(M)$ is said to be *closed* with respect to exterior differentiation if and only if

$$\alpha \in I \Rightarrow d\alpha \in I,$$

or more compactly, if $dI \subset I$. An algebraic ideal which is closed with respect to exterior differentiation is called a *differential ideal*.

A finite collection of forms, $\Sigma = \{\alpha^1, \ldots, \alpha^K\}$ generates an algebraic ideal

$$I_\Sigma = \left\{ \omega \in \Omega(M) \,|\, \omega = \sum_{i=1}^{K} \theta^i \wedge \alpha^i \text{ for some } \theta^i \in \Omega(M) \right\}.$$

We also can talk about the differential ideal generated by Σ. Thus, if S_d denotes the collection of all differential ideals containing Σ it is defined to be the smallest differential ideal containing Σ

$$\mathcal{I}_\Sigma = \bigcap_{I \in S_d} I.$$

Theorem 3.1.1. *Let Σ be a finite collection of forms and let $\mathcal{I}_\Sigma$ be the differential ideal generated by Σ. Define the collection*

$$\Sigma' = \Sigma \cup d\Sigma$$

and denote the algebraic ideal which generates by $I_{\Sigma'}$.

Proof. By definition $\mathcal{I}_\Sigma$ is closed with respect to exterior differentiation, so $\Sigma' \subset \mathcal{I}_\Sigma$. Consequently, $I_{\Sigma'} \subset \mathcal{I}_\Sigma$. The ideal $I_{\Sigma'}$ is closed with respect to exterior differentiation and contains Σ by construction. Therefore, from the definition of $\mathcal{I}_\Sigma$ we have that $\mathcal{I}_\Sigma \subset I_{\Sigma'}$.

□

The associated space and retracting space of an ideal in $\mathcal{I}_\Sigma$ is called *characteristic distribution of Cauchy* and is denoted by $\mathcal{A}(\mathcal{I}_\Sigma)$.

3.2 Exterior Differential Systems

In the previous section we have introduced systems of exterior equations on a vector space V and characterized their solutions as subspaces of V. We are now ready to define a similar notion for a collection of differential forms defined on a manifold M. The basic problem will be to study the integral submanifolds of M which satisfy the constraints represented by the exterior differential system.

Definition 3.2.1 (Exterior Differential System)**.** An *exterior differential system* is a finite collection of equations

$$\alpha^1 = 0, \ldots, \alpha^r = 0,$$

where each $\alpha^i \in \Omega^k(M)$ is a smooth $k-$form. A *solution to an exterior differential system* is any submanifold N of M which satisfies $\alpha^i(x)|_{T_xN} \equiv 0$ for all $x \in N$ and all $i \in \{1, \ldots, r\}$.

An exterior differential system can be viewed pointwise as a system of exterior equations on T_pM. In view of this, one might expect that a solution would be defined as a distribution on the manifold. The drawback with this approach is that most distributions are not integrable, and we want our solution set to be a collection of integral submanifolds. Therefore, we will restrict our solution set to integrable distributions.

Theorem 3.2.1. *Given an exterior differential system*

$$\alpha^1 = 0, \ldots, \alpha^K = 0$$

and the corresponding differential ideal $\mathcal{I}_\Sigma$ generated by the collection of forms

$$\Sigma = \{\alpha^1, \ldots, \alpha^K\},$$

an integral submanifold N of M solves the system of exterior equations if and only if it also solves the equation $\pi = 0$ for each $\pi \in \mathcal{I}_\Sigma$.

Proof. If an integral submanifold N of M is a solution to Σ, then for all $x \in N$ and all $i \in \{1, \ldots, K\}$,

$$\alpha^i(x)|_{T_xN} \equiv 0.$$

Taking the exterior derivative we get

$$d\alpha^i(x)|_{T_xN} \equiv 0.$$

Hence, the submanifold also satisfies the exterior differential system

$$\alpha^1 = 0, \ldots, \alpha^K = 0,\ d\alpha^1 = 0, \ldots, d\alpha^K = 0.$$

By the Theorem 3.1.1 we know that the differential ideal generated by Σ is equal to the algebraic ideal generated by the above system. Therefore, the Theorem 2.3.1 tells us that every solution N to Σ is also a solution for every element of $\mathcal{I}_\Sigma$. Conversely, if N solves the equation $\pi = 0$ for every $\pi \in \mathcal{I}_\Sigma$ then in particular it must solve Σ.

□

This theorem allows us to work either with the generators of an ideal or with the ideal itself. In fact, some authors define exterior differential systems as differential ideals of $\Omega(M)$. Because a set of generators Σ generates both a differential ideal $\mathcal{I}_\Sigma$ and a algebraic ideal I_Σ, we can define two different notions of equivalence for exterior differential systems.

Two exterior differential systems Σ_1 and Σ_2 are said to be *equivalent* if they generate the same algebraic ideal. i.e, $\mathcal{I}_{\Sigma_1} = \mathcal{I}_{\Sigma_2}$. Intuitively, we want to think of two exterior differential systems as equivalent if they have the same solution set. Therefore, we will usually discuss equivalence in the latter sense.

3.3 Pfaffian Exterior Differential Systems

Pfaffian systems are of particular interest because they can be used to represent a set of first-order ordinary differential equations.

Definition 3.3.1 (Paffian System)**.** An exterior differential system of the form

$$\alpha^1 = \alpha^2 = \cdots = \alpha^s = 0,$$

where the α^i are independent 1-forms on a n-dimensional manifold M, is called a *Pfaffian system* of codimension $m - s$. If $\{\alpha^1, \ldots, \alpha^m\}$ is a basis of $\Omega^1(M)$, then the set $\{\alpha^{s+1}, \ldots, \alpha^m\}$ is called a *complement* to the Pfaffian system

An *independence condition* is a 1-form τ that is required to be nonzero along integral curves of the Pfaffian system. That is $\alpha^i(c(t))(c'(t)) = 0$, then $\tau(c(t))(c'(t)) \neq 0$. The 1-forms $\alpha^1, \ldots, \alpha^s$, generate the algebraic ideal

$$\mathcal{I} = \{I\} = \{\sigma \in \Omega(M) : \sigma \wedge \alpha^1 \wedge \ldots \wedge \alpha^s = 0\}.$$

For an ideal generated by a set of 1-forms, each element in the ideal has the form

$$\xi = \sum_{j=1}^{s} a_{ij}\, \theta^j \wedge \alpha^j$$

for some $\theta^j \in \Omega(M)$. The exterior differential system generated by I must be closed under differentiation, thus it contains $\mathcal{I}$ and $d\mathcal{I}$. We will focus mainly in codistributions of 1-forms I which generates the exterior differential system.

It is possible to rephrase Frobenius's Theorem in a concise way using ideals. Let $\mathcal{I}$ be the ideal generated by $\{\alpha^1, \ldots, \alpha^s\}$ and write $d\mathcal{I}$ for the set consisting of the exterior derivative of all elements of $\mathcal{I}$. We say that $\mathcal{I}$ is integrable if there exist functions $h^1, \ldots, h^s$ such that $\mathcal{I}$ is also generated by $\{dh^1, \ldots, dh^s\}$.

Definition 3.3.2 (Frobenius Condition)**.** A set of linearly independent 1-forms $\alpha^1, \ldots, \alpha^s$ in a neighborhood of a point is said to satisfy the Frobenius condition if one of the following equivalent conditions holds:

(a) $\mathcal{I}$ is integrable.

(b) $d\mathcal{I} \subset \mathcal{I}$.

(c) $d\alpha^i \wedge \alpha^1 \wedge \cdots \wedge \alpha^s = 0$ for all $1 \leq i \leq s$.

(d) $d\alpha^i = \sum_{j=1}^{s} \theta^i_j \wedge \alpha^j$ for some $\theta^i_j \in \Omega(M)$, $1 \leq i, j \leq s$.

(e) $d\alpha^i \equiv 0 \mod \mathcal{I}$.

The condition $d\alpha^i \equiv 0 \mod \mathcal{I}$ uses the notion of congruences. Given two forms $\sigma, \omega \in \Omega(M)$, we write $\omega \equiv \sigma \mod \mathcal{I}$ if there exists an exterior form $\eta \in \mathcal{I}$ such that $\omega = \sigma + \eta$. If I is a codistribution, then we write $\omega \equiv \sigma \mod I$ if there exist exterior form $\alpha \in I$ and $\eta \in \Omega(M)$ such that $\omega = \sigma + \eta \wedge \alpha$. It follows that if I is the generator set for an ideal $\mathcal{I}$, then $\omega \mod \mathcal{I} = \omega \mod I$. In the case that $\mathcal{I}$ is generated by 1-forms $\alpha^1, \ldots, \alpha^s$, we will often make use of the relation

$$\omega \equiv 0 \mod \mathcal{I} \iff \omega = \sum_{i=1}^{s} \theta^i \wedge \alpha^i \text{ for some } \theta^i \in \Omega(M).$$

When $d\alpha^i$ is a linear combination of $\alpha^1, \ldots, \alpha^s$, the following expression is frequently used

$$d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^s \quad 1 \leq i \leq s$$

where the *mod* operation is implicitly performed over the algebraic ideal generated by α^i.

Now we can state and proof the Frobenius's Theorem for codistributions.

Theorem 3.3.1 (Frobenius Theorem for Codistributions)**.** *Let I be an algebraic ideal generated by the independent 1-forms $\alpha^1, \ldots, \alpha^{m-r}$ which satisfies the Frobenius condition. Then, in a neighborhood of x there exist functions $h^1, \ldots, h^m$ such that*

$$I = \{\alpha^1, \ldots, \alpha^{m-r}\} = \{dh^{r+1}, \ldots, dh^m\}.$$

Proof. First of all, notice that I is a differential ideal because it satisfies the Frobenius condition. We will denote by $\Delta = \operatorname{span}\{\alpha^1, \ldots, \alpha^{m-r}\} \subset T^*M$. We will prove it by induction on r. Let $r = 1$, then $(\Delta_p)^\perp \subset T_pM$ has dimension 1 for $p \in M$. Relative to a system of local coordinates x^i, for $1 \leq i \leq m$, the equations of the differential system is written in the classical form

$$\frac{dx^1}{X^1(x)} = \cdots = \frac{dx^m}{X^m(x)},$$

where the functions $X^i(x^1, \ldots, x^m)$, not all zero, are the coefficients of a vector field

$$X = \sum_{i=1}^{m} X^i(x) \frac{\partial}{\partial x^i}$$

spanning $(\Delta_p)^\perp$. By the Flow Box Coordinate Theorem we can choose coordinates $h^1, \ldots, h^m$, such that $(\Delta_p)^\perp = \operatorname{span}\{\partial/\partial h^1\}$, then $\Delta_p = \operatorname{span}\{dh^2, \ldots, dh^m\}$. The latter clearly forms a set of generators of I. Notice that in this case the Frobenius condition is void.

Suppose $r \geq 2$ and the theorem to be true for $r-1$. Let x^i, for $1 \leq i \leq m$, be local coordinates such that

$$\alpha^1, \ldots, \alpha^{m-r}, dx^r$$

are linearly independent. The differential system defined by these $m-r+1$ forms also satisfies the Frobenius condition. By the induction hypothesis, there are coordinates $h^1, \ldots, h^m$ so that

$$dh^r, dh^{r+1}, \ldots, dh^m$$

are a set of generators of the corresponding differential ideal. It follows that dx^r is a linear combination of these forms or that x^r is a function of $h^r, \ldots, h^m$. Without loss of generality, we suppose

$$\frac{\partial x^r}{\partial h^r} \neq 0.$$

Since

$$dx^r = \frac{\partial x^r}{\partial h^r} dh^r + \sum_{i=1}^{m-r} \frac{\partial x^r}{\partial h^{r+1}} dh^{r+i},$$

we may now solve for dh^r in terms of dx^r and $dh^{r+1}, \ldots, dh^m$. Since $\alpha^1, \ldots, \alpha^{m-r}$ are linear combinations of $dh^r, \ldots, dh^m$, they can now be expressed in the form

$$\alpha^i = \sum_{j=1}^{m-r} a^i_j dh^{r+j} + b_i dx^r \text{ for } 1 \leq i \leq m-r,$$

where $a^i_j, b_i \in \mathcal{C}^\infty(M)$ for $1 \leq i, j \leq m-r$. Since α^i and dx^r are linearly independent, the matrix (a^i_j) must be non-singular. Hence, we can find a new set of generators for I in the form

$$\tilde{\alpha}^i = dh^{r+i} + g^i dx^r \text{ for } 1 \leq i \leq m-r,$$

where $g^i \in \mathcal{C}^\infty(M)$ for $1 \leq i \leq m-r$, and the Frobenius condition remains satisfied. Exterior differentiation gives

$$d\tilde{\alpha}^i = dg^i \wedge dx^r \equiv \sum_{j=1}^{r-1} \frac{\partial g^i}{\partial h^j} dh^j \wedge dx^r \equiv 0 \mod \tilde{\alpha}^1, \ldots, \tilde{\alpha}^{m-r}.$$

It follows that

$$\frac{\partial g^i}{\partial h^j} = 0 \text{ for } 1 \leq i \leq m-r, \quad 1 \leq j \leq r-1,$$

which means that g^i are functions of $h^r, \ldots, h^m$. Hence, in the h-coordinates, we are studying a system of $m-r$ forms of degree 1 involving only the $m-r+1$ coordinates $h^r, \ldots, h^m$. This reduces to the situation settled at the beginning of this proof. Hence, the induction is complete.

□

Corollary 3.3.2. *Let $y^1, \cdots y^m$ be functions whose differentials are linearly independent from linearly independent 1-forms $\alpha^1, \ldots, \alpha^p$ and satisfying the relative Frobenius conditions*

$$d\alpha^i \wedge \alpha^1 \wedge \cdots \alpha^p \wedge dy^1 \wedge \cdots \wedge dy^m = 0 \quad 1 \le i \le m.$$

Then, setting

$$\alpha = \left(\alpha^1, \cdots, \alpha^p\right)^t, \quad Y = (y^1, \cdots y^m)^t$$

there exists a vector of functions $Z = (z^1, \cdots, z^p)^t$ a $p \times p$ matrix A and a $p \times m$ matrix B, such that

$$\alpha = AdZ + BdY$$

For more general exterior differential systems, we have the following integrability results.

Proposition 3.3.1. *If the Cauchy characteristic distribution $\mathcal{A}(\mathcal{I}_\Sigma)$ of $\mathcal{I}_\Sigma$ has constant dimension r in a neighborhood of x, then the distribution $\mathcal{A}(\mathcal{I}_\Sigma)$ is integrable.*

Theorem 3.3.3. *Let $\mathcal{I}$ be a differential ideal whose retracting space $\mathcal{C}(\mathcal{I})$ has a constant dimension $s = m - r$. There is a neighborhood in which there are coordinates $(x^1, \ldots, x^r; y^1, \ldots, y^m)$ such that $\mathcal{I}$ has a set of generators which are forms in $y^1, \ldots, y^s$ and their differentials.*

Proof. By Proposition 3.3.1 the differential system defined by $\mathcal{C}(\mathcal{I})$, or what is the same, the distribution defined by $\mathcal{A}(\mathcal{I})$, is completely integrable. We may choose coordinates $(x^1, \ldots, x^r; y^1, \ldots, y^s)$ so that the foliation is defined given by

$$y^\sigma = \text{const}, \quad 1 \le \sigma \le s.$$

By the retraction theorem, $\mathcal{I}$ has a set of generators which are forms in $dy^\sigma, 1 \le \sigma \le s$. But their coefficients may involve x^ρ, $1 \le \rho \le r$. The theorem follows when we show that we can choose a new set of generators for $\mathcal{I}$ which are forms in the y^σ coordinates in which the x_ρ do not appear. To exclude the trivial case, we suppose that $\mathcal{I}$ is a proper ideal, so that it contains no non-zero functions.

Let $\mathcal{I}_q$ be the set of q-forms in $\mathcal{I}$, $q = 1, 2, \ldots$. Let $\varphi^1, \ldots, \varphi^p$ be the linearly independent 1-forms in $\mathcal{I}_1$ such that any form in $\mathcal{I}_1$ is a linear combination. Since $\mathcal{I}$ is closed, $d\varphi^i \in \mathcal{I}$, $1 \le i \le p$. For a fixed ρ, we have that $\frac{\partial}{\partial x^\rho} \in \mathcal{A}(\mathcal{I})$, which implies

$$\frac{\partial}{\partial x^\rho} \lrcorner \, d\varphi^i = L_{\partial/\partial x^\rho} \varphi^i \in \mathcal{I}_1,$$

since the left-hand side is of degree 1. It follows that

$$\frac{\partial \varphi^i}{\partial x^\rho} = L_{\frac{\partial}{\partial x^\rho}} \varphi^i = \sum_j a_{ij} \varphi^j, \quad 1 \le i, j \le p \tag{3.1}$$

where the left hand side stands for the form obtained from φ^i by taking partial derivatives of the coefficients with respect to x^ρ.

For this fixed ρ, we regard x^ρ as the variable and $x^1, \ldots, x^{\rho-1}, x^{\rho+1}, \ldots, x^r, y^1, \ldots, y^s$ as parameters. Consider the system of ordinary differential equations

$$\frac{dz^i}{dx^\rho} = \sum_j a_{ij} z^j, \quad 1 \leq i, j \leq p. \tag{3.2}$$

Let $z_i^k, 1 \leq k \leq p$, be a fundamental system of solutions, so that

$$\det\left(z_i^k\right) \neq 0.$$

We shall replace φ^i by the $\widetilde{\varphi}^k$ defined by

$$\varphi^i = \sum z_i^k \widetilde{\varphi}^k. \tag{3.3}$$

By differentiating (3.3) with respect to x^ρ and using (3.1) and (3.2), we get

$$\frac{\partial \widetilde{\varphi}^k}{\partial x^\rho} = 0,$$

so that $\widetilde{\varphi}^k$ does not involve x^ρ. Applying the same process to the other x, we arrive at a set of generators $\mathcal{I}_1$ which are forms in y^σ.

Suppose this process carried out for $\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}$, so that they consist of forms in y^σ. Let $\mathcal{J}_{q-1}$ the ideal generated by for $\mathcal{I}_1, \ldots, \mathcal{I}_{q-1}$. Let $\psi^\alpha \in \mathcal{I}_q$, $1 \leq \alpha \leq r$, linearly independents $\mod \mathcal{J}_{q-1}$, such that any q-form of $\mathcal{I}_q$ is congruent $\mod \mathcal{J}_{q-1}$ to a linear combination of them. By the above argument, such forms include

$$\frac{\partial}{\partial x^\rho} \lrcorner\, d\psi^\alpha = L_{\partial/\partial x^\rho} \psi^\alpha.$$

Hence, we have

$$\frac{\partial \psi^\alpha}{\partial x^\rho} \equiv \sum b_\beta^\alpha \psi^\beta, \quad \mod \mathcal{J}_{q-1}, \quad 1 \leq \alpha, \beta \leq r.$$

By using the above argument, we can replace the ψ^α by $\widetilde{\psi}^\beta$ such that

$$\frac{\partial \widetilde{\psi}^\alpha}{\partial x^\rho} \in \mathcal{J}_{q-1}.$$

This means that we can write

$$\frac{\partial \widetilde{\psi}^\alpha}{\partial x^\rho} = \sum_h \eta_h^\alpha \wedge \omega_h^\alpha,$$

where $\eta_h^\alpha \in \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_{q-1}$ and are, therefore, forms in y^σ. Let θ_h^α defined by

$$\frac{\partial \theta_h^\alpha}{\partial x^\rho} = \omega_h^\alpha.$$

Then, the forms

$$\widetilde{\widetilde{\psi}}^\alpha = \widetilde{\psi}^\alpha - \sum_h \eta_\alpha^h \wedge \theta_\alpha^h$$

do not involve x^ρ, and can be used to replace ψ^α. Applying this process to all $x^\rho, 1 \leq \rho \leq r$, we find a set of generators for $\mathcal{I}_q$, which are forms only in y^σ.

□

3.4 Derived flags

Let $I = \{\alpha^1, \ldots, \alpha^s\}$ be a smooth codistribution on M. The exterior derivative induces a mapping $d : I \to \Omega^2(M)/I$

$$d : \lambda \to d\lambda \mod I \in \Omega^2(M).$$

The mapping d is a linear mapping over $\mathcal{C}^\infty(M)$ such that

$$\begin{aligned} d(f\alpha + g\beta) =& df \wedge \alpha + fd\alpha + dg \wedge \beta + gd\beta \mod I \\ =& fd\alpha + gd\beta \mod I \\ =& fd(\alpha) + gd(\beta). \end{aligned}$$

It follows that the kernel of d is a codistribution on M[1]. We call this subspace, $I^{(1)}$, the first derived flag of the system I

$$I^{(1)} = \ker(d) = \{\lambda \in I : d\lambda \mod I \equiv 0\}.$$

$I^{(1)}$ contains the 1-forms in I which are integrable mod I.

We can represent $I^{(1)}$ using a set of 1-forms, but it is important to note that the basis of $I^{(1)}$ may be not a simple subset of the basis of I. Linear combinations of basis elements must be searched to find a basis derived from the derived system.

Since $I^{(1)}$ is itself a codistribution on M, one may inductively continue this procedure of obtaining derived systems and define

$$I^{(2)} = \{\lambda \in I^{(1)} : d\lambda \equiv 0 \mod I^{(1)}\} \subset I^{(1)}$$

or, in general,

$$I^{(k+1)} = \{\lambda \in I^{(k)} : d\lambda \equiv 0 \mod I^{(k)}\} \subset I^{(k)}.$$

This procedure results in a nested sequence of codistributions

$$I^{(k+1)} \subset I^{(k)} \subset \cdots \subset I^{(1)} \subset I^{(0)}. \tag{3.4}$$

If the dimension of each $I^{(i)}$ is constant, then, this construction terminates for some finite integer N.

Definition 3.4.1 (Derived Length)**.** Let I be an algebraic ideal corresponding to a Pfaffian system. We define the *derived length* of I as the smallest integer N such that

$$I^{(N)} = I^{(N+1)}$$

The derived flag describes the integrability properties of the Pfaffian system generated by I. If I is completely integrable, then by Frobenius's Theorem, we have $I^{(1)} = I^{(0)}$, i.e., the length of the derived flag is zero. In fact, $I^{(N)}$ is always integrable since, by definition, $dI^{(N)} \mod I^{(N)} \equiv 0$. $I^{(N)}$ is the largest integrable subsystem contained in I.

Thus, if $I^{(N)} \neq \{0\}$ then there exist functions $h^1, \ldots, h^r$ such that $\{dh^1, \ldots, dh^r\} \subset I$. As a result, if a Pfaffian system contains an integrable subsystem $I^{(N)} \neq \{0\}$, which

[1] At each point $p \in M$, the kernel of d is a linear subspace of T_p^*M.

is spanned by the 1-forms $dh^1, \dots, dh^r$, then the integral curves of the system are constrained to satisfy the following equations for some constants k_i,

$$dh^i = 0 \Longrightarrow h^i = k_i, \quad \text{for} \quad 1 \le i \le r,$$

or equivalently, trajectories of the system must lie on the manifold,

$$M = \{x \,:\, h^i(x) = k_i \quad \text{for} \quad 1 \le i \le r\}.$$

In particular, this implies that if $I^{(N)} \neq 0$, it is not possible to find an integral curve of the Pfaffian system which connects a configuration $x(t_0) = x_0$ to another configuration $x(t_f) = x_1$ unless the initial and final configurations satisfy

$$h^i(x_0) = h^i(x_1) \quad \text{for} \quad 1 \le i \le r.$$

In the context of control theory, this means that the system is not controllable since there exist functions which provides a foliation of the state space and it is impossible to move from one leaf of the foliation to another. This controllability result is provided by Chow's Theorem.

Theorem 3.4.1 (Chow's Theorem). *Let $I = \{\alpha^1, \dots, \alpha^s\}$ represent a set of constraints and assume that the derived flag of the system exists. Then, there exists a path $x(t)$ between any two point satisfying $\alpha^i(x) \cdot \dot{x} = 0$ for all $1 \le i \le s$ if and only if there exists an integer N such that $I^{(N)} = \{0\}$.*

In control theory, Chow's theorem is usually stated using regular distribution $I^{\perp}$.

Theorem 3.4.2 (Chow's Theorem for Regular Distributions). *Let $\Delta = I^{\perp}$ a regular distribution. Then, for regular systems of the form*

$$\dot{x} = \sum_{i=1}^{k} g_i(x) u_i, \quad g_i \in \Delta$$

there exist admissible controls to steer the system between two given arbitrary points $x_0, x_1 \in U$ if and only if, for some N,

$$(\Delta_N)^{\perp}(x) = T\mathbb{R}^m \cong \mathbb{R}^m$$

for all $x \in U$.

The connection between Chow's theorem for regular distributions and exterior differential systems formulation is made with the following lemma.

Lemma 3.4.1. *If $I^{(0)} = \Delta^{\perp}$, then $I^{(1)} = (\Delta + [\Delta, \Delta])^{\perp}$.*

This lemma allows us to compute the derived flag for a system given the distribution $\Delta = I^{\perp}$. Define the nested set of distributions

$$\Delta = \Delta_0 \subset \Delta_1 \subset \cdots \subset \Delta_k$$

as $\Delta_i = \Delta_{i-1} + [\Delta_{i-1}, \Delta_{i-1}]$, called the *filtration* of Δ_0[2]. This sequence terminates if the dimension of each Δ_i is constant, and it follows from Theorem 3.4.1 that $I^{(i)} = (\Delta_i)^{\perp}$.

[2] Or the coderived coflag of $I^{(0)}$.

4
The Goursat Normal Forms

Now that we have defined an exterior differential system and introduced some tools for analyzing them, we are ready to study some important normal forms for exterior differential systems. We will restrict ourselves to Pfaffian systems. The first normal form which we introduce, the Pfaffian form, is restricted to systems of only one equation. The Engel form applies to two equations on a four-dimensional space, and the Goursat form is for $m-2$ equations on an m-dimensional space. The extended Goursat normal form is defined for systems with codimension greater than two. The Goursat normal forms can be thought of as the generalization of linear systems. Their study will lead us to the study of linearization of control.

4.1 Systems of One Equation

We will first study Pfaffian systems of codimension $m-1$, or systems consisting of a single equation

$$\alpha = 0$$

where α is a 1-form on a manifold M. In some chart (U, x) of a point $p \in M$ the equation can be expressed as

$$a_1(x)dx^1 + a_2(x)dx^2 + \cdots + a_m(x)dx^m = 0.$$

In order to understand the integral manifolds of this equation we will attempt to express α in a normal form by performing a coordinate transformation.

Definition 4.1.1 (Rank of a Form)**.** Let $\alpha \in \Omega^1(M)$. The integer r defined as

$$(d\alpha)^r \wedge \alpha \neq 0$$

$$(d\alpha)^{r+1} \wedge \alpha = 0$$

is called *rank* of α.

The following theorem allows us, under a rank condition, to write α in a normal form.

Theorem 4.1.1 (Pfaff theorem)**.** *Let $\alpha \in \Omega^1(M)$ have a constant rank r in a neighborhood of p. Then there exists a coordinate chart (U, z) such that in these coordinates $\alpha = dz^1 + z^2dz^3 + \cdots + z^{2r}dz^{2r+1}$.*

Proof. Let $\mathcal{I}$ be the differential ideal generated by α. From Theorem 2.3.4 the retracting space of $\mathcal{I}$ has dimension $2r+1$. By the Theorem 3.3.3 there exist local coordinates $y^1, \ldots, y^m$ such that $\mathcal{I}$ has a set of generators in $y^1, \ldots, y^{2r+1}$. Then, by dimension count, any function f_1 of those $2r+1$ coordinates results in

$$(d\alpha)^r \wedge \alpha \wedge df_1 = 0.$$

Now, let $\mathcal{I}_1$ be the ideal generated by $\{df_1, \alpha, d\alpha\}$. If $r = 0$, then the result follows from the Frobenius's Theorem 3.3.1. If $r > 0$, then the forms df_1 and α must be linearly independent, since α is not integrable. Applying Theorem 2.3.4 to $\mathcal{I}_1$, let r_1 be the smallest integer such that

$$(d\alpha)^{r_1+1} \wedge \alpha \wedge df_1 = 0.$$

Clearly, $r_1 + 1 \leq r$. Furthermore, the equality sign must hold because $(d\alpha)^r \wedge \alpha \neq 0$. Applying Theorem 3.3.3 to $\mathcal{I}_1$ there exists a function f_2 such that

$$(d\alpha)^{r-1} \wedge \alpha \wedge df_1 \wedge df_2 = 0.$$

Repeating this process, we find r functions $f_1, f_2, \ldots, f_r$ satisfying

$$\begin{aligned} d\alpha \wedge \alpha \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_r &= 0, \\ \alpha \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_r &\neq 0. \end{aligned}$$

Finally, let I be the ideal $\{df, \ldots, df_r, \alpha, d\alpha\}$. Its retracting space $\mathcal{C}(I_r)$ is of dimension $r+1$. There is a function f_{r+1} such that:

$$\begin{aligned} \alpha \wedge df_1 \wedge df_2 \wedge \cdots \wedge df_r \wedge df_{r+1} &= 0, \\ df_1 \wedge df_2 \wedge \cdots \wedge df_r \wedge df_{r+1} &\neq 0. \end{aligned}$$

By modifying α by a factor, we can write

$$\alpha = df_{r+1} + g_1 df_1 + \cdots + g_r df_r.$$

Because $(d\alpha)^r \wedge \alpha \neq 0$, the functions $f_1, \ldots, f_{r+1}, g_1, \ldots, g_r$ are independent. The result then follows by setting

$$z^1 = f_{r+1} \qquad z^{2i} = g_i \qquad z^{2i+1} = kf_i$$

for $1 \leq i \leq r$.

□

The proof uses a number of tools that are beyond the scope of this work. In the $r = 1$ case, the proof reduces to proving that there exist two functions f_1 and f_2 which satisfy

$$\begin{array}{ccc} d\alpha \wedge \alpha df_1 = 0 & & \alpha \wedge df_1 \neq 0 \\ & \text{and} & \\ \alpha df_1 \wedge df_2 = 0 & & df_1 \wedge df_2 \neq 0. \end{array}$$

Given f_1 and f_2, α can be scaled such that

$$\alpha = df_2 + gdf_1 = dz^1 - z^2 dz^3.$$

The Pfaff theorem guarantees that these equations have a solution (it does not to be unique). A basis of the right null space of this constraints is given by

$$g_1 = \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^3} \quad g_2 = \frac{\partial}{\partial z^2}.$$

The following theorem is similar to Pfaff's theorem and basically expresses the result in a more symmetric form.

Theorem 4.1.2 (Symmetric Version of Pfaff Theorem)**.** *Given any $\alpha \in \Omega^1(M)$ with constant rank r in a neighborhood U of p, then there exist coordinates $z, y^1, \ldots, y^r, x^1, \ldots, x^r$ such that*

$$\alpha = dz + \frac{1}{2}\sum_{i=1}^{r}(y^i dx^i - x^i dy^i).$$

The Pfaffian system $\alpha = 0$ in a manifold M is said to have the local accessibility property if every point $x \in M$ has a neighborhood U such that every point in U can be joined to x by an integral curve. The following theorem answers the question of when this Pfaffian system has the local accessibility property.

Theorem 4.1.3 (Caratheodory Theorem)**.** *The Pfaffian system*

$$\alpha = 0,$$

on α where α a has constant rank, has the local accessibility property if and only if

$$\alpha \wedge d\alpha \neq 0.$$

Proof. The condition above basically says that the rank of α must be greater than or equal to 1. If α has rank 0 then $d\alpha \wedge \alpha = 0$ and, therefore, by the Frobenius's Theorem 3.3.1, we can write,

$$\alpha = dh = 0$$

for some function h. The integral curves are of the form $h = c$ for any arbitrary constant c. Since we can only join points $p, q \in M$ for which $h(p) = h(q)$, we do not have the local accessibility property.
Conversely, let α have rank $r \geq 1$. From Theorem 4.1.2, we can find coordinates $z, x^1, \ldots, x^r, y^1, \ldots, y^r, u^1, \ldots, u^s$ in some neighborhood U, with $2r + s + 1$ as dimension of M, such that

$$\alpha = dz + \frac{1}{2}\sum_{i=1}^{r}(y^i dx^i - x^i dy^i) = 0,$$

and therefore

$$dz = \frac{1}{2}\sum_{i=1}^{r}(y^i dx^i - x^i dy^i).$$

Given any two points $p, q \in U$ we must find integral curve $\gamma : [0,1] \longrightarrow U$ with $c(0) = p$ i $c(1) = q$. Since we are working locally, we can assume that the initial point

p is the origin: $z(p) = x^i(p) = y^i(p) = u^i(p) = 0$. Let the final point q be defined by $z(q) = z^1, x^i(q) = x^{1i}, y^i(q) = y^{1i}, u^i(q) = u^{1i}$. Because the expression of the 1-form does not depend on the u^i coordinates, we can choose the curve tu^{1i} to connect these u^i coordinates of p and q.

In the (x^i, y^i) plane there are many curves$(x^i(t), y^i(t))$ that join the origin with the desired point (x^{1i}, y^{1i}). We need to find one which steers the z coordinate to z^1. In order to satisfy the equation $\alpha = 0$, we must have that

$$dz = \frac{1}{2}\sum_{i=1}^{r}\left(x^i dy^i - y^i dx^i\right).$$

Integrating this equation one gets

$$z(t) = \frac{1}{2}\int_0^t \sum_{i=1}^{r}\left(x^i\frac{dy^i}{dt} - y^i\frac{dx^i}{dt}\right)dt = \frac{1}{2}\sum_{i=1}^{r} A_i,$$

where A_i is the area enclosed by the curve $(x^i(t), y^i(t))$ and the chord joining the origin with (x^{1i}, y^{1i}). In order to reach the point q, the curve $(x^i(t), y^i(t))$ must satisfy $z(1) = z^1$. Geometrically, it is clear that a curve $(x^i(t), y^i(t))$ linking the points p and q while enclosing the area prescribed by z^1 will always exist. Thus, the integral curve $\gamma(t)$ given by

$$(z(t), x^1(t), \ldots, x^r(t), y^1(t), \ldots, y^r(t), tu^1(t), \ldots, tu^s(t))$$

has $c(0) = p$ i $c(1) = q$ and satisfies the equation $\alpha = 0$. Therefore, the system therefore has the local accessibility property.

$\square$

4.2 Systems of Codimension Two

We now consider Pfaffian systems of codimension two. We are again interested in performing coordinate changes so that the generators of these Pfaffian systems are in some normal form.

Theorem 4.2.1 (Engels theorem)**.** *Let I be a dimension two codistribution, spanned by*

$$I = \langle \alpha^1, \alpha^2 \rangle$$

of four variables. If the derived flag satisfies

$$\dim I^{(1)} = 1,$$

$$\dim I^{(2)} = 0,$$

then, there exist coordinate z^1, z^2, z^3, z^4 such that

$$I = \{dz^4 - z^3 dz^1, dz^3 - z^2 dz^1\}.$$

Proof. Choose a basis of I adapted to the derived flag; that is $I^{(0)} = I = \{\alpha^1, \alpha^2\}$, $I^{(1)} = \{\alpha^1\}$ and $I^{(2)} = \{0\}$. Choose α^3 and α^4 to complete the basis. Since $I^{(2)} = \{0\}$ we have

$$d\alpha^1 \wedge \alpha^1 \neq 0,$$

while

$$(d\alpha^1)^2 \wedge \alpha^1 = 0,$$

since it is a 5-form on a 4-dimensional space. Therefore, α^1 has rank 1. By Pfaff's theorem, we know that there exists a coordinate change such that

$$\alpha^1 = dz^4 - z^3 dz^1.$$

Taking the exterior derivative, we have that

$$d\alpha^1 = -dz^3 \wedge dz^1 = dz^1 \wedge dz^3.$$

Now, since $\alpha^1 \in I^{(1)}$, the definition of the first derived system will imply that

$$d\alpha^1 \wedge \alpha^1 \wedge \alpha^2 = 0,$$

and thus

$$dz^1 \wedge dz^3 \wedge \alpha^1 \wedge \alpha^2 = 0.$$

Therefore, α^2 must be a linear combination of dz^1, dz^3 and α^1:

$$\alpha^2 \equiv a(x)dz^3 + b(x)dz^1 \mod \alpha^1.$$

By definition, this means that

$$\alpha^2 + \lambda(x)\alpha^1 = a(x)dz^3 + b(x)dz^1.$$

Now if either $a(x) = 0$ or $b(x) = 0$ it will imply that $d\alpha^2 \wedge \alpha^1 \wedge \alpha^2 = 0$ and thus the flag assumptions are violated because if $I^{(0)} = \{\alpha^1, \alpha^2\}$ and $I^{(1)} = \{\alpha^1\}$ that implies $d\alpha^2 \not\equiv 0 \mod \alpha^1, \alpha^2$. Thus $a(x) \neq 0$, then

$$\frac{1}{a(x)}\alpha^2 + \frac{\lambda(x)}{a(x)}\alpha^1 = dz^3 + \frac{b(x)}{a(x)}dz^1,$$

and if we set $z^2 = -\frac{b(x)}{a(x)}$ and setting

$$\frac{1}{a(x)}\alpha^2 + \frac{\lambda(x)}{a(x)}\alpha^1 = dz^3 - z^2 dz^1,$$

and thus

$$I = \{\alpha^1, \alpha^2\} = \left\{\alpha^1, \frac{1}{a(x)}\alpha^2 + \frac{\lambda(x)}{a(x)}\alpha^1\right\} = \{dz^4 - z^3dz^1, dz^3 - z^2dz^1\}.$$

□

It should be noted that the dimension assumption is only used in the proof so it is guaranteed that $(d\alpha^1)^2 \wedge \alpha^1 = 0$. If α^1 as rank 1, this equality holds by definition.

Corollary 4.2.2. *Let $I = \{\alpha^1, \alpha^2\}$ be a two-dimensional codistribution. If the derived flag satisfies $\dim I^{(1)} = 1$, $\dim I^{(2)} = 0$ and $\alpha^1 \in I^{(1)}$ has rank 1, then there exist coordinates z^1, z^2, z^3, z^4 such that*

$$I = \{dz^4 - z^3dz^1, dz^3 - z^2dz^1\}.$$

Proof. The proof is deduced from the Engel's theorem.

□

4.3 The Goursat Normal Form

Engel's theorem can be generalized to a system with m configuration variables and $m-2$ constraints.

Theorem 4.3.1 (Goursat Normal Form)**.** *Let I be a Pfaffian system spanned by s 1-forms*

$$I = \{\alpha^1, \ldots, \alpha^s\},$$

on a space of dimension $m = s+2$. Suppose that there exists an integrable form π with $\pi \neq 0 \mod I$ satisfying the Goursat congruences,

$$\begin{aligned} &d\alpha^i \equiv -\alpha_{i+1} \wedge \pi \mod \alpha^1, \ldots, \alpha^i, \quad 1 \leq i \leq s-1, \\ &d\alpha^s \not\equiv 0 \mod I. \end{aligned} \tag{4.1}$$

Then there exists a coordinate system $z^1, z^2, \ldots, z^m$ in which the Pfaffian system is in Goursat normal form:

$$I = \{dz^3 - z^2 dz^1, dz^4 - z^3 dz^1, \ldots, dz^m - z^{m-1} dz^1\}.$$

Proof. The Goursat congruences can be expressed as

$$\begin{aligned} d\alpha^1 &\equiv -\alpha^2 \wedge \pi \mod \alpha^1, \\ d\alpha^2 &\equiv -\alpha^3 \wedge \pi \mod \alpha^1, \alpha^2, \\ &\vdots \\ d\alpha^{s-1} &\equiv -\alpha^s \wedge \pi \mod \alpha^1, \alpha^2, \ldots, \alpha^{s-1}, \\ d\alpha^s &\equiv -\alpha^{s+1} \wedge \pi \mod \alpha^1, \alpha^2, \ldots, \alpha^s, \end{aligned}$$

where $\alpha^{s+1} \notin I$. It can be shown that $\{\alpha^{s+1}, \pi\}$ must form a complement to I. This basis satisfies the Goursat congruences and it is adapted to the derived flag of I:

$$\begin{aligned} I^{(0)} &= \{\alpha^1, \alpha^2, \ldots, \alpha^s\}, \\ I^{(1)} &= \{\alpha^1, \ldots, \alpha^{s-1}\}, \\ &\vdots \\ I^{(s-1)} &= \{\alpha^1\}, \\ I^{(s)} &= \{0\}. \end{aligned}$$

From the Goursat congruences,

$$d\alpha^1 \equiv -\alpha^2 \wedge \pi \mod \alpha^1,$$

which means that

$$d\alpha^1 = -\alpha^2 \wedge \pi + \alpha^1 \wedge \eta$$

for some 1-form η. But then we have that

$$\begin{aligned} d\alpha^1 \wedge \alpha^1 &= -\alpha^2 \wedge \pi \wedge \alpha^1 \neq 0, \\ (d\alpha^1)^2 \wedge \alpha^1 &= 0 \end{aligned}$$

which means that α^1 has rank 1. We can therefore apply Pfaff's theorem and suppose that multiplying α^1 by a certain factor if it is necessary, α^1 can be expressed as

$$\alpha^1 = dz^m - z^{m-1}dz^1$$

of some choice of z^1, z^{m-1}, z^m. Furthermore, by Corollary 4.2.2 we can express α^2 as

$$\alpha^2 = dz^{m-1} - z^{m-2}dz^1. \tag{4.2}$$

In these new coordinates we have

$$d\alpha^1 \wedge \alpha^1 = -dz^{m-1} \wedge dz^1 \wedge dz^m.$$

Now, we have that

$$d\alpha^1 \wedge \alpha^1 \wedge \pi = \pi \wedge (-dz^{m-1} \wedge dz^1 \wedge dz^m) = \pi \wedge (-\alpha^2 \wedge \pi \wedge \alpha^1) = 0,$$

and therefore π is a linear combination of dz^1, dz^{m-1}, dz_n. Noting that $dz^{m-1} \equiv z^{m-2}dz^1 \mod \alpha^1, \alpha^2$,

$$\begin{aligned} \pi &= adz^1 + bdz^{m-1} + cdz^m, \\ &= adz^1 + bz^{m-2}dz^1 + cz^{m-1}dz^1 \mod \alpha^1, \alpha^2 \end{aligned}$$

where $\psi = a + bz^{m-2} + cz^{m-1}$ is nonzero, since we have assumed that $\pi \neq 0 \mod I$. From the Goursat congruences we have that

$$d\alpha^2 = -\alpha^3 \wedge \pi \mod \alpha^1, \alpha^2,$$

while from (4.2) we have

$$d\alpha^2 = -dz^{m-2} \wedge dz^1,$$

and thus

$$-dz^{m-2} \wedge dz^1 = -\alpha^3 \wedge \pi \mod \alpha^1, \alpha^2,$$

which means that

$$\alpha^3 = \lambda(x)dz^{m-2} \mod dz^1, \alpha^1, \alpha^2,$$

for a nonzero function $\lambda(x)$. Therefore, we can rewrite this as

$$\alpha^3 = dz^{m-2} - \frac{1}{\lambda(x)}dz^1 \mod dz^1, \alpha^1, \alpha^2,$$

and if we set $z^{m-3} = 1/\lambda(x)$ we have

$$\alpha^3 = dz^{m-2} - z^{m-3}dz^1 \mod \alpha^1, \alpha^2,$$

and we can therefore let

$$\alpha^3 = dz^{m-2} - z^{m-3}dz^1.$$

If we inductively continue this procedure using the Goursat congruences we obtain

$$\begin{aligned} \alpha^4 &= dz^{m-3} - z^{m-4}dz^1, \\ &\vdots \\ \alpha^s &= dz^3 - z^2dz^1. \end{aligned}$$

Now, from the Goursat congruences we have that

$$d\alpha^s \neq 0 \mod I,$$

and, therefore,

$$\alpha^1 \wedge \alpha^2 \wedge \cdots \wedge \alpha^s \wedge d\alpha^s \neq 0.$$

If we substitute the α^i into the above expression we obtain

$$dz^1 \wedge dz^2 \wedge \cdots \wedge dz^m \neq 0,$$

and therefore the function $z^1, \ldots, z^m$ can serve as a local coordinate system.

□

The following example illustrates the power of the Goursat's theorem by applying it in order to linearize a nonlinear system. Note that the integral curves of a system in Goursat normal form are completely determined by two arbitrary functions in one variable and their derivatives. For example, once $z^1(\tau)$ and $z^s(\tau)$ are known, all of the other coordinates are determined from

$$z^i = \frac{\dot{z}^{i+1}(\tau)}{\dot{z}^i(\tau)},$$

where the dot indicates the standard derivative with respect to the independent variable τ. Because of this property, these two coordinates are sometimes referred to as *linearizing outputs for the Pfaffian system.*

Example 4.3.1 (Feedback Linearization by Goursat Normal Form)**.** Consider the following nonlinear system with s configuration variables and a single input

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1, \ldots, x_s, u), \\
\dot{x}_2 &= f_2(x_1, \ldots, x_s, u), \\
&\vdots \\
\dot{x}_s &= f_s(x_1, \ldots, x_s, u).
\end{aligned}$$

Equivalently, we can look at the following Pfaffian system,

$$I = \{dx^i - f_i(x^1, \ldots, x^s, u)dt\}, \quad 1 \leq i \leq s.$$

The system is of codimension 2 since we have s constraints and $s+2$ variables, namely $x^1, \ldots, x^s, u, t$. Assume that the form $\pi = dt$ satisfies the Goursat congruences. Then by Goursat's theorem there exists a coordinate transformation $z = \Phi(x, u, t)$ such that I is generated by

$$I = \{dz^3 - z^2 dz^1,\, dz^4 - z^3 dz^1,\, \ldots,\, dz^{s+2} - z^{s+1} dz^1\}.$$

The annihilating distribution of the above codistribution is

$$\begin{aligned}
\dot{z}^1 &= v_1, \\
\dot{z}^2 &= v_2, \\
\dot{z}^3 &= z^2 v_1, \\
&\vdots \\
\dot{z}^{s+2} &= z^{s+1} v_1,
\end{aligned}$$

which, if we set $v_1 = 1$, is clearly a linear system. If it turns out that the z^1 coordinate corresponds to time in the original coordinates, that is $z^1 = t$, then the connection becomes even more clear. Goursat's theorem can, thus, be used to linearize single-input nonlinear systems that satisfy the Goursat congruences.

4.4 Converting Systems to Chained Form

Chained form is dual to the Goursat normal form presented above. That is, a system with constraints in Goursat normal form

$$I = \{dz^3 - z^2 dz^1, dz^4 - z^3 dz^1, \ldots, dz^m - z^{m-1} dz^1\}$$

can always be written as a control system in chained form by choosing

$$\begin{aligned} g_1 &= \frac{\partial}{\partial z^1} + z^2 \frac{\partial}{\partial z^3} + \cdots + z^{m-1} \frac{\partial}{\partial z^m} \\ g_2 &= \frac{\partial}{\partial z^2} \end{aligned}$$

which form a basis of the distribution annihilated by I. Thus, we can formulate the problem of finding a basis for the constraints, which is in Goursat form, as the problem of finding a feedback transformation to convert a system to chained form.

The Goursat congruences are somewhat unsatisfying since they require existence of a 1-form π. Necessary and sufficient conditions for the existence of such a π, and hence converting a set of constraints into Goursat normal form.

So, let $I = \{\alpha^1, \ldots, \alpha^s\}$ be a codistribution of $\mathbb{R}^m$ and write $\Delta = I^\perp$ for the distribution which spans the null space of the codistribution. We define two nested sets of distributions

$$\begin{array}{lcllcl} E_0 & = & \Delta & F_0 & = & \Delta \\ E_1 & = & E_0 + [E_0, E_0] & F_1 & = & F_0 + [F_0, F_0] \\ & \vdots & & & \vdots & \\ E_{i+1} & = & E_i + [E_i, E_i] & F_{i+1} & = & F_i + [F_i, F_0]. \end{array} \tag{4.3}$$

Under the assumption that each distribution is constant rank, the two sequences have finite length (possibly different).

The filtration $\{F_i\}$ is the one which usually appears in the context of nonlinear controllability and beedback linearization. In particular, F_i consists of all brackets up to order i. The distribution E_i also contains all brackets of order i, but may contain additional Lie products of higher order. This is due to the iterative construction of F_i. The filtration E_i is precisely the sequence of distributions which is perpendicular to the derived flag of $I = \Delta^\perp$.

The following theorem allows us to completely characterize the set of systems which are equivalent to a system in chained (or Goursat) form in the case that the relative growth vector of the system is $\sigma = (2, 1, \ldots, 1)$. We will apply this results in the chapter about N-Trailer.

Theorem 4.4.1. *Given a 2-dimensional distribution $\Delta = I^{\perp}$, define E_i and F_i as in (4.3). Suppose that E_i and F_i satisfy*

$$\dim E_i = \dim F_i = i + 2 \quad 0 \leq i \leq m - 2.$$

Then, there exists a local basis $\{\alpha^1, \ldots, \alpha^s\}$ and a 1-form π such that the Goursat congruences are satisfied.

5
Procedures

In this section, we will give a series of steps and explanations needed to be followed to find the behavior of the state variables of a given system. In this paper, we consider driftless control systems with two inputs over a manifold M, i.e.; systems of the form

$$\dot{x} = g_1(x)\, u_1 + g_2(x)\, u_2,$$

$x \in M$, called *nonholonomic systems* or *driftless systems* over a m-dimensional manifold M. The associated distribution to this type of systems is generated by the vector fields $g_1,\, g_2 \in \mathfrak{X}(M)$

$$\Delta = \langle g_1, g_2 \rangle.$$

The dual of this distribution is a subspace of the cotangent space T^*M defined, in this case, as follows:

$$\Delta^{\perp} = \{\omega \in \Lambda^1(M) \,:\, i_g(\omega) = 0,\, \forall g \in \Delta\rangle$$

where the 1-forms have to be linearly independents. Notice that since $\dim \Delta = 2$, then $\dim \Delta^{\perp} = m - 2$. Usually, we will work with $M = \mathbb{R}^{m+2}$. By the definition (3.3.1), the associated Pffafian system to our control systems is

$$\alpha^1 = \alpha^2 = \cdots = \alpha^m = 0$$

that is a system of codimension 2.

In the previous chapter, we saw how to express the basis elements of the codistribution in the Goursat normal form when the Pfaffian system is of codimension 2 or greater than 2 respectively. Thus, being following the constructive demonstration of the Pfaffian and Engel's theorems or following the developed theory about the Goursat normal form, given a Pfaffian system on $\mathbb{R}^{m+2}$, we are able to find chains of integrators so that the ideal generated by the 1-forms belonging on the codistribution is expressed as

$$I = \{\alpha^1, \alpha^2, \dots, \alpha^m\} = \{dz_3 - z_2 dz_1, dz_4 - z_3 dz_1, \dots, dz_{m+2} - z_{m+1} dz_1\}.$$

Once found the change of the 1-forms to the Goursat normal form, we want to seek for two generic vector fields $\bar{g}_1$ and $\bar{g}_2$ such that the contraction with all the 1-forms is zero, i.e.;

$$i_{\bar{g}_j}(dz_{i+1} - z_i dz_1) = 0$$

for $i = 2, \dots, m+1$. The solutions are

$$\begin{aligned}\bar{g}_1 =& \frac{\partial}{\partial z_1} + z_2 \frac{\partial}{\partial z_3} + \cdots + z_{m+1} \frac{\partial}{\partial z_{m+2}} \\ \bar{g}_2 =& \frac{\partial}{\partial z_2}.\end{aligned}$$

Often, the system found by doing the contraction of the fields with the 1-forms and the system obtained by derivating the variables $\{z\}$ are not the same. To achieve the last one being in the canonical Goursat form, it should be necessary to do a feed-back. Finally, we will establish the diffeomorphism that matches the state variables $\{x_1, \dots, x_{m+2}\}$ and $\{z_1, \dots, z_{m+2}\}$.

Since we can express variables z as functions of x variables, like $z_i = f_i(x_1, \dots, x_{m+2})$, and taking into account that the derivate of the coordinate x_i is the i-th component of $\dot{x} = g_1(x)\, u_1 + g_2(x)\, u_2$,

$$\dot{x}_i = g_1^i(x) u_1 + g_2^i(x) u_2,$$

we can define the derivate of $z_i = f_i(x_1, \dots, x_{m+2})$ as

$$\dot{z}_i = \sum_{j=1}^{m+2} \frac{\partial f_i}{\partial x_j} \dot{x}_j = \sum_{j=1}^{m+2} \frac{\partial f_i}{\partial x_j} (g_1^j u_1 + g_2^j u_2).$$

Therefore,

$$\begin{cases} \dot{z}_1 = \displaystyle\sum_{i=1}^{m+2} \frac{\partial f_1}{\partial x_i} (g_1^i u_1 + g_2^i u_2) \\ \dot{z}_2 = \displaystyle\sum_{i=1}^{m+2} \frac{\partial f_2}{\partial x_i} (g_1^i u_1 + g_2^i u_2) \\ \quad \vdots \\ \dot{z}_{m+1} = \displaystyle\sum_{i=1}^{m+2} \frac{\partial f_{m+1}}{\partial x_i} (g_1^i u_1 + g_2^i u_2) \\ \dot{z}_{m+2} = \displaystyle\sum_{i=1}^{m+2} \frac{\partial f_{m+2}}{\partial x_i} (g_1^i u_1 + g_2^i u_2). \end{cases}$$

Then, we define two feedback laws that give us the new controls $\bar{u}_1$ and $\bar{u}_2$

$$\bar{u}_1 = \sum_{i=1}^{m+2} \frac{\partial f_1}{\partial x_i}(g_1^i u_1 + g_2^i u_2)$$
$$\bar{u}_2 = \sum_{i=1}^{m+2} \frac{\partial f_2}{\partial x_i}(g_1^i u_1 + g_2^i u_2).$$

Then, the system expressed in the new state variables becomes

$$\begin{cases} \dot{z}_1 = \bar{u}_1 \\ \dot{z}_2 = \bar{u}_2 \\ \dot{z}_3 = z_2 \bar{u}_1 \\ \quad \vdots \\ \dot{z}_{m+1} = z_m \bar{u}_1 \\ \dot{z}_{m+2} = z_{m+1} \bar{u}_1. \end{cases} \tag{5.1}$$

and we call it *system in the canonical form associated to the Goursat form.* Notice that sometimes it is convenient to add new state variables to achieve the same dimensions.

It is immediate to see that

$$y_1 = z_1 \quad y_2 = z_{m+2}$$

are the flat outputs of the system (5.1), because one can express the variables $\{z\}$ depending on the flat outputs and its derivatives, let's see it:

$$\begin{cases} z_2 = \dfrac{\dot{z}_1}{\bar{u}_1} = \dfrac{\dot{y}_2}{\dot{y}_1} \\ z_3 = \dfrac{\dot{z}_2}{\bar{u}_1} = z_3(\dot{y}_1, \ddot{y}_1, \dot{y}_2, \ddot{y}_2) \\ \vdots \\ z_{m+1} = \dfrac{\dot{z}_m}{\bar{u}_1} = z_{m+1}(\dot{y}_1, \ldots, y_1^{(m)}, \dot{y}_2, \ldots, y_2^{(m)}). \end{cases}$$

To consider the diffeomorphism between the new variables and

$$\{y_1, \dot{y}_1, \ldots, y_1^{(m)}, y_2, \ldots, y_2^{(m)}\}$$

we have to consider the prolongation of m new state variables

$$z_{m+i+3} = \frac{d^i}{dt^i}\bar{u}_1 = \bar{u}_1^{(i)} \text{ for } 0 \le i \le m-1,$$

and two feedback laws

$$
\begin{aligned}
v_1 &= \frac{d^{m+1}}{dt^{m+1}} y_1(t) = \bar{u}_1^{(m)} \\
v_2 &= \bar{u}_2.
\end{aligned}
$$

The goal to be achieved in a system, given initial and final conditions to the state variables, is to find motor controls that at each instant of time the solution trajectories of the original system pass through c_i and c_f.

We will impose then, the conditions c_i and c_f to the original state variables. Through the diffeomorphism $\{x\} \leftrightarrow \{z\}$ we will find the corresponding initial and final conditions for $\{z\}$ that will be denoted by $\overline{c_i}$ and $\overline{c_f}$. With this data and adding conditions to $z_{m+3}, \ldots, z_{2m+2}$, we find the conditions that have to be satisfied by the flat outputs and its derivatives thanks to the diffeomorphism $\{z\} \leftrightarrow \{y\}$ and that will be denoted by $\hat{c_i}$ i $\hat{c_f}$.

Given $2m+2$ initial and final conditions[1], in total $4m+4$ conditions, there exist two unique polynomial of degree $2m+1$ denoted by $P_{2m+1}(t)$ and $Q_{2m+1}(t)$ such that

$$
y_1(t) = P_{2m+1}(t), \quad y_2(t) = Q_{2m+1}(t).
$$

Imposing the above conditions, the interpolation polynomials are determined and, consequently, the flat outputs expression involving the time is found. Clearly, its derivatives will be depend also on time.

We have commented above that the variables $\{z\}$ can be expressed involving the flat outputs and its derivatives that involve the time. The flat output system becomes

$$
\begin{cases}
y_1^{(m+1)} = \dfrac{d^{m+1}}{dt^{m+1}} y_1 = w_1 \\
y_2^{(m+1)} = \dfrac{d^{m+1}}{dt^{m+1}} y_2 = w_2
\end{cases}
$$

which is at the same time

$$
\begin{cases}
y_1^{(m+1)} = \dfrac{d}{dt} y_1^{(m)} = \dfrac{d}{dt} \bar{u}_1^{(m-1)} = \dfrac{d}{dt} z_{2m+2} = v_1 \\
y_2^{(m+1)} = \dfrac{d}{dt} y_2^{(m)} = \dfrac{d^m}{dt^m} \dot{y}_2 = \dfrac{d^m}{dt^m} \dot{z}_{m+2} = \dfrac{d^m}{dt^m} (z_{m+1} z_{m+3}) = \alpha + \beta v_1 + \gamma v_2.
\end{cases}
$$

Therefore,

$$
\begin{aligned}
w_1 &= v_1 \\
w_2 &= \alpha + \beta v_1 + \gamma v_2.
\end{aligned}
$$

Finally, we find $\bar{u}_1$ and $\bar{u}_2$ as a function of v_1 and v_2, and we find the original controls u_1 and u_2 solving the system

$$
\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^{m+2} \dfrac{\partial f_1}{\partial x_i} g_1^i & \sum_{i=1}^{m+2} \dfrac{\partial f_1}{\partial x_i} g_2^i \\ \sum_{i=1}^{m+2} \dfrac{\partial f_2}{\partial x_i} g_1^i & \sum_{i=1}^{m+2} \dfrac{\partial f_2}{\partial x_i} g_2^i \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.
$$

[1] Notice that for the flat outputs y_1 and y_2 we have $2m+2$ initial and final conditions.

6
Planar Space Robot

Consider a simplified model of a planar robot, as shown in Figure 6.1. This robot consists of two arms connected to a central body via revolution joints. If the robot is free-floating, then the law of conservation of angular momentum implies that moving the arms causes the central body to rotate. In the case that the angular momentum is zero, this conservation law can be viewed as a Pfaffian constraint on the system. Let M and I represent the mass and inertia of the central body and let m represent the mass of the arms, which we take to be concentrated at the tips. The revolution joints are located at a distance r from the middle of the central body and the links attached to these joints have length l.

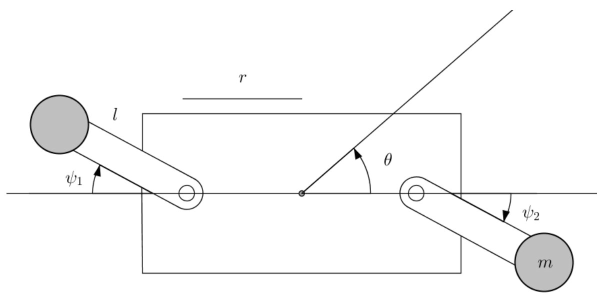

Figure 6.1: A simplified model of planar space robot.

We let (x_1, y_1) and (x_2, y_2) represent the position of the ends of each of the arms (in terms of θ, ψ_1 and ψ_2). Let θ be the angle of the central body with respect to the horizontal, ψ_1 and ψ_2 the angles of the left arm and right arms with respect to the central body, and $p \in \mathbb{R}^2$ the location of a point on the central body (say the center of mass). The kinetic energy of the system (See [2, pages 334–335]) has the form

$$\begin{aligned} K &= \frac{1}{2}(M+2m)\|\dot{p}\|^2 + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}m(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m(\dot{x}_2^2 + \dot{y}_2^2) \\ &= \frac{1}{2}(M+2m)\|\dot{p}\|^2 + \frac{1}{2}\begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\theta} \end{bmatrix}^{\perp} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \dot{\psi}_1 \\ \dot{\psi}_2 \\ \dot{\theta} \end{bmatrix}, \end{aligned}$$

where a_{ij} can be calculated as

$$\begin{aligned}
a_{11} &= a_{22} = ml^2 \\
a_{12} &= 0 \\
a_{13} &= ml^2 + mr\cos(\psi_1) \\
a_{23} &= ml^2 + mr\cos(\psi_2) \\
a_{33} &= I + 2ml^2 + 2mr^2 + 2mrl\cos(\psi_1) + 2mrl\cos(\psi_2).
\end{aligned}$$

Note that the kinetic energy of the system is independent of the variable θ. It, therefore, follows from Lagrange's equations (See [4]) that in the absence of external forces,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta} = 0.$$

Thus, the quantity $\dfrac{\partial L}{\partial \dot{\theta}}$ is a constant of the motion. This is precisely the angular momentum, α, of the system:

$$\alpha = a_{13}\dot{\psi}_1 + a_{23}\dot{\psi}_2 + a_{33}\dot{\theta}.$$

If the initial angular momentum is zero, then conservation of angular momentum ensures that the angular momentum stays zero, giving the following constraint equation

$$a_{13}(\psi)\dot{\psi}_1 + a_{23}(\psi)\dot{\psi}_2 + a_{33}(\psi)\dot{\theta} = 0. \tag{6.1}$$

Since the actuated variables are the hinge angles of the left and right arm, we choose as inputs $u_1 = \dot{\psi}_1$ and $u_2 = \dot{\psi}_2$. Using these in Eq. (6.1) and setting $q = (\psi_1, \psi_2, \theta)^{\perp}$, we get

$$\dot{q} = g_1(q)u_1 + g_2(q)u_2$$

where

$$g_1(q) = \begin{bmatrix} 1 \\ 0 \\ -\dfrac{a_{13}}{a_{33}} \end{bmatrix} \qquad g_2(q) = \begin{bmatrix} 0 \\ 1 \\ -\dfrac{a_{23}}{a_{33}} \end{bmatrix}.$$

Let $x = (x_1, x_2, x_3)^{\perp} = (\psi_1, \psi_2, \theta)^{\perp}$, then Eq. (6.1) is written as

$$\alpha = a_{13}(x)\dot{x}_1 + a_{23}(x)\dot{x}_2 + a_{33}(x)\dot{x}_3.$$

In the x's variables, the original system is written as $\dot{x} = g_1(x)u_1 + g_2(x)u_2$ which can be expressed as

$$\begin{cases} \dot{x}_1 = u_1 \\ \dot{x}_2 = u_2 \\ \dot{x}_3 = -\dfrac{a_{13}}{a_{33}}u_1 - \dfrac{a_{23}}{a_{33}}u_2. \end{cases} \tag{6.2}$$

The exterior derivative of α is

$$d\alpha = -2mrl\sin(x_1)\,dx_1 \wedge dx_3 - 2mrl\sin(x_2)\,dx_2 \wedge dx_3.$$

Let's find the rank of α:

$$d\alpha \wedge \alpha = 2m^2 rl\,(r\sin(x_1 - x_2) + l^2(\sin(x_1) - \sin(x_2)))\,dx_2 \wedge dx_2 \wedge dx_3 \neq 0$$

for all $x_1, x_2 \neq k\pi$, $k \in \mathbb{Z}$. We know that $d\alpha \wedge \alpha$ is a 3-form in a 3-dimensional space, therefore $(d\alpha)^r \wedge \alpha = 0$ for all $r \geq 2$. So, the rank of α is 1.

Now, we can apply the Pfaff theorem and rewrite α as $\alpha = dz_3 - z_2 dz_1$. It's easy to check that

$$\begin{cases} z_1 = x_3, \\ z_2 = -a_{33} = -I - 2ml^2 - 2mr^2 - 2mrl\cos(x_1) - 2mrl\cos(x_2)), \\ z_3 = ml^2(x_1 + x_2) + mr(\sin(x_1) + \sin(x_2)), \end{cases}$$

is the desired change of variables. To express the system in the new variables, we have to find $\bar{g}_1, \bar{g}_2, \bar{u}_1$ and $\bar{u}_2$ such that $\dot{z} = \bar{g}_1\bar{u}_1 + \bar{g}_2\bar{u}_2$. The vector fields $\bar{g}_i$ must satisfy $i_{\bar{g}_i}(\alpha) = 0$, so they are

$$\begin{aligned} \bar{g}_1 &= \frac{\partial}{\partial z_2} = (0,1,0)^{\perp} \\ \bar{g}_2 &= \frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_3} = (1,0,z_2)^{\perp}. \end{aligned} \tag{6.3}$$

Our new system is written as

$$\dot{z} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \bar{u}_1 + \begin{pmatrix} 1 \\ 0 \\ z_2 \end{pmatrix} \bar{u}_2 + .$$

By construction of the change of variables that transforms x into z we know that

$$\begin{cases} \dot{z}_1 = \dot{x}_3 = -\dfrac{a_{13}}{a_{33}}u_1 - \dfrac{a_{23}}{a_{33}}u_2 = \bar{u}_2 \\ \dot{z}_2 = 2mrl\sin(x_1)\dot{x}_1 + 2mrl\sin(x_2)\dot{x}_2 = 2mrl\sin(x_1)u_1 + 2mrl\sin(x_2)u_2 = \bar{u}_1 \\ \dot{z}_3 = a_{13}\dot{x}_1 + a_{23}\dot{x}_2 = a_{13}u_1 + a_{23}u_2 = z_2\bar{u}_2. \end{cases}$$

Therefore, our new controls $\bar{u}_1$ and $\bar{u}_2$ are

$$\begin{aligned} \bar{u}_1 &= 2mrl\sin(x_1)u_1 + 2mrl\sin(x_2)u_2, \\ \bar{u}_2 &= -\frac{a_{13}}{a_{33}}u_1 - \frac{a_{23}}{a_{33}}u_2. \end{aligned}$$

Now, we are looking for the flat outputs $y_1(z,\bar{u})$ and $y_2(z,\bar{u})$. The fact that $\dot{z}_1, \dot{z}_2$ and $\dot{z}_3$ depend on $z_2, \bar{u}_1$ and $\bar{u}_2$, we can take as flat outputs $y_1 = z_1$ and $y_2 = z_3$. First of all, we find the control $\bar{u}_2$ as a function of $\dot{y}_1$

$$\dot{y}_1 = \dot{z}_1 = \bar{u}_2 \Longrightarrow \bar{u}_2 = \dot{y}_1.$$

Then, we can find z_2

$$\dot{y}_2 = \dot{z}_3 = z_2\bar{u}_2 = z_2\dot{y}_1 \Longrightarrow z_2 = \frac{\dot{y}_2}{\dot{y}_1}.$$

The z variables depend on the feedback laws and their derivatives like $z = z(y_1, \dot{y}_1, y_2, \dot{y}_2)$, but we cannot define a diffeomorphism yet. We must prolong the system adding a new state variable

$$z_4 = \bar{u}_2$$

and two new control laws

$$v_1 = \bar{u}_1, \quad v_2 = \dot{\bar{u}}_2.$$

Now, our system can be written as

$$\begin{cases} \dot{z}_1 = z_4 \\ \dot{z}_2 = v_1 \\ \dot{z}_3 = z_2 z_4 \\ \dot{z}_4 = v_2. \end{cases}$$

Therefore, the change of variables in the prolonged system is

$$\begin{cases} z_1 = y_1 \\ z_2 = \dot{y}_2/\dot{y}_1 \\ z_3 = y_2 \\ z_4 = \dot{y}_1. \end{cases}$$

Now, we got a diffeomorphism between $\{y_1, \dot{y}_1, y_2, \dot{y}_2\}$ and $\{z_1, z_2, z_3, z_4\}$ given by

$$\begin{cases} y_1 = z_1 \\ \dot{y}_1 = z_4 \\ y_2 = z_3 \\ \dot{y}_2 = z_2 z_4 \end{cases}$$

We have to check for which values this diffeomorphism exists and avoid the singularities when we impose the initial condition values. The determinants of the change of variables are the following

$$|Jz| = \begin{vmatrix} 0 & 0 & 1 \\ 2mrl\sin(x_1) & 2mrl\sin(x_2) & 0 \\ a_{13}(x) & a_{23}(x) & 0 \end{vmatrix} = 2m^2rl^3(\sin(x_1) - \sin(x_2)) + 2m^2r^2l\sin(x_1 - x_2)).$$

So, if $x_1 \neq x_2$ and $x_1, x_2 \neq k\pi, k \in \mathbb{Z}$, then the inverse exist. For the y variables,

$$|Jy| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & z_4 & 0 & z_2 \end{vmatrix} = -z_4,$$

which is invertible if $z_4 \neq 0$. The feedback law is given by

$$\begin{aligned} w_1 &= \ddot{y}_1 = \dot{z}_4 = v_2, \\ w_2 &= \ddot{y}_2 = \dot{z}_2 z_4 + z_2 \dot{z}_4 = z_4 v_1 + z_2 v_2. \end{aligned}$$

Inversely, we can compute the controls v_1 and v_2 as a function of $y_1, \dot{y}_1, y_2, \dot{y}_2, w_1$ and w_2 as

$$\begin{aligned} v_1 &= \frac{\dot{y}_1 w_2 - \dot{y}_2 w_1}{(\dot{y}_1)^2}, \\ v_2 &= w_1. \end{aligned}$$

Now we can express $\bar{u}_1$ and $\bar{u}_2$ as a function of z_1, z_2, z_3, z_4, v_1 and v_2 as

$$\begin{aligned} \bar{u}_1 &= v_1, \\ \dot{\bar{u}}_2 &= v_2. \end{aligned}$$

Finally, the initial controls u_1 and u_2 can be reached solving the system

$$\begin{pmatrix} \bar{u}_1 \\ \bar{u}_2 \end{pmatrix} = \begin{pmatrix} 2mrl\sin(x_1) & 2mrl\sin(x_2) \\ -\dfrac{a_{13}}{a_{33}} & -\dfrac{a_{23}}{a_{33}} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

So, u_1 and u_2 are

$$\begin{aligned} u_1 &= \frac{a_{23}}{2mrl(a_{23}\sin(x_1) - a_{13}\sin(x_2))}\bar{u}_1 + \frac{a_{33}\sin(x_2)}{a_{23}\sin(x_1) - a_{13}\sin(x_2)}\bar{u}_2, \\ u_2 &= -\frac{a_{13}}{2mrl(a_{23}\sin(x_1) - a_{13}\sin(x_2))}\bar{u}_1 - \frac{a_{33}\sin(x_1)}{a_{23}\sin(x_1) - a_{13}\sin(x_2)}\bar{u}_2. \end{aligned} \tag{6.4}$$

Consider $m = 1$, $I = 1$, $l = 2$, $r = 3l/4$, $t_0 = 0$ and $t_f = 1$ and take as initial and final conditions of x the values

$$\begin{aligned} x(0) =& (\psi_1(0), \psi_2(0), \theta(0)) = \left(\frac{7\pi}{8}, \frac{-\pi - 1}{8}, \frac{\pi}{2}\right) \\ x(1) =& (\psi_1(1), \psi_2(1), \theta(1)) = \left(\frac{5\pi}{8}, \frac{-3\pi + 1}{8}, \frac{3\pi}{4}\right). \end{aligned}$$

First of all, we must transform the initial and final conditions, $x(0)$ and $x(1)$, in terms of z variables. Since $z_4 = \bar{u}_2$ we can take as initial and final condition the values that we want. So, taking $z_4(0) = 1$ and $z_4(1) = 2$, the initial and final conditions in z variables are

$$\begin{aligned} z(0) =&(z_1(0), z_2(0), z_3(0), z_4(0)) = (1.570796327, -13.17048378, 8.756480043, 1)\\ z(1) =&(z_1(1), z_2(1), z_3(1), z_4(1)) = (2.356194490, -14.17319166, 3.723971711, 2). \end{aligned}$$

Finally, we transform the initial and final conditions of z in terms of $y = (y_1,\, \dot{y}_1,\, y_2,\, \dot{y}_2)$ as follows

$$\begin{aligned} y(0) =&(y_1(0), \dot{y}_1(0), y_2(0), \dot{y}_2(0)) = (1.570796327, 1, 8.756480043, -13.17048378)\\ y(1) =&(y_1(1), \dot{y}_1(1), y_2(1), \dot{y}_2(1)) = (2.356194490, 2, 3.723971711, -28.34638332)\,. \end{aligned}$$

Consider $P_3(t) = a_3t^3 + a_2t^2 + a_1t + a_0$ such that $P_3(t) = y_1(t)$. Let's find the coefficients of $P_3(t)$.

$$\begin{aligned} y_1(t) &= a_3t^3 + a_2t^2 + a_1t + a_0\\ \dot{y}_1(t) &= 3a_3t^2 + 2a_2t + a_1 \end{aligned}$$

For $t = 0$:

$$\begin{aligned} a_0 &= 1.570796327\\ a_1 &= 1. \end{aligned}$$

For $t = 1$:

$$\begin{aligned} a_3 + a_2 &= 2.3561945 - 2.570796327\\ 3a_3 + 2a_2 &= 2 - 1. \end{aligned}$$

Solving the linear system we find

$$\begin{aligned} a_2 &= -1.643805511\\ a_3 &= 1.429203674. \end{aligned}$$

Therefore,

$$y_1(t) = P_3(t) = 1.429203674t^3 - 1.643805511t^2 + t + 1.570796327. \tag{6.5}$$

Analogously, we proceed in the same way with $y_2(t) = Q_3(t) = b_3t^3 + b_2t^2 + b_1t + b_0$.

$$\begin{aligned} y_2(t) &= b_3t^3 + b_2t^2 + b_1t + b_0\\ \dot{y}_2(t) &= 3b_3t^2 + 2b_2t + b_1. \end{aligned}$$

For $t = 0$:

$$\begin{aligned} b_0 &= 8.756480043\\ b_1 &= -13.17048378. \end{aligned}$$

For $t = 1$:

$$\begin{aligned} b_3 + b_2 &= 3.723971711 - 4.414003737\\ 3b_3 + 2b_2 &= -28.34638332 + 13.17048378. \end{aligned}$$

Solving the linear system,

$$\begin{aligned} b_2 &= 39.589825884 \\ b_3 &= -31.451850436. \end{aligned}$$

Therefore,

$$y_2(t) = -31.451850436t^3 + 39.589825884t^2 - 13.17048378t + 8.756480043. \tag{6.6}$$

Now, we must find the feedback as a function of time

$$\begin{aligned} w_1 =& \frac{d^2}{dt^2} y_1(t) = \frac{d^2}{dt^2} P_3 = 8.575222044t - 3.287611022 \\ w_2 =& \frac{d^2}{dt^2} y_2(t) = \frac{d^2}{dt^2} Q_3 = -188.711102616t + 79.179651768. \end{aligned}$$

As a consequence, the controls v_1 and v_2 have the expression

$$\begin{aligned} v_1 =& \frac{\dot{y}_1 w_2 - \dot{y}_2 w_1}{(\dot{y}_1)^2} = \frac{-29.28719717t^2 - 75.77127978t + 35.88022413}{(4.287611022t^2 - 3.287611022t + 1)^2} \\ v_2 =& w_1 = 8.575222044t - 3.287611022. \end{aligned}$$

Finally, we obtain the expressions of $\bar{u}_1(t)$ and $\bar{u}_2(t)$ as a function of v_1 and v_2. For $\bar{u}_2$ we know that it satisfies the following equation,

$$\dot{\bar{u}}_2 = v_2(t) \Longrightarrow \bar{u}_2 = \dot{y}_1(t).$$

So, $\bar{u}_1(t)$ and $\bar{u}_2(t)$ are

$$\begin{aligned} \bar{u}_1(t) =& \frac{-29.28719717t^2 - 75.77127978t + 35.88022413}{(4.287611022t^2 - 3.287611022t + 1)^2} \\ \bar{u}_2(t) =& 4.287611022t^2 - 3.287611022t + 1. \end{aligned}$$

Undoing the feedback in the controls $\bar{u}_1$ and $\bar{u}_2$, we find the expression of the initial controls solving the system (6.4).

Before finding the controls $u_1(t)$ and $u_2(t)$, we can integrate (6.2) using the numerical method Runge-Kutta 45 implemented in Matlab.

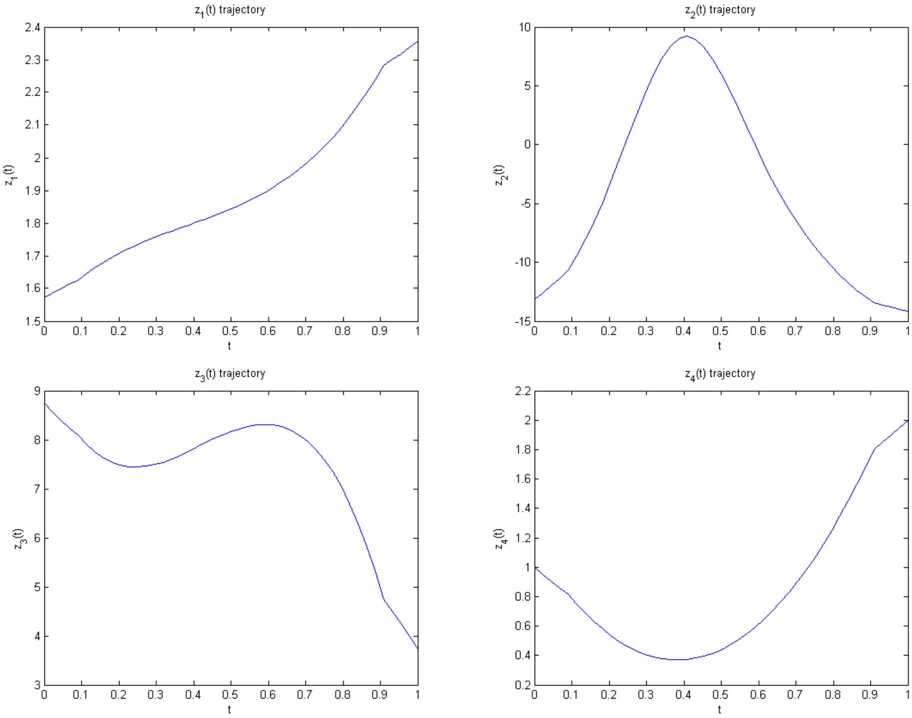

Figure 6.2: Trajectories of the z variables $z_1(t)$, $z_2(t)$, $z_3(t)$ and $z_4(t)$ respectively.

7 Mobile Robot with a Trailer

In this chapter, we will derive the kinematic model of a mobile robot with a trailer and then find the flat outputs of the system. The two-wheeled mobile robot is differentially driven and the trailer is attached at the center O of the mobile robot through a rotational joint as Figure 7.1 shows. In cartesian coordinates, the system's configuration is given by

$$q = (x_1, y_1, \theta_1, \theta_0)^T,$$

where x_1, y_1 are the position of the midpoint C of the trailer's axle. θ_1 and θ_0 are the heading angles of the trailer and the robot, respectively. L is the distance between the center of the mobile robot and the midpoint of the trailer's axle. Figure 7.1 shows the schematic of the system and its configuration. From the geometric relationship, the center position of the mobile robot is given as $x_0 = x_1 + L\cos(\theta_1)$, $y_0 = y_1 + L\sin(\theta_1)$.

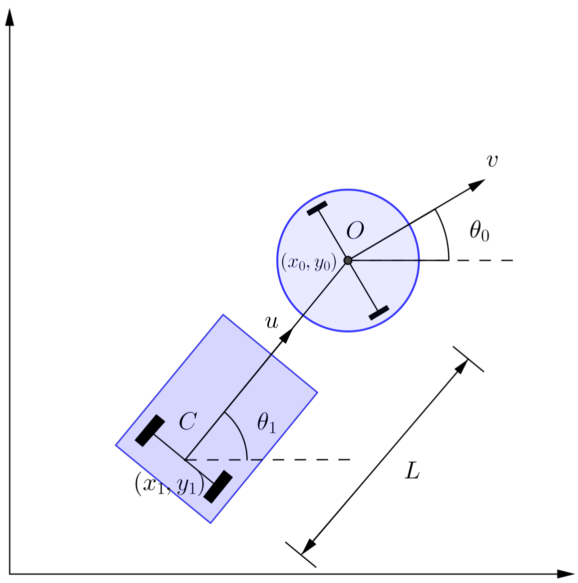

Figure 7.1: A differentially driven mobile robot with a trailer in Cartesian space described by $(x_1, y_1, \theta_1, \theta_0)$.

From the assumption of no-slip condition on the wheels of the robot and the trailer, the instantaneous velocities at C and O along their respective axles become zero. One gets nonholonomic constraints of the form

$$C(q)\dot{q} = 0,$$

where

$$C(q) = \begin{pmatrix} \sin(\theta_0) & -\cos(\theta_0) & -L\cos(\theta_0-\theta_1) & 0 \\ \sin(\theta_1) & -\sin(\theta_1) & 0 & 0 \end{pmatrix}.$$

When a matrix $S(q)$ spans the null space of $C(q)$, it is possible to define velocity vector $\nu(t)$ such that

$$\dot{q} = S(q)\nu(t). \tag{7.1}$$

Hence, if we represent the velocity vector ν as the heading speed v and the turning speed $\dot{\theta}_0$ of the robot, or $\nu = (v,\ \dot{\theta}_0)^T$, we can find that the matrix $S(q)$ can be written as

$$S(q) = \begin{pmatrix} \cos(\theta_0-\theta_1)\cos(\theta_1) & 0 \\ \cos(\theta_0-\theta_1)\sin(\theta_1) & 0 \\ \sin(\theta_0-\theta_1)/L & 0 \\ 0 & 1 \end{pmatrix}.$$

Therefore, $S(q)$ represents the kinematic model of the system.

If we define $x = (x_1,\ x_2,\ x_3,\ x_4)^T = (x_1,\ y_1,\ \theta_1,\ \theta_0)^T$, $u_1 = v$ and $u_2 = \dot{\theta}_0$, we can rewrite the system (7.1) as

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} \cos(x_3)\cos(x_4-x_3) \\ \sin(x_3)\cos(x_4-x_3) \\ \sin(x_4-x_3)/L \\ 0 \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} u_2. \tag{7.2}$$

The vector fields of the system (7.2) are

$$g_1 = \begin{pmatrix} \cos(x_3)\cos(x_4-x_3) \\ \sin(x_3)\cos(x_4-x_3) \\ \sin(x_4-x_3)/L \\ 0 \end{pmatrix} \text{ and } g_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

that define the distribution $\Delta = \langle g_1, g_2 \rangle$. The system (7.2) with the distribution Δ is controllable.

Now, we want to find the flat outputs using the Engels theorem. The annihilator of Δ is $I = \{\alpha_1, \alpha_2\}$ where

$$\alpha_1 = -\tan(x_3)dx_1 + dx_2 \text{ and } \alpha_2 = \frac{\tan(x_3-x_4)}{L\cos(x_3)}dx_1 + dx_3.$$

The derived flags of the ideal I are

$$I^{(0)} = \{\alpha_1, \alpha_2\},\ I^{(1)} = \{\alpha_1\},\ I^{(2)} = \{0\}.$$

We want to express I as $I = \{\alpha_1, \alpha_2\} = \{dz_4 - z_3 dz_1, dz_3 - z_2 dz_1\}$. It is easy to check that

$$\begin{cases} z_1 = x_1 \\ z_3 = \tan(x_3) \\ z_4 = x_2 \end{cases}$$

holds that $\alpha_1 = dz_4 - z_3 dz_1$. Now, we know that

$$\alpha_2 + \lambda(x)\alpha_1 = a(x)dz_3 + b(x)dz_1,$$

but α_2, dz_3 and dz_1 do not depend on dx_2. It implies that $\lambda(x) = 0$ and

$$\alpha_2 = a(x)dz_3 + b(x)dz_1.$$

Imposing that this equality holds term by term

$$\frac{\tan(x_3 - x_4)}{L\cos(x_3)}dx_1 + dx_3 = \frac{a(x)}{\cos^2(x_3)}dx_3 + b(x)dx_1,$$

we obtain that

$$a(x) = \cos^2(x_3) \text{ and } b(x) = \frac{\tan(x_3 - x_4)}{L\cos(x_3)}.$$

Following the proof of Engel's theorem, we know that $z_2 = -\dfrac{b(x)}{a(x)}$. Now, I can be expressed like $I = \{dz_4 - z_3 dz_1, dz_3 - z_2 dz_1\}$ using

$$\begin{cases} z_1 = x_1, \\ z_2 = -\dfrac{\tan(x_3 - x_4)}{L\cos^3(x_3)}, \\ z_3 = \tan(x_3), \\ z_4 = x_2. \end{cases} \tag{7.3}$$

The next step is to look for two vector fields $\bar{g}_1$ and $\bar{g}_2$, two controls $\bar{u}_1$ and $\bar{u}_2$ such that $\dot{z} = \bar{g}_1\bar{u}_1 + \bar{g}_2\bar{u}_2$, and

$$i_{\bar{g}_k}(dz_4 - z_3 dz_1) = 0 \text{ and } i_{\bar{g}_k}(dz_3 - z_2 dz_1) = 0 \text{ for } k = 1, 2.$$

This vector fields are

$$\begin{aligned} \bar{g}_1 &= \frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_3} + z_3\frac{\partial}{\partial z_4} = (1, 0, z_2, z_3)^T \\ \bar{g}_2 &= \frac{\partial}{\partial z_2} = (0, 1, 0, 0)^T. \end{aligned} \tag{7.4}$$

So, it means that

$$\dot{z} = \bar{g}_1\bar{u}_1 + \bar{g}_2\bar{u}_2 = \begin{pmatrix} 1 \\ 0 \\ z_2 \\ z_3 \end{pmatrix}\bar{u}_1 + \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}\bar{u}_2.$$

In these new variables the system (7.2) is expressed as

$$\begin{cases} \dot{z}_1 = \cos(x_3)\cos(x_4 - x_3)u_1 = \bar{u}_1 \\ \dot{z}_2 = -\dfrac{\sin(x_4 - x_3)(3\cos(x_3 - x_4)\sin(x_3 - x_4)\sin(x_3) + \cos(x_3))}{L^2\cos^2(x_3 - x_4)\cos^4(x_3)}u_1 \\ \qquad + \dfrac{1}{L\cos^2(x_3 - x_4)\cos^3(x_3)}u_2 = \bar{u}_2 \\ \dot{z}_3 = \dfrac{\sin(x_4 - x_3)}{L\cos^2(x_3)}u_1 = z_2\bar{u}_1 \\ \dot{z}_4 = \sin(x_3)\cos(x_4 - x_3)u_1 = z_3\bar{u}_1. \end{cases} \tag{7.5}$$

So, our new controls are of the form $\bar{u} = b(x)u$,

$$\begin{aligned}
\bar{u}_1 =& \cos(x_3)\cos(x_4 - x_3)u_1 \\
\bar{u}_2 =& -\frac{\sin(x_4 - x_3)(3\cos(x_3 - x_4)\sin(x_3 - x_4)\sin(x_3) + \cos(x_3))}{L^2\cos^2(x_3 - x_4)\cos^4(x_3)}u_1 \\
&+ \frac{1}{L\cos^2(x_3 - x_4)\cos^3(x_3)}u_2.
\end{aligned}$$

Later on, we are looking for the flat outputs $y_1(z, \bar{u})$ and $y_2(z, \bar{u})$. The fact that $\dot{z}_1, \dot{z}_2, \dot{z}_3$ and $\dot{z}_4$ depend on $z_2, z_3, \bar{u}_1$ and $\bar{u}_2$ allows us to take as flat outputs $y_1 = z_1$ and $y_2 = z_4$. First of all, we find the control $\bar{u}_1$ in function of $\dot{y}_1$

$$\dot{y}_1 = \dot{z}_1 = \bar{u}_1 \Rightarrow \bar{u}_1 = \dot{y}_1,$$

then we can find z_3, z_2 and $\bar{u}_2$ in terms of $\dot{y}_1$ and $\dot{y}_2$

$$\begin{cases}
\dot{y}_2 = \dot{z}_4 = z_3\bar{u}_1 = z_3\dot{y}_1 \\
\dot{z}_3 = \dfrac{\ddot{y}_2\dot{y}_1 - \ddot{y}_1\dot{y}_2}{(\dot{y}_1)^2} = z_2\bar{u}_1 = z_2\dot{y}_1 \\
\dot{z}_2 = \dfrac{\dddot{y}_2(\dot{y}_1)^2 - \dddot{y}_1\dot{y}_1\dot{y}_2 - 3\ddot{y}_2\ddot{y}_1\dot{y}_1 + 3\dot{y}_2(\ddot{y}_1)^2}{(\dot{y}_1)^4} = \bar{u}_2.
\end{cases}$$

Now, $z_1, z_2, z_3, z_4, \bar{u}_1$ and $\bar{u}_2$ can be expressed in terms of y_1, y_2 and their derivatives as follows

$$\begin{cases}
z_1 = y_1, \\
z_2 = \dfrac{\ddot{y}_2\dot{y}_1 - \ddot{y}_1\dot{y}_2}{(\dot{y}_1)^3}, \\
z_3 = \dfrac{\dot{y}_2}{\dot{y}_1}, \\
z_4 = y_2.
\end{cases} \tag{7.6}$$

The z variables depend on the feedback laws and their derivatives like $z = z(y_1, \dot{y}_1, \ddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2)$, but we cannot define a diffeomorphism yet. We must prolong the system adding two new state variables

$$z_5 = \bar{u}_1, \quad z_6 = \dot{\bar{u}}_1$$

and two new control laws

$$v_1 = \ddot{\bar{u}}_1, \quad v_2 = \bar{u}_2.$$

Then, our system can be written as

$$\begin{cases}
\dot{z}_1 = z_5 \\
\dot{z}_2 = v_2 \\
\dot{z}_3 = z_2 z_5 \\
\dot{z}_4 = z_3 z_5 \\
\dot{z}_5 = z_6 \\
\dot{z}_6 = v_1
\end{cases} \tag{7.7}$$

Now, we got a diffeomorphism between $\{y_1, \dot{y}_1, \ddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2\}$ and $\{z_1, z_2, z_3, z_4, z_5, z_6\}$. We have to check for which values this diffeomorphism exists and avoid the singularities when we impose the initial condition values. The determinants of the change of variables are the following

$$|Jz| = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -\dfrac{\cos(x_3)(1+\tan^2(x_3-x_4)) + 3\sin(x_3)\tan(x_3-x_4)}{L\cos^4(x_3)} & \dfrac{1+\tan^2(x_3-x_4)}{L\cos^3(x_3)} \\ 0 & 0 & 1+\tan^2(x_3) & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

$$= -\frac{1}{L\cos^2(x_3-x_4)\cos^5(x_3)}.$$

So, if $x_3 - x_4 \neq \frac{k\pi}{2}$ and $x_3 \neq \frac{k\pi}{2}$ for $k \in \mathbb{N}$, then the inverse exists.

$$|Jy| = \begin{vmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & z_5 & 0 & z_3 & 0 \\ 0 & z_5^2 & z_6 & 0 & 2z_2z_5 & z_3 \end{vmatrix} = -z_5^3.$$

So, for all point with $z_5 \neq 0$ the inverse exists.

The feedback is of the form

$$\begin{aligned} w_1 &= \frac{d}{dt}\ddot{y}_1 = v_1 \\ w_2 &= \frac{d}{dt}\ddot{y}_2 = \alpha + \beta v_1 + \gamma v_2. \end{aligned}$$

Let's find α, β and γ:

$$\begin{aligned} \frac{d}{dt}\ddot{y}_1 &= \frac{d}{dt}(z_2z_5^2 + z_3z_6) = \frac{d}{dt}(\dot{z}_3u_1 + \dot{u}_1z_3) = \ddot{z}_3u_1 + 2\dot{z}_3\dot{u}_1 + z_3\ddot{u}_1 \\ &= (\dot{z}_2z_5 + \dot{z}_5z_2)u_1 + 2\dot{z}_3\dot{u}_1 + z_3\ddot{u}_1 \\ &= v_2z_5^2 + 3z_2z_5z_6 + v_1z_3. \end{aligned}$$

So,

$$\begin{aligned} \alpha &= 3z_2z_5z_6 \\ \beta &= z_3 \\ \gamma &= z_5^2. \end{aligned}$$

Which implies

$$v_2 = \frac{w_2 - \alpha - \beta w_1}{\gamma}.$$

Let's $L = 1$, $t_0 = 0$ and $t_f = 1$ and take as initial and final conditions of x the values

$$\begin{aligned} x(0) &= (x_1(0), x_2(0), \theta_1(0), \theta_0(0)) = \left(0, 0, 0, \frac{\pi}{4}\right) \\ x(1) &= (x_1(1), x_2(1), \theta_1(1), \theta_0(1)) = \left(1, 1, \frac{\pi}{4}, \frac{\pi}{4}\right). \end{aligned}$$

First of all, we must transform the initial and final conditions, $x(0)$ and $x(1)$, in terms of z variables. Since $z_5 = \bar{u}_1$ and $z_6 = \dot{\bar{u}}_1$, we can take as initial and final condition whatever values we want. So, taking $z_5(0) = z_5(1) = 1$ and $z_6(0) = z_6(1) = 0$, the initial and final conditions in z variables are

$$\begin{aligned} z(0) =& (z_1(0), z_2(0), z_3(0), z_4(0)) = (0, 1, 0, 0, 1, 0) \\ z(1) =& (z_1(1), z_2(1), z_3(1), z_4(1)) = (1, 0, 1, 1, 1, 0) \,. \end{aligned}$$

Finally, we transform the initial and final conditions of z in terms of $y = (y_1,\ \dot{y}_1,\ \ddot{y}_1,\ y_2,\ \dot{y}_2,\ \ddot{y}_2)$ as follows

$$\begin{aligned} y(0) =& (y_1(0), \dot{y}_1(0), \ddot{y}_1(0), y_2(0), \dot{y}_2(0), \ddot{y}_2(0)) = (0, 1, 0, 0, 1, 0) \\ y(1) =& (y_1(1), \dot{y}_1(1), \ddot{y}_1(1), y_2(1), \dot{y}_2(1), \ddot{y}_2(1)) = (1, 1, 0, 1, 0, 0) \,. \end{aligned}$$

Consider $P_5(t) = a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0$ such that $P_5(t) = y_1(t)$. Let's find the coefficients of $P_5(t)$.

$$\begin{aligned} y_1(t) &= a_5 t^5 + a_4 t^4 + a_3 t^3 + a_2 t^2 + a_1 t + a_0 \\ \dot{y}_1(t) &= 5a_5 t^4 + 4a_4 t^3 + 3a_3 t^2 + 2a_2 t + a_1 \\ \ddot{y}_1(t) &= 20a_5 t^3 + 12a_4 t^2 + 6a_3 t + 2a_2. \end{aligned}$$

For $t = 0$:

$$\begin{aligned} a_0 &= 0 \\ a_1 &= 1 \\ a_2 &= 0. \end{aligned}$$

For $t = 1$:

$$\begin{aligned} 1 &= a_5 + a_4 + a_3 + 1 \\ 1 &= 5a_5 + 4a_4 + 3a_3 + 1 \\ 0 &= 20a_5 + 12a_4 + 6a_3. \end{aligned}$$

Solving the linear system we find:

$$\begin{aligned} a_3 &= 0 \\ a_4 &= 0 \\ a_5 &= 0, \end{aligned}$$

therefore,

$$y_1(t) = P_5(t) = t. \tag{7.8}$$

Analogously, we proceed in the same way with $y_2(t) = Q_5(t) = b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + b_0$.

$$\begin{aligned} y_2(t) &= b_5 t^5 + b_4 t^4 + b_3 t^3 + b_2 t^2 + b_1 t + b_0 \\ \dot{y}_2(t) &= 5b_5 t^4 + 4b_4 t^3 + 3b_3 t^2 + 2b_2 t + b_1 \\ \ddot{y}_2(t) &= 20b_5 t^3 + 12b_4 t^2 + 6b_3 t + 2b_2. \end{aligned}$$

For $t = 0$:

$$\begin{aligned} b_0 &= 0 \\ b_1 &= 1 \\ b_2 &= 0. \end{aligned}$$

For $t = 1$:

$$\begin{aligned} 1 &= b_5 + b_4 + b_3 + 1 \\ 0 &= 5b_5 + 4b_4 + 3b_3 + 1 \\ 0 &= 20b_5 + 12b_4 + 6b_3. \end{aligned}$$

Solving the linear system:

$$\begin{aligned} b_3 &= 4 \\ b_4 &= -7 \\ b_5 &= 3, \end{aligned}$$

therefore,

$$y_2(t) = 3t^5 - 7t^4 + 4t^3 + t. \tag{7.9}$$

Now, we must find the feedback as a function of time

$$\begin{aligned} w_1 =& \frac{d^3}{dt^3} y_1(t) = \frac{d^3}{dt^3} P_5(t) = 0 \\ w_2 =& \frac{d^3}{dt^3} y_2(t) = \frac{d^3}{dt^3} Q_5(t) = 180t^2 - 168t + 24. \end{aligned}$$

As a consequence, the controls v_1 and v_2 have the expression

$$\begin{aligned} v_1 =& \frac{d^3}{dt^3} y_1(t) = \frac{d^3}{dt^3} P_5 = 0 \\ v_2 =& \frac{w_2 - \alpha - \beta w_1}{\gamma} = 180t^2 - 168t + 24, \end{aligned}$$

where $\alpha = 0$, $\beta = 15t^4 - 28t^3 + 12t^2 + 1$ and $\gamma = 1$.

Finally, we obtain the expressions of $\bar{u}_1(t)$ and $\bar{u}_2(t)$ in function of v_1 and v_2. For $\bar{u}_1$, we know that it satisfies the following ordinary differential equation,

$$\ddot{\bar{u}}_1 = v_1(t), \quad \bar{u}_1(0) = 1, \quad \dot{\bar{u}}_1(0) = 0.$$

So, $\bar{u}_1(t)$ and $\bar{u}_2(t)$ are

$$\begin{aligned} \bar{u}_1(t) =& 1 \\ \bar{u}_2(t) =& 180t^2 - 168t + 24. \end{aligned}$$

Undoing the feedback in the controls $\bar{u}_1$ and $\bar{u}_2$, we find the expression of the initial controls

$$\begin{aligned} u_1(t) =& = \frac{1}{\cos(x_3)\cos(x_3 - x_4)} \\ u_2(t) =& - \frac{\sin(x_3 - x_4)(3\cos(x_3 - x_4)\sin(x_3 - x_4)\sin(x_3) + \cos(x_3))}{\cos(x_3 - x_4)\cos^2(x_3)} \\ &+ \cos^2(x_3 - x_4)\cos^3(x_3)(180t^2 - 168t + 24). \end{aligned}$$

Before finding the controls $u_1(t)$ and $u_2(t)$, we can integrate (7.2) using the numerical method Runge-Kutta 45 implemented in Matlab.

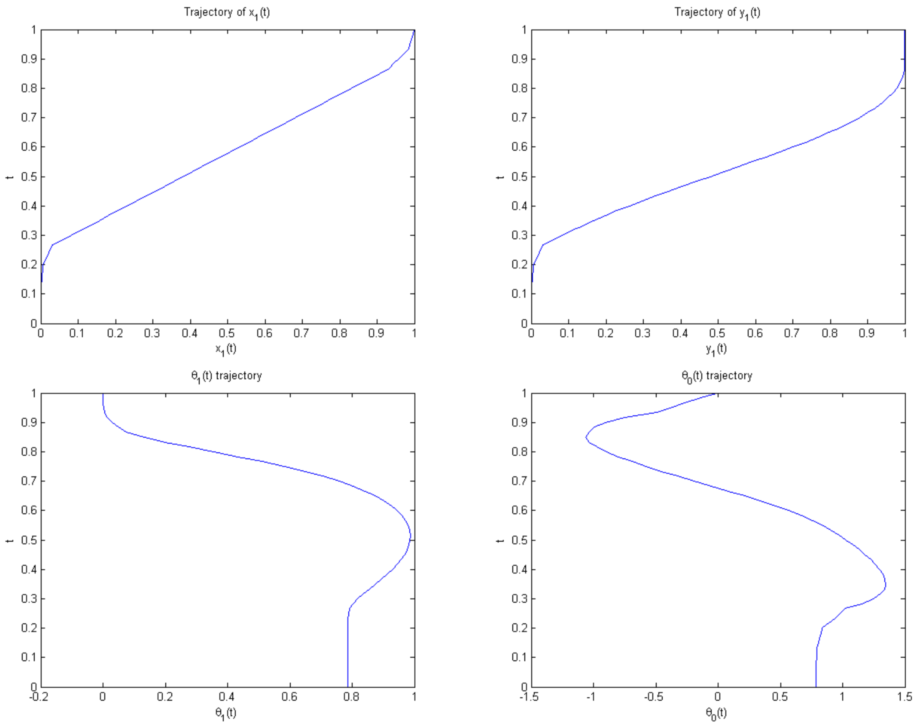

Figure 7.2: Trajectories of the state variables $x_1(t)$, $y_1(t)$,$\theta_1(t)$ and $\theta_0(t)$ respectively.

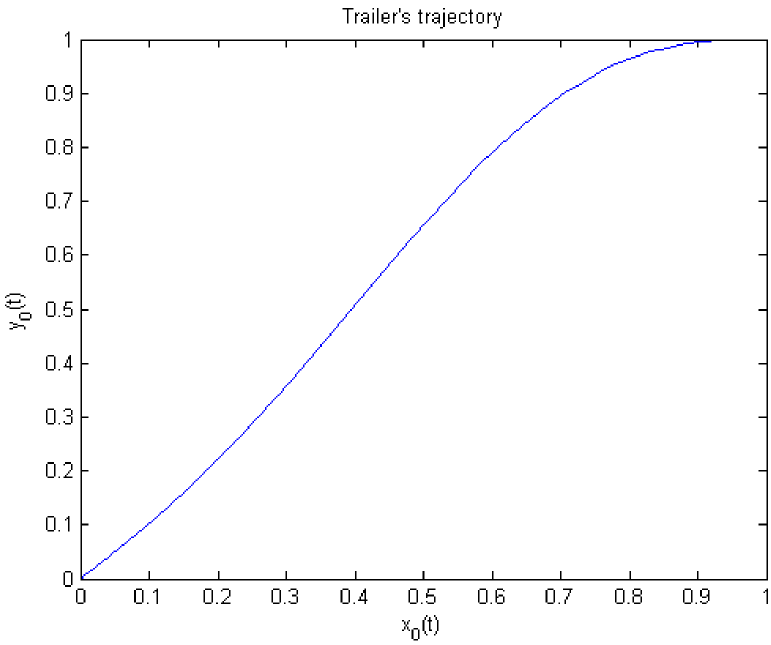

Figure 7.3: The graphic shows the trajectory of the trailer, given by $(x_1(t), y_1(t))$.

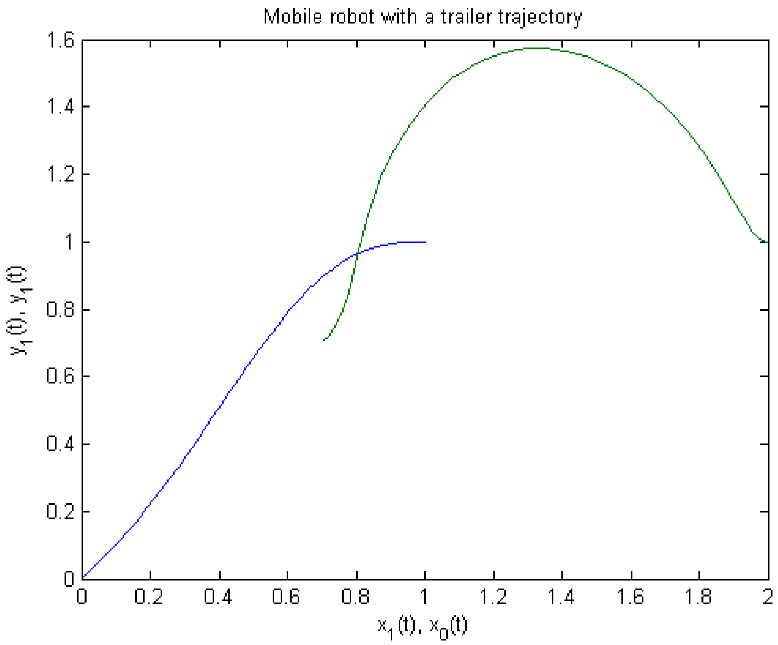

Figure 7.4: The graphic in blue line represents the trajectory of the trailer and the green line the trajectory of the mobile, given by $x_0(t) = x_1(t) + \cos(\theta_1(t))$ and $y_0(t) = y_1(t) + \sin(\theta_1(t))$.

8
The N-Trailer Pfaffian System

8.1 The System of Rolling Constraints and Its Derived Flags

Consider a single-axle mobile robot with n trailers attached, as sketched in Figure 8.1.

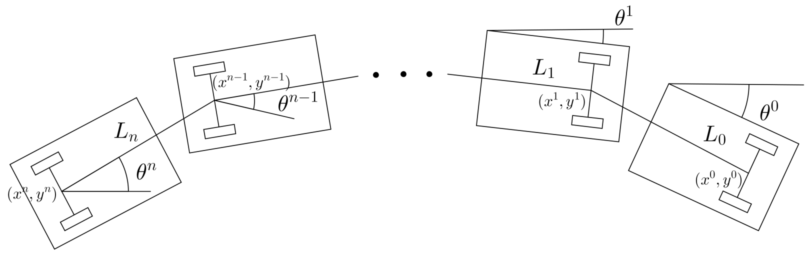

Figure 8.1: The N-Trailer.

Each trailer is attached to the body in front of it by a rigid bar, and the rear set of wheels of each body is constrained to roll without slipping. The trailers are assumed to be identical, with possibly different link length L_i. The x, y coordinates of the mid-point between the two wheels on the ith axle are refereed to as (x^i, y^i) and the hitch angles (all measured with respect to the horizontal) are given by θ^i. The connections between the bodies give rise to the following relations:

$$\begin{aligned} x^{i-1} &= x^i - L_i \cos(\theta^i), \\ y^{i-1} &= y^i - L_i \sin(\theta^i), \end{aligned} \tag{8.1}$$

for $i = 1, 2, \ldots, n$. Thus, it follows that the space parameterized by coordinates

$$(x^0, y^0, \theta^0, \ldots, x^n, y^n, \theta^n) \in \mathbb{R}^{2n+2} \times (\mathbb{S}^1)^{n+1}$$

is not reachable. These constraints (8.1) are holonomic and will reduce the dimension of the configuration space, since the position (x^i, y^i) for $i \geq 1$ can be expressed in terms of x^0, $y^0, \theta^0, \ldots, \theta^i$. By symmetry, (x^i, y^i) for $i < n$ also can be expressed in terms of x^n, y^n, θ^n, θ^{n-1}, $\ldots$, θ^i. For our purposes it is useful to use as configuration space variables the x, y coordinates of a point on the nth trailer and the $n+1$ hitch angles: x^n, y^n, θ^n, $\ldots$, θ^0 because the calculations that follow are vastly simplified. We will refer to the state space or configuration space as $x = (x^n, y^n, \theta^n, \ldots, \theta^0)$. We have assumed that the bodies are connected between the midpoints of the two sets of rear wheels; it should be noted that if the trailers are hitched behind the rear axle, the equations will not simplify as shown here.

The wheels of the robot and trailers are constrained to roll without slipping; this implies that the velocity of each body in the direction perpendicular to its wheels must be zero. We model each pair of rear wheels as a single wheel at the midpoint of the axle and state the nonslipping condition in terms of coordinates, beginning with the nth trailer

$$\dot{x}^n \sin(\theta^n) - \dot{y}^n \cos(\theta^n) = 0. \tag{8.2}$$

Equation (8.2) models the fact that the velocity perpendicular to the wheels is zero. In the language of 1-forms, we write this as

$$\alpha^1(x^n, y^n, \theta^n, \ldots, \theta^0) = \sin(\theta^n)dx^n - \cos(\theta^n)dy^n. \tag{8.3}$$

To write the other rolling constraints, we define v^i to be the velocity of the ith trailer. The direction of motion of the $(i+1)$st trailer and consequently the direction of v^{i+1}, if its wheels are rolling without slipping, is along the direction of the hitch joining the $(i+1)$st body to the ith body. Since the bodies are linked together by rigid rods, it follows that the projection of v^i onto the line of the hitch is equal to v^{i+1}. Thus, we have that

$$v^{i+1}(x) = \cos(\theta^{i+1} - \theta^i)v^i(x). \tag{8.4}$$

Also, we have that the velocity of the nth trailer v^n is given by

$$v^n(x) = \cos(\theta^n)\dot{x}^n + \sin(\theta^n)\dot{y}^n. \tag{8.5}$$

In the sequel we will need to use (8.5) as a 1-form (i.e., we will need to use $v^n dt$) ans we denote this by abuse of notation as

$$v^n(x) = \cos(\theta^n)dx^n + \sin(\theta^n)dy^n. \tag{8.6}$$

We may now recursively write down the rolling without slipping constraints for all the trailers. The velocity of each trailer has a component due to the velocity v^{i+1} of previous trailer and a component $L_{i+1}\dot{\theta}^{i+1}$ due to the rotation of the hitch. The relative geometry of this situation is illustrated in Figure 2. The component of v^{i+1} in the direction perpendicular to the wheel base is $v^{i+1}\sin(\theta^i - \theta^{i+1})$ and the component of $L_{i+1}\dot{\theta}^{i+1}$ in this direction is $L_{i+1}\dot{\theta}^{i+1}\cos(\theta^i - \theta^{i+1})$. If the ith trailer rolls without slipping then must have

$$L_{i+1}\dot{\theta}^{i+1}\cos(\theta^i - \theta^{i+1}) - v^{i+1}\sin(\theta^i - \theta^{i+1}) = 0. \tag{8.7}$$

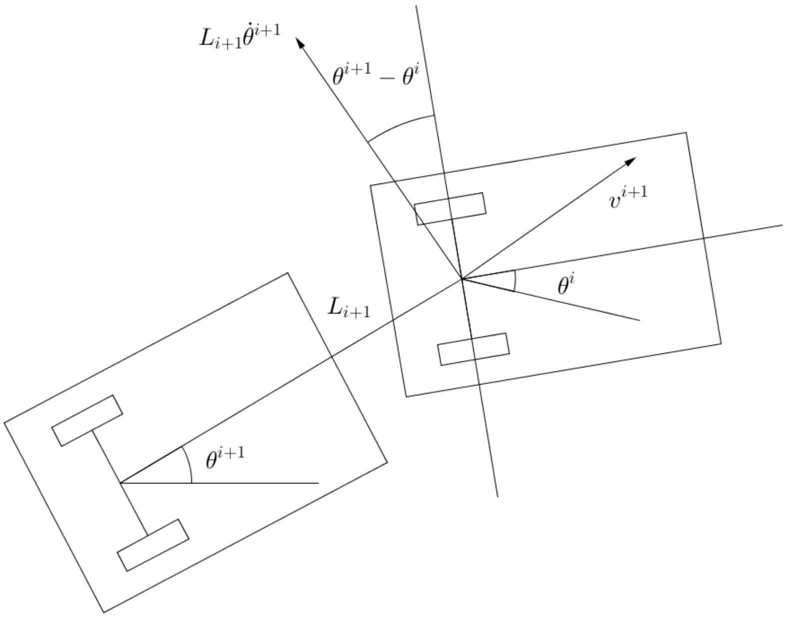

Figure 8.2: Showing the definition of the angles and velocities of the ith trailer.

Dividing through (8.7) by $\cos(\theta^i - \theta^{i+1})$ yields the form constraint for $0 \le i \le n-1$, which we write as $\alpha^{n-i+1}(x)\dot{x} = 0$, where α^{n-i+1} has the expression, in coordinates,

$$\alpha^{n-i+1}(x) = L_{i+1}d\theta^{i+1} - \tan(\theta^i - \theta^{i+1})v^{i+1}. \tag{8.8}$$

Note that we have used the 1-form version v^{i+1} in (8.8) and that there will be a singularity in the constraint when $\theta^i - \theta^{i+1} = \pm\pi/2$, or one of the trailers is jack-knifed.

The forms $\alpha^1(x)$, $\alpha^2(x)$, $\ldots$, $\alpha^{n+1}(x)$ represents the constraints that the wheels of the nth, $(n-1)$st, ..., zeroth trailer (i.e., the cab), respectively, roll without slipping. They are given by formulas given by (8.8) with the recursion relations in (8.4). Thus, the Pfaffian system for the N-trailer problem is generated by

$$I = \text{span } \{\alpha^1, \alpha^2, \ldots, \alpha^{n+1}\}. \tag{8.9}$$

The following theorem gives the derived flags associated with this Pfaffian system.

Theorem 8.1.1 (Derived Flag for the N-Trailer Pfaffian System)**.** *Consider the Pfaffian system of the N-trailer system (8.9) with the 1-forms α^i defined by (8.8) and (8.3). The 1-forms α^i are adapted to the derived flag in the following sense*

$$\begin{aligned}
I^{(0)} &= \textit{span } \{\alpha^1, \alpha^2, \ldots, \alpha^{n+1}\} \\
I^{(1)} &= \textit{span } \{\alpha^1, \alpha^2, \ldots, \alpha^{n}\} \\
&\vdots \\
I^{(n)} &= \textit{span } \{\alpha^1\} \\
I^{(n+1)} &= \{0\}.
\end{aligned} \tag{8.10}$$

Proof. The proof is by recursion starting from the bottom of the flag of (8.10). Indeed for the first step, we compute $d\alpha^1$

$$\begin{aligned} d\alpha^1 &= \cos(\theta^n)d\theta^n \wedge dx^n + \sin(\theta^n)d\theta^n \wedge dy^n \\ &= -v^n \wedge d\theta^n. \end{aligned}$$

From (8.6) it follows that $d\alpha^1 \neq 0 \mod \alpha^1$. This establishes the last two steps of the derived flag above. For the preceding step, we note that the form α^2 is given by

$$\alpha^2 = L_n d\theta^n - \tan(\theta^n - \theta^{n-1})v^n.$$

This yields that $d\theta^n$ is proportional to $v^n \mod \alpha^2$. Consequently, we have that $d\alpha^1 = -v^n \wedge d\theta^n$ is equal $0 \mod \alpha^2$. This establishes that

$$\begin{aligned} I^{(n-1)} &= \text{span } \{\alpha^1, \alpha^2\} \\ I^{(n)} &= \text{span } \{\alpha^1\} \\ I^{(n+1)} &= \{0\}. \end{aligned} \tag{8.11}$$

We need to show that $d\alpha^i = 0 \mod \alpha^1, \ldots, \alpha^{i-1}, \alpha^i$. To verify this, it is useful to have the following preliminary lemma.

Lemma 8.1.1. *For the 1-forms v^i we have that*

$$dv^{n-i} \equiv 0 \mod \alpha^1, \ldots, \alpha^{i+2}. \tag{8.12}$$

Proof. Start first with

$$dv^n = -\sin(\theta^n)d\theta^n \wedge dx^n + \cos(\theta^n)d\theta^n \wedge dy^n \equiv 0 \mod \alpha^1.$$

Thus $dv^n \equiv 0 \mod \alpha^1, \alpha^2$. From $v^{n-1} = v^n \sec(\theta^n - \theta^{n-1})$ it follows that

$$dv^{n-1} = \sec(\theta^n - \theta^{n-1})dv^n + \sec(\theta^n - \theta^{n-1})\tan(\theta^n - \theta^{n-1})v^n \wedge (d\theta^n - d\theta^{n-1}).$$

This first term is zero mod α^1 since $dv^n \equiv 0 \mod \alpha^1$. The second term is zero mod α^2 since v^n is proportional to $d\theta^n$ mod α^2, and the third term is zero mod α^3 since v^n is proportional to $\theta^{n-1} \mod \alpha^3$. Thus, we have that

$$dv^{n-1} \equiv 0 \mod \alpha^1, \alpha^2, \alpha^3.$$

Proceeding recursively, we have that

$$dv^{n-i} \equiv 0 \mod \alpha^1, \alpha^2, \ldots, \alpha^{i+2}$$

which completes the proof of the lemma.

□

We will also need to make use of the relation

$$d\theta^{n-i+2} \equiv v^n \mod \alpha^i \tag{8.13}$$

which follows directly from the definition of the α^i in (8.8) and the linear dependence of the 1-forms v^i, given in (8.4).

Continuing with the proof of the theorem, we now begin the calculation of

$$\begin{aligned} d\alpha^i = & -\sec^2(\theta^{n-i+2} - \theta^{n-i+1})(d\theta^{n-i+2} - d\theta^{n-i+1}) \wedge v_{n-i+2} \\ & - \tan(\theta^{n-i+2} - \theta^{n-i+1}) dv_{n-i+2}. \end{aligned}$$

This expression has three terms. By (8.12), we have that $dv_{n-i+2} \equiv 0 \mod \alpha^1, \ldots, \alpha^i$. Also by the proportionality of $d\theta^i$ to v^n (8.13) and the linear dependence of the v^i's (8.4), we have that $d\theta^{n-i+2} \wedge v_{n-i+2} \equiv 0 \mod \alpha^i$ and $d\theta^{n-i+2} \wedge v_{n-i+2} \equiv 0 \mod \alpha^{i-1}$. Thus, we have that $d\alpha^i \equiv 0 \mod \alpha^1, \ldots, \alpha^i$ which implies that the derived flag has the form $I^{(n-i+1)} = \{\alpha^1, \ldots, \alpha^i\}$, as stated.

□

We note that the $I^{(n+1)} = \{0\}$ implies that the N-trailer system is completely controllable by Chow's Theorem.

8.2 Conversion to Goursat Normal Form

In the preceding section, we have shown that the basis of the constraints $\alpha^1, \ldots, \alpha^{n+1}$ defined in (8.3) and (8.8) is adapted to its derived flag in the sense of (8.10). It remains to check whether the α^i satisfy the Goursat congruences and if they do, to find a transformation that puts them into Goursat canonical form.

Theorem 8.2.1 (Goursat Congruences for the N-Trailer System)**.** *Consider the Pfaffian system associated with the N- trailer system (8.9) with the 1-forms α^i defined in (8.3) and (8.8). There exist a change of basis of the 1-forms α^i to $\bar{\alpha}^i$ which preserves the adapted structure, and a 1-form π such that the Goursat congruences are satisfied*

$$\begin{aligned} & d\bar{\alpha}^i \equiv -\bar{\alpha}^{i+1} \wedge \pi \mod \bar{\alpha}^1, \ldots, \bar{\alpha}^i \quad i = 1, \ldots, n \\ & d\bar{\alpha}^{n+1} \neq 0 \mod I. \end{aligned}$$

The 1-form which satisfies these congruences is given by $\pi = \cos(\theta^n)dx^n + \sin(\theta^n)dy^n = v^n$, and it is equivalent to the velocity form of the nth trailer.

Proof. The outline for the proof is first to determine a suitable 1-form π from the first Goursat congruence, $d\alpha^1 \equiv -\alpha^2 \wedge \pi$. Then, we construct the new basis elements $\bar{\alpha}^i$ one at a time such that satisfy the rest of the congruences. For this example, we find that these new basis elements are multiples of the original basis elements, and since the original basis is adapted to the derived flag, the new basis is also adapted.

We determine π by completing the basis of $\{\alpha^1, \ldots, \alpha^{n+1}\}$ with

$$\begin{aligned} \alpha^{n+2} &= \cos(\theta^n)dx^n + \sin(\theta^n)dy^n \\ \alpha^{n+3} &= d\theta^0. \end{aligned}$$

Note that $\alpha^{n+2} = v^n$, the velocity form of the last trailer. We then set $\pi = \lambda_1\alpha^{n+2} + \lambda_2\alpha^{n+3}$ and solve λ_1, λ_2 using

$$d\alpha^1 \equiv -\alpha^2 \wedge \pi \mod \alpha^1.$$

Calculating the exterior derivative of α^1

$$\begin{aligned} d\alpha^1 &= \cos(\theta^n)d\theta^n \wedge dx^n + \sin(\theta^n)d\theta^n \wedge dy^n \\ &= d\theta^n \wedge v^n \end{aligned} \tag{8.14}$$

and then examining $\alpha^2 \wedge \pi$

$$\alpha^2 \wedge \pi = (L_n d\theta^n - \tan(\theta^n - \theta^{n-1})v^n) \wedge (\lambda_1 v^n + \lambda_2 d\theta^0)$$

we see if we choose $\lambda_1 = 1$, $\lambda_2 = 0$, then

$$\alpha^2 \wedge \pi = L_n d\theta^n \wedge v^n = L_n d\alpha^1.$$

We note here that we could have chosen $\lambda_1 = -1/L_n$, but instead we will define a new basis element $\bar{\alpha}^2 = -(1/L_n)\alpha^2$. Then the 1-form $\pi = v^n$ will satisfy

$$d\alpha^1 = -\bar{\alpha}^2 \wedge \pi.$$

We now continue this procedure to find the rest of the transformed basis. Taking the exterior derivative of $\bar{\alpha}^2$

$$d\bar{\alpha}^2 = \frac{1}{L_n}\sec^2(\theta^n - \theta^{n-1})(d\theta^n - d\theta^{n-1}) \wedge v^n - \frac{1}{L_n}\tan(\theta^n - \theta^{n-1})dv^n$$

and noting that

$$\begin{aligned} v^n \wedge d\theta^n &\equiv 0 \quad \bmod \bar{\alpha}^2 \\ dv^n &\equiv 0 \quad \bmod \alpha^1 \end{aligned}$$

it can be seen that

$$d\bar{\alpha}^2 \equiv -\frac{1}{L_n}\sec^2(\theta^n - \theta^{n-1})d\theta^{n-1} \wedge v^n \quad \bmod \alpha^1, \bar{\alpha}^2.$$

Also, since

$$\alpha^3 \wedge \pi = L_{n-1} d\theta^{n-1} \wedge v^n$$

a choice of

$$\bar{\alpha}^3 = \frac{1}{L_n L_{n-1}}\sec^2(\theta^n - \theta^{n-1})\alpha^3$$

will result in the congruence

$$d\bar{\alpha}^2 \equiv -\bar{\alpha}^3 \wedge \pi \quad \bmod \alpha^1, \bar{\alpha}^2.$$

Since the new basis we are defining is merely a scaled version of the original basis, mod-ing out by α^i or $\bar{\alpha}^i$ is equivalent.

In general, we assume that $\bar{\alpha}^i$ has been defined as

$$\bar{\alpha}^i = \frac{(-1)^{i-1}}{L_n \cdots L_{n-i+2}}\sec^{i-1}(\theta^{n-1} - \theta^n)\sec^{i-2}(\theta^{n-2} - \theta^{n-1}) \cdots \sec^2(\theta^{n-i+3} - \theta^{n-i+2})\alpha^i.$$

Using the congruences

$$\begin{aligned} d\theta^{n-i} \wedge d\theta^{n-i+1} &\equiv 0 \quad \bmod \alpha^{i+2}, \alpha^{i+3} \\ d\theta^{n-i} \wedge v^n &\equiv 0 \quad \bmod \alpha^{i+2} \\ dv^{n-i} &\equiv 0 \quad \bmod \alpha^1, \ldots, \alpha^{i+2} \end{aligned}$$

we can show that

$$\begin{aligned}
d\bar{\alpha}^i \equiv & \frac{(1)^{i-1}}{L_n \cdots L_{n-i+2}} \sec^{i-1}(\theta^{n-1}-\theta^n)\sec^{i-2}(\theta^{n-2}-\theta^{n-1})\cdots\sec^2(\theta^{n-i+3}-\theta^{n-i+2}) \\
& \cdot \sec^2(\theta^{n-i+2}-\theta^{n-i+1})d\theta^{n-i+1}\wedge v_{n-i+2} \mod \alpha^1, \bar{\alpha}^2, \ldots, \bar{\alpha}^i \\
\equiv & \frac{(1)^{i-1}}{L_n \cdots L_{n-i+2}} \sec^{i}(\theta^{n-1}-\theta^n)\sec^{i-1}(\theta^{n-2}-\theta^{n-1})\cdots\sec^3(\theta^{n-i+3}-\theta^{n-i+2}) \\
& \cdot \sec^2(\theta^{n-i+2}-\theta^{n-i+1})d\theta^{n-i+1}\wedge v^n \mod \alpha^1, \bar{\alpha}^2, \ldots, \bar{\alpha}^i \\
\equiv & -\bar{\alpha}^{i+1}\wedge v^n \mod \alpha^1, \bar{\alpha}^2, \ldots, \bar{\alpha}^i.
\end{aligned}$$

All that remains now is to demonstrate that

$$d\bar{\alpha}^{n+1} \not\equiv 0 \mod I.$$

From the above analysis, we know

$$\begin{aligned}
d\bar{\alpha}^{n+1} \equiv & \frac{(-1)^n}{L_n \cdots L_1} \sec^{n+1}(\theta^{n-1}-\theta^n)\cdots\sec^3(\theta^2-\theta^1) \\
& \cdot \sec^2(\theta^1-\theta^0)d\theta^0\wedge v^n \mod \alpha^1, \bar{\alpha}^2, \ldots, \bar{\alpha}^{n+1}
\end{aligned}$$

which is nonzero.

□

8.3 Conversion to Chained Form

In Chapter 4, we described a method for converting the N-trailer exterior differential system into Goursat normal form. Recalling that the dual of Goursat normal form is a chained form, we now show how a similar procedure can be used to transform the nonholonomic control system corresponding to the N-trailer system into a chained canonical form.

We note that an exterior differential system on $\mathbb{R}^n$ of codimension two, given by

$$I = \{\alpha^1(x), \ldots, \alpha^{n-2}(x)\}$$

is the dual to a two-input nonholonomic control system

$$\Sigma: \quad \dot{x} = g_1(x)u_1 + g_2(x)u_2 \tag{8.15}$$

where the vector fields $g_j(x)$ span a 2-dimensional distribution Δ which is annihilates by the 1-forms α^i

$$\alpha^i(x)\cdot g_j(x) = 0.$$

When we transform an exterior differential system into Goursat normal form, we only perform a coordinate transformation $z = f(x)$. There is no input *per se* to a formal exterior differential system, although we can speak of the two degrees of freedom of the system, given by the distribution $\Delta = I^\perp$. The procedure for transforming a nonholonomic control system such as (8.15) into a chained form requires both a coordinate transformation and state feedback. Although for the most general case, and a state feedback is given by

$$\bar{u} = a(x) + b(x)u$$

for drift-less nonholonomic systems it is easily seen that $a(x) = 0$[1]. The purpose of the state feedback $\bar{u} = b(x)u$ is therefore to transform the basis of the distribution Δ into chained form in the new coordinate system

$$\begin{aligned} \bar{g}_1(z) &= \frac{\partial}{\partial z_1} + z_2\frac{\partial}{\partial z_3} + \cdots + z_{n-1}\frac{\partial}{\partial z_n} \\ \bar{g}_2(z) &= \frac{\partial}{\partial z_2}. \end{aligned} \tag{8.16}$$

Proposition 8.3.1. *Consider an N-trailer system with $n+1$ rolling constraints*

$$\begin{aligned} \alpha^1 &= \sin(\theta^n)dx^n - \cos(\theta^n)dy^n = 0 \\ \alpha^{n-i+1} &= L_{i+1}d\theta^{i+1} - \tan(\theta^{i+1} - \theta^i)v^{i+1} = 0 \textit{ for } i = 0, \ldots, n-1, \end{aligned}$$

where the v^i are specified in (8.4). A basis for the distribution Δ which is annihilated by these 1-forms $\{\alpha^1, \ldots, \alpha^{n+1}\}$ is given by

$$g_1 = \begin{bmatrix} \cos(\theta^n) \\ \sin(\theta^n) \\ \frac{1}{L_n}\tan(\theta^{n-1} - \theta^n) \\ \vdots \\ \frac{1}{L_1}\prod_{i=2}^{n}\sec(\theta^{i-1} - \theta^i)\tan(\theta^0 - \theta^1) \\ 0 \end{bmatrix} \quad g_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Proof. The proof of this proposition requires the constraints α^i to be written out in coordinates $(x^n, y^n, \theta^n, \ldots, \theta^0)$, and then it can be checked that the two given vector fields, g_1 and g_2, are in the null space of this set of constraints. Since $\alpha^{n-i+1} = L_{i+1}d\theta^{i+1} - \tan(\theta^{i+1} - \theta^i)v^{i+1}$, $v^i = \sec(\theta^n - \theta^{n-1})\sec(\theta^{n-1} - \theta^{n-2})\cdots\sec(\theta^{n-i+1} - \theta^{n-i})v^n$ and $v^n = \cos(\theta^n)dx^n + \sin(\theta^n)dy^n$, we know that

$$\alpha^{n-i+1} = L_{i+1}d\theta^{i+1} - \tan(\theta^{i+1} - \theta^i)\left(\prod_{j=0}^{i}\cos(\theta^{n-j} - \theta^{n-j-1})\right)(\cos(\theta^n)dx^n + \sin(\theta^n)dy^n).$$

Then, is a tedious calculation check that $\alpha^{n-i+1}(x) \cdot g_j(x) = 0$.

□

Although there are many different choices of g_1, g_2 which will span Δ, the two which we have picked are natural in the sense that when the nonholonomic control system is written as

$$\dot{x} = g_1(x)u_1 + g_2(x)u_2$$

[1] If this were not the case, the state feedback would add a drift term to a drift-less system and could not result in a chained form.

the input functions have the physical meaning $u_1 = v^n$ in the linear velocity of the nth trailer, and $u_2 = w$ is the rotational velocity of the lead cab (i.e., the cab). From a practical point of view, we have control only in the velocity v^0 of the lead car given in terms of v^n by

$$v^0 = \sec(\theta^n - \theta^{n-1})\sec(\theta^{n-1} - \theta^{n-2})\cdots\sec(\theta^1 - \theta^2)\sec(\theta^0 - \theta^1)v^n.$$

This is merely an input transformation, and will not change any of the properties of the chained-form system.

We will now derive the coordinate transformation and the changes of inputs required to put the system into chained form, as was discussed in Chapter 4. Recall that a system in chained canonical form is defined to be

$$\begin{aligned}
\dot{z}^1 &= \bar{u}_1 \\
\dot{z}^2 &= \bar{u}_2 \\
\dot{z}^3 &= z_2\bar{u}_1 \\
&\vdots \\
\dot{z}^{n+3} &= z_{n+2}\bar{u}_1.
\end{aligned}$$

We note that the functions $z_1(t)$ and $z_{n+3}(t)$ will completely define all the state variables of a chained-form system. These functions are referred to by as flat outputs since the other $n+1$ states and the two inputs can be determined from the equations

$$\begin{aligned}
\bar{u}_1 &= \dot{z}^1 \\
\bar{u}_2 &= \dot{z}^2 \\
z_i &= \dot{z}^{i+1}/\bar{u}_1.
\end{aligned} \tag{8.17}$$

Consequently, a coordinate transformation into chained form is completely defined by the first and last coordinates of the chain z_1 and z_{n+2}, as functions of the original coordinates x, along with (8.17) [2]. It does need to be checked that the transformation which results from (8.17) is a valid diffeomorphism.

8.4 Coordinates from the Last Trailer

Now, we have to show that the 1-forms α^i do satisfy the Goursat congruences, we can follow the steps of the proof of Goursat Normal Form Theorem to find the coordinate transformation. First of all, applying Pfaff Theorem to the 1-form α^1, we look for possibly nonunique functions f_1, f_2 which satisfy (8.18), namely

$$\begin{array}{ccc}
d\alpha^1 \wedge \alpha^1 \wedge df_1 = 0 & & \alpha^1 \wedge df_1 \neq 0 \\
 & \text{and} & \\
\alpha^1 \wedge df_1 \wedge df_2 = 0 & & df_1 \wedge df_2 \neq 0.
\end{array} \tag{8.18}$$

[2]The fact that such a transform exists follows from our having verified the Goursat congruences for the α^i in the previous subsection.

Since $\alpha^1 = \sin(\theta^n)dx^n - \cos(\theta^n)dy^n$ and $d\alpha^1 = -v^n \wedge d\theta^n$, it follows that $d\alpha^1 \wedge \alpha^1 = -dx^n \wedge dy^n \wedge d\theta^n$. Thus, f_1 may be chosen to be any function of x^n, y^n, θ^n exclusively. We now proceed to explain the coordinates from the last trailer.

If we choose $f_1 = x^n$, then the second equation of (8.18) becomes

$$\cos(\theta^n)dx^n \wedge dy^n \wedge df_2 = 0$$

with the proviso that $df_1 \wedge df_2 \neq 0$. A nonunique choice of f_2 is $f_2 = y^n$. For the change of coordinates, we have

$$\begin{aligned} z_1 &= f_1(x) = x^n \\ z_{n+3} &= f_2(x) = y^n. \end{aligned}$$

The 1-form $\alpha^1 = 0$ may be written by dividing through by $\cos(\theta^n)$ as

$$\alpha^1 = dy^n - \tan(\theta^n)dx^n = dz_{n+3} - z_{n+2}dz_1$$

so that $z_{n+2} = \tan(\theta^n)$. By the proof of Engel's theorem, we now need to find a, b such that

$$\begin{aligned} \alpha^2 &\equiv a dz_{n+2} + b dz_1 \mod \alpha^1 \\ &\equiv a \sec^2(\theta^n)d\theta^n + b dx^n \mod \alpha^1. \end{aligned}$$

But $\alpha^2 = L_n d\theta^n - \tan(\theta^n - \theta^{n-1})v^n$. Hence, we have that

$$a = \frac{L_n}{\sec^2(\theta^n)}, \quad b = \frac{-\tan(\theta^n - \theta^{n-1})}{\cos(\theta^n)}$$

and we may write

$$\alpha^2 \equiv dz_{n+2} + \frac{b}{a}dz_1.$$

Now, we define

$$z_{n+1} = -\frac{b}{a} = \frac{\tan(\theta^n - \theta^{n-1})\cos(\theta^n)}{L_n}.$$

The remaining coordinates are found by solving the equations

$$\alpha^i = dz_{n-i+4} - z_{n-i+3}dz_1 \mod \alpha^1, \ldots, \alpha^{i-1}$$

for $i \geq 2$.

The corresponding input transformation is

$$\begin{aligned} \bar{u}_1 &= \dot{z}^1 = \cos(\theta^n)v^n \\ &= \cos(\theta^n)\cos(\theta^{n-1} - \theta^n)\cos(\theta^{n-2} - \theta^{n-1}) \cdots \cos(\theta^0 - \theta^1)v^0. \end{aligned}$$

The other input $\bar{u}_2 = \dot{z}^2$ is a complicated function of x, v^0, w for the general case with n trailers; however, it is easily verified that $\partial\bar{u}_2/\partial w \neq 0$, implying that the input transformation $\bar{u} = b(x)u$ is nonsingular. The remaining coordinates $z = f(x)$ are defined using (8.17). But in the proof of Theorem 8.2.1 we define the 1-form

$$\bar{\alpha}^i = \frac{(-1)^{i-1}}{L_n \cdots L_{n-i+2}} \sec^{i-1}(\theta^{n-1} - \theta^n)\sec^{i-2}(\theta^{n-2} - \theta^{n-1}) \cdots \sec^2(\theta^{n-i+3} - \theta^{n-i+2})\alpha^i$$

as a rescaling of α^i. Then, the coordinate z_i is the coefficient of dx^n in $\bar{\alpha}^{n+3-i}$.

After obtain z_i we know from (8.17) that

$$\begin{aligned} \bar{u}_1 &= \dot{z}^1 \\ \bar{u}_2 &= \dot{z}^2 \\ z_i &= \dot{z}^{i+1}/\bar{u}_1. \end{aligned}$$

Since the functions $z_1(t)$ and $z_{n+3}(t)$ define completely all the state variables, our flat outputs are

$$y_1 = z_1 \qquad y_2 = z_{n+3} \tag{8.19}$$

because $\dot{z}_1, \ldots, \dot{z}_{n+3}$ depends on $z_2, z_3, \ldots, z_{n+2}, \bar{u}_1$ and $\bar{u}_2$. Using (8.17) we can find $z_2, \ldots, z_{n+2}$ as functions of y_1, y_2 and their derivatives. So, the z variables depend on the feedback laws and their derivatives like

$$z_i = z_i(\dot{y}_1, \ldots, y_1^{(n+1)}, \dot{y}_2, \ldots, y_2^{(n+1)}) \text{ for } 2 \leq i \leq n+2,$$

but we cannot define a diffeomorphism yet. We must prolong the system adding $n+1$ new state variables defined as

$$z_{n+i+3} = \frac{d^{i-1}\bar{u}_1}{dt^{i-1}} = \bar{u}_1^{(i-1)} \tag{8.20}$$

for $i = 1, \ldots n+1$, and two new control laws

$$v_1 = \frac{d^{n+2}}{dt^{n+2}} y_1 = \bar{u}_1^{(n+1)}, \quad v_2 = \bar{u}_2. \tag{8.21}$$

Now, our system is written as

$$\begin{array}{ccc} \dot{z}_1 & = & z_{n+4} \\ \dot{z}_2 & = & v_1 \\ \dot{z}_3 & = & z_2 z_{n+4} \\ & \vdots & \\ \dot{z}_{n+3} & = & z_{n+2} z_{n+4} \\ \dot{z}_{n+4} & = & z_{n+5} \\ \dot{z}_{n+5} & = & z_{n+6} \\ & \vdots & \\ \dot{z}_{2n+3} & = & z_{2n+4} \\ \dot{z}_{2n+4} & = & v_2. \end{array} \tag{8.22}$$

It should be noted that this coordinate transformation is only defined locally. Since its definition requires a division by $\bar{u}_1$, if any of the factors in $\bar{u}_1$ are zero, the transformation is undefined for that configuration. For example, if $\theta^n = \pi/2$, corresponding to the last trailer being at right-angles with the coordinate frame, this coordinate transformation is no longer valid. In addition, if the ith trailer is jack-knifed, that is to say, for some $1 \leq i \leq n$, $\theta^i - \theta^{i-1} = \pm\pi/2$, the coordinate transformation is also singular.

Notice that if we define $\phi = \theta^0 - \theta^1$, the system of the N-trailer is equivalent to a system of $(N-1)$-trailers pulled by a car, where ϕ is the angle of the directional wheels of the car. So, we will consider this configuration because it is useful for future implementations.

8.4.1 The 2-Trailer

Consider the system of the 2-trailer, defined by the variables $(x^2, y^2, \theta^2, \theta^1, \theta^0) \in \mathbb{R}^5$ and the 1-forms

$$\begin{aligned}
\alpha^1 &= \tan(\theta^2)dx^2 - dy^2 \\
\alpha^2 &= -\tan(\theta^1 - \theta^2)\cos(\theta^2)dx^2 - \tan(\theta^1 - \theta^2)\sin(\theta^2)dy^2 + L_2 d\theta^2 \\
\alpha^3 &= -\frac{\tan(\theta^0 - \theta^1)\cos(\theta^2)}{\cos(\theta^1 - \theta^2)}dx^2 + \frac{\tan(\theta^0 - \theta^1)\sin(\theta^2)}{\cos(\theta^1 - \theta^2)}dy^2 + L_1 d\theta^1.
\end{aligned}$$

Defining the $\bar{\alpha}^i$ as

$$\begin{aligned}
\bar{\alpha}^1 &= \alpha^1 \\
\bar{\alpha}^2 &= -\frac{1}{L_2}\alpha^2 \\
\bar{\alpha}^3 &= \frac{1}{L_2 L_1}\sec^2(\theta^2 - \theta^1)\alpha^3.
\end{aligned}$$

we obtain that

$$\begin{aligned}
\bar{\alpha}^1 &= \tan(\theta^2)dx^2 - dy^2 \\
\bar{\alpha}^2 &= \frac{\tan(\theta^1 - \theta^2)\cos(\theta^2)}{L_2}dx^2 + \frac{\tan(\theta^1 - \theta^2)\sin(\theta^2)}{L_2}dy^2 - d\theta^2 \\
\bar{\alpha}^3 &= \frac{\tan(\theta^0 - \theta^1)\cos(\theta^2)}{L_2 L_1\cos^3(\theta^1 - \theta^2)}dx^2 + \frac{\tan(\theta^0 - \theta^1)\sin(\theta^2)}{L_1 L_2\cos^3(\theta^1 - \theta^2)}dy^2 - \frac{1}{L_2\cos^2(\theta^1 - \theta^2)}d\theta^1.
\end{aligned}$$

Taking $z_1 = x^2$, $z_5 = y^2$ and using that z_i is the dx^2 coefficient of $\bar{\alpha}^{5-i}$ for $i = 2, 3, 4$, we have that

$$\begin{cases}
z_1 = x^2 \\
z_2 = \dfrac{\tan(\theta^0 - \theta^1)\cos(\theta^2)}{L_2 L_1\cos^3(\theta^1 - \theta^2)} \\
z_3 = \dfrac{\tan(\theta^1 - \theta^2)\cos(\theta^2)}{L_2} \\
z_4 = \tan(\theta^2) \\
z_5 = y^2.
\end{cases}$$

Using (8.17) and (8.19), we take as flat outputs $y_1 = z_1$ and $y_2 = z_5$. Since we cannot define a diffeomorphism, we must prolong the system adding 3 new state variables defined in (8.20)

$$z_6 = \bar{u}_1, \quad z_7 = \bar{u}_1^{(1)}, \quad z_8 = \bar{u}_1^{(2)}$$

and two feedback laws defined in (8.21)

$$\begin{aligned}
v_1 =& \frac{d^4}{dt^4}y_1(t) = \frac{d^3}{dt^3}\bar{u}_1 = \bar{u}_1^{(3)} \\
v_2 =& \bar{u}_2.
\end{aligned}$$

Now, our system is written as (8.22)

$$\begin{cases} \dot{z}_1 = z_6 \\ \dot{z}_2 = v_2 \\ \dot{z}_3 = z_2 z_6 \\ \dot{z}_4 = z_3 z_6 \\ \dot{z}_5 = z_4 z_6 \\ \dot{z}_6 = z_7 \\ \dot{z}_7 = z_8 \\ \dot{z}_8 = v_1. \end{cases}$$

The diffeomorphism between z variables and the flat outputs is given by

$$\begin{cases} y_1 = z_1 \\ y_1^{(1)} = z_6 \\ y_1^{(2)} = z_7 \\ y_1^{(3)} = z_8 \\ y_2 = z_5 \\ y_2^{(1)} = z_4 z_6 \\ y_2^{(2)} = z_3 z_6^2 + z_4 z_7 \\ y_2^{(3)} = z_2 z_6^3 + 3 z_3 z_6 z_7 + z_4 z_8. \end{cases}$$

The feedback is of the form

$$\begin{aligned} w_1 =& v_1 \\ w_2 =& 6 z_2 z_6^2 z_7 + 3 z_3 z_7^2 + 4 z_3 z_6 z_8 + z_4 v_1 + z_6^3 v_2, \end{aligned}$$

and it implies that

$$\begin{aligned} v_1 =& w_1 \\ v_2 =& \frac{w_2 - 6 z_2 z_6^2 z_7 - 3 z_3 z_7^2 - 4 z_3 z_6 z_8 - z_4 w_1}{z_6^3}. \end{aligned}$$

Let $t = 0$ and $t_f = 1$ be the initial and final time, and impose the initial and final conditions

$$\begin{aligned} x(0) = (x^2(0), y^2(0), \theta^2(0), \theta^1(0), \theta^0(0)) = (1, 1, 0, 0, 0) \\ x(1) = (x^2(1), y^2(1), \theta^2(1), \theta^1(1), \theta^0(1)) = \left(0, 0, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}\right). \end{aligned}$$

Then, the trajectories of the 2-trailer are shown in the following figures.

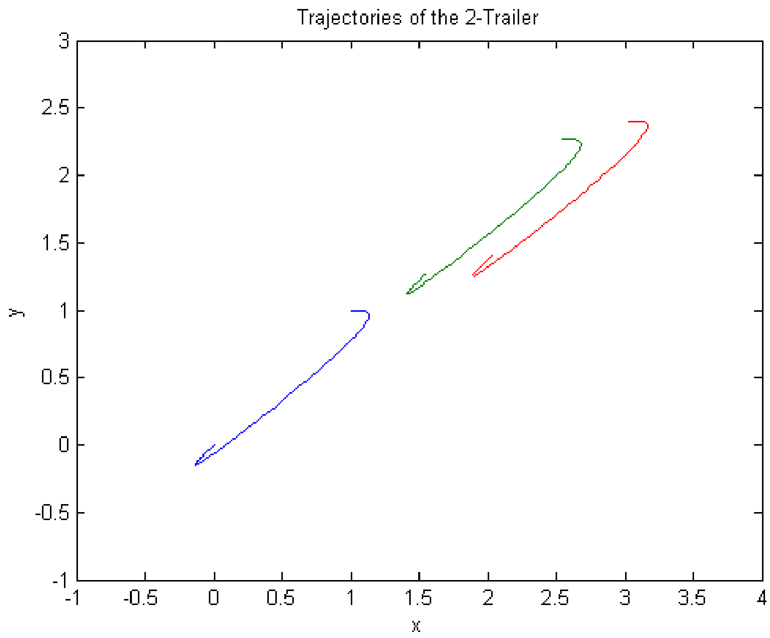

Figure 8.3: The trajectories of the 2-trailer, where the blue curve is the trajectory of the rear wheels of the trailer, the green curve the rear wheels of the cab and the red curve the trajectories of directional wheels.

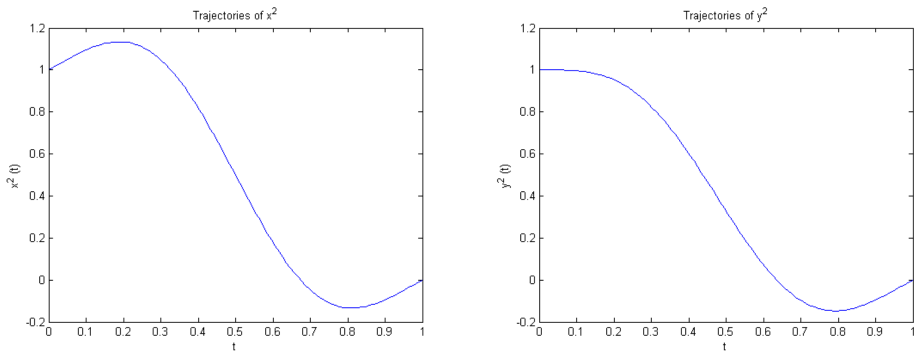

Figure 8.4: Trajectories of the state variables $x^2(t)$ and $y^2(t)$ respectively.

8.4.2 The 3-Trailer

Consider the system of the 3-trailer, defined by the variables $(x^3, y^3, \theta^3, \theta^2, \theta^1, \theta^0) \in \mathbb{R}^6$ and the 1-forms

$$
\begin{aligned}
\alpha^1 &= \tan(\theta^3)dx^3 - dy^3 \\
\alpha^2 &= \tan(\theta^2 - \theta^3)\cos(\theta^3)dx^3 + \tan(\theta^2 - \theta^3)\sin(\theta^3)dy^3 - L_3 d\theta^3 \\
\alpha^3 &= \frac{\tan(\theta^1 - \theta^2)\cos(\theta^3)}{\cos(\theta^2 - \theta^3)}dx^3 + \frac{\tan(\theta^1 - \theta^2)\sin(\theta^3)}{\cos(\theta^2 - \theta^3)}dy^3 - L_2 d\theta^2 \\
\alpha^4 &= \frac{\tan(\theta^0 - \theta^1)\cos(\theta^3)}{\cos(\theta^1 - \theta^2)\cos(\theta^2 - \theta^3)}dx^3 + \frac{\tan(\theta^0 - \theta^1)\sin(\theta^3)}{\cos(\theta^1 - \theta^2)\cos(\theta^2 - \theta^3)}dy^3 - L_1 d\theta^1.
\end{aligned}
$$

Defining the $\bar{\alpha}^i$ as

$$
\begin{aligned}
\bar{\alpha}^1 &= \alpha^1 \\
\bar{\alpha}^2 &= -\frac{1}{L_2}\alpha^2 \\
\bar{\alpha}^3 &= \frac{1}{L_2 L_1}\sec^2(\theta^2-\theta^1)\alpha^3 \\
\bar{\alpha}^4 &= \frac{\sec^3(\theta^2-\theta^3)\sec^2(\theta^1-\theta^2)}{L_3 L_2 L_1}\alpha^4.
\end{aligned}
$$

we obtain that

$$
\begin{aligned}
\bar{\alpha}^1 &= \tan(\theta^3)dx^3 - dy^3 \\
\bar{\alpha}^2 &= \frac{\tan(\theta^2-\theta^3)\cos(\theta^3)}{L_3}dx^3 + \frac{\tan(\theta^2-\theta^3)\sin(\theta^3)}{L_3}dy^3 - d\theta^3 \\
\bar{\alpha}^3 &= \frac{\tan(\theta^1-\theta^2)\cos(\theta^3)}{L_3 L_2\cos^3(\theta^2-\theta^3)}dx^3 + \frac{\tan(\theta^1-\theta^2)\sin(\theta^3)}{L_3 L_2\cos^3(\theta^2-\theta^3)}dy^3 - \frac{1}{L_3\cos^2(\theta^2-\theta^3)}d\theta^2 \\
\bar{\alpha}^4 &= \frac{\tan(\theta^0-\theta^1)\cos(\theta^3)}{L_3 L_2 L1\cos^4(\theta^2-\theta^3)\cos^3(\theta^1-\theta^2)}dx^3 + \frac{\tan(\theta^0-\theta^1)\sin(\theta^3)}{L_3 L_2 L_1\cos^4(\theta^2-\theta^3)\cos^3(\theta^1-\theta^2)}dy^3 \\
&- \frac{1}{L_3 L_2\cos^3(\theta^2-\theta^3)\cos^2(\theta^1-\theta^2)}d\theta^1.
\end{aligned}
$$

Taking $z_1 = x^3$, $z_6 = y^3$ and using that z_i is the dx^2 coefficient of $\bar{\alpha}^{6-i}$ for $i = 2, \dots, 5$, we have that

$$
\begin{cases}
z_1 = x^3 \\
z_2 = \dfrac{\tan(\theta^0-\theta^1)\cos(\theta^3)}{L_3 L_2 L_1\cos^4(\theta^2-\theta^3)\cos^3(\theta^1-\theta^2)} \\
z_3 = \dfrac{\tan(\theta^1-\theta^2)\cos(\theta^3)}{L_3 L_2\cos^3(\theta^2-\theta^3)} \\
z_4 = \dfrac{\tan(\theta^2-\theta^3)\cos(\theta^3)}{L_3} \\
z_5 = \tan(\theta^3) \\
z_6 = y^3.
\end{cases}
$$

Using (8.17) and (8.19), we take as flat outputs $y_1 = z_1$ and $y_2 = z_6$. Since we cannot define a diffeomorphism, we must prolong the system adding 4 new state variables defined in (8.20)

$$
z_7 = \bar{u}_1, \quad z_8 = \bar{u}_1^{(1)}, \quad z_9 = \bar{u}_1^{(2)}, \quad z_{10} = \bar{u}_1^{(3)}
$$

and two feedback laws defined in (8.21)

$$
\begin{aligned}
v_1 =& \frac{d^5}{dt^5}y_1(t) = \frac{d^4}{dt^4}\bar{u}_1 = \bar{u}_1^{(4)} \\
v_2 =& \bar{u}_2.
\end{aligned}
$$

Now, our system is written as (8.22)

$$\begin{cases} \dot{z}_1 = z_7 \\ \dot{z}_2 = v_2 \\ \dot{z}_3 = z_2 z_7 \\ \dot{z}_4 = z_3 z_7 \\ \dot{z}_5 = z_4 z_7 \\ \dot{z}_6 = z_5 z_7 \\ \dot{z}_7 = z_8 \\ \dot{z}_8 = z_9 \\ \dot{z}_9 = z_{10} \\ \dot{z}_{10} = v_1. \end{cases}$$

The diffeomorphism between z variables and the flat outputs is given by

$$\begin{cases} y_1 = z_1 \\ y_1^{(1)} = z_7 \\ y_1^{(2)} = z_8 \\ y_1^{(3)} = z_9 \\ y_1^{(4)} = z_{10} \\ y_2 = z_6 \\ y_2^{(1)} = z_5 z_7 \\ y_2^{(2)} = z_2 z_7^2 + z_5 z_8 \\ y_2^{(3)} = z_3 z_7^3 + 3 z_4 z_7 z_8 + z_5 z_9 \\ y_2^{(4)} = z_2 z_7^4 + 6 z_3 z_7^2 z_8 + 3 z_4 z_8^2 + 4 z_4 z_7 z_9 + z_5. \end{cases}$$

The feedback is of the form

$$\begin{aligned} w_1 =& v_1 \\ w_2 =& 10 z_2 z_7^3 z_8 + 15 z_3 z_7 z_8^2 + 10 z_3 z_7^2 z_9 + 10 z_4 z_8 z_9 + 5 z_4 z_7 z_{10} + z_5 v_1 + z_7^4 v_2, \end{aligned}$$

and it implies that

$$\begin{aligned} v_1 =& w_1 \\ v_2 =& \frac{w_2 - 10 z_2 z_7^3 z_8 + 15 z_3 z_7 z_8^2 + 10 z_3 z_7^2 z_9 + 10 z_4 z_8 z_9 + 5 z_4 z_7 z_{10} - z_5 w_1}{z_7^4}. \end{aligned}$$

Let $t = 0$ and $t_f = 1$ be the initial and final time, and impose the initial and final conditions

$$\begin{aligned} x(0) &= (x^2(0), y^2(0), \theta^3(0), \theta^2(0), \theta^1(0), \theta^0(0)) = (1, 1, 0, 0, 0, 0) \\ x(1) &= (x^2(1), y^2(1), \theta^3(1), \theta^2(1), \theta^1(1), \theta^0(1)) = \left(0, 0, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{4}\right). \end{aligned}$$

Then, the trajectories of the 3-trailer are shown in the following pictures.

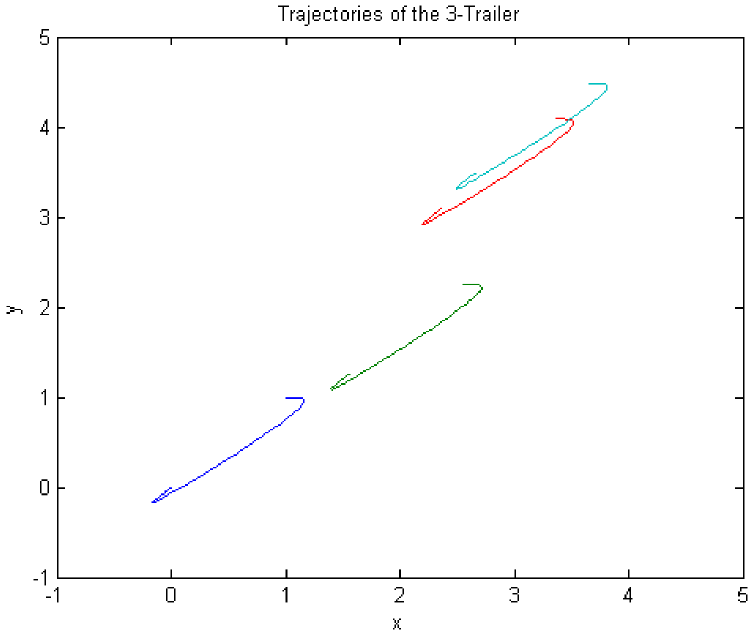

Figure 8.5: The trajectories of the 3-trailer, where the blue curve is the trajectory of the rear wheels of the second trailer, the green curve the rear wheels of the first trailer, the red curve the rear wheels of the cab and the cyan curve the trajectories of directional wheels.

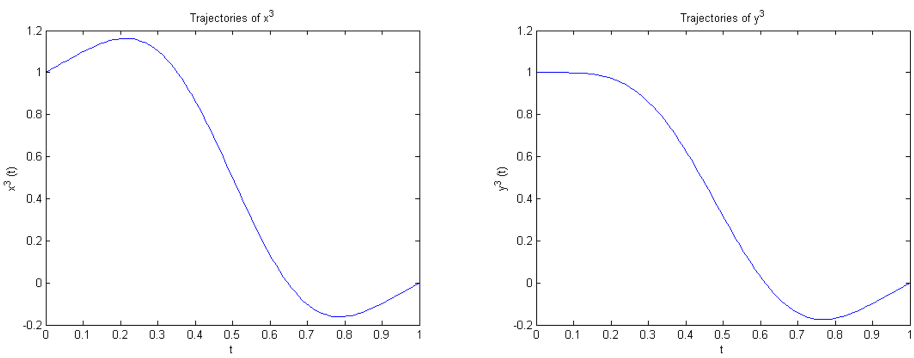

Figure 8.6: Trajectories of the state variables $x^3(t)$ and $y^3(t)$ respectively.

9
Conclusions

The work presents three robotics systems solved using differential flatness. The first robotic system consists in a simplified planar space robot with two arms, which is solved using Pfaff's theorem and feedback linearization. After that, Engel's theorem has been applied to a mobile robot with a trailer to establish a feedback linearization.

Finally, we presented the N-Trailer system viewed from the last trailer. In order to apply feedback linearization, we converted the system into Goursat normal form and later into chained form. It has been proved that the N-Trailer system can always be transformed into Goursat normal form and then, into chained form. Later on, we introduced coordinates from the last trailer that allow us to find the new state variables. Then, these new coordinates are used in the feedback linearization process.

We observe that differential flatness considerably simplifies the development of control design via feedback linearization. It is a powerful tool when we work with systems of m+2 state variables and two inputs, because that ensures us that we can apply Goursat normal form, which cannot be applied always, and therefore, convert the system into chained form.

However, we have to remark that the proposed method will find a path between any start and goal points in chained form coordinates, but there is no guarantee that this path, when transformed back into original variables, will avoid transformation singularities. This must be checked for every path.

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