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Interval-Valued Fuzzy Ideals Generated by an Interval-Valued Fuzzy Subset in Ordered Semigroups

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Abstract

In this paper, we define the concept of interval-valued fuzzy left (right, two sided, interior, bi-) ideal in ordered semigroups. We show that the interval-valued fuzzy subset $\overline{J}$ is an interval-valued fuzzy left (right, two sided, interior, bi-) ideal generated by an interval-valued fuzzy subset $\overline{A}$ iff $J^-$and $J^+$are fuzzy left (right, two sided, interior, bi-) ideals generated by $A^-$and A^+ respectively.

Keywords. Interval-valued fuzzy subsemigroup, Interval-valued fuzzy left (right, two-sided) ideal, Interval-valued fuzzy interior ideal, Interval-valued fuzzy bi-ideal.

1 Introduction and Preliminaries

Interval-valued fuzzy subsets were proposed thirty years ago as a natural extension of fuzzy sets by L.A.Zadeh [10]. Interval-valued fuzzy subsets have many applications in several areas. In [10], Zadeh also constructed a method of approximate inference using his interval-valued fuzzy subsets. In [9], Al Narayanan and T. Manikantan introduced the notions of interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in semigroups. The concept of fuzzy sets on ordered semigroups has been introduced by N.Kehayopulu and M.Tsingelis in [5]. In [6], they have discussed fuzzy bi-ideals in ordered semigroups and they discuss fuzzy interior ideals in ordered semigroups in [7].

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In this note we have introduced the concept of an interval-valued fuzzy left (right, two-sided, interior, bi-) ideal generated by an interval-valued fuzzy subset in ordered semigroups. Some characterizations of such generated interval-valued fuzzy ideals are also discussed.

An ordered semigroup is an ordered set S at the same time a semigroup such that $a \leq b \Rightarrow ax \leq bx$ and $xa \leq xb$ for all $a, b, x \in S$.

A non-empty subset A of an ordered semigroup S is called a subsemigroup of S if $A^2 \subseteq A$.

A non-empty subset A of an ordered semigroup S is called a right (resp. left) ideal of S if

1. $AS \subseteq A$ $(resp.\ SA \subseteq A)$
2. $a \in A$, $S \ni b \leq a \Rightarrow b \in A$.

A is called an ideal of S if it is both a right and a left ideal of S (cf. e.g, [1, 2]).

A subsemigroup A of S is called an interior ideal of S [7] if

1. $SAS \subseteq A$.
2. $a \in A$, $S \ni b \leq a \Rightarrow b \in A$.

A subsemigroup A of S is called a bi-ideal of S [6] if

1. $ASA \subseteq A$.
2. $a \in A$, $S \ni b \leq a \Rightarrow b \in A$.

An ordered semigroup S is called regular [4] if for each element a of S, there exists an element x in S such that $a \leq axa$.

A fuzzy subset f of an ordered semigroup S is a function from S to the unit interval $[0, 1]$ [5].

Let S be an ordered semigroup. A fuzzy subset f of S is called a fuzzy subsemigroup of S if $f(xy) \geq \min\{f(x), f(y)\}$ for all $x, y \in S$.

A fuzzy subset f of S is called a fuzzy left (resp. right) ideal of S if
1) $x \leq y \implies f(x) \geq f(y)$
2) $f(xy) \geq f(y)$ (resp. $f(xy) \geq f(x)$).

If f is both a fuzzy left ideal and a fuzzy right ideal of S, then f is called a fuzzy ideal of S or a fuzzy two sided ideal of S [1].

Equivalently, f is called a fuzzy ideal of S if
1) $x \leq y \implies f(x) \geq f(y)$
2) $f(xy) \geq \sup\{f(x), f(y)\}$.

A fuzzy subsemigroup f of S is called a fuzzy interior ideal of S [7] if
1) $x \leq y \implies f(x) \geq f(y)$
2) $f(xyz) \geq f(y)$.

A fuzzy subsemigroup f of S is called a fuzzy bi-ideal of S [6] if
1) $x \leq y \implies f(x) \geq f(y)$
2) $f(xyz) \geq \min\{f(x), f(z)\}$.

2 Interval-valued fuzzy ideals generated by an interval-valued fuzzy subset in ordered semigroups

An interval number on $[0,1]$, say $\overline{a}$, is a closed subinterval of $[0,1]$, that is, $\overline{a} = [a^-, a^+]$ where $0 \leq a^- \leq a^+ \leq 1$. Let $D\ [0,1]$ denotes the family of all closed subintervals of $[0,1]$, $\overline{0} = [0,0]$ and $\overline{1} = [1,1]$.

Now we define " $\leq$ ", " $=$ ", "Min^i", "Max^i" in case of two elements in $D\ [0,1]$. Consider two elements $\overline{a} = [a^-, a^+]$ and $\overline{b} = [b^-, b^+]$ in $D\ [0,1]$. Then

i) $\overline{a} \leq \overline{b}$ iff $a^- \leq b^-$ and $a^+ \leq b^+$
ii) $\overline{a} = \overline{b}$ iff $a^- = b^-$ and $a^+ = b^+$
iii) $Min^i\{\overline{a}, \overline{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}]$
iv) $Max^i\{\overline{a}, \overline{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}]$.

Let X be a set. A mapping $\overline{A} : X \longrightarrow D[0,1]$ is called an interval valued fuzzy subset (briefly, an $i-v$ fuzzy subset) of X, where $\overline{A}(x) = [A^-(x), A^+(x)]$ for all $x \in X$, A^- and A^+ are fuzzy subsets in X such that $A^-(x) \leq A^+(x)$ for all $x \in X$.

Let $\overline{A}$ and $\overline{B}$ be two $i-v$ fuzzy subsets of X. Define the relation " $\subseteq$ " between $\overline{A}$ and $\overline{B}$ as follows:

$\overline{A} \subseteq \overline{B}$ iff $\overline{A}(x) \leq \overline{B}(x)$ for all $x \in X$, *i.e* $A^-(x) \leq B^-(x)$ and $A^+(x) \leq B^+(x)$ for all $x \in X$.

Let S be an ordered semigroup with identity element 1 and $IF(S)$ denotes the set of all $i-v$ fuzzy subsets of S.

Definition 1 *Let A be a fuzzy subset of an ordered semigroup S with identity element 1. Then the smallest fuzzy left (right, two sided, interior, bi-) ideal of S containing A is called a fuzzy left (right, two sided, interior, bi-) ideal of S generated by A denoted by $< A >_L$ $\{(< A >_R, < A >, < A >_I, < A >_B\}$ respectively.*

Definition 2 *An $i-v$ fuzzy subset $\overline{A}$ of an ordered semigroup S is called an $i-v$ fuzzy subsemigroup of S if for all $x, y \in S$, $\overline{A}(xy) \geq Min^i\{\overline{A}(x), \overline{A}(y)\}$.*

Definition 3 *An $i-v$ fuzzy subset $\overline{A}$ of an ordered semigroup S is called an $i-v$ fuzzy left (resp. right) ideal of S if for all $x, y \in S$*

i) $x \leq y \Rightarrow \overline{A}(x) \geq \overline{A}(y)$
ii) $\overline{A}(xy) \geq \overline{A}(y)$ $(resp.\ \overline{A}(xy) \geq \overline{A}(x))$.

An $i-v$ fuzzy subset $\overline{A}$ in S is called an interval valued fuzzy two sided ideal of S if it is both an $i-v$ fuzzy left ideal and an $i-v$ fuzzy right ideal of S.

Definition 4 *An $i-v$ fuzzy subsemigroup $\overline{A}$ of an ordered semigroup S is called an $i-v$ fuzzy interior ideal of S if for all $x, y, z \in S$*

i) $x \leq y \Rightarrow \overline{A}(x) \geq \overline{A}(y)$
ii) $\overline{A}(xyz) \geq \overline{A}(y)$.

Definition 5 *An $i-v$ fuzzy subsemigroup $\overline{A}$ of an ordered semigroup S is called an $i-v$ fuzzy bi-ideal of S if for all $x, y, z \in S$*

i) $x \leq y \Rightarrow \overline{A}(x) \geq \overline{A}(y)$
ii) $\overline{A}(xyz) \geq Min^i\{\overline{A}(x), \overline{A}(z)\}$.

Theorem 1 *An interval valued fuzzy subset $\overline{A}$ of an ordered semigroup S is an $i-v$ fuzzy subsemigroup (left ideal, right ideal, two sided ideal, interior ideal, bi-ideal) of S iff A^- and A^+ are fuzzy subsemigroups (left ideals, right ideals, interior ideals, bi-ideals) of S.*

Proof. Let $\overline{A}$ be an $i-v$ fuzzy subsemigroup of S.
Then for all $x, y \in S$,
$\overline{A}(xy) \geq Min^i\{\overline{A}(x), \overline{A}(y)\}$, where
$Min^i\{\overline{A}(x), \overline{A}(y)\} = [min\{A^-(x), A^-(y)\}, min\{A^+(x), A^+(y)\}]$. Thus
$\overline{A}(xy) \geq [min\{A^-(x), A^-(y)\}, min\{A^+(x), A^+(y)\}]$. Hence
$[A^-(xy), A^+(xy)] \geq [min\{A^-(x), A^-(y)\}, min\{A^+(x), A^+(y)\}]$. Thus
$A^-(xy) \geq min\{A^-(x), A^-(y)\}$ and $A^+(xy) \geq min\{A^+(x), A^+(y)\}]$.
Hence A^-, A^+ are fuzzy subsemigroups of S.
The converse is straightforward.
Now suppose $\overline{A}$ is an $i-v$ fuzzy left ideal of S.
Then for all $x, y \in S$, $x \leq y \Rightarrow \overline{A}(x) \geq \overline{A}(y)$ and $\overline{A}(xy) \geq \overline{A}(y)$.
Now $\overline{A}(x) \geq \overline{A}(y) \Rightarrow [A^-(x), A^+(x)] \geq [A^-(y), A^+(y)]$.
That is $A^-(x) \geq A^-(y)$ and $A^+(x) \geq A^+(y)$.
$\overline{A}(xy) \geq \overline{A}(y) \Rightarrow [A^-(xy), A^+(xy)] \geq [A^-(y), A^+(y)]$.
Thus $A^-(xy) \geq A^-(y)$ and $A^+(xy) \geq A^+(y)$.
Hence A^- and A^+ are fuzzy left ideals of S.
The converse is straightforward. Similarly we can prove for other cases. □

Definition 6 *Let $\overline{A} \in IF(S)$. Then the smallest $i-v$ fuzzy left (right, two sided) ideal of S containing $\overline{A}$ is called an $i-v$ fuzzy left (right, two sided) ideal of S generated by $\overline{A}$, denoted by $<\overline{A}>_L$ $(<\overline{A}>_R, <\overline{A}>)$ respectively.*

Theorem 2 *Let $\overline{A} \in IF(S)$, then $<\overline{A}>_L = \overline{J}$,*

where $\overline{J} = [J^-, J^+]$ such that
$$J^-(x) = \sup_{\substack{x \leq x_1x_2 \\ x_1, x_2 \in S}} A^-(x_2) \text{ and } J^+(x) = \sup_{\substack{x \leq x_1x_2 \\ x_1, x_2 \in S}} A^+(x_2) \text{ for all } x \in S.$$

Proof. For all $a \in S$
$J^-(a) = \sup_{a \leq x_1x_2} A^-(x_2) \geq A^-(a)$, since $a = 1.a \Rightarrow A^-(a) \leq J^-(a)$.
Similarly $A^+(a) \leq J^+(a) \Rightarrow \overline{A}(a) = [A^-(a) \quad A^+(a)] \leq \overline{J}(a) = [J^-(a) \quad J^+(a)]$.
Thus $\overline{A} \subseteq \overline{J}$.
Now we show that $\overline{J}$ is an interval-valued fuzzy left ideal of S, for this we have to show $x \leq y \Rightarrow \overline{J}(x) \geq \overline{J}(y)$ and $\overline{J}(xy) \geq \overline{J}(y)$ for all $x, y \in S$.
Let $x, y \in S$, such that $x \leq y$.
If $y \leq x_3x_4$ then $x \leq x_3x_4$.
Hence $J^-(y) = \sup_{y \leq x_3x_4} A^-(x_4) \leq \sup_{x \leq x_1x_2} A^-(x_2) = J^-(x)$.

Similarly $J^+(x) \geq J^+(y) \;\; \Rightarrow \;\; \overline{J}(x) \geq \overline{J}(y)$.
Now if $y \leq x_1x_2$ then $xy \leq (xx_1)\, x_2$.
Hence $J^-(y) = \sup\limits_{y \,\leq\, x_1x_2} A^-(x_2) \leq \sup\limits_{xy \,\leq\, x_3x_4} A^-(x_4) = J^-(xy)$.
Similarly $J^+(xy) \geq J^+(y)$. Hence $\overline{J}(xy) \geq \overline{J}(y)$.
Let $\overline{B}$ be any $i-v$ fuzzy left ideal of S such that $\overline{B} \supseteq \overline{A}$.
Then for all $a \in S \quad B^-(a) \geq A^-(a)$ and $B^+(a) \geq A^+(a)$.
Now

$$\begin{aligned} J^-(a) &= \sup_{a \leq a_1a_2} A^-(a_2) \\ &\leq \sup_{a \leq a_1a_2} B^-(a_2) \\ &\leq \sup_{a \leq a_1a_2} B^-(a_1a_2) \leq B^-(a). \end{aligned}$$

Similarly $B^+(a) \geq J^+(a)$ for all $a \in S \Rightarrow \overline{J} \subseteq \overline{B}$. Hence $<\overline{A}>_L = \overline{J}$. $\square$

Theorem 3 *Let* $\overline{A} \in IF(S)$, *then* $<\overline{A}>_R = \overline{J}$,

where $\overline{J} = [J^-, J^+]$ such that
$J^-(x) = \sup\limits_{\substack{x \,\leq\, x_1x_2 \\ x_1,x_2 \,\in\, S}} A^-(x_1)$ and
$J^+(x) = \sup\limits_{\substack{x \,\leq\, x_1x_2 \\ x_1,x_2 \,\in\, S}} A^+(x_1)$ for all $x \in S$.

Proof. The proof is similar to the proof of Theorem 2. $\square$

Theorem 4 *Let* $\overline{A} \in IF(S)$. *An interval-valued fuzzy subset* $\overline{J}$ *is an* $i-v$ *fuzzy left (right) ideal of* S *generated by* $\overline{A}$ *if and only if* J^- *and* J^+ *are fuzzy left (right) ideals of* S *generated by* A^- *and* A^+ *respectively.*

Proof. Suppose $\overline{J}$ is an $i-v$ fuzzy left ideal of S generated by $\overline{A}$.

Then by Theorem 1, J^+ and J^- are fuzzy left ideals of S.

Since $\overline{A} \subseteq \overline{J}$, we have $A^- \subseteq J^-$ and $A^+ \subseteq J^+$.

If B is a fuzzy left ideal of S containing A^-, then define $\overline{B} : S \to D[0,1]$ by $\overline{B}(x) = [B^-(x),\, B^+(x)]$, where $B^-(x) = B(x)$ for all $x \in S$ and $B^+(x) = 1$ for all $x \in S$.

Since B^- and B^+ are fuzzy left ideals of S, by Theorem 1, $\overline{B}$ is an $i-v$ fuzzy left ideal of S.

Clearly $\overline{A} \subseteq \overline{B}$ so $\overline{J} \subseteq \overline{B} \Rightarrow J^- \subseteq B^- = B \Rightarrow J^-$ is fuzzy left ideal of S generated by A^-.

Similarly we can show that J^+ is fuzzy left ideal of S generated by A^+.

Conversely, assume that J^- and J^+ are fuzzy left ideals of S generated by A^- and A^+ respectively.

Then by Theorem 1, $\overline{J}$ is an $i-v$ fuzzy left ideal of S containing $\overline{A}$.

If $\overline{B}$ is an $i-v$ fuzzy left ideal of S containing $\overline{A}$, then $A^- \leq B^-$ and $A^+ \leq B^+$.

Since B^- and B^+ are fuzzy left ideals of S, we have $J^- \leq B^-$ and $J^+ \leq B^+$.

Hence $\overline{J} \subseteq \overline{B}$. Thus $\overline{J}$ is an $i-v$ fuzzy left ideal of S generated by $\overline{A}$. $\square$

Theorem 5 *Let* $\overline{A} \in IF(S)$, *then* $<< \overline{A} >_L >_R = < \overline{A} > = << \overline{A} >_R >_L$.

Proof. By Theorem 3, $<< \overline{A} >_L >_R$ is an $i - v$ fuzzy right ideal of S.
Clearly, $<< \overline{A} >_L >_R = [<< A^- >_L >_R, << A^+ >_L >_R]$.
For all $x, y \in S$,

$$<< A^- >_L >_R (xy) = \sup_{xy \leq a_1 a_2} < A^- >_L (a_1) = \sup_{xy \leq a_1 a_2} \sup_{a_1 \leq z_1 z_2} A^-(z_2)$$

and

$$<< A^- >_L >_R (y) = \sup_{y \leq y_1 y_2} < A^- >_L (y_1) = \sup_{y \leq y_1 y_2} \sup_{y \leq w_1 w_2} A^-(w_2).$$

Obviously,
$<< A^- >_L >_R (xy) \geq << A^- >_L >_R (y)$.
Similarly we have $<< A^+ >_L >_R (xy) \geq << A^+ >_L >_R (y)$.
It follows that,

$$\begin{aligned} << \overline{A} >_L >_R (xy) &= [<< A^- >_L >_R (xy), << A^+ >_L >_R (xy)] \\ &\geq [<< A^- >_L >_R (y), << A^+ >_L >_R (y)] \\ &= << \overline{A} >_L >_R (y). \end{aligned}$$

Hence $<< \overline{A} >_L >_R$ is an $i - v$ fuzzy left ideal of S.
So $<< A^- >_L >_R$ is an $i - v$ fuzzy ideal of S.
Since $\overline{A} \subseteq < \overline{A} >_L \subseteq << \overline{A} >_L >_R$, we have $<< \overline{A} >_L >_R \supseteq \overline{A}$.
Suppose $\overline{B}$ is any $i - v$ fuzzy ideal of S such that $\overline{B} \supseteq \overline{A}$.
Since $< \overline{A} >_L$ is a smallest $i - v$ fuzzy left ideal of S containing $\overline{A}$, we have, $\overline{B} \supseteq < \overline{A} >_L$. Also $\overline{B} \supseteq << \overline{A} >_L >_R$, since $<< \overline{A} >_L >_R$ is a smallest $i - v$ fuzzy left ideal of S containing $< \overline{A} >_L$. This shows that $<< \overline{A} >_L >_R$ is a smallest $i - v$ fuzzy left ideal of S containing $\overline{A}$. Therefore $<< \overline{A} >_L >_R = < \overline{A} >$.
Similarly we can prove that $<< \overline{A} >_R >_L = < \overline{A} >$.
Hence $<< \overline{A} >_L >_R = < \overline{A} > = << \overline{A} >_R >_L$. □

Definition 7 *Let* $\overline{A} \in IF(S)$. *Then the smallest* $i - v$ *fuzzy interior ideal of* S *containing* $\overline{A}$ *is called an* $i - v$ *fuzzy interior ideal of* S *generated by* $\overline{A}$, *denoted by* $< \overline{A} >_I$.

Theorem 6 *Let* $\overline{A} \in IF(S)$, *then* $< \overline{A} >_I = \overline{J}$, *where* $\overline{J} = [J^-, J^+]$ *such that*

$$J^-(x) = \sup_{\substack{x \leq x_1 x_2 x_3 \\ x_1, x_2, x_3 \in S}} A^-(x_2)$$

$$J^+(x) = \sup_{\substack{x \leq x_1 x_2 x_3 \\ x_1, x_2, x_3 \in S}} A^+(x_2) \quad \text{for all } x \in S.$$

Proof. For all $a \in S$, we have
$J^-(a) = \sup_{a \leq x_1 x_2 x_3} A^-(x_2) \geq A^-(a)$ because $a \leq 1a1$.

Similarly $J^+(a) \geq A^+(a)$.
Therefore $\overline{J}(a) = [J^-(a)\ ,\ J^+(a)] \geq [A^-(a)\ ,\ A^+(a)] = \overline{A}(a)$.
Let $x, y \in S$, such that $x \leq y$.
If $y \leq x_1x_2x_3$ then $x \leq x_1x_2x_3$. Hence
$J^-(y) = \sup\limits_{y \leq x_1x_2x_3} A^-(x_2) \leq \sup\limits_{x \leq x_4x_5x_6} A^-(x_5) = J^-(x)$.
Similarly $J^+(y) \leq J^+(x)$.
Hence $\overline{J}(x) = [J^-(x)\ , J^+(x)] \geq [J^-(y)\ ,\ J^+(y)] = \overline{J}(y)$.
Also for all $x, y, z \in S$,
if $y \leq a_1a_2a_3$ then $xyz \leq (xa_1)a_2(a_3z)$. Hence
$J^-(y) = \sup\limits_{y \leq a_1a_2a_3} A^-(a_2) \leq \sup\limits_{xyz \leq b_1b_2b_3} A^-(b_2) = J^-(xyz)$.
Similarly $J^+(y) \leq J^+(xyz)$.
Thus $\overline{J}(y) = [J^-(y), J^+(y)] \leq [J^-(xyz), J^+(xyz)] = \overline{J}(xyz)$.
This shows that $\overline{J}$ is an $i-v$ fuzzy interior ideal of S containing $\overline{A}$.
Let $\overline{B}$ be any $i-v$ fuzzy interior ideal of S such that $\overline{B} \supseteq \overline{A}$.
Then for all $a \in S$,

$$\begin{aligned} J^-(a) &= \sup_{a \leq a_1a_2a_3} A^-(a_2) \\ &\leq \sup_{a \leq a_1a_2a_3} B^-(a_2) \\ &\leq \sup_{a \leq a_1a_2a_3} B^-(a_1a_2a_3) \leq B^-(a). \end{aligned}$$

Similarly $J^+(a) \leq B^+(a)$. Hence $\overline{J}(a) \leq \overline{B}(a)$.
This shows that $\overline{J}$ is the smallest $i-v$ fuzzy interior ideal of S containing $\overline{A}$, that is $<\overline{A}>_I = \overline{J}$. □

Theorem 7 *Let $\overline{A} \in IF(S)$. An interval valued fuzzy subset $\overline{J}$ is an $i-v$ fuzzy interior ideal of S generated by $\overline{A}$ iff J^- and J^+ are fuzzy interior ideals of S generated by A^- and A^+ respectively.*

Proof. The proof is similar to the proof of Theorem 4. □

Definition 8 *An $i-v$ fuzzy subsemigroup $\overline{A}$ of S is called an $i-v$ fuzzy submonoid of S if $\overline{A}(1) \geq \overline{A}(x)$ for all $x \in S$.*

Theorem 8 *Let $\overline{A}$ be an $i-v$ fuzzy submonoid of S then $<\overline{A}>_B = \overline{J}$ where*

$$J^-(x) = \sup_{x \leq x_1x_2x_3} min\{A^-(x_1), A^-(x_3)\} \text{ and}$$

$$J^+(x) = \sup_{x \leq x_1x_2x_3} min\{A^+(x_1), A^+(x_3)\} \quad \text{for all } x \in S.$$

Proof. For all $x \in S$,

$$\begin{aligned} J^-(x) &= \sup_{x \leq x_1x_2x_3} min\{A^-(x_1), A^-(x_3)\} \\ &\geq min\{A^-(1), A^-(x)\} = A^-(x), \end{aligned}$$

because $x \leq 1.1.x$ and $\overline{A}(1) \geq \overline{A}(x)$ for all $x \in S$.
Similarly we have $J^{+}(x) \geq A^{+}(x)$.
Therefore $\overline{J}(x) \geq \overline{A}(x)$ and so $\overline{J} \supseteq \overline{A}$.
Let $x, y \in S$ such that $x \leq y$.
If $y \leq x_1x_2x_3$ then $x \leq x_1x_2x_3$. Hence

$$
\begin{aligned}
J^{-}(y) &= \sup_{y \leq x_1x_2x_3} min\{A^{-}(x_1), A^{-}(x_3)\} \\
&\leq \sup_{x \leq a_1a_2a_3} min\{A^{-}(a_1), A^{-}(a_3)\} \\
&= J^{-}(x).
\end{aligned}
$$

Similarly we have $J^{+}(x) \geq J^{+}(y)$.
Thus $\overline{J}(x) \geq \overline{J}(y)$.
Also for all $x, y, z \in S$.
If $x \leq x_1x_2x_3$ and $z \leq z_1z_2z_3$, then $xyz \leq (x_1x_2x_3)y(z_1z_2z_3)$. Hence

$$
\begin{aligned}
J^{-}(xyz) &= \sup_{xyz \leq a_1a_2a_3} min\{A^{-}(a_1), A^{-}(a_3)\} \\
&\geq \sup_{\substack{xyz \leq x_1(x_2x_3yz_1z_2)z_3 \\ x \leq x_1x_2x_3,\ z \leq z_1z_2z_3}} min\{A^{-}(x_1), A^{-}(z_3)\}.
\end{aligned}
$$

We can write $A^{-}(x_1) \geq min\{A^{-}(x_1), A^{-}(x_3)\}$,
$A^{-}(z_3) \geq min\{A^{-}(z_1), A^{-}(z_3)\}$. It follows that

$$
\begin{aligned}
J^{-}(xyz) &\geq \sup_{\substack{xyz \leq x_1(x_2x_3yz_1z_2)z_3 \\ x \leq x_1x_2x_3,\ z \leq z_1z_2z_3}} min\{min\{A^{-}(x_1), A^{-}(x_3)\}, min\{A^{-}(z_1), A^{-}(z_3)\}\} \\
&= min\{\sup_{x \leq x_1x_2x_3} min\{A^{-}(x_1), A^{-}(x_3)\}, \sup_{z \leq z_1z_2z_3} min\{A^{-}(z_1), A^{-}(z_3)\}\} \\
&= min\{J^{-}(x), J^{-}(z)\}.
\end{aligned}
$$

Similarly we have $J^{+}(xyz) \geq min\{J^{+}(x), J^{+}(z)\}$.

$$
\begin{aligned}
\text{Therefore } \overline{J}(xyz) &= [J^{-}(xyz), J^{+}(xyz)] \\
&\geq [min\{J^{-}(x), J^{-}(z)\}, min\{J^{+}(x), J^{+}(z)\}] \\
&= Min^{i}\{\overline{J}(x), \overline{J}(z)\}
\end{aligned}
$$

and so, $\overline{J}(xyz) \geq Min^{i}\{\overline{J}(x), \overline{J}(z)\}$.
Taking $y = 1$, we have $\overline{J}(xz) \geq Min^{i}\{\overline{J}(x), \overline{J}(z)\}$.
This shows that $\overline{J}$ is an $i-v$ fuzzy bi-ideal of S.
Let $\overline{B}$ be an $i-v$ fuzzy bi-ideal of S such that $\overline{B} \supseteq \overline{A}$.
Then for all $a \in S$, we have

$$
\begin{aligned}
J^{-}(a) &= \sup_{a \leq a_1a_2a_3} min\{A^{-}(a_1), A^{-}(a_3)\} \\
&\leq \sup_{a \leq a_1a_2a_3} min\{B^{-}(a_1), B^{-}(a_3)\} \\
&\leq \sup_{a \leq a_1a_2a_3} B^{-}(a_1a_2a_3) \leq B^{-}(a).
\end{aligned}
$$

Similarly we have $J^{+}(a) \leq B^{+}(a)$. Thus $\overline{J} \subseteq \overline{B}$.
Hence $\overline{J}$ is a smallest $i-v$ fuzzy bi-ideal of S containing $\overline{A}$.
That is $<\overline{A}>_B = \overline{J}$. □

Theorem 9 *Let $\overline{A}$ be an $i-v$ fuzzy submonoid of S. Then an $i-v$ fuzzy subset $\overline{J}$ is an $i-v$ fuzzy bi-ideal of S generated by $\overline{A}$ if and only if J^{-} and J^{+} are fuzzy bi-ideals of S generated by A^{-} and A^{+} respectively.*

Proof. The proof is similar to the proof of Theorem 4. □

Theorem 10 *Let S be a regular ordered semigroup and $\overline{A} \in IF(S)$,*

then $< \overline{A} >_B = \overline{J}$, where
$J^-(x) = \sup_{x \leq x_1 x_2 x_3} min\{A^-(x_1), A^-(x_3)\}$ and

$J^+(x) = \sup_{x \leq x_1 x_2 x_3} min\{A^+(x_1), A^+(x_3)\}$ for all $x \in S$.

Proof. From the proof of Theorem 8, it is enough to prove that $\overline{J} \supseteq \overline{A}$.
For all $x \in S$, we have

$$J^-(x) = \sup_{x \leq x_1 x_2 x_3} min\{A^-(x_1), A^-(x_3)\} \geq \sup_{x \leq xax} min\{A^-(x), A^-(x)\} = A^-(x)$$

Similarly we have $J^+(x) \geq A^+(x)$. Therefore $\overline{J}(x) \geq \overline{A}(x)$ and so $\overline{J} \supseteq \overline{A}$. □

Remark 1 *Theorem 9 is also true for an $i-v$ fuzzy subset $\overline{A}$ of a regular ordered semigroup S.*

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