Mathware & Soft Computing 15 (2008) 143-157

Aggregation Operators and Lipschitzian Conditions

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Abstract

Lipschitzian aggregation operators with respect to the natural T- indistinguishability operator E_T and their powers, and with respect to the residuation $\overrightarrow{T}$ with respect to a t-norm T and its powers are studied.

A t-norm T is proved to be E_T-Lipschitzian and $\overrightarrow{T}$-Lipschitzian, and is interpreted as a fuzzy point and a fuzzy map as well.

Given an Archimedean t-norm T with additive generator t, the quasi-arithmetic mean generated by t is proved to be the most stable aggregation operator with respect to T.

Keywords: Aggregation Operator, Residuation, T-indistinguishability Operator, Lipschitzian condition.

1 Introduction

Lipschitzian aggregation operators have been studied in [5] [6] [16] by considering the usual metric on the unit interval. In this paper we study the Lipschitzian condition of aggregation operators with respect to the natural indistinguishability operator E_T and their powers E_T^p so that an aggregation operator h is E_T^p-Lipschitzian when for all $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in [0,1]$

$$T(E_T^p(x_1, y_1), ..., E_T^p(x_n, y_n)) \leq E_T(h(x_1, x_2, ..., x_n), h(y_1, y_2, ..., y_n)).$$

This means that from similar inputs we obtain similar aggregations.

An aggregation operator h will be called $\overrightarrow{T}^p$-Lipschitzian if for all $x_1, x_2, ..., x_n$, $y_1, y_2, ..., y_n \in [0,1]$

$$T(\overrightarrow{T}^p(x_1|y_1), ..., \overrightarrow{T}^p(x_n|y_n)) \leq \overrightarrow{T}(h(x_1, x_2, ..., x_n)|h(y_1, y_2, ..., y_n)),$$

meaning that if from x_i we can infer y_i for all $i = 1, 2, ..., n$, from the aggregation $h(x_1, x_2, ..., x_n)$, the aggregation $h(y_1, y_2, ..., y_n)$ can be inferred.

When T is the Lukasiewicz t-norm, the E_T-Lipschitzian condition coincides with the 1-Lipschitzian condition with the usual metric on [0,1] and the definition of [16] is recovered.

It is worth noticing the relation between the E_T-Lipschitzian condition of an aggregation operator h and its extensionality with respect to the integral powers $T(\overbrace{E_T, ..., E_T}^{n\ times})$.

The t-norm T can be seen as a fuzzy point and a fuzzy map as well.

Also, if T is a continuous Archimedean t-norm with an additive generator t and m_t the quasi-arithmetic mean generated by t ($m_t(x_1, x_2, ..., x_n) = t^{-1}\left(\frac{t(x_1)+t(x_2)+...+t(x_n)}{n}\right)$), then m_t is the most stable aggregation operator with respect to T.

The paper includes a section with some basic concepts in fuzzy reasoning that can be interpreted as Lipschitzian conditions.

2 Preliminaries

This section recalls the definition of the power $x_T^{(n)}$ of an element x of the unit interval with respect to the t-norm T and generalizes it to irrational exponents. Also the definition of aggregation operator is recalled.

For the sake of simplicity we will assume continuity for t-norms throughout the paper.

Since a t-norm T is associative, we can extend it to an n-ary operation in the standard way:

$$T(x) = x$$
$$T(x_1, x_2, ...x_n) = T(x_1, T(x_2, ..., x_n)).$$

In particular, following the notation in [15], $T(\overbrace{x, x, ..., x}^{n\ times})$ will be denoted by $x_T^{(n)}$.

The n-th root $x_T^{(\frac{1}{n})}$ of x with respect to T is defined by

$$x_T^{(\frac{1}{n})} = sup\{z \in [0,1] \mid z_T^{(n)} \leq x\}$$

and for $m, n \in N$, $x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)}$.

Lemma 2.1. [15] *If* $k, m, n \in N,\ k, n \neq 0$ *then* $x_T^{(\frac{km}{kn})} = x_T^{(\frac{m}{n})}$.

Lemma 2.2. *Let* $x_1, ..., x_n \in (0,1]$ *and* $n \in N$. $T(x_{1_T}^{(\frac{1}{n})}, ..., x_{n_T}^{(\frac{1}{n})}) \neq 0$.

The powers $x_T^{(\frac{m}{n})}$ can be extended to irrational exponents in a straightforward way.

Definition 2.3. *If $r \in R^+$ is a positive real number, let $\{a_n\}_{n\in N}$ be a sequence of rational numbers with $lim_{n\to\infty} a_n = r$. For any $x \in [0,1]$, the power $x_T^{(r)}$ is*

$$x_T^{(r)} = lim_{n\to\infty} x_T^{(a_n)}.$$

Continuity assures the existence of this limit and its independence from the selection of the sequence $\{a_n\}_{n\in N}$.

Proposition 2.4. *Let T be an Archimedean t-norm with additive generator t, $x \in [0,1]$ and $r \in R^+$. Then*

$$x_T^{(r)} = t^{[-1]}(rt(x)).$$

where $t^{[-1]}$ is the pseudo-inverse of t.

Proof. Due to continuity of t we need to prove it only for positive rational numbers r.

If r is a positive integer, then trivially $x_T^{(r)} = t^{[-1]}(rt(x))$.

If $r = \frac{1}{n}$ with $n \in N$, then $x_T^{(\frac{1}{n})} = z$ with $z_T^{(n)} = x$ or $t^{[-1]}(nt(z)) = x$ and $x_T^{(\frac{1}{n})} = t^{[-1]}\left(\frac{t(x)}{n}\right)$.

For a rational number $\frac{m}{n}$,

$$x_T^{(\frac{m}{n})} = \left(x_T^{(\frac{1}{n})}\right)_T^{(m)} = t^{[-1]}\left(mt\left(x_T^{(\frac{1}{n})}\right)\right) =$$

$$t^{[-1]}\left(mt\left(t^{[-1]}\left(\frac{t(x)}{n}\right)\right)\right) = t^{[-1]}\left(\frac{m}{n}t(x)\right).$$

□

Finally, let us recall the definition of aggregation operator.

Definition 2.5. [5] *An aggregation operator is a map $h : \bigcup_{n\in N}[0,1]^n \to [0,1]$ satisfying*

1. $h(0,...,0) = 0$ and $h(1,...,1) = 1$
2. $h(x) = x \ \forall x \in [0,1]$
3. $h(x_1,...,x_n) \leq h(y_1,...,y_n)$ if $x_1 \leq y_1,...,x_n \leq y_n$ (*monotonicity*).

The restriction of h to $[0,1]^n$ will be denoted by $h_{(n)}$ so that a global aggregation operator h can be split into the family of n-ary operators $(h_{(n)})_{n\in N}$.

For example, for a t-norm T, $T_{(n)}(x_1,...,x_n)$ is simply $T(x_1,...,x_n)$

3 $\overrightarrow{T}$-Lipschitzian aggregation operators

Definition 3.1. *The residuation $\overrightarrow{T}$ of a t-norm T is defined by*

$$\overrightarrow{T}(x|y) = sup\{\alpha \in [0,1] \mid T(x,\alpha) \leq y\}.$$

The residuation $\overrightarrow{T}$ of a t-norm T can be seen as a fuzzy implication. It satisfies the following properties.

Proposition 3.2. *Let $\overrightarrow{T}$ be the residuation of a t-norm T. $\overrightarrow{T}$ satisfies for all $x, y, z \in [0,1]$*

1. $x \leq z \Longrightarrow \overrightarrow{T}(x|y) \geq \overrightarrow{T}(z|y)$
2. $y \leq z \Longrightarrow \overrightarrow{T}(x|y) \leq \overrightarrow{T}(x|z)$
3. $x \leq y \Longrightarrow \overrightarrow{T}(x|y) = 1$
4. $\overrightarrow{T}(1|y) = y$
5. $\overrightarrow{T}(x|x) = 1$
6. $\overrightarrow{T}(x|y) \geq y$.

In particular, $\overrightarrow{T}(0|0) = \overrightarrow{T}(0|1) = \overrightarrow{T}(1|1) = 1$ and $\overrightarrow{T}(1|0) = 0$, so that it coincides with the classical implication in the crisp values.

Note that given $x, y \in [0,1]$, either $\overrightarrow{T}(x|y)$ or $\overrightarrow{T}(y|x)$ equal 1.

Proposition 3.3. *Let T be an Archimedean t-norm with additive generator t. For all $p > 0$, $\overrightarrow{T}^p$ can be calculated for all $x, y \in [0,1]$ by*

$$\overrightarrow{T}^p(x|y) = t^{[-1]}(pt(\overrightarrow{T}(x|y))).$$

Example 3.4.

1. *If T is the Lukasiewicz t-norm, then $\overrightarrow{T}^p(x|y) = Max(0, Min(1-p(x-y), 1))$ for all $x, y \in [0,1]$.*
2. *If T is the product t-norm, then $\overrightarrow{T}^p(x|y) = Min\left(\left(\frac{y}{x}\right)^p, 1\right)$ for all $x, y \in [0,1]$.*

Proposition 3.5. *Let T-be a t-norm and $p, q > 0$. $\overrightarrow{T}^p \leq \overrightarrow{T}^q$ if and only if $p \geq q$.*

In general, $\overrightarrow{T}^p$ do not satisfy the conditions for being a fuzzy implication. For example, for the Lukasiewicz t-norm, $\overrightarrow{T}^p(1|0) = Max(0, 1-p)$, which is different from 0 for $p < 1$.

Lemma 3.6. *Let a, b be elements of the unit interval with $a \leq b$, T a t-norm and $p > 0$. Then $a_T^{(p)} \leq b_T^{(p)}$.*

Proof. If p is a positive integer, then clearly $a_T^{(p)} \leq b_T^{(p)}$.

If $p = \frac{1}{n}$ with n a positive integer, then again $a_T^{(p)} \leq b_T^{(p)}$, since the supremum of all $z \in [0,1]$ with $z_T^{(p)} \leq a$ will be smaller than the supremum of all $z \in [0,1]$ with $z_T^{(p)} \leq b$.

Therefore $a_T^{(p)} \leq b_T^{(p)}$ for all positive rational p. Due to the continuity of T, the result can be extended to all real $p > 0$. □

Proposition 3.7. *Let T be a t-norm. Then $\overrightarrow{T}^{p}$ satisfies properties 1., 2., 3., 5. of Proposition 3.2.*

Proof. It is a consequence of the monotonicity of the powers with respect to a t-norm. □

Nevertheless, properties 4. and 6. of Proposition 3.2 may be violated by $\overrightarrow{T}^{p}$. The next two propositions give conditions in which these properties are satisfied.

Proposition 3.8. *Let T be an Archimedean t-norm. $\overrightarrow{T}^{p}$ satisfies property 6. of Proposition 3.2 if and only if $p \leq 1$.*

Proof. Let t be an additive generator of T. Then

$$\overrightarrow{T}^{p}(x|y) = t^{[-1]}(pt(\overrightarrow{T}(x|y))) \geq y$$

if and only if

$$pt(\overrightarrow{T}(x|y)) \leq t(y)$$

that is equivalent to

$$t(\overrightarrow{T}(x|y)) \leq \frac{t(y)}{p}$$

and to

$$\overrightarrow{T}(x|y) \geq t^{[-1]}\left(\frac{t(y)}{p}\right).$$

$\frac{t(y)}{p} \geq t(y)$ and $t^{[-1]}\left(\frac{t(y)}{p}\right) \leq y$ if and only if $p \leq 1$. So

$$\overrightarrow{T}(x|y) \geq y \geq t^{[-1]}\left(\frac{t(y)}{p}\right)$$

if and only if $p \leq 1$. □

Proposition 3.9. *Let T be an Archimedean t-norm. $\overrightarrow{T}^{p}$ satisfies property 4. of Proposition 3.2 if and only if T is strict or T is non-strict and $p \geq 1$.*

Proof.

$$\overrightarrow{T}^p (1|0) = t^{[-1]}(pt(\overrightarrow{T} (1|0))) = t^{[-1]}(pt(0))$$

If T is strict, then $t(0) = \infty$ and $t^{[-1]}(\infty) = 0$.

If T is non-strict, then $t^{[-1]}(pt(0)) = 0$ if and only if $p \geq 1$. □

We can now define the Lipschitzian condition of an aggregation operator with respect to $\overrightarrow{T}^p$.

Definition 3.10. *Given a t-norm T and $p > 0$, an aggregation operator h is $\overrightarrow{T}^p$-Lipschitzian if for all $x_1, x_2, ..., x_n, y_1, y_2, ..., y_n \in [0,1]$*

$$T(\overrightarrow{T}^p (x_1|y_1), ..., \overrightarrow{T}^p (x_n|y_n)) \leq \overrightarrow{T} (h(x_1, x_2, ..., x_n)|h(y_1, y_2, ..., y_n)).$$

The interpretation of this property is that if from x_i we can infer y_i for all $i = 1, 2, ..., n$, from the aggregation $h(x_1, x_2, ..., x_n)$, the aggregation $h(y_1, y_2, ..., y_n)$ can be inferred.

If $x_i \leq y_i$ for all $i = 1, 2, ..., n$, then the property is satisfied trivially $(1 \leq 1)$. If $x_i \geq y_i$ for all $i = 1, 2, ..., n$, then the property coincides with E_T^p-Lipschitzianity (see Section 4).

Proposition 3.12 states that a t-norm T is a $\overrightarrow{T}$-Lipschitzian aggregation operator.

Lemma 3.11. *Let T be a continuous t-norm. Then, for all $x, y \in [0,1]$ with $x \geq y$*

$$T(x, \overrightarrow{T} (x|y)) = y.$$

Proposition 3.12. *Let T be a continuous t-norm. Then T is a $\overrightarrow{T}$-Lipschitzian aggregation operator.*

Proof. We must prove

$$T(\overrightarrow{T} (x_1|y_1), ..., \overrightarrow{T} (x_n|y_n)) \leq \overrightarrow{T} (T(x_1, ..., x_n)|T(y_1, ..., y_n))$$

or, equivalently,

$$T(\overrightarrow{T} (x_1|y_1), ..., \overrightarrow{T} (x_n|y_n), x_1, ..., x_n) \leq T(y_1, ..., y_n).$$

But,

if $x_i \leq y_i$, then $T(\overrightarrow{T} (x_i|y_i), x_i) \leq y_i$ and

If $x_i \geq y_i$, then $T(\overrightarrow{T} (x_i|y_i), x_i) = y_i$. □

Note that if $x_i \geq y_i$ for all $i = 1, ...n$, then $T(\overrightarrow{T} (x_1|y_1), ..., \overrightarrow{T} (x_n|y_n)) = \overrightarrow{T} (T(x_1, ..., x_n)|T(y_1, ..., y_n))$. Since for every t-norm different from the Minimum $\overrightarrow{T}^p < \overrightarrow{T}^q$ if $p > q$, we have that $T \neq Min$ is not $\overrightarrow{T}^p$-Lipschitzian for $p < 1$.

4 E_T-Lipschitzian aggregation operators

Lipschitzian aggregation operators with respect to the natural T- indistinguishability operator E_T and their powers are a special kind of aggregation operators that generalize the definition of [16]. Their interest lays in the fact that they are stable operators in the sense that the similarity between the aggregation of two n-tuples is bounded by the similarity between them.

It is interesting to point out that the Lipschitzian condition is equivalent to extensionality (Proposition 5.6).

Among other results, it will be proved that a t-norm T is E_T-Lipschitzian.

Also quasi-arithmetic means are proved to be the most stable aggregation operators.

Definition 4.1. *The natural T-indistinguishability operator E_T associated to a given t-norm T is the fuzzy relation on* [0,1] *defined by*

$$E_T(x, y) = T(\overrightarrow{T}\ (x|y), \overrightarrow{T}\ (y|x)) = Min(\overrightarrow{T}\ (x|y), \overrightarrow{T}\ (y|x)).$$

Note that $\overrightarrow{T}\ (x|y) = E_T(x, y)$ if and only if $x \geq y$.

Example 4.2.

1. *If T is an Archimedean t-norm with additive generator t, then $E_T(x, y) = t^{-1}(|t(x) - t(y)|)$ for all $x, y \in [0, 1]$.*

2. *If T is the Lukasiewicz t-norm, then $E_T(x, y) = 1 - |x - y|$ for all $x, y \in [0, 1]$.*

3. *If T is the Product t-norm, then* $E_T(x, y) = \begin{cases} \frac{Min(x,y)}{Max(x,y)} & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$

4. *If T is the Minimum t-norm, then* $E_T(x, y) = \begin{cases} Min(x, y) & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$

E_T is indeed a special kind of (one-dimensional) T-indistinguishability operator (Definition 4.3) [4] and in a logical context where T plays the role of the conjunction, E_T is interpreted as the bi-implication associated to T [8].

The general definition of T-indistinguishability operator is as follows.

Definition 4.3. *Given a t-norm T, a T-indistinguishability operator E on a set X is a fuzzy relation $E : X \times X \to [0, 1]$ satisfying for all $x, y, z \in X$*

1. $E(x, x) = 1$ *(Reflexivity)*

2. $E(x, y) = E(y, x)$ *(Symmetry)*

3. $T(E(x, y), E(y, z)) \leq E(x, z)$ *(T-transitivity).*

Proposition 4.4. *Let E be a T indistinguishability operator on a set X. The fuzzy relation E^n defined by*

$$E^n(x,y) = T(\overbrace{E(x,y),...,E(x,y)}^{n\ times})\ \forall x,y \in X$$

is a T-indistinguishability operator.

Proposition 4.5. [11] *Let E be a T-indistinguishability operator on a set X. $E^{\frac{1}{n}}$ is a T-indistinguishability operator on X.*

Corollary 4.6. *Let E be a T-indistinguishability operator on a set X. $E^{\frac{m}{n}}$ is a T-indistinguishability operator on X.*

Proof. Propositions 4.4. and 4.5. □

Corollary 4.7. *Let E_T be the natural T-indistinguishability operator on* [0,1] *associated to T. $E_T^{\frac{m}{n}}$ is a T-indistinguishability operator.*

Continuity of the t-norm T allows us to extend the powers of a T- indistinguishability operator to positive irrational numbers in the same way as in Definition 2.3.

Example 4.8.

1. *If T is a continuous Archimedean t-norm with additive generator t, then $E_T^p(x,y) = t^{[-1]}(p|t(x) - t(y)|)$ for all $x,y \in [0,1]$.*
2. *If T is the Lukasiewicz t-norm, then $E_T^p(x,y) = Max(0, 1 - p|x-y|)$ for all $x,y \in [0,1]$.*
3. *If T is the Product t-norm, then $E_T^p(x,y) = \begin{cases} \frac{Min(x^p,y^p)}{Max(x^p,y^p)} & \text{if } x \neq y \\ 1 & \text{otherwise.} \end{cases}$*
4. *If T is the Minimum t-norm, then $E_T^p(x,y) = E_T(x,y)$ for all $x,y \in [0,1]$.*

Proposition 4.9. *Let T-be a t-norm and $p,q > 0$. $E_T^p \leq E_T^q$ if and only if $p \geq q$.*

The Lipschitzian condition for an aggregation operator with respect to a T-indistinguishability operator E is defined as follows.

Definition 4.10. *Let E be a T-indistinguishability operator on $[0,1]$. h is E-Lipschitzian if and only if $\forall n \in N$, $\forall x_1,...,x_n,y_1,...,y_n \in [0,1]$*

$$T(E(x_1,y_1),...,E(x_n,y_n)) \leq$$

$$E_T(h(x_1,...,x_n), h(y_1,...,y_n)).$$

Next Proposition shows that a t-norm T is an E_T-Lipschitzian aggregation operator.

Proposition 4.11. *Let T be a continuous t-norm. Then T is an E_T-Lipschitzian aggregation operator.*

Proof. It is a consequence of 4.1 and Proposition 3.12 □

If T is a continuous Archimedean t-norm, the E_T^p-Lipschitzian property translates to a classical Lipschitzian condition.

Proposition 4.12. *Let T be a continuous Archimedean t-norm with additive generator t, $p \in [0,1]$ and h an aggregation operator. h is E_T^p-Lipschitzian if and only if $\forall n \in N$, $\forall x_1, ..., x_n, y_1, ..., y_n \in [0,1]$*

$$p|t(x_1) - t(y_1)| + ... + p|t(x_n) - t(y_n)| \geq$$

$$|t\,(h(x_1, ..., x_n)) - t\,(h(y_1, ..., y_n))\,| \quad (1).$$

Last Proposition says that for all $n \in N$ the map $H : [0, t(0)]^n \to [0, t(0)]$ defined by

$$H(x_1, ..., x_n) = t(h(t^{-1}(x_1), ..., t^{-1}(x_n)))$$

is a p-Lipschitzian map.

Also note that if T is the Lukasiewicz t-norm, then (1) is the definition of the Lipschitz property in [16], so that Definition 4.10 contains the one in [16] as a particular case.

If an aggregation operator h is E_T^p-Lipschitzian, it may happen that for different values of n the corresponding n-ary operators $h_{(n)}$ may satisfy the Lipschitzian conditions for different values of p ([5] p. 12).

Definition 4.13. *An aggregation operator is sub idempotent if and only if for all $x \in [0,1]$ and $n \in N$, $h(\overbrace{x, ..., x}^{n\ times}) \leq x$*

Proposition 4.14. *Let $T \neq Min$ be a t-norm, h a sub idempotent aggregation operator and $n \in N$. If $h_{(n)}$ is E_T^p-Lipschitzian, then $p \geq \frac{1}{n}$.*

Proof. If $h_{(n)}$ is E_T^p-Lipschitzian, then in particular, for $x \in X$

$$T((\overbrace{E_T^p(1,x), ..., E_T^p(1,x)}^{n\ times} \leq E_T(h(\overbrace{1, ..., 1}^{n\ times}), h(\overbrace{x, ..., x}^{n\ times}))$$

and so

$$x_T^{(pn)} \leq h(\overbrace{x, ..., x}^{n\ times}) \leq x$$

which holds if and only if $pn \geq 1$ or equivalently, if and only of $p \geq \frac{1}{n}$ □

If T is a non-strict continuous Archimedean t-norm the sub idempotent property can be dropped.

Proposition 4.15. *Let T be a non-strict continuous Archimedean t-norm with additive generator t, h an aggregation operator and $n \in N$. If $h_{(n)}$ is E_T^p-Lipschitzian, then $p \geq \frac{1}{n}$.*

Proof. Putting $x_i = 1$ and $y_i = 0$ for all $i = 1, ..., n$ in Proposition 4.12, we get

$$p|t(1) - t(0)| + ... + p|t(1) - t(0)| \geq |t(1) - t(0)|.$$

$$npt(0) \geq t(0)$$

or

$$p \geq \frac{1}{n}.$$

□

In [5] it has been proved that the arithmetic mean is the only aggregation operator h whose n-ary maps $h_{(n)}$ are $\frac{1}{n}$-Lipschitzian. Proposition 4.18 generalizes this result to arbitrary quasi-arithmetic means.

Next Proposition is well known.

Proposition 4.16. [1], [15] *m is a quasi-arithmetic mean in* [0,1] *if and only if there exists a continuous monotonic map $t : [0, 1] \to [-\infty, \infty]$ such that for all $n \in N$ and $x_1, ..., x_n \in [0, 1]$*

$$m(x_1, ...x_n) = t^{-1}\left(\frac{t(x_1) + ... + t(x_n)}{n}\right).$$

m is continuous if and only if Ran $t \neq [-\infty, \infty]$.

t will be called a generator of m and if m is generated by t we will denote it by m_t.

Proposition 4.17. [11] *The map assigning to every continuous Archimedean t-norm T with generator t the mean m_t generated by t is a bijection between the set of continuous Archimedean t-norms and the set of continuous quasi-arithmetic means.*

Proposition 4.18. *Let T be a continuous Archimedean t-norm with additive generator t and m_t the quasi-arithmetic mean generated by t.*

- *(a) For every $n \in N$ $m_{t(n)}$ is E_T^p-Lipschitzian if and only if $p \geq \frac{1}{n}$.*
- *(b) m_t is the only aggregation operator fulfilling (a)*

5 The Lipschitzian condition

As it is pointed out in the introduction, the Lipschitzian condition is interesting not only for aggregation operators, but it appears naturally in many branches of fuzzy reasoning. Many important properties like the extensionality with respect to a fuzzy relation, the definition of fuzzy maps [8], vague operations [7] or the extensionality of a crisp map [7], only to mention a few of them, have a nice Lipschitzian interpretation. Also Lipschitzian conditions on linguistic modifiers and t-norms have been studied in [17], [18].

Let us have a look at some of these interpretations.

Proposition 5.1. [21] *Let μ be a fuzzy subset of X and T a continuous t-norm. The fuzzy relation E_μ on X defined for all $x, y \in X$ by*

$$E_\mu(x, y) = E_T(\mu(x), \mu(y))$$

is a T-indistinguishability operator on X.

Definition 5.2. *Let E be a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E if and only if for all $x, y \in X$*

$$T(E(x, y), \mu(y)) \leq \mu(x).$$

Proposition 5.3. *Let E be a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E if and only if for all $x, y \in X$*

$$E(x, y) \leq E_\mu(x, y).$$

This is a Lipschitzian condition and if T is Archimedean with additive generator t, then we get the following result.

Corollary 5.4. *Let T be an Archimedean t-norm with additive generator t and E a T-indistinguishability operator on a set X. A fuzzy subset μ of X is extensional with respect to E if and only if for all $x, y \in X$*

$$t(E(x, y)) \geq |t(\mu(x)) - t(\mu(y))|.$$

In particular, if $X = [0, 1]$ and $E = E_T^p$, then μ is extensional if an only if

$$p|t(x) - t(y)| \geq |t(\mu(x)) - t(\mu(y))|.$$

The Lipschitzian condition of an aggregation operator can also be translated to extensionality with respect to a T-indistinguishability operator.

If $E_1, ..., E_n$ are T-indistinguishability operators defined on the universes $X_1, ...,$ X_n respectively, a T-indistinguishability operator on $X_1 \times ... \times X_n$ can be defined in the following way.

Proposition 5.5. *Let $E_1, ..., E_n$ be T-indistinguishability operators on $X_1, ..., X_n$ respectively. Then the fuzzy relation $T(E_1, ..., E_n)$ on $X_1 \times ... \times X_n$ defined for all $(x_1, ..., x_n), (y_1, ..., y_n) \in X_1 \times ... \times X_n$ by*

$$T(E_1, ..., E_n)((x_1, ..., x_n), (y_1, ..., y_n)) = T(E_1(x_1, y_1), ..., E_n(x_n, y_n))$$

is a T-indistinguishability operator on $X_1 \times ... \times X_n$.

Proposition 5.6. *Let E be a T-indistinguishability on $[0, 1]$ and h an aggregation operator. h is E-Lipschitzian if and only if $h_{(n)}$ (as a fuzzy subset of $[0, 1]^n$) is extensional with respect to $T(\overbrace{E, ..., E}^{n\ times})$ for all $n \in N$.*

Proof. Proposition 5.3. □

Fuzzy maps fuzzify the concept of map between two universes X and Y in which two T-indistinguishability operators are defined. A fuzzy map R is compatible with the two T-indistinguishability operators in the following sense.

Definition 5.7. *Let E, F be two T-indistinguishability operators on X and Y respectively and R a fuzzy set of $X \times Y$ (i.e.: $R : X \times Y \rightarrow [0,1]$). R is a fuzzy map from X to Y if and only if for all $x, x' \in X$, $y, y' \in Y$*

- *(a)* $T(E(x,x'), F(y,y'), R(x,y)) \leq R(x',y')$
- *(b)* $T(R(x,y), R(x,y')) \leq F(y,y')$.

Property (a) says that R is a fuzzy subset of $X \times Y$ extensional with respect to the T-indistinguishability $T(E,F)$ on $X \times Y$ and therefore it is a Lipschitzian condition.

$T_{(n)}$ can be seen as a fuzzy map from $[0,1]^{n-1}$ into $[0,1]$.

Proposition 5.8. *Let T be a continuous t-norm. $T_{(n)}$ is a fuzzy map from $[0,1]^k$ to $[0,1]^{n-k}$ endowed with the T indistinguishability operators $T(\overbrace{E_T, ..., E_T}^{k \ times})$ and $T(\overbrace{E_T, ..., E_T}^{n-k \ times})$ respectively.*

Extensional crisp maps $f : X \rightarrow Y$ with respect to two T-indistinguishability operators E and F defined on X and Y respectively are interesting since they generate a fuzzy map from X to Y in a very natural way [8].

Definition 5.9. *A crisp map $f : X \rightarrow Y$ is called extensional with respect to E and F, T-indistinguishability operators on X and Y respectively, if and only if*

$$E(x_1, x_2) \leq F(f(x_1), f(x_2)).$$

This is again a Lipschitzian condition.

Corollary 5.10. *Let T be an Archimedean t-norm with additive generator t, $X = Y = [0,1]$, and E_T^p and E_T^q powers of the natural T-indistinguishability operator E_T. $f : [0,1] \rightarrow [0,1]$ is extensional with respect to E_T^p and E_T^q if and only if*

$$\frac{p}{q}|t(x_1) - t(x_2)| \geq |t(f(x_1)) - t(f(x_2))|.$$

$T_{(n)}$ is also a fuzzy point of $[0,1]^n$

Definition 5.11. *Let E be a T-indistinguishability operator on a set X and μ a fuzzy subset of X. μ is a fuzzy point of X with respect to E if and only if for all $x, y \in X$*

$$T(\mu(x), \mu(y)) \leq E(x,y).$$

Proposition 5.12. *Let T be a continuous t-norm. $T_{(n)}$ is a fuzzy point of $[0,1]^n$ with respect to $T(\overbrace{E_T, ..., E_T}^{n \ times})$.*

Proof. We have to prove that

$$T(T(x_1, ..., x_n), T(y_1, ..., y_n))$$

$$\leq T(E_T(x_1, y_1), ..., E_T(x_n, y_n))$$

which is an immediate consequence of

$$T(x_i, y_i) \leq E_T(x_i, y_i) \text{ for all } i = 1, ..., n.$$

□

6 Concluding Remarks

In this paper Lipschitzian aggregation operators with respect to the natural T-indistinguishability operator E_T and their powers, and with respect to the residuation $\overrightarrow{T}$ and its powers have been studied.

It has been proved that a t-norm T is $\overrightarrow{T}$- and E_T-Lipschitzian, and a fuzzy point and a fuzzy map as well.

Quasi-arithmetic means m_t play an important role since they are the most stable aggregation operator with respect to T, meaning that the corresponding n-ary operators $m_{t(n)}$ are $E_T^{\frac{1}{n}}$-Lipschitzian maps.

Lipschitzian properties are not only interesting for aggregation operators, but in almost any part of fuzzy reasoning (see the references of section 5 and [14]

Acknowledgments

Research partially supported by DGICYT project number TIN2006-14311.

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