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On the Threshold of Bounded Pseudo-Distances

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Abstract

This paper deals with the relationship between bounded pseudo-distances and its associated W_φ-indistinguishabilities, from which the idea of threshold of transitivity comes. By the way, bounded pseudo-distances are characterized.
Keywords: T-indistinguishabilities, bounded-distances, threshold.

1 Introduction

1.1 Distance and indistinguishability, as well as threshold, are important concepts in the experimental sciences and, in particular, in Computational Intelligence. Concerning the concept of a threshold, for which there is not a completely satisfactory definition, it can be said that:

- It is a fixed point or value where an abrupt change is observed,
- It is the point that must be exceeded to begin producing an effect or result or to elicit a response,
- It is the lowest point at which a stimulus begins to produce a sensation,
- It is the minimal stimulus that produces excitation of any structure, eliciting a motor response, etc.

These descriptions cover most of the cases where the concept of threshold applies. Following the Webster's dictionary, a threshold is "the point at which a

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stimulus is of sufficient intensity to begin to produce an effect". In that sense, below a value t in a numerical scale measuring the intensity of some input, it does not produce any effect, but as soon as the intensity surpasses the value t the input's effect is detected.

1.2 In many problems in Computational Intelligence concerning the similarity of certain elements, when measured by a numerical index of similarity $S(x, y) \in [0, 1]$ associated to each pair of these elements (like in Case-Based Reasoning), it appears the following question: What can be said on $S(a, c)$ when it is $0 < S(a, b)$ and $0 < S(b, c)$? Namely, when it does be $0 < S(a, c)$? Equivalently, if $0 < r \leq S(a, b)$ and $0 < r \leq S(b, c)$, when it exists $t(r) > 0$ such that $0 < t(r) \leq S(a, c)$? This problem can be called that of "large transitivity", and if R_S is the set of values r which satisfy large transitivity for S, then $t_S = \inf R_S$ is the minimum value for which this last inequality holds. It can be called the large transitivity threshold for S.

Sometimes S is taken to be $S(x, y) = 1 - d(x, y)$ with d a bounded distance. In these cases $0 < r \leq S(x, y)$ is equivalent to $d(x, y) \leq 1 - r < 1$.

1.3 When the index $S : X \times X \longrightarrow [0, 1]$ is either min-transitive or prod-transitive [7], respectively,

- $\min(S(a, b), S(b, c)) \leq S(a, c)$
- $S(a, b) \cdot S(b, c) \leq S(a, c)$,

for all $a, b, c \in X$, from $0 < r \leq S(a, b)$, $0 < r \leq S(b, c)$, follows

- $0 < r = \min(r, r) \leq \min(S(a, b), S(b, c)) \leq S(a, c)$
- $0 < r^2 \leq S(a, b) \cdot S(b, c) \leq S(a, c)$,

and then $t_S = \inf(0, 1] = 0$ for min and $t_S = \inf(0, 1] = 0$ for prod, is the corresponding threshold of large transitivity for the two kind of indexes, a threshold that actually is non informative. If S is W-transitive, with $W(x, y) = \max(0, x + y - 1)$ the Lukasiewicz t-norm, from $W(S(a, b), S(b, c)) \leq S(a, c)$, for all a, b, c in X, what follows is

$$W(r, r) = \max(0, 2r - 1) \leq W(S(a, b), S(b, c)) \leq S(a, c),$$

and it could be $W(r, r) = 0$ with $r > 0$. Since $W(r, r) = 0$ happens if and only if $r \leq 0.5$, a threshold only exists if $r > 0.5$. That is, if $0.5 < r \leq S(a, b)$, and $0.5 < r \leq S(b, c)$, it is $0 < t(r) = 2r - 1 \leq S(a, c)$. If the intensity of the link between a and b, and of that between b and c is greater than r, then $t(r) = 2r - 1 > 0$ and is $S(a, c) \in [2r - 1, 1]$. In this case $t_S = \inf(\frac{1}{2}, 1] = \frac{1}{2}$.

It will be proved that S is a W-indistinguishability if and only if $d = 1 - S$ is a pseudo-distance bounded by 1. Hence, to every pseudo-distance bounded by $a > 0$ it is associated the W-indistinguishability $S = 1 - \frac{d}{a}$, that allows to define a threshold for d from that of S. At this point it should be noticed that $0 < r \leq S(x, y)$ is equivalent to $d(x, y)/a \leq 1 - r < 1$.

1.4 This paper tries to study the threshold of transitivity of W_φ-indistinguishabilities, as well as the link between such indexes and bounded pseudo-distances and, in particular, to define a threshold for this last coming from that of the W_φ-indistinguishabilities. By the way, bounded pseudo-distances are characterized.

2 Basic Tools

2.1 A pseudo-distance in a set X is a mapping $d : X \times X \longrightarrow \mathbb{R}^+$ such that

1. $d(x, x) = 0$, for all x in X,
2. $d(x, y) = d(y, x)$, for all x, y in X,
3. $d(x, y) + d(y, z) \geq d(x, z)$, for all x, y, z in X.

A distance is a pseudo-distance such that $d(x, y) = 0$ if and only if $x = y$. A pseudo-distance is bounded by $a > 0$ if $d(X \times X) \subset [0, a]$. Every bounded pseudo-distance is equivalent to a pseudo-distance bounded by 1, in the sense of "d is a pseudo-distance bounded by a if and only if the function $1/a \cdot d$ is a pseudo-distance bounded by 1", whose proof is immediate. Hence given a pseudo-distance d bounded by 1 and $a > 0$, the function $a \cdot d$ is a pseudo-distance bounded by a.

2.2 A strong-negation is a function $N : [0, 1] \longrightarrow [0, 1]$ such that

1. $N(0) = 1$,
2. if $x < y$, then $N(y) < N(x)$,
3. $N(N(x)) = x$, for all x in $[0, 1]$.

An order-automorphism of $[0, 1]$ is a function $\varphi : [0, 1] \longrightarrow [0, 1]$ such that

1. $\varphi(0) = 0, \varphi(1) = 1$,
2. if $x < y$, then $\varphi(y) < \varphi(x)$.

The functions $N_\varphi : [0, 1] \longrightarrow [0, 1]$ defined by $N_\varphi(x) = \varphi^{-1}(1-\varphi(x))$ are strong negations and (see [6]) for all strong-negation N there are order-automorphisms φ such that $N = N_\varphi$. Of course, both functions N and φ are bijective, continuous and N verifies $N(1) = N(N(0)) = 0$, and $N^{-1} = N$.

2.3 For what concerns the definitions and properties of t-norms (T), and t-conorms (S), see [1].

The three well known basic continuous t-norms are $T = \min$, $T = \text{prod}$, and $T(x, y) = W(x, y) = \max(0, x + y - 1)$ (Lukasiewicz t-norm). The t-norm min is the biggest of all them, since $T(x, y) \leq T(x, 1) = x$, $T(x, y) \leq T(1, y) = y$ imply $T(x, y) \leq min(x, y)$. For all order-automorphism φ, the function $T_\varphi = \varphi^{-1} \circ T \circ (\varphi \times \varphi)$ is a t-norm if and only if T is a t-norm, and T_φ is continuous if and only if T is continuous. Hence, for all continuous t-norm T there is the family of continuous t-norms $F(T) = \{T_\varphi : \varphi \text{ an automorphism}\}$, and in particular, there is the Lukasiewicz family

$$W_\varphi(x, y) = \varphi^{-1}(W(\varphi(x), \varphi(y))) = \varphi^{-1}(\max(0, \varphi(x) + \varphi(y) - 1)).$$

Neither $\min_\varphi(= \min)$, nor $\text{prod}_\varphi(= \varphi^{-1}(\varphi(x) \cdot \varphi(y)))$, have zero-divisors, but the t-norms W_φ do have such kind of elements: it is, $W_\varphi(x, y) = 0$ if and only if $\varphi(x) + \varphi(y) - 1 \leq 0$, or if and only if $y \leq N_\varphi(x)$.

2.4 A function $E : X \times X \longrightarrow [0, 1]$ is a T-indistinguishability (see [7, 10]) on the set X,4 if it verifies

1. $E(x, x) = 1$, for all x in X,
2. $E(x, y) = E(y, x)$, for all x, y in X,
3. $T(E(x, y), E(y, z)) \leq E(x, z)$, for all x, y, z in X.

If E is a T-indistinguishability on $[0, 1]$, for any $f : X \longrightarrow [0, 1]$, the function E_f defined by $E_f(x, y) = E(f(x), f(y))$ is a T-indistinguishability on X.

Examples of T-indistinguishabilities are given by

$$E_T(x, y) = \min(J_T(x, y), J_T(y, x)),$$

with $J_T(x, y) = \sup\{z \in [0, 1] : T(z, x) \leq y\}$. For example,

– From $J_{\min}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ y & \text{if } x > y \end{cases}$, is $E_{\min}(x, y) = \begin{cases} 1 & \text{if } x = y \\ \min(x, y) & \text{if } x \neq y \end{cases}$

– From $J_{\text{prod}}(x, y) = \begin{cases} 1 & \text{if } x \leq y \\ \frac{y}{x} & \text{if } x > y \end{cases}$, is $E_{\text{prod}}(x, y) = \begin{cases} 1 & \text{if } x = y \\ \min(\frac{x}{y}, \frac{y}{x}) & \text{if } x \neq y \end{cases}$

– From $J_{W_\varphi}(x, y) = \varphi^{-1}(\min(1, 1 - \varphi(x) + \varphi(y))$, is

$$E_{W_\varphi}(x, y) = \varphi^{-1}(1 - \varphi(|x - y|)).$$

Theorem 2.1. *$E : X \times X \longrightarrow [0, 1]$ is a T-indistinguishability if and only if there exists a family $\mathcal{F}$ of functions $f : X \longrightarrow [0, 1]$, such that*

$$E(x, y) = \inf\{E_T(f(x), f(y)) : f \in \mathcal{F}\}.$$

Proof. See [7, 10]. □

Hence, for all finite family $\mathcal{F} = \{f_1, \ldots, f_n\}$ of functions $f_i : X \longrightarrow [0, 1]$, the T-indistinguishability $E(x, y) = \min\{E_T(f_i(x), f_i(y) : 1 \leq i \leq n\}$ is said to be a finitely generated T-indistinguishability. For example,

$$\begin{aligned} E(x, y) &= \min\{\varphi^{-1}(1 - \varphi(|f_i(x) - f_i(y)|)) : 1 \leq i \leq n\} \\ &= \varphi^{-1}(1 - \max_{1 \leq i \leq n}(|f_i(x) - f_i(y)|)), \end{aligned}$$

is a finitely generated W_φ-indistinguishability.

2.5 Remarks.

2.5.1 As it is easy to prove, an order-automorphism φ of $[0,1]$ is sub-additive ($\varphi(x+y) \leq \varphi(x)+\varphi(y)$), if and only if the order-automorphism φ^{-1} is super additive ($\varphi^{-1}(x)+\varphi^{-1}(y) \leq \varphi^{-1}(x+y)$).

2.5.2 If d is a pseudo-distance on X bounded by 1, and the order-automorphism φ is sub-additive, the function $d_\varphi = \varphi \circ d$ is also a pseudo-distance on X bounded by 1.

2.5.3 The order-automorphisms $\varphi(x) = x^n (n = 2, 3, \ldots)$ are super-additive, and consequently the order-automorphisms $\varphi^{-1} = \sqrt[n]{x} (n = 2, 3, \ldots)$ are sub-additive.

3 Bounded Pseudo-distances and W_φ-indistinguishabilities

Theorem 3.1. *Let it be a function* $d : X \times X \longrightarrow [0,1]$. *If for some super-additive order-automorphism* φ *on* $[0,1]$, *the function* $E_\varphi(x,y) = N_\varphi(d(x,y))$ *is a* W_φ*-indistinguishability, d is a pseudo-distance bounded by 1.*

Proof. It is $d(x,y) = N_\varphi(E(x,y))$. Hence, $d(x,x) = 0$ and $d(x,y) = d(y,x)$ for all x, y in X. From,

$$\begin{aligned} W_\varphi(E_\varphi(x,y), E_\varphi(y,z)) &= \varphi^{-1}(\max(0, \varphi(E_\varphi(x,y)) + \varphi(E_\varphi(y,z)) - 1)) \\ &= \varphi^{-1}(\max(0, 1 - \varphi(d(x,y)) - \varphi(d(y,z)))) \\ &\leq E_\varphi(x,z) \\ &= N_\varphi(d(x,z)) \\ &= \varphi^{-1}(1 - \varphi(d_\varphi(x,z)), \end{aligned}$$

follows $\max(0, 1 - \varphi(d(x,y)) + \varphi(d(y,z))) \leq 1 - \varphi(d_\varphi(x,z))$. Hence,

$$\varphi(d(x,z)) \leq \varphi(d(x,y)) + \varphi(d(y,z)) \leq \varphi(d(x,y) + d(y,z)),$$

since φ is super-additive. Finally, $d(x,z) \leq d(x,y) + d(y,z)$. □

Theorem 3.2. *Let it be* φ *a sub-additive order-automorphism of* $[0,1]$, *and d a pseudo-distance on X bounded by 1. The function* $E_\varphi(x,y) = N_\varphi(d(x,y))$ *is a* W_φ*-indistinguishability.*

Proof. Obviously, $E_\varphi(x,x) = 0$, and $E_\varphi(x,y) = E_\varphi(y,x)$. From $d(x,z) \leq d(x,y) + d(y,z)$, follows

$$\varphi(d(x,z)) \leq \varphi(d(x,y) + d(y,z)) \leq \varphi(d(x,y)) + \varphi(d(y,z)),$$

since φ is sub-additive. Then, $1 - \varphi(d(x,y)) - \varphi(d(y,z)) \leq 1 - \varphi(d(x,z))$, and $\max(0, 1 - \varphi(d(x,y)) - \varphi(d(y,z))) \leq 1 - \varphi(d(x,z))$. Hence,

$$\begin{aligned} \varphi^{-1}(\max(0, 1 - \varphi(d(x,y)) - \varphi(d(y,z)))) &\leq \varphi^{-1}(1 - \varphi(d(x,z))) \\ &= N_\varphi(d(x,z)) \\ &= E_\varphi(x,y) \end{aligned}$$

By the other side,

$$\begin{aligned} W_\varphi(E_\varphi(x,y), E_\varphi(y,z)) &= \varphi^{-1}(\max(0, \varphi(E_\varphi(x,y)) + \varphi(E_\varphi(y,z)) - 1)) \\ &= \varphi^{-1}(\max(0, 1 - \varphi(d(x,y)) - \varphi(d(y,z)))). \end{aligned}$$

That is, $W_\varphi(E_\varphi(x,y), E_\varphi(y,z)) \leq E_\varphi(x,z)$, for all x, y, z in X. □

Corollary 1. *d is a pseudo-distance bounded by 1 if and only if $E = 1 - d$ is a W-indistinguishability.*

Proof. The order-automorphism $\varphi = \mathrm{Id}$ is both sub-additive and super-additive. Hence, if E is a W-indistinguishability, with N_{Id}, is $d = N_{\mathrm{Id}} \circ E = 1 - E$ a pseudo-distance bounded by 1, by theorem 3.1 and if d is a pseudo-distance, then $E = N_{\mathrm{Id}} \circ d = 1 - d$ is a W-indistinguishability by theorem 3.2. □

Theorem 3.3 (Characterization of bounded pseudo-distances)**.** *The only pseudo-distances bounded by $a > 0$ are those defined by*

$$d(x,y) = a \cdot \sup\{|f_i(x) - f_i(y)| : i \in I\},$$

for some family of functions $f_i : X \longrightarrow [0,1], i \in I$.

Proof. Since $\frac{1}{a} \cdot d$ is a pseudo-distance bounded by 1, $E = 1 - \frac{d}{a}$ is a W-indistinguishability by corollary 1. Then, by theorem 2.1 there is a family of functions $\{f_i : i \in I\}$ such that $E(x,y) = \inf_{i \in I} E_W(f_i(x), f_i(y)) = \inf_{i \in I}\{1 - |f_i(x) - f_i(y)|\} = 1 - \sup_{i \in I} |f_i(x) - f_i(y)|$. Then,

$$d(x,y) = a \cdot (1 - E(x,y)) = a \cdot \sup_{i \in I} |f_i(x) - f_i(y)|.$$

□

A bounded pseudo-distance is finitely generated if I is a finite set. Hence, the only finitely-generated bounded pseudo-distances are those of the form

$$d(x,y) = a \cdot \max_{1 \leq i \leq n} |f_i(x) - f_i(y)|,$$

for all x, y in X. Notice that the euclidean distance in $X = [0,1]$, $d(x,y) = |x - y|$ is finitely generated by the single function $f = \mathrm{Id}$.

The family $\{f_i : i \in I\}$ can be taken as giving some "measurements" of the objects in X, relatively to the attributes or properties they can show. For example, if $X = \{x_1, x_2, x_3, x_4\}$, and the attributes on considerations are A_1 and A_2, with

$$f_i(x_j) = \text{degree up to which } x_j \text{ is } A_i (1 \leq i \leq 2, 1 \leq j \leq 4),$$

the corresponding pseudo-distance can be obtained once known the 2×4 numbers $f_i(x_j) \in [0,1]$. In the case given by the table 1, it results the distance bounded by 1:

	x_1	x_2	x_3	x_4
f_1	0.7	0.5	0.7	0.4
f_2	0.8	0.4	0.6	0.5

Table 1: Two generating functions

$$\begin{aligned}
&d(x_1, x_1) = d(x_2, x_2) = d(x_3, x_3) = d(x_4, x_4) = \max(0, 0) = 0\\
&d(x_1, x_2) = d(x_2, x_1) = \max(|0.7 - 0.5|, |0.8 - 0.4|) = 0.4\\
&d(x_1, x_3) = d(x_3, x_1) = \max(0, 0.2) = 0.2\\
&d(x_1, x_4) = d(x_4, x_1) = 0.3\\
&d(x_2, x_3) = d(x_3, x_2) = 0.2\\
&d(x_2, x_4) = d(x_4, x_2) = 0.1\\
&d(x_3, x_4) = d(x_4, x_3) = 0.3
\end{aligned}$$

It is easy to check that d is a distance. For example, with the triplet (x_2, x_3, x_4) is $d(x_2, x_3) + d(x_3, x_4) = 0.2 + 0.3 = 0.5 \geq 0.2 = d(x_2, x_4)$. Notice that the function f_1 is not injective.

In particular, functions f_i can be probabilities in which case, if $X = \{x_1, \ldots, x_n\}$, it should be $\sum_{j=1}^{n} f_i(x_j) = 1$, for all $i \in I$.

Theorem 3.4. *A finitely generated bounded pseudo-distance with at least an injective function, is a distance.*

Proof. It is clear because $d(x, y) = 0 = a \cdot \max_{1 \leq i \leq n} |f_i(x) - f_i(y)|$ if and only if $f_i(x) = f_i(y)$ for all i, and then $x = y$. □

The reciprocal result of this theorem is not true, since it is possible to have distances from a family of non-injective functions. For example, with $X = \{x_1, x_2, x_3\}$ and two non-injective functions f_1, f_2 with $f_1(x_1) = f_1(x_2) = 0.5$, $f_1(x_3) = 0.6$ and $f_2(x_1) = 0.3$, $f_2(x_2) = f_2(x_3) = 0.4$, it results the bounded distance given by $d(x_1, x_2) = d(x_1, x_3) = d(x_2, x_3) = 0.1$.

3.1 Remark. To every pseudo-distance on X bounded by $a > 0$, it is associated the family of W_φ-indistinguishabilities on X, $E_{W_\varphi} = N_\varphi \circ (\frac{d}{a})$, for all sub-additive order-automorphism φ of $[0, 1]$. In each case, the more adequate φ for the problem under consideration should be selected.

4 The Threshold of Transitivity of a W_φ-Indistinguishability

If $E : X \times X \longrightarrow [0, 1]$ is T-transitive, from $0 < r \leq E(x, y)$, $0 < r \leq E(y, z)$, it follows $0 \leq T(r, r) \leq T(E(x, y), E(y, z)) \leq E(x, z)$. That is, $0 \leq T(r, r) \leq E(x, z)$. For both $T = \min$ and $T = \text{prod}_\varphi$, it is $0 < T(r, r) \leq E(x, z)$, but for $T = W_\varphi$ it could be $W_\varphi(r, r) = 0$, in which case E fails to be largely transitive. Since $W_\varphi(r, r) = 0$, is equivalent to $2\varphi(r) - 1$, or $r \leq \varphi^{-1}(0.5)$, it suffices to take $r > \varphi^{-1}(0.5)$ to have,

$$0 < r \leq E(x,y), \text{ and } 0 < r \leq E(y,z), \text{ imply } 0 < \varphi^{-1}(2\varphi(r)-1) \leq E(x,z).$$

Let us call $\inf\{r \in [0,1] : r > \varphi^{-1}(0.5)\} = \varphi^{-1}(0.5)$ the threshold of transitivity of E. Notice that from $r \leq 1$, or $\varphi(r) \leq 1$, it follows $2\varphi(r) - 1 \leq \varphi(r)$, that is $\varphi^{-1}(2\varphi(r)-1) \leq r$. Hence, $\varphi^{-1}(2\varphi(r)-1) \in (0, r]$, provided $r > \varphi^{-1}(0.5)$.
Example. If $X = \{x_1, \ldots, x_2\}$, and $f_s : X \longrightarrow [0,1]$, with $1 \leq s \leq m$, the function (see [3]),

$$E(x_i, x_j) = \frac{\sum_{s=1}^{n} \min(f_s(x_i), f_s(x_j))}{\max(\sum_{s=1}^{n} f_s(x_i), \sum_{s=1}^{n} f_s(x_j))},$$

whose values are in $[0,1]$, is W_φ-transitive with $\varphi(x) = x^2$. Hence, its threshold of transitivity is $\varphi^{-1}(0.5) = \sqrt{0.5} = 0.7071$ and, consequently, it suffices to take $r = 0.7072$ to have $\varphi^{-1}(2\varphi(r)-1) = \sqrt{2 \cdot 0.7072^2 - 1} = 0.000264$, and

$$\text{If } 0.7072 \leq E(x,y), \text{ and } 0.7072 \leq E(y,z), \text{ then } 0.000264 \leq E(x,z).$$

Observe that with $r = 0.8$ it results $\varphi^{-1}(2\varphi(r)-1) = \sqrt{0.28} = 0.529$.

5 The Threshold of a Bounded Pseudo-Distance

If $d : X \times X \longrightarrow \mathbb{R}^+$ is a pseudo-distance bounded by $a > 0$, for each sub-additive order-automorphism φ, the corresponding W_φ-indistinguishability

$$E_\varphi(x,y) = N_\varphi(\frac{d(x,y)}{a}),$$

has the threshold of transitivity $\varphi^{-1}(0.5)$. Then it suffices to take $r > \varphi^{-1}(0.5)$ to be sure that if $0 < r \leq E_\varphi(x,y)$, and $0 < r \leq E_\varphi(y,z)$, it is $0 < \varphi^{-1}(2\varphi(r)-1) \leq E_\varphi(x,z)$.

Hence, if $d(x,y) \leq aN_\varphi(r)$, and $d(y,z) \leq aN_\varphi(r)$, then

$$d(x,z) \leq aN_\varphi(\varphi^{-1}(2\varphi(r)-1)) = a\varphi^{-1}(2(1-\varphi(r)).$$

Since, $d(x,y) \leq d(x,y) + d(y,z) \leq 2aN_\varphi(r)$ and $d(x,z) \leq a$, it follows

$$d(x,z) \leq a \cdot \min(1, 2N_\varphi(r), \varphi^{-1}(2(1-\varphi(r))).$$

Then, for each $r > \varphi^{-1}(0.5)$, the number

$$\delta(\varphi, r) = a \cdot \min(1, 2N_\varphi(r), \varphi^{-1}(2(1-\varphi(r))),$$

can be called the φ-threshold of the bounded pseudo-distance d.

Notice that with $\varphi = \text{Id}$ and $r > \varphi^{-1}(0.5) = 0.5$, is

$$\delta(\text{Id}, r) = a \cdot \min(1, 2(1-r), 2(1-r)) = a \min(1, 2(1-r)) = 2a(1-r).$$

That is, if $d(x,y) \leq a(1-r)$ and $d(y,z) \leq a(1-r)$, is $d(x,z) \leq 2a(1-r)$.

5.1 Remark. The function $\delta(\varphi, r)$ is decreasing for r : If $r < s$, since $2N_\varphi(s) < 2N_\varphi(r)$, and $\varphi^{-1}(2(1-\varphi(s))) < \varphi^{-1}(2(1-\varphi(r)))$, it follows $\delta(\varphi, s) < \delta(\varphi, r)$. In particular, from $\varphi^{-1}(0.5) < r$ it follows

$$\begin{aligned}\delta(\varphi, r) &< \delta(\varphi, \varphi^{-1}(0.5)) \\ &= a \cdot \min(1, 2N_\varphi(\varphi^{-1}(0.5)), \varphi^{-1}(2(1-\varphi(\varphi^{-1}(0.5))))) \\ &= a \cdot \min(1, 2\varphi^{-1}(0.5), 1) \\ &= 2a\varphi^{-1}(0.5).\end{aligned}$$

Then, it is always $\delta(\varphi, r) < 2a\varphi^{-1}(0.5)$. Note that it is $\delta(\varphi, r) = 0$ if and only if $r = 1$.

Notice that $0 < r \leq E(x, y)$ is equivalent to $d(x, y) \leq aN_\varphi(r) < a$. Hence $\varphi^{-1}(0.5) < E_\varphi(x, y)$, is equivalent to $d(x, y) \leq a \cdot \varphi^{-1}(0.5)$.

5.2 Examples.

5.2.4 If d is a pseudo-distance bounded by 1, with $\varphi(x) = \sqrt{x}$, is $\varphi^{-1}(0.5) = 0.5^2 = 0.25$. Taking, for example, $r = 0.26$ it results:

$$\begin{aligned}d(x, y) &\leq N_\varphi(0.26) = (1-\sqrt{0.26})^2 = 0.2402 \\ d(y, z) &\leq N_\varphi(0.26) = 0.2402,\end{aligned}$$

and

$$\begin{aligned}d(x, z) &\leq \min(1, 2N_\varphi(0.26), \varphi^{-1}(2(1-\varphi(0.26)))) \\ &= \min(1, 0.4804, 0.9608) = 0.4804.\end{aligned}$$

That is, $\delta(\varphi, 0.26) = 0.4804$, a value that is less than $2a\varphi^{-1}(0.5) = 0.5$.

5.2.5 Take $d(x, y) = |x - y|$ in $[0, 1]$, with $\varphi = \text{Id}$. It is $N_\varphi = 1 - \text{Id}$, and $2a\varphi^{-1}(0.5)$=1.

With $r = 0.6$ it results $d(x, y) \leq 1 - 0.6 = 0.4$, $d(y, z) \leq 0.4$, and $d(x, z) \leq \min(1, 0.8, 2(1 - 0.6)) = 0.8$. With $r = 0.51$ is $d(x, y) \leq 1 - 0.51 = 0.49$, $d(y, z) \leq 0.49$, and $d(x, z) \leq \min(1, 0.98, 2(1 - 0.51)) = 0.98$. Notice that this value is, as it was pointed out, less than $2a\varphi^{-1}(0.5) = 1$.

5.2.6 With the same distance of 5, take $\varphi(x) = \sqrt{x}$. Then,

$$E_\varphi(x, y) = N_\varphi(|x - y|) = (1 - \sqrt{|x - y|})^2,$$

and $\varphi^{-1}(0.5) = 0.5^2 = 0.25$.

With $r = 0.26$,

$$d(x, y) \leq (1 - \sqrt{0.26})^2 = (1 - 0.51)^2 = 0.2401, d(y, z) \leq 0.2401,$$

and $d(x, z) \leq \min(2 \cdot 0.2401, (2(1-\sqrt{0.26}))^2) = 0.4802$. With $r = 0.6$, $d(x, y) \leq (1-\sqrt{0.6})^2 = 0.0506$, $d(y, z) \leq 0.0506$, and $d(x, z) \leq \min(1, 2 \cdot 0.2401, (2(1-\sqrt{0.6}))^2) = 04802$. Notice that this value is less than $2a\varphi^{-1}(0.5) = 2 \cdot 1 \cdot 0.25 = 0.5$.

5.2.7 What happens with $r \leq \varphi^{-1}(0.5)$? In example 5.2.5, take $r = 0.4$. It results $d(x,y) \leq 1 - 0.4 = 0.6, d(y,z) \leq 0.6$, and $d(x,z) \leq \min(1, 1.2, 2(1-0.4)) = 1$, an unfruitful result, since it is always $d(x,z) \leq 1$. It results a non-informative conclusion.

5.2.8 It should be pointed out that index E in the example of section 4, is also W-transitive. Hence, it has also the threshold $\varphi^{-1}(0.5) = \text{Id}^{-1}(0.5) = 0.5$. Nevertheless, since E is applied (see [4]) with the threshold 0.7, that was found experimentally, this means that E is used as W_φ-transitive with $\varphi(x) = x^2$. Then, in such application the "separation or distinction" between the objects $x_1, \ldots, x_n$, to which E applies is measured with the pseudo-distance

$$\begin{aligned} d(x_i, x_j) &= N_\varphi(E(x_i, x_j)) \\ &= \varphi^{-1}(1 - \varphi(E(x_i, x_j))) \\ &= \sqrt{1 - (\frac{\sum_s \min(f_s(x_i), f_s(x_j))}{\max(\sum_s f_s(x_i), \sum_s f_s(x_j))},)^2}. \end{aligned}$$

Hence, $0.7 < E(x_i, x_j)$ means $d(x_i, x_j) < N_\varphi(0.7) = \sqrt{1 - 0.7^2} = 0.7142$. That is, two objects x_i, x_j are taken as indistinguishable as soon as its separation is less than 0.7142.

5.2.9 In $X = [0,1]$, the distance $d(x,y) = \frac{|x-y|}{1+|x-y|} \in [0,1]$ is bounded by 0.5 since from $|x-y| \leq 1$ follows $2|x-y| \leq 1 + |x-y|$. Hence, with $\varphi = \text{Id}$ the corresponding W-indistinguishability is $E(x,y) = 1 - d(x,y) = \frac{1}{1+|x-y|}$. Then, with $r = 0.52 > \varphi^{-1}(0.5) = 0.5$, if $d(x,y) < aN_\varphi(r) = 0.5(1 - 0.52) = 0.24$, and $d(y,z) < 0.24$, it results $d(x,z) < 0.5 \min(1, 2 \cdot 0.52, 2(1-0.52)) = 0.48$ that, of course, is less than $2 \cdot 0.5 \cdot 0.5 = 0.5$.

With $r = 0.8$, is $d(x,y) < 0.5 \cdot 0.2 = 0.1$, $d(y,z) < 0.1$, and $d(x,z) < 0.5 \min(1, 2 \cdot 0.2, 2 \cdot 0.2) = 0.2$.

6 The Maximum Threshold of a Bounded Pseudo-Distance

It was shown in section 5 that $\delta(\varphi, r)$, the threshold for a given φ and r, never surpasses the value $2a\varphi^{-1}(0.5)$. But what if we consider all sub-additive order-automorphisms of $[0,1]$? Is there an upper bound for all the possible values $\delta(\varphi, r)$ for each pseudo-distance bounded by a?

The set $A = \{\varphi^{-1}(0.5) : \varphi \in \text{SO}\}$, with SO the set of all sub-additive order-automorphism of $[0,1]$, has a supremum since it is always $\varphi^{-1}(0.5) < 1$. Hence, $\sup A \leq 1$. Let us call $\alpha = \sup A$.

Theorem 6.1. $\alpha = 0.5$

Proof. It is evident than $0.5 \leq \alpha$ because $\text{Id} \in \text{SO}$. But also $\alpha \leq 0.5$ because if $\varphi \in \text{SO}$, then φ^{-1} is super-additive, so

$$2\varphi^{-1}(0.5) = \varphi^{-1}(0.5) + \varphi^{-1}(0.5) \leq \varphi^{-1}(2 \cdot 0.5) = \varphi^{-1}(1) = 1$$

and hence for all $\varphi \in \mathrm{SO}$ it holds $\varphi^{-1}(0.5) \leq 0.5$ and then $\alpha \leq 0.5$. □

Because of all this, the final supremum for $\delta(\varphi, r)$, for all $\varphi \in \mathrm{SO}$ and, each time with $r > \varphi^{-1}(0.5)$, is $\sup\{2a\varphi^{-1}(0.5) : \varphi \in \mathrm{SA}\} = 2a\alpha = a$, the "diameter" of the pseudo-distance.

7 Conclusions

As it was said in section 1, the goal of this paper is to partially deal with large transitivity, that is, to study when an index of similarity $S : X \times X \longrightarrow [0,1]$ verifies the property:

if $0 < S(x,y)$, and $0 < S(y,z)$, then it exists $t > 0$ such that $t \leq S(x,z)$,

for all x, y, z in X. When t is the minimum number verifying $0 < t \leq S(x,z)$, it is called the threshold of transitivity of S.

What is here considered is the special case of the indices S obtained by

$$S(x,y) = N_\varphi(\frac{d(x,y)}{a}),$$

with N_φ the strong negation given by an automorphism φ of $[0,1]$, and d a pseudo-distance on X bounded by $a > 0$. In this case, from the threshold of transitivity of S, the function indicating the degree of indistinguishability between x and y, it is deduced a threshold for d, the function indicating the degree of separation between x and y.

As a consequence of the equivalence between bounded pseudo-distances and W_φ-indistinguishabilities, and by means of the known characterization of these last indexes of similarity, a characterization of bounded pseudo-distances is obtained throughout a family of "measurements" in $[0,1]$ of the considerated objects.

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