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Logical Aggregation Based on Interpolative Boolean Algebra

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Abstract

Explicit inclusion of logic in the process of aggregation (information fusion) is very important in real problems from many points of view such as adequacy and transparency. In this paper aggregation is treated as a logical and/or pseudo-logical operation based on interpolative Boolean algebra (IBA). IBA is a real-valued ([0, 1]-valued) realization of Boolean algebra. As classical two-valued realization of Boolean algebra is the base of classical two-valued logic, IBA is the base of real-valued logic.
Keywords: Aggregation, Logical aggregation, Interpolative realization of Boolean algebra, Generalized Boolean polynomial, Pseudo Boolean polynomial, Generalized Choquet integral, OWA.

1 Introduction

A very important problem in many fields of applications[1] is the aggregation (fusion) of many partial aspects – attributes (characteristics, demands, goals, symptoms, criteria, perceptions...) into one global representative aspect. In the existing practice the weighted sum of partial aspects is used most often as an aggregation tool. From logical point of view this approach is trivial. The weighted sum is additive and for all effects of interest which are not additive in their nature it is inadequate. In multi-attribute decision making community this problem was recognized [2, 9] and as a solution they use techniques of capacity theory [1] known in fuzzy community as fuzzy measure and fuzzy integrals [9]. In these approaches additivity is relaxed by monotonicity. As a consequence, the possible domain of application of this approach from logical point of view is much wider. But monotonicity is still a superfluously strong constraint since many of logical functions are non monotone in their nature. A generalized discrete Choquet integral [5] is defined for a generalized measure – non monotone in a general case. This approach includes

[1]Information fusion, decisions making, negotiation, classification, pattern recognition, scene analyze, data mining, machine learning, diagnostics, forecasting, intelligent agents, financial engineering, information retrieval, intelligent information system, intelligent systems,...

all logical and/or pseudo-logical functions but for only one arithmetic operator for interpolation intention, *min* function.

In this paper logical aggregation based on *interpolative Boolean algebra* [7] (IBA) is introduced. IBA is real-valued ([0, 1]-valued) realization of Boolean algebra. As a consequence, all logical and/or pseudo-logical functions can be treated as well as with generalized Choquet integral but with all possible interpolative operators.

IBA is based on structure functionality principle as the new paradigm. To any element of finite Boolean algebra there corresponds uniquely its structure (content, relation of inclusion). Structure of any attribute - element of Boolean algebra domain determines which atomic Boolean elements are included in it and/or which are not included in it. *Structure functional principle* says that the structure of any combined element of Boolean algebra can be determined (computed) directly on the base of structures of its components. Structure functional is an algebraic (value irrelevant) principle. *Truth functional principle* is a value realization of structure functional principle (its figure on a value level) which is valid (in the sense that it preserves all Boolean axioms and theorems) only for a two-valued realization of Boolean algebra.

Technically IBA is based on *generalized Boolean polynomials* (GBP) [6, 7]. GBP uniquely corresponds to any element of Boolean algebra and/or any Boolean function can be transformed into corresponding GBP. Atomic GBPs correspond to atomic elements of analyzed Boolean algebra of attributes. GBP of any element of analyzed Boolean algebra is equal to the sum of relevant atomic GBPs. Which atomic elements are relevant for combined attribute – element of Boolean algebra of attributes is defined by its structure.

Linear convex combination of GBP – pseudo GBP is in the new approach the most general aggregation function.

GBP is described in Section 2. In section 3 is described pseudo GBP. A representative example of logical aggregation is given in Section 4.

2 Generalized Boolean Polynomial

GBP is a polynomial whose variables are free elements of Boolean algebra and operators are standard $+$ and $-$, and generalized product $\otimes$. A set of feasible generalized product is a subclass of T-norms which satisfies an additional constraint – probability consistence. In the new approach a generalized product has a crucially different role; it is only an arithmetic operator, contrary to conventional fuzzy approaches where T-norms have the role of algebraic operator. IBA determines the procedure of transforming an analyzed element of Boolean algebra and/or a Boolean function into GBP directly.

Generalized Boolean polynomial (GBP) in logical aggregation has a role of logical combined attribute (property, characteristic, aspect...). Variables of GBP are elements from the analyzed set of primary attributes $\Omega = \{a_1, ..., a_n\}$. Primary attributes have the following characteristic: no one of primary attributes can be expressed as a Boolean function of the remaining primary attributes from Ω.

Boolean algebraic domain is the set $BA(\Omega)$ of all the possible logical combined attributes generated by the set of primary attributes Ω using two binary operators $\bigcup$, $\bigcap$ join and meet, respectively, and one unary operator C complementation or negation. The following structure is Boolean algebraic structure of attributes:

$$\langle BA(\Omega), \cup, \cap, C \rangle .$$

Set $BA(\Omega)$ is defined by the following expression:

$$BA(\Omega) = \mathrm{P}(\mathrm{P}(\Omega)) .$$

Boolean algebra domain of attributes $BA(\Omega)$is a partially ordered set. A partial order is based on the algebraic (value irrelevant) relation of inclusion.

Definition 1 *Element $\varphi \in BA(\Omega)$ of analyzed Boolean algebra is included in another element$\psi \in BA(\Omega)$,$\varphi \subset \psi$, if and only if $\varphi \cap \psi = \varphi$, or concisely:*

$$\varphi \subset \psi \quad \Leftrightarrow \quad \varphi \cap \psi = \varphi. \tag{1}$$

Relation of inclusion as algebraic (value irrelevant) property has its following implication on all possible value realizations:

$$\varphi \subset \psi \quad \Rightarrow \quad \varphi^v \le \psi^v. \tag{2}$$

Where: $\varphi^v, \ \psi^v \in [0, \ 1]$ are generalized value realizations of$\varphi, \psi \in BA(\Omega)$.

Definition 2 *Atomic attributes $\alpha(S)$, $\quad (S \in \mathrm{P}(\Omega))$ are the simplest elements of Boolean algebra domain of attributes $BA(\Omega)$ in the sense that they do not include in themselves anything except for a trivial Boolean constant* ***0****. The atomic attributes of $BA(\Omega)$ are described by the following expressions:*

$$\alpha(S) = \bigcap_{a_i \in S} a_i \bigcap_{a_j \in \Omega \setminus S} C a_j, \tag{3}$$

$(S \in \mathrm{P}(\Omega))$.

Example 1 *Atomic attributes of Boolean algebra generated by $\Omega = \{a, b\}$are given in the following table:*

S	$\alpha(S)$
$\emptyset$	$Ca \cap Cb$
$\{a\}$	$a \cap Cb$
$\{b\}$	$Ca \cap b$
$\{a, b\}$	$a \cap b$

Atomic elements $\alpha(S)$, $\ S \in \mathrm{P}(\Omega)$ of analyzed Boolean algebra have the following properties:

1. they are *mutually disjoint*:

$$\alpha(S_i) \cap \alpha(S_j) = \begin{cases} \underline{0} & i \neq j \\ \alpha(S_i) & i = j \end{cases};$$

2. they *cover universe* $\bar{1}$:

$$\bigcup_{S \in \mathrm{P}(\Omega)} \alpha(S) = \bar{1},$$

$(\underline{0},\ \bar{1},\ \alpha(S) \in BA(\Omega), \quad S \in \mathrm{P}(\Omega))$.

Any combined attribute – element of Boolean algebra domain of attributes includes in itself relevant attributes. Which attributes are relevant for analyzed combined attribute determines its structure (content, relation of inclusion of atomic attributes).

Definition 3 *Structural function σ_φ of analyzed attribute (element of Boolean algebra domain of attributes) $\varphi \in BA(\Omega)$, is set function which maps: $\sigma_\varphi : \mathrm{P}(\Omega) \to \{0,\ 1\}$ and it is defined by the following expression:*

.

$$\sigma_\varphi(S) = \begin{cases} 1, & \alpha(S) \subseteq \varphi; \quad (\alpha(S) \cap \varphi = \alpha(S)) \\ 0, & \alpha(S) \not\subseteq \varphi; \quad (\alpha(S) \cap \varphi = \underline{0}) \end{cases}; \tag{4}$$

$(S \in \mathrm{P}(\Omega); \quad \underline{0},\ \varphi \in BA(\Omega))$.

Example 2 *The structural functions of atomic attributes of Boolean algebra, which is generated by the following set of primary attributes $\Omega = \{a, b\}$ are given in the following table:*

S	$\alpha(S)$	$\sigma_{\alpha(S)}(\emptyset)$	$\sigma_{\alpha(S)}(\{a\})$	$\sigma_{\alpha(S)}(\{b\})$	$\sigma_{\alpha(S)}(\{a,b\})$
$\emptyset$	$Ca \cap Cb$	1	0	0	0
$\{a\}$	$a \cap Cb$	0	1	0	0
$\{b\}$	$Ca \cap b$	0	0	1	0
$\{a,b\}$	$a \cap b$	0	0	0	1

From definition of structural function it follows that structural function of primary attribute $a_i \in \Omega$ is given by the following expression

$$\sigma_{a_i}(S) = \begin{cases} 1, & a_i \in S \\ 0, & a_i \notin S \end{cases}; \tag{5}$$

$(a_i \in \Omega, \quad S \in \mathrm{P}(\Omega))$.

Example 3 *The structural functions of primary variables $\Omega = \{a, b\}$ are given in the following table:*

a_i	$\sigma_{a_i}(\emptyset)$	$\sigma_{a_i}(\{a\})$	$\sigma_{a_i}(\{b\})$	$\sigma_{a_i}(\{a,b\})$
a	0	1	0	1
b	0	0	1	1

For structure as an algebraic characteristic (value irrelevant), the structure functional principle is valid:

Definition 4 *Structure of any combined element of finite Boolean algebra can be determined (calculated) directly on the basis of structures of its components using the following equations:*

$$\begin{array}{l} \sigma_{\varphi\cup\psi}(S) = \sigma_\varphi(S) \vee \sigma_\psi(S) \\ \sigma_{\varphi\cap\psi}(S) = \sigma_\varphi(S) \wedge \sigma_\psi(S) \\ \sigma_{C\varphi}(S) = \neg\sigma_\varphi(S) \end{array} \tag{6}$$

Where: $(\varphi, \psi \in BA(\Omega), \quad S \in \mathrm{P}(\Omega))$

$$\begin{array}{ccc} \wedge & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{array}; \qquad \begin{array}{ccc} \vee & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{array}; \qquad \begin{array}{cc} & \neg \\ 0 & 1 \\ 1 & 0 \end{array}.$$

Any element of Boolean algebra $\varphi \in BA(\Omega)$ is a logical combined attribute and it can be represented by the corresponding disjunctive normal form:

$$\begin{aligned} \varphi(a_1, ..., a_n) &= \bigcup_{S\in\mathrm{P}(\Omega)|\sigma_\varphi(S)=1} \alpha(S)(a_1, ..., a_n) \\ &= \bigcup_{S\in\mathrm{P}(\Omega)|\sigma_\varphi(S)=1} \left(\bigcap_{a_i\in S} a_i \bigcap_{a_j\in\Omega\setminus S} Ca_j\right) \end{aligned}, \tag{7}$$

$(S \in \Omega, \quad \varphi, \psi \in BA(\Omega))$.

From structural functionality principle it follows that:

$$\begin{aligned} (\varphi\cup\psi)(a_1, ..., a_n) &= \bigcup_{S\in\mathrm{P}(\Omega)|\sigma_{\varphi\cup\psi}(S)=1} \alpha(S)(a_1, ..., a_n) \\ &= \bigcup_{S\in\mathrm{P}(\Omega)|\sigma_\varphi(S)\vee\sigma_\psi(S)=1} \alpha(S)(a_1, ..., a_n) \end{aligned} \tag{8}$$

$$\begin{aligned} (\varphi\cap\psi)(a_1, ..., a_n) &= \bigcup_{S\in\mathrm{P}(\Omega)|\sigma_{\varphi\cap\psi}(S)=1} \alpha(S)(a_1, ..., a_n) \\ &= \bigcup_{S\in\mathrm{P}(\Omega)|\sigma_\varphi(S)\wedge\sigma_\psi(S)=1} \alpha(S)(a_1, ..., a_n) \end{aligned} \tag{9}$$

$$\begin{aligned} (C\varphi)(a_1, ..., a_n) &= \bigcup_{S\in\mathrm{P}(\Omega)|\sigma_{C\varphi}(S)=1} \alpha(S)(a_1, ..., a_n) \\ &= \bigcup_{S\in\mathrm{P}(\Omega)|\sigma_\varphi(S)=0} \alpha(S)(a_1, ..., a_n) \\ &= 1 - \varphi(a_1, ..., a_n) \end{aligned} \tag{10}$$

Any attribute has its value realization on a valued level. Values of primary attributes reflect initial (or primary) perception of analyzed problem and they take values from a real unit interval [0, 1]. Any logical combined property is defined by corresponding generalized Boolean polynomial – GBP.

In atomic and as a consequence in any GBP figurate as operators standard addition (+); standard subtraction (–) and generalized product ($\otimes$).

The generalized product is defined as a subclass of T-norms [4] by the following definition.

Definition 5 *Generalized product $\otimes$ is any function which maps*

$$\otimes : [0,\ 1] \times [0,\ 1] \to [0,\ 1]$$

and it satisfies the following axioms:

1. $a_i^v \otimes a_j^v = a_j^v \otimes a_i^v$
2. $a_i^v \otimes \left(a_j^v \otimes a_k^v\right) = \left(a_i^v \otimes a_j^v\right) \otimes a_k^v$
3. $a_i^v \leq a_j^v \quad \Rightarrow \quad a_i^v \otimes a_k^v \leq a_j^v \otimes a_k^v$
4. $a_i^v \otimes 1 = a_i^v$
5. $\sum\limits_{K \in \mathrm{P}(\Omega \setminus S)} (-1)^{|K|} \bigotimes\limits_{a_i \in S \cup K} a_i^v \geq 0, \qquad \forall S \in \mathrm{P}(\Omega)$

   $(\Omega = \{a_1, ..., a_n\}, \quad a_i^v \in [0,\ 1],\ i = 1, \ldots, n)$

Analyzed GBP is equal to the sum of relevant atomic GBP-s. Atomic GBP is defined by the following definition:

Definition 6 *Atomic generalized Boolean polynomials are defined by the following expressions:*

$$\alpha^{\otimes}(S)(a_1^v, ..., a_n^v) = \sum_{K \in \mathrm{P}(\Omega \setminus S)} (-1)^{|K|} \bigotimes_{a_i \in K \cup S} a_i^v \tag{11}$$

$(S \in \mathrm{P}(\Omega), \quad a_i \in \Omega, \quad a_i^v \in [0,\ 1],\ \ i = 1, .., n)$.

Example 4 *: Atomic Boolean polynomials for the case when set of primary attributes is $\Omega = \{a, b\}$, are given in the following table:*

***Table 1**: Example of Atomic Boolean polynomials*

S	$\alpha(S)$	$\alpha^{\otimes}(S)(a^v, b^v)$
$\emptyset$	$Ca \cap Cb$	$1 - a^v - b^v + a^v \otimes b^v$
$\{a\}$	$a \cap Cb$	$a^v - a^v \otimes b^v$
$\{b\}$	$Ca \cap b$	$b^v - a^v \otimes b^v$
$\{a, b\}$	$a \cap b$	$a^v \otimes b^v$

Axiom number 5 in Definition 5 is additional to the set of axioms (1.-4.) immanent to T-norms and it has a role to ensure that the values of atomic Boolean polynomials are non-negative: $\alpha^{\otimes}(S) \geq 0, \quad (S \in \mathrm{P}(\Omega))$. As a consequence Boolean polynomials of all other elements of Boolean algebra are non negative in any value realization.

Example 5 *In the case $\Omega = \{a, b\}$ generalized product, according to axioms of non-negativity can be in the following interval:*

$$\max(a + b - 1,\ 0) \leq a \otimes b \leq \min(a,\ b).$$

Claim: *In spite of formal similarity between T-norms and generalized products, their roles are crucially different: while a T-norm in conventional fuzzy approaches has the role of a logical (and/or algebraic) operator a generalized product $\otimes$ is only an arithmetic operator on a value level without any influence on algebra.*

Any element $\varphi \in BA(\Omega)$ of Boolean algebra of attributes can be represented by a corresponding generalized Boolean polynomial $\varphi^{\otimes}(a_1^v, ..., a_n^v)$, which can be defined in the following way:

Definition 7 *GBP which corresponds to φ analyzed element of Boolean algebra $BA(\Omega)$ is equal to the sum of relevant atomic GBP-s:*

$$\begin{aligned} \varphi^{\otimes}(a_1^v, ..., a_n^v) &= \sum_{S \in \mathrm{P}(\Omega) | \sigma_{\varphi}(S) = 1} \alpha^{\otimes}(S)(a_1^v, ..., a_n^v), \\ &= \sum_{S \in \mathrm{P}(\Omega)} \sigma_{\varphi}(S)\ \alpha^{\otimes}(S)(a_1^v, ..., a_n^v), \end{aligned} \tag{12}$$

$(a_i^v \in [0,\ 1], \quad a_i \in \Omega)$.
Which atoms are relevant for analyzed element of Boolean algebra domain is determined by its structural function.

A generalized Boolean polynomial $\varphi^{\otimes}(a_1^v, ..., a_n^v)$ enables calculation of the value of a corresponding attribute $\varphi \in BA(\Omega)$ (element of Boolean algebra) for an analyzed object.

A generalized Boolean polynomial given by the expression (3) can be represented in the following way:

$$\varphi^{\otimes}(a_1^v, ..., a_n^v) = \sum_{S \in \mathrm{P}(\Omega)} \sigma_{\varphi}(S) \sum_{K \in \mathrm{P}(\Omega \setminus S)} (-1)^{|K|} \bigotimes_{a_i \in K \cup S} a_i^v, \tag{13}$$

$(\varphi \in BA(\Omega), \quad a_i^v \in [0,\ 1], \quad a_i \in \Omega)$.

A generalized Boolean polynomial can be represented as a scalar product of the following two vectors: (a) *structural vector* and (b) *vector of atomic Boolean polynomials,* which are defined by the following two definitions:

Definition 8 *Structural vector $\vec{\sigma}_\varphi$ of analyzed Boolean algebra element$\varphi \in BA(\Omega)$ – logical attribute is the following $1 \times N$vector:*

$$\vec{\sigma}_\varphi = [\sigma_\varphi(S) \,|S \in \mathrm{P}(\Omega)] \tag{14}$$

Where: $\Omega = \{a_1, ..., a_n\}$, $N = 2^{|\Omega|}$, and $\sigma_\varphi(S)$, $(S \in \mathrm{P}(\Omega))$is defined by Definition 3.

Definition 9 *Vector of atomic Boolean polynomials $\vec{\alpha}^{\otimes}(a_1^v, ..., a_n^v)$is the following $N \times 1$ vector:*

$$\vec{\alpha}^{\otimes}(a_1^v, ..., a_n^v) = \left[\alpha^{\otimes}(S)(a_1^v, ..., a_n^v)\,|S \in \mathrm{P}(\Omega)\right]^T \tag{15}$$

Where: $\left(a_i \in \Omega, \quad a_i^v \in [0,\ 1],\ i = 1, .., n; \quad N = 2^{|\Omega|}\right).$

So, a generalized Boolean polynomial is a scalar product of the above defined two vectors:

$$\varphi^{\otimes}(a_1^v, ..., a_n^v) = \vec{\sigma}_\varphi \vec{\alpha}^{\otimes}(a_1^v, ..., a_n^v), \tag{16}$$

where: $\varphi \in BA(\Omega)$, $a_i^v \in [0,\ 1]$, $a_i \in \Omega$.

For structural vectors all Boolean axioms are valid: Associativity, Commutativity, Absorption, Distributivity, Excluded middle and Contradiction

$$\begin{array}{ll}
\vec{\sigma}_{\varphi\cup(\psi\cup\phi)} = \vec{\sigma}_{(\varphi\cup\psi)\cup\phi}, & \vec{\sigma}_{\varphi\cap(\psi\cap\phi)} = \vec{\sigma}_{(\varphi\cap\psi)\cap\phi}; \\
\vec{\sigma}_{\varphi\cup\psi} = \vec{\sigma}_{\psi\cup\varphi}, & \vec{\sigma}_{\varphi\cap\psi} = \vec{\sigma}_{\psi\cap\varphi}; \\
\vec{\sigma}_{\varphi\cup(\varphi\cap\psi)} = \vec{\sigma}_{\varphi}, & \vec{\sigma}_{\varphi\cap(\varphi\cup\psi)} = \vec{\sigma}_{\varphi}; \\
\vec{\sigma}_{\varphi\cup(\psi\cap\phi)} = \vec{\sigma}_{(\varphi\cup\psi)\cap(\varphi\cup\phi)}, & \vec{\sigma}_{\varphi\cap(\psi\cup\phi)} = \vec{\sigma}_{(\varphi\cap\psi)\cup(\varphi\cap\phi)}; \\
\vec{\sigma}_{\varphi\cup C\varphi} = \vec{1}, & \vec{\sigma}_{\varphi\cap C\varphi} = \vec{0}.
\end{array}$$

respectively; and all Boolean theorems: Idempotency, Boundedness, 0 and 1 are complements, De Morgan's laws and Involution:

$$\begin{array}{ll}
\vec{\sigma}_{\varphi\cup\varphi} = \vec{\sigma}_{\varphi}, & \vec{\sigma}_{\varphi\cap\varphi} = \vec{\sigma}_{\varphi}; \\
\vec{\sigma}_{\varphi\cup 0} = \vec{\sigma}_{\varphi}, & \vec{\sigma}_{\varphi\cap 1} = \vec{\sigma}_{\varphi}; \\
\vec{\sigma}_{\varphi\cup 1} = \vec{1}, & \vec{\sigma}_{\varphi\cap 0} = \vec{0}; \\
\vec{\sigma}_{C0} = \vec{1}, & \vec{\sigma}_{C1} = \vec{0}; \\
\vec{\sigma}_{C(\varphi\cup\psi)} = \vec{\sigma}_{C\varphi\cap C\psi}, & \vec{\sigma}_{C(\varphi\cap\psi)} = \vec{\sigma}_{C\varphi\cup C\psi}; \\
\vec{\sigma}_{CC\varphi} = \vec{\sigma}_{\varphi}; &
\end{array}$$

respectively; where: $\varphi, \psi, \phi \in BA(\Omega)$.

So, the structure of a Boolean algebra element preserves Boolean properties in a generalized case described by Boolean polynomials.

As a consequence for any two elements of Boolean algebra $\varphi,\ \psi \in BA(\Omega)$ the following equations are valid:

$$\begin{aligned}
(\varphi \cap \psi)^{\otimes}(a_1^v, ..., a_n^v) &= \vec{\sigma}_{\varphi\cap\psi}\vec{\alpha}^{\otimes}(a_1^v, ..., a_n^v), \\
&= (\vec{\sigma}_\varphi \wedge \vec{\sigma}_\psi)\,\vec{\alpha}^{\otimes}(a_1^v, ..., a_n^v).
\end{aligned} \tag{17}$$

$$
\begin{aligned}
(\varphi \cup \psi)^{\otimes}(a_1^v,...,a_n^v) &= \vec{\sigma}_{\varphi\cup\psi}\vec{\alpha}^{\otimes}(a_1^v,...,a_n^v), \\
&= (\vec{\sigma}_{\varphi} \vee \vec{\sigma}_{\psi})\,\vec{\alpha}^{\otimes}(a_1^v,...,a_n^v).
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
(C\varphi)^{\otimes}(a_1^v,...,a_n^v) &= \vec{\sigma}_{C\varphi}\vec{\alpha}^{\otimes}(a_1^v,...,a_n^v), \\
&= \left(\vec{1} - \vec{\sigma}_{\varphi}\right)\vec{\alpha}^{\otimes}(a_1^v,...,a_n^v), \\
&= 1 - (\varphi)^{\otimes}(a_1^v,...,a_n^v).
\end{aligned} \tag{19}
$$

Actually, Boolean polynomial maps a corresponding element of Boolean algebra into its value from the real unit interval [0, 1] on the value level so that a partial order on the value level is preserved. Since a partial order is based on Boolean laws, they are preserved on the value level in a general case too, contrary to other approaches.

3 Generalized Pseudo-Boolean Polynomial

Pseudo-Boolean polynomial is defined on the following way:

Definition 10 *A pseudo-Boolean polynomial is a linear convex combination of analyzed elements of IBA – generalized Boolean polynomials*

$$
\begin{aligned}
&\pi_{\mu}^{\otimes}(a_1^v,...,a_n^v) = \sum_{i=1}^{m} w_i \varphi_i^{\otimes}(a_1^v,...,a_n^v), \\
&\sum_{i=1}^{m} w_i = 1; \qquad w_j \geq 0, \quad j = 1,...,m.
\end{aligned} \tag{20}
$$

$$\left(a_i \in \Omega, \quad a_i^v \in [0,\ 1],\ i = 1,..,n; \quad \varphi_j \in BA(\Omega) \quad j = 1,...,m\right).$$

From the definition of generalized Boolean polynomials, an *interpolative pseudo-Boolean polynomial* is given by the following expression:

$$
\begin{aligned}
\pi_{\mu}^{\otimes}(a_1^v,...,a_n^v) &= \sum_{i=1}^{m} w_i \sum_{S\in \mathrm{P}(\Omega)} \sigma_{\varphi_i}(S) \sum_{C\in \mathrm{P}(\Omega\setminus S)} (-1)^{|C|} \bigotimes_{a_i\in S\cup C} a_i^v \\
&= \sum_{S\in \mathrm{P}(\Omega)} \mu(S) \sum_{C\in \mathrm{P}(\Omega\setminus S)} (-1)^{|C|} \bigotimes_{a_i\in S\cup C} a_i^v .
\end{aligned} \tag{21}
$$

Definition 11 *Generalized measure μ of interpolative pseudo-Boolean polynomial $\pi_{\mu}^{\otimes}$ is a set function*

$$\mu : \mathrm{P}(\Omega) \to [0,\ 1], \quad \Omega = \{a_1,...,a_n\}$$

defined by the following expression, [8]:

$$\mu(S) = \sum_{i=1}^{m} w_i \sigma_{\varphi_i}(S), \quad . \tag{22}$$

$$\left(S \in \mathrm{P}(\Omega), \quad \varphi_i \in BA(\Omega); \quad \sum_{j=1}^{m} w_j = 1, \quad w_i \geq 0, \quad i = 1, ..., m \right).$$

$\sigma_{\varphi_i}, \quad i = 1, ..., m$ *are structure functions of the corresponding Boolean functions* $\varphi_i \in BA(\Omega), \quad i = 1, ..., m$.

Vector of generalized measure is a linear convex combination of corresponding structural functions:

$$\vec{\mu} = \sum_{i=1}^{m} w_i \vec{\sigma}_{\varphi_i}. \tag{23}$$

Interpolative pseudo-Boolean polynomial can be represented as a scalar product of the following two vectors:

$$\begin{aligned} \pi_{\mu}^{\otimes}(a_1^v, ..., a_n^v) &= \vec{\mu}\vec{\alpha}^{\otimes}(a_1^v, ..., a_n^v) \\ &= \sum_{i=1}^{m} w_i \vec{\sigma}_{\varphi_i} \vec{\alpha}^{\otimes}(a_1^v, ..., a_n^v) \end{aligned} \tag{24}$$

The characteristics of pseudo-Boolean polynomial depend on the generalized product, and its structure function. Structure functions can be classified into: (a) additive, (b) monotone and (c) generalized $((a) \subset (b) \subset (c))$.

4 Logical Aggregation

A starting point is a finite set of primary attributes $\Omega = \{a_1, ..., a_n\}$. The task of logical aggregation (LA) [8] is the fusion of primary quality attribute values into one resulting globally representative value using logical tools. In a general case LA has two steps: (1) Normalization of primary attributes' values:

$$\cdot^v : \Omega \to [0,\ 1].$$

The result of normalization is a generalized logical and/or [0, 1] value of analyzed primary attribute, and
(3) Aggregation of normalized values of primary attributes into one resulting value by a pseudo-logical function as a logical aggregation operator:

$$Aggr : [0,\ 1]^n \to [0,\ 1].$$

A Boolean logical function φ is transformed into a corresponding generalized Boolean polynomial (GBP), [7], $\varphi^{\otimes} : [0,\ 1]^n \to [0,\ 1]$. Actually, to any element of Boolean algebra of attributes $\varphi_i \in BA(\Omega)$ there corresponds uniquely GBP$\varphi_i^{\otimes}(a_1^v, ..., a_n^v)$. GBP is defined by expression (3) and/or (11).

A pseudo-logical function is a linear convex combination of generalized Boolean polynomials [7], defined by expression (12) and/or (13).

Definition 12 *Operator of logical aggregation in a general case is a pseudo-logical function:*

$$Agg_{\mu}^{\otimes}\left(a_1^v,...,a_n^v\right) = \pi_{\mu}^{\otimes}\left(a_1^v,...,a_n^v\right) \tag{25}$$

or

$$Agg_{\mu}^{\otimes}\left(a_1^v,...,a_n^v\right) = \sum_{S\in \mathrm{P}(\Omega)} \mu\left(S\right) \sum_{C\in \mathrm{P}(\Omega\backslash S)} (-1)^{|C|} \bigotimes_{a_i\in S\cup C} a_i^v \tag{26}$$

Aggregation measure is a structural function of pseudo-logical function – logical aggregation operator (14). So, *Aggregation measure* is a set function $\mu : \mathrm{P}(\Omega) \to [0,\ 1]$, which in a general case is not a monotone function (generalized capacity), defined by the following expression:

$$\mu\left(S\right) = \sum_{i=1}^{m} w_i \sigma_{\varphi_i}\left(S\right), \tag{27}$$

$\left(S \in \mathrm{P}\left(\Omega\right), \quad \sum\limits_{i=1}^{m} w_i = 1, \quad w_i \geq 0, \quad \varphi_i \in BA\left(\Omega\right), \quad i = 1,...,m\right).$

As a consequence, a logical aggregation operator depends on the chosen: (a) measure of aggregation and (b) operator of generalized product. By a corresponding choice of the measure of aggregation μ and generalized product $\otimes$ the known aggregation operators can be obtained as special cases:

Weighted sum

For the aggregation measure and generalized product:

$$\mu_{add}\left(S\right) = \sum_{i=1}^{n} w_i \sigma_{a_i}^v\left(S\right), \quad S \in \mathrm{P}\left(\Omega\right); \quad \otimes := \min.$$

Logical aggregation operator is a weighted sum:

$$Agg_{\mu_{add}}^{\min}\left(a_1^v,...,a_n^v\right) = \sum_{a_i\in\Omega} w_i a_i^v$$

Arithmetic mean

For the aggregation measure and generalized product:

$$w_i = \frac{1}{n}, \qquad \mu_{mean}\left(S\right) = \frac{|S|}{|\Omega|}; \quad \otimes := \min$$

Logical aggregation operator is the arithmetic mean:

$$Agg_{\mu_{mean}}^{\min}\left(a_1^v,...,a_n^v\right) = \frac{1}{n}\sum_{a_i\in\Omega} a_i^v$$

K-th attribute only

For the aggregation measure and generalized product:

$$w_i = \begin{cases} 1 & i = k \\ 0 & i \neq k \end{cases} ; \qquad \mu_k(S) = \begin{cases} 1 & a_k \in S \\ 0 & a_k \notin S \end{cases} ; \qquad \otimes := \min$$

Logical aggregation operator is the k-th attribute only:

$$Agg_{\mu_k}^{\otimes}(a_1^v, ..., a_n^v) = a_k^v$$

Minimal value of attributes

For the aggregation measure and generalized product:

$$\mu_{AND}(S) = \begin{cases} 1, & S = \Omega \\ 0, & S \neq \Omega \end{cases} ; \quad \otimes := \min .$$

Logical aggregation operator is the *min* function

$$Agg_{\mu_{AND}}^{\min}(a_1^v, ..., a_n^v) = \min\{a_1^v, ..., a_n^v\} .$$

Maximal value of attributes

For the aggregation measure and generalized product:

$$\mu_{OR}(S) = \begin{cases} 1, & S \neq \emptyset \\ 0, & S = \emptyset \end{cases} ; \quad \otimes ;= \min$$

Logical aggregation operator is the *max* function

$$Agg_{\mu_{OR}}^{\min}(a_1^v, ..., a_n^v) = \max\{a_1^v, ..., a_n^v\} .$$

OWA-ordered weight aggregation

For the aggregation measure and generalized product:

$$\mu_{OWA}(S) = \begin{cases} 0, & S = \emptyset \\ \sum\limits_{i=1}^{m} w_i, & |S| = m \end{cases} ; \quad \otimes := \min$$

Logical aggregation operator is an OWA aggregation operator

$$Agg_{\mu_{OWA}}^{\min}(a_1^v, ..., a_n^v) = OWA(a_1^v, ..., a_n^v) .$$

OWA is defined by the following expression:

$$OWA(a_1^v, ..., a_n^v) = \sum_{i=1}^{n} w_i a_{(i)}^v$$

$$a_{(1)}^v \le a_{(2)}^v \le ... \le a_{(n)}^v, \quad \sum_{i=1}^{n} w_i = 1, \quad w_i \ge 0.$$

k-th order statistics
For the aggregation measure and generalized product:

$$\mu_{k^{th}}(S) = \begin{cases} 0, & |S| < k \\ 1, & |S| \ge k \end{cases} \; ; \quad \otimes := \min$$

Logical aggregation operator is the k-th order statistics

$$Agg_{\mu_{k^{th}}}^{\min}(a_1^v, ..., a_n^v) = a_{(k)}^v,$$

where:

$$a_{(1)}^v \le a_{(2)}^v \le ... \le a_{(n)}^v.$$

Discrete Choquet integral
For any monotone aggregation measure μ_{mon} and generalized product:

$$\mu_{mon}, \qquad \otimes := \min$$

Logical aggregation operator is a discrete Choquet integral:

$$Agg_{\mu_{mon}}^{\otimes}(a_1^v, ..., a_n^v) = C_{\mu_{mon}}(a_1^v, ..., a_n^v).$$

A discrete Choquet integral is defined by the following expression:

$$C_{\mu_{mon}}(a_1^v, ..., a_n^v) = \sum_{k=1}^{n} \left(a_{(k)}^v - a_{(k-1)}^v\right) \mu_{mon}\left(A_{(k)}\right),$$

where:

$$a_{(1)}^v \le ... \le a_{(n)}^v; \quad A_{(k)} = \left\{a_{(k)}^v, ..., a_{(n)}^v\right\}.$$

Comment: *All above mention kinds of logical aggregation are special case of discrete Choquet integral.*
Generalized Discrete Choquet integral
For a measure which can be nonmonotone μ and ***min*** for generalized product:

$$\mu, \qquad \otimes := \min$$

Logical aggregation operator is a generalized discrete Choquet integral:

$$Agg_{\mu}^{\otimes}(a_1^v, ..., a_n^v) = GC_{\mu}(a_1^v, ..., a_n^v).$$

A generalized discrete Choquet integral [5] is defined by the following expression:

$$GC_{\mu}\left(a_1^v,...,a_n^v\right) = \sum_{k=1}^{n+1}\left(a_{(k)}^v - a_{(k-1)}^v\right)\mu\left(A_{(k)}\right)$$

where:

$$\begin{array}{l} 0 = a_{(0)} \le a_{(1)}^v \le ... \le a_{(n)}^v \le a_{(n+1)}^v = 1; \\ A_{(k)} = \left\{a_{(k)}^v, ..., a_{(n)}^v\right\}, \quad A_{(n+1)} = \emptyset, \quad (\mu(\emptyset) \neq 0, \quad \textit{in general case}) \end{array}.$$

5 Example of Logical Aggregation Application

In this section a modified example from [3] is analyzed.

Example 6 Objects A, B, C and D are described by quality attributes, whose values are from a real unit interval [0, 1], given in the following table:

Table 2: Values of quality attributes

Object	a	b	c
A	.75	.9	.3
B	.75	.8	.4
C	.3	.65	.1
D	.3	.55	.2

An object should be compared on the base of a global quality. A global quality is actually the aggregation of attributes so the following aspects should be incorporated: (a) the average value of quality attributes and (b) if the analyzed object is good by attribute a then attribute c is more important than b and if analyzed object is not good by attribute a then attribute b is more important than c.
A partial demand (a) is given by the following trivial expression:

$$\frac{a+b+c}{3}$$

A partial demand (b) is given by the following logical expression:

$$\varphi(a,b,c) = (a \cap c) \cup (Ca \cap b) \tag{28}$$

A generalized Boolean polynomial of logical expression (16) is:

$$\begin{aligned} \varphi^{\otimes}(a,b,c) &= ((a \cap c) \cup (Ca \cap b))^{\otimes} \\ &= b + a \otimes c - a \otimes b \end{aligned} \tag{29}$$

A possible logical aggregation operator is:

$$\begin{aligned} Aggr^{\otimes}(a,b,c) &= \frac{1}{2}\frac{a+b+c}{3} + \frac{1}{2}\varphi^{\otimes}(a,b,c) \\ &= \frac{1}{2}\frac{a+b+c}{3} + \frac{1}{2}(b + a \otimes c - a \otimes b) \end{aligned} \tag{30}$$

A corresponding measure of aggregation is:

$$\mu = \frac{1}{6}\left(\sigma_a + \sigma_b + \sigma_c\right) + \frac{1}{2}\left(\sigma_a \wedge \sigma_c\right) \vee \left(C\sigma_a \wedge \sigma_b\right).$$

or given as a table:

Table 3: Measure of aggregation

S	$\mu(S)$
$\emptyset$	0
$\{a\}$	1/6
$\{b\}$	2/3
$\{c\}$	1/6
$\{a,b\}$	5/6
$\{a,c\}$	5/6
$\{b,c\}$	1/3
$\{a,b,c\}$	1

It is clear that the measure is non-monotone since$\mu(\{b\}) \geq \mu(\{b,c\})$, and as a consequence it is not possible to use a standard Choquet integral.
In the case $\otimes :=$ min function $\varphi^{\min}(a,b,c)$ is actually a generalized discrete Choquet integral and its values are given in the following table:

Table 4: Values for $\otimes :=$ min

Object	$\varphi^{\min}(a,b,c)$
A	.45
B	.45
C	.45
D	.45

So, these results without discrimination are not adequate.

In the case when a generalized product is an ordinary product,$\otimes := *$, quitting conventional approaches, the corresponding values of function $\varphi^{*}(a,b,c)$ are given in the following table:

Table 5: Values for $\otimes := *$

Object	$\varphi^{*}(a,b,c)$
A	.450
B	.500
C	.485
D	.445

The values of aggregation function, for given aggregation measure, table 1, and for $\otimes := *$, are presented in the following table:

Table 6: Values of resulting aggregation function

Object	$Aggr^{*}(a,b,c)$
A	.5500
B	.5750
C	.4175
D	.3725

These results completely reflect all specified demands.

6 Conclusion

The aggregation of different attributes, aspects, partial goals - into one representative global criterion is a very important task in many fields of real applications as well as for theoretical purposes. Conventional aggregation tools are very often inadequate. Logical aggregation in a general case is a weighted sum of partial demands. Partial demands for aggregation usually are logical demands which can be adequately described only by logical expressions. Therefore, aggregation in a general case is a generalized pseudo-logic function. In this paper logical aggregation is based on interpolative Boolean algebra (IBA). IBA is a real valued ([0, 1]-valued) realization of Boolean algebra. The new approach treats logical functions – partial aggregation demand, as a generalized Boolean polynomial (GBP). GBP can process values from the whole real unit interval [0, 1]. Logical aggregation has multiple advantages among others from the stand point of its possibility and interpretability. It is interesting that conventional aggregation operators are only a special case of logical aggregation operators and, as a consequence of using LA, one can do much more in an adequate direction than before.

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