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Convergences in Perfect BL-algebras*

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Abstract

The aim of the paper is to investigate some concepts of convergence in the class of perfect BL-algebras. Similarity convergence was developed by G. Georgescu and A. Popescu in the case of the residuated lattices, while the convergence with a fixed regulator was studied by Š. Černák for lattice-ordered groups and MV-algebras and by the author for residuated lattices. In this paper we study the similarity convergence and the convergence with a fixed regulator for the perfect BL-algebras. The main result is the construction of Cauchy completion of a perfect BL-algebra.

Keywords Perfect BL-algebras, Archimedean BL-algebras, Radical, Locally Archimedean, Similarity convergence, Convergence regulator, Cauchy completion

1 Introduction

A variety of papers has been written on the subject of convergence in ordered structures. Order convergence in a lattice-ordered group is studied in [21] and [22], while L-convergence is presented in [1] and convergence in [2]. Š. Černák studied the convergence with a fixed regulator for abelian ℓ-groups in [8] and for Archimedean ℓ-groups in [6]. Order convergence in MV-algebras is presented in [17], α-convergence was investigated in [20] and various kinds of Cauchy completions of MV-algebras are studied in [3]. Using the Mundici functor, Š. Černák [7] extended the convergence with fixed regulator from abelian l-groups to MV-algebras. For the class of perfect MV-algebras, order convergence has been presented in [14] and the convergence with a fixed regulator was treated by the author in [10], based on the Di Nola-Lettieri functors $\mathcal{D}$ and Δ. Order convergence on Łukasiewicz-Moisil algebras was studied in [16]. In the case of residuated lattices, two concepts of convergence have been developed: similarity convergence by G.Georgescu and A.Popescu in [18] and the convergence with a fixed regulator by the author in [9]. BL-algebras are

*2000 *Mathematics Subject Classification 03G10, 03G25, 06D35* fuzzy structures constructed from continuous t-norms and they were introduced by Hájek in [19] as algebraic counterparts for his Basic Logic. In this paper we investigate the similarity convergence and the convergence with a fixed regulator in the case of perfect BL-algebras. Because the Archimedean property plays a crucial role for various kinds of convergence in the ordered structures, we are interested in investigating the convergences on non Archimedean structures. Similarly with the case of the perfect MV-algebras we introduce the locally Archimedean property of a perfect BL-algebras. The main results state that any perfect BL-algebra has a Cauchy completion and any locally Archimedean BL-algebra has a v-Cauchy completion.

2 Preliminaries on BL-algebras and perfect BL-algebras

Definition 2.1. A BL-algebra is an algebra $(A, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ of the type $(2, 2, 2, 2, 0, 0, 0)$ satisfying the following conditions $(x, y, z \in A)$:
(B_1) $(A, \vee, \wedge, 0, 1)$ is a bounded lattice;
(B_2) $(A, \otimes, 1)$ is a commutative monoid;
(B_3) $x \otimes y \leq z$ iff $x \leq y \rightarrow z$;
(B_4) $x \wedge y = x \otimes (x \rightarrow y)$;
(B_5) $(x \rightarrow y) \vee (y \rightarrow x) = 1$.

We will refer to a BL-algebra $(A, \vee, \wedge, \otimes, \rightarrow, 0, 1)$ by its univers A. A BL-algebra is not trivial if $0 \neq 1$. For any BL-algebra A, the reduct $L(A)$ is a bounded distributive lattice ([23]). A BL-algebra A is called *BL-chain* or *linear* BL-algebra if it is totally ordered, i.e. a BL-algebra such that its lattice order is total.
For any $x \in A$ we define $\bar{x} = x \rightarrow 0$.
For any $n \in N, x \in A$ we put $x^0 = 1$ and $x^{n+1} = x^n \otimes x = x \otimes x^n$.
The *order* of $x \in A$, denoted $ord(x)$ is the smallest $n \in N$ such that $x^n = 1$.
If no such n exists, then $ord(x) = \infty$. A BL-algebra is called *locally finite* if all non unit elements of A have finite order.
The following proposition provides some rules of calculus in a BL-algebra.

Proposition 2.2. *([12], [13], [19]) In any BL-algebra A the following properties hold:* (1) $x \otimes y \leq x \wedge y \leq x, y$;
(2) $x \leq y$ *implies* $x \otimes z \leq y \otimes z$;
(3) $x \leq y$ *iff* $x \rightarrow y = 1$;
(4) $1 \rightarrow x = x$, $x \rightarrow x = 1$, $x \leq y \rightarrow x$, $x \rightarrow 1 = 1$;
(5) $x \otimes \bar{x} = 0$;
(6) $x \otimes y = 0$ *iff* $x \leq \bar{y}$;
(7) $x \vee y = 1$ *implies* $x \otimes y = x \wedge y$;
(8) $\bar{1} = 0$, $\bar{0} = 1$, $x \leq \bar{\bar{x}}$, $\bar{\bar{\bar{x}}} = \bar{x}$;
(9) $x \rightarrow (y \rightarrow z) = (x \otimes y) \rightarrow z$;
(10) $(x \rightarrow y) \rightarrow (\rightarrow z) = x \wedge y) \rightarrow z$;
(11) $x \leq y$ *implies* $z \rightarrow x \leq z \rightarrow y$, $y \rightarrow z \leq x \rightarrow z$ *and* $\bar{y} \leq \bar{x}$;

(12) $x \otimes (y \vee z) = (x \otimes y) \vee (x \otimes z)$;
(13) $x \otimes (y \wedge z) = (x \otimes y) \wedge (x \otimes z)$;
(14) $x \to (y \wedge z) = (x \to y) \wedge (x \to z)$;
(15) $(y \wedge z) \to x = (y \to x) \vee (z \to x)$;
(16) $(x \vee y) \to z = (x \to z) \wedge (y \to z)$;
(17) $x \to y \leq (y \to z) \to (x \to z)$;
(18) $x \to y \leq (z \to x) \to (z \to y)$;
(19) $x \to y \leq (x \otimes z) \to (y \otimes z)$;
(20) $(x \to y) \otimes (y \to z) \leq x \to z$;
(21) $(x \wedge y)^{-} = \bar{x} \vee \bar{y}$ *and* $(x \vee y)^{-} = \bar{x} \wedge \bar{y}$;
(22) $x \otimes y)^{-} = x \to \bar{y}$;
(23) $(x \vee y) = [(x \to y) \to y] \wedge [(y \to x) \to x]$.

Definition 2.3. ([19]) A *filter* of a BL-algebra A is a nonempty subset $F \subseteq A$ such that for all $x, y \in A$:
(F_1) $x, y \in F$ implies $x \otimes y \in F$;
(F_2) $x \in F$ and $x \leq y$ implies $y \in F$.

Definition 2.4. ([25]) A *deductive system* of a BL-algebra A is a subset $D \subseteq A$ such that:
(D_1) $1 \in D$;
(D_2) for all $x, y \in A$, $x, x \to y \in D$ implies $y \in D$.
A deductive system D is called:
$a)$ *proper* - if $0 \notin D$;
$b)$ *prime* - if D is proper and for all $x, y \in A, x \vee y \in D$ implies $x \in D$ or $y \in D$;
$c)$ *maximal* - if D is proper and for any deductive system E such that $D \subseteq E \subseteq A$, either $E = D$ or $E = A$;
$d)$ *primary* - if D is proper and for all $x, y \in A, (x \otimes y)^{-} \in P$ implies $(x^n)^{-} \in P$ or $(y^n)^{-} \in P$ for some $n \in N$;
$e)$ *Boolean* - if for all $x \in A, x \vee \bar{x} \in D$;
$f)$ *implicative* - if for all $x, y, x \in A$, $x \to (\bar{z} \to y) \in D$ and $y \to z \in D$ imply $x \to z \in D$.
A BL-algebra A is called *local* if it has a unique maximal deductive system.

Proposition 2.5. *([23]) If A is a BL-algebra and $F \subseteq A$, then the following are equivalent:*
(i) *F is a filter of A;*
(ii) *F is a deductive system of A.*

For every subset $X \subseteq A$, the smallest filter of A which contains X, i.e. the intersection of all filters F of A such that $X \subseteq F$ is called the filter *generated by* X and it is denoted by [X).

Proposition 2.6. *([12], [13]) If $X \subseteq A$, then*
$[X) = \{y \in A | y \geq x_1 \otimes x_2 \otimes ... \otimes x_n$ *for some* $n \geq 1$ *and* $x_1, x_2, ..., x_n\}$.

Remarks 2.7. ([12], [13])
(1) If X is a filter of A, then $[X) = X$;

(2) If $X = x$ we write [x) instead of $[\{x\})$ and $[x) = \{y \in A \mid y \geq x^n$ for some $n \geq 1\}$.
$[x)$ is called *principal* filter.
(3) If X is a filter of A and $x \in A$, then
$F(x) = [F \cup \{x\}) = \{y \in A \mid y \geq (f_1 \otimes x)^{n_1} \otimes (f_2 \otimes x)^{n_2} \otimes ... \otimes (f_m \otimes x)^{n_m}$ for some

$$m \geq 1, n_1, n_2, ..., n_m \geq 0, f_1, f_2, ..., f_m \in F\}.$$

If D is a proper deductive system of A, consider $D^* = \{x \in A \mid x \leq \bar{y}$ for some $y \in D\}$. We have $D \cap D^* = \oslash$. Indeed, if $x \in D \cap D^* = \oslash$ then $x \in D$ and $x \leq \bar{y}$ for some $y \in D$. It follows that $\bar{x} \geq \bar{\bar{y}} > y$. Because D is deductive system we get $\bar{x} \in D$.
Hence, $0 = x \otimes \bar{x} \in D$, which is a contradiction.
We also denote by $D(A) = \{x \in A \mid x^n > 0$ for all $n \in \mathbb{N}\}$.

Proposition 2.8. *([26]) In any BL-algebra A the following are equivalent:*
(i) *$D(A)$ is a deductive system;*
(ii) *$[D(A))$ is a proper deductive system;*
(iii) *A is local;*
(iv) *$D(A)$ is the unique maximal deductive system of A;*
(v) *For all $x, y \in A$ and $n \in \mathbb{N}, n \geq 1$, we have $x^n, y^n > 0$ iff $x^n \otimes y^n > 0$.*

According to [24], if D is a deductive system of the BL-algebra A we define the equivalence relation $\approx$ on A by:

$$x \approx y \ iff \ (x \to y) \otimes (y \to x) \in D.$$

Then, the quotient algebra A/D becomes a BL-algebra with the natural operations induced from those of A. Denoting by x/D the equivalence class of x, then $x/D = 1/D$ iff $x \in D$.

Proposition 2.9. *([15], [26]) Let A be a BL-algebra. Then:*
(1) *A is local iff for all $x \in A, ord(x) < \infty$ or $ord(\bar{x}) < \infty$;*
(2) *A deductive system P of A is primary iff A/P is local ;*
(3) *Any prime deductive system is primary ;*
(4) *BL-algebra A is local iff every proper deductive system of A is primary ;*
(5) *The subset $A_0 = \{x \in A | \bar{x} = 0\}$ is a proper deductive system of A and $A_0 \subseteq D(A)$.*

Definition 2.10. ([26]) A local BL-algebra is called *perfect* if for any $x \in A$, $ord(x) < \infty$ iff $ord(\bar{x}) < \infty$.

Proposition 2.11. *([26]) Let A be a local BL-algebra. The following are equivalent:*
(i) *A is perfect*
(ii) *$A = D(A) \cup D(A)^*$.*

Definition 2.12. ([5]) The intersection of the maximal deductive systems of the BL-algebra A is called the *radical* of A and it is denoted by $Rad(A)$.
Obviously, $Rad(A)$ is a deductive system of A.

Proposition 2.13. *([13])* $Rad(A) = \{x \in A | (x^n)^- \leq x$ *for any* $n \in \mathbb{N}\}$.

Denote $Rad(A)^- = \{\bar{x} \mid x \in Rad(A)\ \}$. One can easy check that if $x \in Rad(A)$, then $\bar{x} \in Rad(A)^-$ and if $x \in Rad(A)^-$, then $\bar{x} \in Rad(A)$.

Definition 2.14. ([5] A nonunit element $x \in A$ is called *infinitesimal* if $x^n \geq \bar{x}$ for any $n \in N$. We will denote by *Infinit(A)* the set of all infinitesimal elements of A.

Proposition 2.15. *([5]) Let* A *be a local BL-algebra and* $x \in A, x \neq 1$. *The following are equivalent:*
(i) $x \in Infinit(A)$;
(ii) $x \in Rad(A)$.

Corollary 2.16. $Rad(A) = Infinit(A) \cup \{1\}$.

Proposition 2.17. *The local BL-algebra* A *is perfect iff* $A = Rad(A) \cup Rad(A)^*$.

Proof. Because A is a local BL-algebra, By Proposition 2.8 it follows that $D(A)$ is the unique maximal deductive system of A. Hence, $Rad(A) = D(A)$. By Proposition 2.11 we get $A = Rad(A) \cup Rad(A)^*$. □

From the proof of the above proposition it follows that $Rad(A)^* = D(A)^*$.

Proposition 2.18. *In any BL-algebra* A *the following are equivalent:*
(i) $Rad(A) = \{1\}$;
(ii) $x^n \geq \bar{x}$ *for any* $n \in \mathbb{N}$ *implies* $x = 1$;
(iii) $x^n \geq \bar{y}$ *for any* $n \in \mathbb{N}$ *implies* $x \vee y = 1$;
(iv) $x^n \geq \bar{y}$ *for any* $n \in \mathbb{N}$ *implies* $x \to y = y$ *and* $y \to x = x$.

Proof. $(i) \Leftrightarrow (ii)$ follows from Proposition 2.15 ;
$(ii) \Rightarrow (iii)$. Let $x, y \in A$ such that $x^n \geq \bar{y}$ for any $n \in \mathbb{N}$. By Proposition 2.2 and by the hypothesis we have:

$$(x \vee y)^- = \bar{x} \wedge \bar{y} \leq \bar{y} \leq x^n \leq (x \vee y)^-,$$

hence $(x \vee y)^n \geq (x \vee y)^-$ for any $n \in \mathbb{N}$. Thus, by the hypothesis we get $x \vee y = 1$.
$(iii) \Rightarrow (ii)$. Consider $x \in A$ such that $x^n \geq \bar{y}$ for any $n \in \mathbb{N}$.
By (iii), taking $y = x$ we get $x \vee x = 1$, hence $x = 1$.
$(ii) \Rightarrow (iv)$. Let $x, y \in A$ such that $x^n \geq \bar{y}$ for any $n \in \mathbb{N}$.
Similarly as in (i) $\Rightarrow$ (ii), if $x, y \in A$ we have $(x \vee y)^n \geq (x \vee y)^-$ for any $n \in \mathbb{N}$, hence, by the hypothesis, we get $x \vee y = 1$.
By Proposition 2.2 we have $x \vee y = [(x \to y) \to y] \wedge [(y \to x) \to x]$.
Since $x \vee y = 1$, it follows that $[(x \to y) \to y] \wedge [(y \to x) \to x] = 1$,
hence $(x \to y) \to y = 1$ and $(y \to x) \to x = 1$.
From $(x \to y) \to y = 1$ we have $x \to y \leq y$ and taking in consideration that $y \leq x \to y$ we obtain $x \to y = y$. Similarly, $y \to x = x$.
(iv) $\Rightarrow$ (ii). Consider $x \in A$ such that $x^n \geq \bar{x}$ for any $n \in \mathbb{N}$. By the hypothesis we obtain $x \to x = x$, hence $x = 1$. □

Definition 2.19. A BL-algebra is called *Archimedean* or *semisimple* if one of the equivalent above conditions is satisfied.

Definition 2.20. A perfect BL-algebra A is called *locally Archimedean* whenever from $x, y \in Rad(A)$ such that $x^n \geq y$ for all $n \in \mathbb{N}$, it follows that $x = 1$.

Example 2.21. Define on the real unit $A = [0, 1]$ the operations:

$$x \otimes y = min\{x, y\}$$

$$x \to y = \begin{cases} 1, \text{ if } x \leq y \\ y, \text{ otherwise} \end{cases}$$

Then $(A, \leq, min, max, \otimes, \to, 0, 1)$ is BL-algebra called *Gödel structure*.
Obviously, $\bar{x} = x \to 0 = 0$ and $x^n = x$ for all $x \in A$.
It follows that $Rad(A) = (0, 1]$ and $Rad(A)^* = \{0\}$, hence A is a perfect BL-algebra, but it is not Archimedean. A is not locally Archimedean (indeed, for $x = 1/2$, $y = 1/3$ we have $x^n > y$ for all $n \in \mathbb{N}$, but $x \neq 1$).

Example 2.22. Define on the real unit $A = [0, 1]$ the operation $\otimes$ as usual multiplication of real numbers and the operation

$$x \to y = \begin{cases} 1, \text{ if } x \leq y \\ y/x, \text{ otherwise} \end{cases}$$

Then $(A, \leq, min, max, \otimes, \to, 0, 1)$ is a BL-algebra called the *product structure*.
Obviously, $\bar{x} = x \to 0 = 0$ for all $x \in A$.
It follows that $Rad(A) = (0, 1]$ and $Rad(A)^* = \{0\}$, hence A is a perfect BL-algebra, but it is not Archimedean. One can easy check that A locally Archimedean.

In a BL-algebra A we define the *distance function* $d : A \times A \to A$ by

$$d(x, y) = (x \to y) \wedge (y \to x).$$

Proposition 2.23. *([18]). The distance function fulfills the following properties:*
(1) $d(x, y) = d(y, x)$;
(2) $d(x, y) = 1$ *iff* $x = y$;
(3) $d(x, 1) = x$;
(4) $d(x, 0) = \bar{x}$;
(5) $d(x, z) \otimes d(z, y) \leq d(x, y)$;
(6) $d(x, y) \leq d(x \otimes u, y \otimes u)$;
(7) $d(x, u) \otimes d(y, v) \leq d(x \otimes y, u \otimes v)$;
(8) $d(x, u) \otimes d(y, v) \leq d(x \to x, v \to u)$;
(9) $d(x, u) \wedge d(y, v) \leq d(x \wedge y, u \wedge v)$;
(10) $d(x, u) \wedge d(y, v) \leq d(x \vee y, u \vee v)$;
(11) *If* $x, y \in [x', y']$ *then* $d(x', y') \leq d(x, y)$.

Let $(x_n)_n$ be a sequence in a BL-algebra A. If $(x_n)_n$ is increasing we denote $(x_n)_n \uparrow$. Similarly, if $(x_n)_n$ is decreasing we denote $(x_n)_n \downarrow$.
If $(x_n)_n$ is increasing, $\vee_n x_n$ exists and $\vee_n x_n = x$, we denote $(x_n)_n \uparrow x$.
Similarly, if $(x_n)_n$ is decreasing, $\wedge_n x_n$ exists and $\wedge_n x_n = x$, we denote $(x_n)_n \downarrow x$.

Proposition 2.24. *([18]) (1-sphere propery) If $(x_n)_n$ and $(y_n)_n$ are two sequences in A such that $(x_n)_n \uparrow 1$ and $(y_n)_n \uparrow 1$, then $(x_n * y_n)_n \uparrow 1$.*

3 Convergence in perfect BL-algebras

In this section we will introduce the concept of convergence in perfect BL-algebras in the same way it was introduced in [18] for the case of residuated-lattices.

Lemma 3.1. *If $(x_n)_n$ is a sequence in a perfect BL-algebra A and $(x_n)_n \uparrow 1$, then there is $n_0 \in \mathbb{N}$ such that $x_n \in Rad(A)$ for any $n \geq n_0$.*

Proof. Since $Rad(A)$ is a deductive system and $(x_n)_n$ is an increasing sequence, it suffices to show that there is $n_0 \in \mathbb{N}$ such that $x_n \in Rad(A)$ for all $n \geq n_0$.
Assume $x_n \notin Rad(A)$ for all $n \in \mathbb{N}$. If $Rad(A) \neq \{1\}$, then there is an element $y \in Rad(A)$, $y < 1$ such that $y \geq x_n$ for all $n \in \mathbb{N}$ which is a contradiction with the fact that $\vee_n x_n = 1$. If $Rad(A) = \{1\}$, then $Rad(A)^- = \{0\}$ so $x_n = 0$ for all $n \in \mathbb{N}$ which is again a contradiction. □

Definition 3.2. Let $(x_n)_n$ be a sequence in an arbitrary BL-algebra A. Then $(x_n)_n$ *converges* to $x \in A$ if there is a sequence $(s_n)_n$ such that $(s_n)_n \uparrow 1$ and $d(x_n, x) \geq s_n$ for all $n \in \mathbb{N}$. We will denote $x_n \rightarrow_s x$.

Proposition 3.3. *Let $(x_n)_n$ be a sequence in an arbitrary BL-algebra A. If $x_n \rightarrow_s x_1$ and $x_n \rightarrow_s x_2$, then $x_1 = x_2$.*

Proof. There exist $(s_n)_n, (t_n)_n \subseteq A$ such that $(s_n)_n \uparrow 1$, $(t_n)_n \uparrow 1$ and $d(x_n, x_1) \geq s_n$, $d(x_n, x_2) \geq t_n$ for all $n \in \mathbb{N}$. We have: $d(x_1, x_2) \geq d(x_1, x_n) \otimes d(x_n, x_2) \geq s_n \otimes t_n$ for all $n \in \mathbb{N}$. But, by the 1-sphere property $(s_n * t_n)_n \uparrow 1$, so $d(x_1, x_2) = 1$, hence $x_1 = x_2$. □

Proposition 3.4. *Let $(x_n)_n, (y_n)_n \subseteq A$ such that $x_n \rightarrow_s x$ and $y_n \rightarrow_s y$. Then:*
(1) $x_n \otimes y_n \rightarrow_s x \otimes y$;
(2) $x_n \vee y_n \rightarrow_s x \vee y$;
(3) $x_n \wedge y_n \rightarrow_s x \wedge y$;
(4) $(x_n \rightarrow y_n) \rightarrow_s (x \rightarrow y)$;
(5) $\bar{x}_n \rightarrow_s \bar{x}$.
Moreover, if $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.

Proof. $(1) - (5)$ follow from Propositions 2.23.
If $x_n \leq y_n$, then $x_n \rightarrow y_n = 1$ for all $n \in \mathbb{N}$, so $(x_n \rightarrow y_n) \rightarrow_s 1$.
But, by (4) $(x_n \rightarrow y_n) \rightarrow_s (x \rightarrow y)$ and applying Proposition 3.3 we get $x \rightarrow y = 1$.
Thus, $x \leq y$. □

Proposition 3.5. *Let A be an arbitrary BL-algebra and $(x_n)_n \subseteq Rad(A)$. If $x_n \rightarrow_s x$, then $x \in Rad(A)$.*

Proof. Since $x_n \in Rad(A)$ we have $x_n^k \geq \bar{x}_n$ for all $n, k \in \mathbb{N}$. By Proposition 3.4 we get $x^k \geq \bar{x}$ for all $n \in \mathbb{N}$, that is, $x \in Rad(A)$. □

Corollary 3.6. *Let A be a perfect BL-algebra. If $(x_n)_n \subseteq Rad(A)^-$ and $x_n \to_s x$, then $x \in Rad(A)^-$.*

Proof. By the above proposition, taking in consideration that $(\bar{x}_n)_n \subseteq Rad(A)$. □

Proposition 3.7. *Let A be a perfect BL-algebra and $(x_n)_n \subseteq A$, $x_n \to_s x$. If $x \in Rad(A)$, then there is n_0 such that $x_n \in Rad(A)$ for all $n \geq n_0$.*

Proof. If for any $n \in \mathbb{N}$ there exists $k_n \geq n$ such that $x_{k_n} \notin Rad(A)$, we get a sequence $(x_{k_n}) \in Rad(A)^-$ which is convergent to $x \in Rad(A)$. This is a contradiction with Corollary 3.6. □

Corollary 3.8. *Let A be a perfect BL-algebra and $(x_n)_n \in A$ such that $x_n \to_s x$. If $x \in Rad(A)^-$, then there is $n_0 \in \mathbb{N}$ such that $x_n \in Rad(A)^-$ for all $n \geq n_0$.*

Definition 3.9. A sequence $(x_n)_n \subseteq A$ is called *Cauchy sequence* if there is a sequence $(s_n)_n$ such that $(s_n)_n \uparrow 1$ and $d(x_n, x_{n+p}) \geq s_n$ for all $n, p \in \mathbb{N}$.
A BL-algebra A is *Cauchy complete* if any Cauchy sequence is convergent.

Proposition 3.10. *Let $(x_n)_n, (y_n)_n$ be two Cauchy sequences in a BL-algebra A. Then, the sequences $(x_n \wedge y_n)_n$, $(x_n \vee y_n)_n$, $(x_n \otimes y_n)_n$, $(x_n \to y_n)_n$, $(\bar{x}_n)$ are Cauchy.*

Proof. Since $(x_n)_n$ and $(y_n)_n$ are Caucy sequences, there exist two sequences $(s_n)_n \uparrow 1$ and $(t_n)_n \uparrow 1$ such that $d(x_n, x_{n+p}) \geq s_n$ and $d(y_n, y_{n+p}) \geq t_n$ for all $n, p \in \mathbb{N}$.
By Proposition 2.23 we have

$$d(x_n \wedge y_n, x_{n+p} \wedge y_{n+p}) \geq d(x_n, x_{n+p}) \wedge d(y_n, y_{n+p}) \geq s_n \otimes t_n.$$

Using the 1-sphere property it follows that $(x_n \wedge y_n)_n$ is a Cauchy sequence.
The rest of assertions in the proposition follow similarly. □

Proposition 3.11. *Let A be a perfect BL-algebra. If $(x_n)_n \subseteq A$ is a Cauchy sequence, then there is $n_0 \in \mathbb{N}$ such that $\{x_n \mid n \geq n_0\} \subseteq Rad(A)$ or $\{x_n \mid n \geq n_0\} \subseteq Rad(A)^-$.*

Proof. Since $(x_n)_n$ is a Cauchy sequence, then there exists $(s_n)_n \uparrow 1$ such that $d(x_n, x_{n+p}) \geq s_n$ for all $n, p \in \mathbb{N}$. By Lemma 3.1, there is $n_0 \in \mathbb{N}$ such that $x_n \in Rad(A)$ for any $n \geq n_0$. Assume there is $n \geq n_0$ and $p \in \mathbb{N}$ such that $x_n \in Rad(A)^-$ and $x_{n+p} \in Rad(A)$. We have:

$$x_n = d(x_n, 1) \geq d(x_n, x_{n+p}) \otimes d(x_{n+p}, 1) \geq s_n \otimes x_{n+p} \in Rad(A),$$

hence $x_n \in Rad(A)$ which is a contradiction.
If there is $n \geq n_0$ and $p \in \mathbb{N}$ such that $x_n \in Rad(A)$ and $x_{n+p} \in Rad(A)^-$ we get:

$$x_{n+p} = d(x_{n+p}, 1) \geq d(x_{n+p}, x_n) \otimes d(x_n, 1) \geq s_n \otimes x_n \in Rad(A),$$

hence $x_{n+p} \in Rad(A)$ which is again a contradiction. □

Theorem 3.12. *Any complete BL-algebra is Cauchy complete.*

Proof. Let $(x_n)_n$ be a Cauchy sequence in the complete BL-algebra A, i.e. there exists a sequence $(s_n)_n$ such that $(s_n)_n \uparrow 1$ and $d(x_n, x_{n+p}) \geq s_n$ for all $n, p \in \mathbb{N}$. Since A is complete lattice, there exists $x = x_{m_0} = \vee_{n \leq 1} x_n$. Hence, for all $n \in \mathbb{N}$ we have $d(x_n, x) = d(x_n, x_{m_0}) \geq s_n$. Thus, $x_n \rightarrow_s x$ and it follows that A is Cauchy complete. □

Definition 3.13. Let $A \hookrightarrow B$ be an embedding of the perfect BL-algebras A and B. We say that B is a *Cauchy completion* of A if:
$a)$ B is a Cauchy complete BL-algebra ;
$b)$ for each $x \in Rad(B)$ there exist two sequences $(x_n)_n, (s_n)_n \subseteq Rad(A)$ such that $(s_n)_n \uparrow 1$ and $d(x_n, x) \geq s_n$ for all $n \in \mathbb{N}$.

For two Cauchy sequences $(x_n)_n$ and $(y_n)_n$ of a BL-algebra A we define:

$$(x_n)_n \equiv (y_n)_n \ iff \ d(x_n, y_n) \rightarrow_s 1.$$

The relation $\equiv$ is an equivalence relation on the set $C(A)$ of all Cauchy sequences of the BL-algebra A.
Let $A^* = C(A)/_{\equiv}$ and $[(x_n)_n]$ the equivalence class of the sequence $(x_n)_n$.
Then A^* is a BL-algebra with respect to the component-wise operations.
If [x] is the class of the constant sequence $(x, x, x, \ldots)$, than the map $x \rightarrow [(x)]$ is an embedding of BL-algebras $A \hookrightarrow A^*$.

Theorem 3.14. *[18] A^* is a complete lattice and for any $x \in A^*$ there is $(x_n)_n \subseteq A$ such that $x_n \rightarrow_s x$.*

Proposition 3.15. *Let $A \hookrightarrow B$ be an embedding of the perfect BL-algebras A and B. Assume that for each $x \in B$ there exist two sequences $(x_n)_n, (s_n)_n \subseteq A$ such that $(s_n)_n \uparrow 1$ and $d(x_n, x) \geq s_n$ for all $n \in \mathbb{N}$. Then, for each $x \in Rad(B)$ there exist two sequences $(x_n)_n, (s_n)_n \subseteq Rad(A)$ such that $(s_n)_n \uparrow 1$ and $d(x_n, x) \geq s_n$ for all $n \in \mathbb{N}$.*

Proof. Assume $x \in Rad(B) \subseteq B$. Then, there exist two sequences $(x_n)_n, (s_n)_n \subseteq A$ such that $(s_n)_n \uparrow 1$ and $d(x_n, x) \geq s_n$ for all $n \in \mathbb{N}$. According to Lemma 3.1 we can assume that $s_n \in Rad(A)$ for all $n \in \mathbb{N}$. Then we have:

$$x_n = d(x_n, 1) \geq d(x_n, x) \otimes d(x, 1) \geq x \otimes s_n \in Rad(B).$$

It follows that $x_n \in A \cap Rad(B) = Rad(A)$. □

Theorem 3.16. *A^* is a Cauchy completion of A.*

Proof. It follows from Theorem 3.14 and Proposition 3.15. □

4 Convergence with a fixed regulator in perfect BL-algebras

Definition 4.1. Let A be a BL-algebra and $0 < v \in A$. The sequence $(x_n)_n \subseteq A$ is said to be *v-convergent* to an element $x \in A$ (or x is *v-limit* of $(x_n)_n$) denoted by $x_n \to_v x$, if there is $q \in \mathbb{N}$ such that $d(x_n, x)^p \geq v^q$ for all $p, n \in \mathbb{N}$.

Proposition 4.2. *Let $(x_n)_n, y_n)_n \subseteq A$ such that $x_n \to_v x$ and $y_n \to_v y$. Then:*
(1) $x_n \otimes y_n \to_v x \otimes y$;
(2) $x_n \vee y_n \to_v x \vee y$;
(3) $x_n \wedge y_n \to_v x \wedge y$;
(4) $(x_n \to y_n) \to_v (x \to y)$;
(5) $\bar{x}_n \to_v \bar{x}$;
(6) $a \otimes x_n \to_v a \otimes x$ *for any* $a \in A$.

Proof. By the hypothesis there are $q_1, q_2 \in \mathbb{N}$ such that $d(x_n, x)^p \geq v^{q_1}$ and $d(x_n, x)^p \geq v^{q_2}$ for all $p, n \in \mathbb{N}$.
(1) By Proposition 2.23, for all $p \in \mathbb{N}$ we have:

$$d(x_n \wedge y_n, x \wedge y)^p \geq d(x_n, x)^p \wedge d(y_n, y)^p \wedge v^{q_1} \wedge v^{q_2}.$$

Taking $q = max(q_1, q_2)$ we get $d(x_n \wedge y_n, x \wedge y)^p \geq v^q$ for all $p, n \in \mathbb{N}$, hence

$$x_n \otimes y_n \to_v x \otimes y;$$

$(2), (3), (4)$ follow similarly, applying Proposition 2.23 ;
(5) By Proposition 2.23 we have

$$d(x_n^-, \overline{x}) = d(x_n \to 0, x \to 0) \geq d(x_n, x) * d(0, 0) = d(x_n, x).$$

Thus, $x_n^- \to_v \overline{x}$;
(6) It follows immediately from $d(a * x_n, a * x) \geq d(x_n, x)$. □

Proposition 4.3. *Let A be a BL-algebra and $(x_n)_n \subseteq Rad(A)$, $0 < v \in Rad(A)$. If $x_n \to_v x$, then $x \in Rad(A)$.*

Proof. Since $x_n \to_v x$, there is $q \in \mathbb{N}$ such that $d(x_n, x)^p \geq v^q$ for all $p, n \in \mathbb{N}$.
We have:

$$x = d(x, 1) \geq d(x_n, x) \otimes d(x_n, 1) \geq v^q \otimes x_n.$$

Because $Rad(A)$ is a deductive system of A and $v^q, x_n \in Rad(A)$ it follows that $v^q \otimes x_n \in Rad(A)$ and then $x \in Rad(A)$. □

Proposition 4.4. *Let A be a locally Archimedean BL-algebra. If $(x_n)_n \subseteq Rad(A)$ and $0 < v \in Rad(A)$, then the v-limit of the sequence $(x_n)_n$ is unique.*

Proof. Suppose $x_1, x_2 \in Rad(A)$ such that $x_n \to_v x_1$ and $x_n \to_v x_2$, that is, there exist $q_1, q_2 \in \mathbb{N}$ such that $d(x_n, x)^p \geq v^{q_1}$ and $(d(x_n, x)^p \geq v^{q_2}$ for all $p, n \in \mathbb{N}$. Taking $q = q_1 + q_2$ we have $d(x_1, x_2)^p \geq d(x_1, x_n)^p \otimes d(x_n, x_2)^p \geq v^q$ for all $p \in \mathbb{N}$. Because A is locally Archimedean we obtain $d(x_1, x_2) = 1$, hence $x_1 = x_2$. □

Proposition 4.5. *Let A be a locally Archimedean BL-algebra. If $(x_n)_n, (y_n)_n \subseteq Rad(A)$ and $0 < v \in Rad(A)$ such that $x_n \to_v x$, $y_n \to_v y$ and $x_n \leq y_n$ for all $n \in \mathbb{N}$, then $x \leq y$.*

Proof. Since $x_n \leq y_n$, we have $x_n \to y_n = 1 \to_v 1$. By Proposition 3.2 it follows that $(x_n \to y_n) \to_v (x \to y)$ and by Proposition 3.4 we get $x \to y = 1$. Thus, $x \leq y$. □

Example 4.6. $a)$ The constant sequence $(x, x, x, \ldots)$ v-converges to x for any $0 < v \in A$
$b)$ Consider the BL-algebra in Example 2.21. If $x_n = \frac{n-1}{n}$, then $x_n \to_1 1$;
$c)$ If a BL-algebra A is not locally Archimedean, then the v-limit is not unique for any $0 < v \in A$. Indeed, let's consider the non locally Archimedean BL-algebra A in Example 2.21, the constant sequence $(x_n)_n = (\frac{1}{3})_n$ and $v = \frac{1}{3} \in A$. One can check that $x_n \to_v 1$ and $x_n \to_v \frac{1}{3}$ which means that the v-limit of the sequence $(x_n)_n$ above defined is not unique.

Definition 4.7. Let $0 < v \in A$. The sequence $(x_n)_n$ is said to be *v-fundamental* or *v-Cauchy* sequence if there is $q \in \mathbb{N}$ such that $d(x_n, x_m)^p \geq v^q$ for all $p, m, n \in \mathbb{N}, m \geq n$.
A BL-algebra A is *v-Cauchy complete* if any v-Cauchy sequence is v-convergent.

Proposition 4.8. *Let $0 < v \in A$. If the sequence $(x_n)_n$ is v-convergent in a BL-algebra A, then $(x_n)_n$ is v-Cauchy.*

Proof. Supose that $x_n \to_v x$ and let's consider $p, m, n \in \mathbb{N}, m \geq n$.
We have $d(x_n, x_m)^p \geq d(x_n, x)^p \otimes d(x_m, x)^p$. Since $x_n \to_v x$ and $x_m \to_v x$, there exist $q_1, q_2 \in \mathbb{N}$ such that $d(x_n, x)^p \geq v^{q_1}$ and $d(x_m, x)^p \geq v^{q_2}$ for all $p, m, n \in \mathbb{N}$.
Taking $q = q_1 + q_2$ we get $d(x_n, x_m)^p \geq v^q$ for all $p, m, n \in \mathbb{N}, m \geq n$, hence the sequence $(x_n)_n$ is v-Cauchy. □

Proposition 4.9. *Let $(x_n)_n, (y_n)_n$ be two v-Cauchy sequences in the BL-algebra A. Then, the sequences $(x_n \wedge y_n)_n$, $(x_n \vee y_n)_n$, $(x_n \otimes y_n)_n$, $(x_n \to y_n)_n$, $(\bar{x}_n)$ are v-Cauchy.*

Proof. It follows from Propositions 2.23 and 2.24. □

Definition 4.10. Let $A \hookrightarrow B$ be an embedding of the perfect BL-algebras A and B. We say that B is a *v-Cauchy completion* of A if:
$a)$ B is a v-Cauchy complete BL-algebra ;
$b)$ for each $x \in Rad(B)$ there exists a sequence $(x_n)_n \subseteq Rad(A)$ such that $x_n \to_v x$.

For two Cauchy sequences $(x_n)_n$ and $(y_n)_n$ of a BL-algebra A we define:

$$(x_n)_n \equiv (y_n)_n \ iff \ d(x_n, y_n) \to_v 1.$$

The relation $\equiv$ is an equivalence relation on the set $C(A)$ of all Cauchy sequences of the BL-algebra A.
Let $A^* = C(A)/_{\equiv}$ and $[(x_n)_n]$ the equivalence class of the sequence $(x_n)_n$.

Then A^* is an BL-algebra with respect to the component-wise operations.
If [x] is the class of the constant sequence $(x, x, x, \ldots)$, than the map $x \to [(x)]$ is an embedding of BL-algebras $A \hookrightarrow A^*$.

Theorem 4.11. *[9] A^* is a complete lattice and for any $x \in A^*$ there is $(x_n)_n \subseteq A$ such that $x_n \to_v x$.*

Proposition 4.12. *Let $A \hookrightarrow B$ be an embedding of the perfect BL-algebras A and B and $0 < v \in Rad(A)$. Assume that for each $x \in B$ there exists a sequence $(x_n)_n \subseteq A$ such that $x_n \to_v x$. Then, for each $x \in Rad(B)$ there exists a sequence $(x_n)_n \subseteq Rad(A)$ such that $x_n \to_v x$.*

Proof. Assume $x \in Rad(B) \subseteq B$. Then, there exists a sequence $(x_n)_n \subseteq A$ such that $x_n \to_v x$. Then we have:

$$x_n = d(x_n, 1) \geq d(x_n, x) \otimes d(x, 1) \geq x \otimes v^q \in Rad(B).$$

It follows that $x_n \in A \cap Rad(B) = Rad(A)$. □

Theorem 4.13. *A^* is a v-Cauchy completion of A.*

Proof. It follows from Theorem 4.11 and Proposition 4.12. □

5 Conclusions

The variety of types of convergences in multi-valued logic algebras is still a very actual domain of research. As conclusions of this paper we emphasize two main points:
1. On the technical level, we showed that the study of the connection between the convergences in BL-algebras and those in perfect BL-algebras is quite different than the corresponding connection in the case of MV-algebras. In the case of perfect MV-algebras a crucial result is the categorical equivalence between the category of perfect MV-algebras and the category of abelian ℓ-groups established by A. Di Nola and A. Lettieri ([11]). This equivalence allows us to extend to perfect MV-algebras the results established for abelian ℓ-groups. The lack of this kind of equivalence for BL-algebras makes the study of convergences more difficult in these structures, more precisely, we should develop a different theory of convergences for BL-algebras.
On the other hand, based on the isomorphism between an arbitrary MV-algebra and some subalgebra of a perfect MV-algebra established by L.P. Belluce and A. Di Nola in [4], some results in perfect MV-algebras can be transferred to an arbitrary MV-algebra. In the case of BL-algebras and perfect BL-algebras, the theories of convergences work parallel and the results can not be transferred from one level to the other.
2. From the point of view of importance of convergences in multiple-valued logic algebras, the convergences investigated in this paper seem to be the appropriate ones for the study of continuous Bosbach states on BL-algebras. More precisely, one

can define the s-continuous and v-continuous state on a BL-algebra corresponding to the s-convergence and respectively v-convergence. It is also a very interesting problem to study if there is a way to extend a Bosbach state on the BL-algebra A to an s-continuous or v-continuous Bosbach state on the corresponding Cauchy completion of A.

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