Scalar cardinalities for divisors of a fuzzy cardinality

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Abstract

In this paper, we investigate the value of the scalar cardinality for the finite fuzzy sets, whose fuzzy cardinality is known as a rational part of the cardinality of another finite fuzzy set, which is defined as generalized natural numbers and their division.

Keywords: Cardinality of fuzzy sets, fuzzy algebra, generalized natural numbers, natural language.

1 Introduction

Statements as "near 1000 elements, "almost 130 young men" are imprecise assertions which can be referred to crisp sets as the set of students of a University or to fuzzy sets as the set of young people in a town. The natural language make use of quantitative informations or queries as "Half of the elements of the set A belong to the subset B", "Twenty per cent of young people are very tall", that need a generalization of the classical cardinality theory for sets.

In the subject literature, several approaches to cardinality of fuzzy sets can be found (see [4], [7], [8]), that we shall classify in two groups. For the first group, in [8] we can found the definitions and the properties of the cardinalities as generalized natural numbers (*gnn*), even their arithmetic operations and order. If a *gnn* describes the cardinality of a finite fuzzy set (*ffs*) "A", then $\alpha = \alpha_- \cap \alpha_+$ where α_- is a nonincreasing function $\mathbf{N} \longrightarrow [\mathbf{0}\ \mathbf{1}]$ with a finite number of nonzero values and α_+ is a nondecreasing function $\mathbf{N} \longrightarrow [0\ 1]$ with a finite number of values differents to 1. Moreover, $\alpha(k) = \alpha_-(k)$ if $0 \leq k < z_A$ and $\alpha(k) = \alpha_+(k)$ if $z_A \leq k \leq n$, for a certain and characteristic number z_A. So, in [8] we can see that the calculation of the **division** between *gnn* , if there exists, is not an algorithm, but the solution of an equation given by the definition and the properties of the multiplication.

For the second group, Wygralak ([9]) sets an axiomatic definition for a wide subgroup, the *scalar cardinalities*, and among them there is the Zadeh and the

Dubois cardinalities. Also, he gets the general expression of its values as a a function of the values rangered by the same fuzzy set.

For the generalized natural numbers, the arithmetic operations are defined by means the modified extension principle, which allows the effective calculation of the result in the cases of addition and multiplication, but in the cases of substraction and division it is necessary to solve an equation that can have no solution. In [1] we have seen that if a *gnn* α describes the cardinality of a *ffs* A and $1/p$ is a rational number it is possible to define a *gnn* β, that describes the cardinality of a subset B of A, such that we can consider as a "pth-part" of A as the values of z_B, |SuppA|, and other characteristic numbers show. In [3] we have seen that the found value is the solution, if there exists, of the division's equation.For this solution of a rational part of a fuzzy cardinality, what about the scalar cardinality?. In this work we investigate the answer to this question.

2 Preliminay Remarks

Each fuzzy set in a universe $\mathbf{M}$ is characterized by a function $A : \mathbf{M} \longrightarrow [0\ 1]$, which is called the membership function of that fuzzy set. As usual, the set "suppA" $= \{x \in \mathbf{M} : A(x) \neq 0\}$. The fuzzy sets referred in this paper are *finite*, i.e. "suppA" is a finite crisp set(*fcs*). If C is a finite crisp set, then $|C|$ denote the "cardinal" of C, "the number of elements which belong to C". $B \subset A$, iff $: B(x) \leq A(x), \forall x \in \mathbf{M}$.

Besides, we will use the following definitions(see [3, 5, 8]):

$$A_t = \{x \in \mathbf{M} : A(x) \geq t\}; t \in [0\ 1] \tag{1}$$

$$[A]_i = sup\{t \in [0\ 1] : |A_t| \geq i\}; i \in \mathbf{N}. \tag{2}$$

$$m = |A_1|; n = |suppA| \tag{3}$$

2.1 Lemma

Let A,B, finite fuzzy sets. The following properties hold true:

a) $[A]_i$ is nonincreasing with respect to i.
b) $[A]_i = 1$, for each $i \leq m$
c) $[A]_i = 0$, for each $i > n$
d) $0 < [A]_i < 1$, for each $m < i \leq n$

2.2 Definition

Let A a fuzzy set , the "*cardinality*" of A is defined ([7],[8]) as a fuzzy set $|A|$ in N as follows:

$$|A|(i) = min\{[A]_i, 1 - [A]_{i+1}\} \tag{4}$$

Let us define:

$$z_A = min\{i \in \mathbf{N} : [A]_i + [A]_{i+1} \leq 1\} \tag{5}$$

2.3 Remark

By definition:

$$|A|(i) = 1 - [A]_{i+1} \text{ if } i < z_A \tag{6}$$

$$|A|(i) = [A]_i \text{ if } i \geq z_A \tag{7}$$

2.4 Lemma

$$|A|(i) \text{ is nondecreasing, for } i < z_A \tag{8}$$

$$|A|(i) \text{ is nonincreasing, for } i \geq z_A \tag{9}$$

a) If $i < z_A, \text{then} [A]_i > 0.5, \text{and} [A]_i \leq 0.5$ if $i > z_A$
b) If $i < z_A - 1, \text{then} |A|(i) < 0.5, \text{and} |A|(i) \leq 0.5$ if $i > z_A$

2.5 Lemma

Let A,B, finite fuzzy sets. The following conditions are equivalents:
a) $\forall i \in \mathbf{N}$:

$$|A|(i) = |B|(i) \tag{10}$$

b)$\forall i \in \mathbf{N}$:

$$[A]_i = [B]_i \tag{11}$$

c) $\forall t \in [0\ 1]$:

$$|A_t| = |B_t| \tag{12}$$

and the *ffs* A, B will be called "equipotents"

2.6 Definition([8]):

Let A,B be finite fuzzy sets, if:

$$||A|(i) - |B|(i)| \leq 1 - s, \tag{13}$$

then we will say that the *ffs* A, B are "s - equipotents", which will be denoted: $(A \sim^s B)$

2.7 Remark

If $B \subset A$, then the inequality $|A|(i) \leq |B|(i), \forall i \in \mathbf{N}$ does not fulfill, but:
a) $|A|(i) \leq |B|(i)$, if $i < min\{z_A, z_B\}$
b) $|B|(i) \leq |A|(i)$, if $i > max\{z_A, z_B\}$

2.8 Definition ([8])

A generalized natural number(*gnn*)is a function $\alpha : \mathbf{N} \longrightarrow [\mathbf{0}\ \mathbf{1}]$ such that.

$$\alpha = \alpha_- \cap \alpha_+$$

where $\{\alpha_-(i)\}$ is nonincreasing and $\{\alpha_+(i)\}$ is nondecreasing.

2.9 Definition ([8])

We will say that α *describes the cardinality* of a *ffs* $A : \mathbf{M} \longrightarrow [\mathbf{0}\ \mathbf{1}]$ if, $\forall i \in \mathbf{N}$:

$$\alpha(i) = min\{[A]_i, 1 - [A]_{i+1}\} \tag{14}$$

2.10 Remark

if we represent a function $F : \mathbf{N} \longrightarrow [\mathbf{0}\ \mathbf{1}]$ by $F = (a_0, a_1, \cdots, a_k, (a))$,where "(a)" means $F(i) = a, \forall i > k$, then, if the *gnn* α *describes* the cardinality of a *fss* A, we have:

$$\alpha = (1 - [A]_1, 1 - [A]_2, \cdots, 1 - [A]_z, [A]_z, \cdots, [A]_n, (0)) \tag{15}$$

and,even:

$$\alpha = (0, \cdots, 0, 1 - [A]_{m+1}, \cdots, 1 - [A]_z, [A]_z, \cdots, [A]_n, (0)) \tag{16}$$

2.11 Remark

So, the descomposition of each *gnn* α which describes the cardinality of a *fss* A : $\alpha = \alpha_- \cap \alpha_+$, with:

$$\alpha_- = ([A]_1, [A]_2, \cdots, [A]_z, [A]_{z+1}, \cdots, [A]_n, (0)) \tag{17}$$

$$\alpha_+ = (1 - [A]_1, 1 - [A]_2, \cdots, 1 - [A]_z, 1 - [A]_{z+1}, \cdots, 1 - [A]_n, (1)) \tag{18}$$

and we have , for each couple of *gnn*, $\alpha, \beta : \alpha = \beta$ if and only if $: \alpha_- = \beta_-$ and $\alpha_+ = \beta_+$

2.12 Remark

For each natural number p, also we will call p to the *gnn* that is defined as follows:

$$p(i) = p, \text{ if } i = p, \text{ and } p(i) = 0, \text{otherwise} \tag{19}$$

So, p describes the cardinal of a finite crisp set $C : \mathbf{M} \longrightarrow [\mathbf{0}\ \mathbf{1}]$, such that there exists p elements $\{m_j\}; j = 1, \cdots, p$, with $C(m_j) = 1$ and $C(m) = 0$, if $m \neq m_j, j = 1, \cdots, p$.

In this case, we have:$|suppC| = p; |C_t| = p$ if $0 < t \leq 1$, hence $[C]_i = 1$ if $1 \leq i \leq p$ and $[C]_i = 0$, if $i > p$.

So, we can say that $p(i) = p_-(i) \wedge p_+(i)$, where:

$$p_-(i) = 1 \text{ if } i \leq p, \text{ and } p_-(i) = 0, \text{ if } i > p \tag{20}$$

$$p_+(i) = 0 \text{ if } i < p, \text{ and } p_+(i) = 1, \text{ if } i \geq p \tag{21}$$

2.13 Definition ([8]):Arithmetical operations with generalized natural numbers

Each monotonic arithmetical operation on natural numbers is extended to *gnn* by means of the *modified extension principle*. In particular , the multiplication is defined as follows:

$$\alpha * \beta = (\alpha_- * \beta_-) \cap (\alpha_+ * \beta_+) \tag{22}$$

where:

$$(\alpha_- * \beta_-)(k) = \bigvee (\alpha_-(i) \wedge \beta_-)(j); i * j \geq k) \tag{23}$$

$$(\alpha_+ * \beta_+)(k) = \bigvee (\alpha_+(i) \wedge \beta_+(j); i * j \leq k) \tag{24}$$

2.14 Remark

1) In order to gain a major simplicity we will write $\alpha\beta$ instead $\alpha * \beta$

2) The extension principle is sufficient for the adition

$$\alpha + \beta(k) = sup\{min(\alpha(i), \beta(j)), i + j = k\} \tag{25}$$

whereas, for the multiplication it is not sufficient because it would not be possible the product of prime numbers.

3) For the substraction and the division, only it is possible their definition as the solution, if it exists, of an equation:

2.15 Definition : Division

Let α, β be two *gnn*. If there exists a *gnn* δ , such that. $\alpha = \beta\delta$, then we will say that the "quotient" of α and β exists and it is δ . We shall write $\alpha/\beta = \delta$

2.16 Remark

if α/β exists and α, β are different of the *gnn* 0, then the quotient is unique.

2.17 Definition([9])

A function sc:FFS $\longrightarrow$ $[0 \ \ +\infty]$ is called a *Scalar Cardinality* if the following postulates are satisfied for each $a, b \in [0 \ 1], x, y \in M, A \in$ FFS and each finite family $\{A_e \in \text{FFS}, e \in J\}$;

1.- $a \leq b \Rightarrow \text{sc}(a/x) \leq \text{sc}(b/x)$, where $a/x \in$ FFS, such that: $a/x(x) = a, a/x(z) = 0, z \neq x$.

2.- if$A_e \wedge A_{e'} = T$, when$e \neq e'$, thensc$(\bigcup_{e\in J} A_e) = \sum_{e \in J} \text{sc}(A_e)$.

3.- $A \in \text{FCS} \Rightarrow \text{sc}(A) = |\text{suppA}|$

where "T" is the ffs such that $T(x) = 0; \forall x \in M$

2.18 Proposition [9])

A function sc:FFS $\longrightarrow$ $[0 \ +\infty]$ is a *scalar cardinality* if and only if:

$$\text{sc(A)} = \sum_{x \in \text{suppA}} f(A(x)) \tag{26}$$

For each A $\in$ FFS, where f: $[0 \ 1] \longrightarrow [0 \ 1]$ is a function such that:

i) $f(0) = 0; f(1) = 1$

ii) $a \leq b \Rightarrow f(a) \leq f(b)$

2.19 Definition ([2])

Let A be a fuzzy set. If $p > 0$ is a natural number and B is a fuzzy subset of A, then:

$$|B| = 1/p|A| \tag{27}$$

iff:

$$|B|(i) = |A|(pi), \tag{28}$$

for each $i \in N$

2.20 Proposition([2])

If $z_A/p \in \mathbf{N}$, then $z_B = z_A/p$ and it is possible to define exactly the cardinality of B from the cardinality of A, iff:

$$[A]_{z_A-p+1} = [A]_{z_A}$$

For instance, for $p = 2$:

i	$[A]_i$	$\|A\|(i)$	$\|B\|(i)$	$[B]_i$
0	1	0	0	1
1	1	0.1	0.2	1
2	0.9	0.2	0.4	0.8
3	0.8	0.2	0.6	0.6
4	0.8	0.4	0.3	0.3
5	0.6	0.4	0.2	0.2
6	0.6	0.6	0	0
7	0.4	0.4	0	0
8	0.3	0.3	0	0
9	0.3	0.3	0	0
10	0.2	0.2	0	0
11	0.1	0.1	0	0
12	0	0	0	0

where $p = 2; z_A = 6; z_B = 3|suppA| = 11; |suppB| = 5$

3 The Cardinality $\beta(i) = \alpha(pi)$ as the Solution of the Division α/p

3.1 Proposition

The *gnn* β, defined by $\beta(k) = \alpha(pk)$, is the solution(if there exists) of the division α/p

Proof:

By definitions 2.13 and 2.15, the solution , if there exists, of the equation $\alpha(k) = (px)(k), \forall i \in \mathbf{N}$ is the generalized natural number:

$$x(k) = x_-(k) \wedge x_+(k)$$

where $x_-(k), x_+(k)$ are the solution of the following equations:

$$\begin{aligned} \alpha_-(k) &= (p_-x_-)(k) = \\ &= sup\{min(p_-(i), x_-(j)), ij \geq k\} = \\ &= \bigvee\{(p_-(i) \wedge x_-(j)), ij \geq k\} \end{aligned} \tag{29}$$

$$\begin{aligned}\alpha_+(k) &= (p_+x_+)(k) = \\ &= sup\{min(p_+(i), x_+(j)), ij \leq k\} = \\ &= \bigvee\{(p_+(i) \wedge x_+(j)), ij \leq k\}\end{aligned} \tag{30}$$

3.2 Proposition

The equation "$\alpha_- = p_-x_-$" has a solution if the following conditions is satisfied:

$$\begin{aligned}\alpha_-(0) &\geq \alpha_-(1) = \alpha_-(2) = \cdots = \alpha_-(p) \\ &\geq \alpha_-(p+1) = \alpha_-(p+2) = \cdots = \alpha_-(2p) \\ &\geq \alpha_-(2p+1) = \cdots = \alpha_-(3p) \\ &\geq \alpha_-(3p+1) = \cdots\end{aligned} \tag{31}$$

and this solution is:

$$x_-(i) = \alpha_-(pi) \tag{32}$$

In an analogous way, the equation "$\alpha_+ = p_+x_+$" has a solution if the following condition is satisfied:

$$\begin{aligned}\alpha_+(0) &= \alpha_+(1) = \alpha_+(2) = \cdots = \alpha_+(p-1) \\ &\leq \alpha_+(p) = \alpha_+(p+1) = \cdots = \alpha_+(2p-1) \\ &\leq \alpha_+(2p) = \cdots = \alpha_+(3p-1) \\ &\leq \alpha_+(3p) = \cdots\end{aligned} \tag{33}$$

and this solution is:

$$x_+(i) = \alpha_+(pi) \tag{34}$$

Hence, the complete solution is:

$$\begin{aligned}x(i) &= x_-(i) \wedge x_+(i) = \alpha_-(pi) \wedge \alpha_+(pi) \\ &= \alpha(pi)\end{aligned} \tag{35}$$

3.3 Remark:

In a general case the cardinal α does not satisfy the conditions 31 and 33, but we can define the cardinal β by:

$$\beta(i) = \alpha(pi), i = 0, 1, \cdots \tag{36}$$

and even; for $i = 0, 1, 2, \cdots$:

$$\beta_-(i) = \alpha_-(pi), \beta_-(i) = \alpha_-(pi) \tag{37}$$

3.4 Proposition:

If the cardinal β is defined as in the above remark, then:

$$
\begin{aligned}
&|(p\beta)_-(k) - \alpha_-(k)| \leq \\
&\alpha_-(pE(k/p)) - \alpha_-(p(E(p/k)+1))
\end{aligned}
\tag{38}
$$

$$
\begin{aligned}
|(p\beta)_+(k) - \alpha_+(k)| &\leq \\
\alpha_+(p(E(k/p)+1)) &- \alpha_+(pE(p/k))
\end{aligned}
\tag{39}
$$

Proof:
For each k, $pE(k/p) \leq k \leq p(E(p/k)+1)$, hence:

$$
\begin{aligned}
\alpha_-(p(E(k/p))| &\geq \alpha_-(k) \\
&\geq \alpha_-(p(E(p/k)+1)),
\end{aligned}
\tag{40}
$$

because $\{\alpha_-(j)\}$ is nonincreasing.But the *gnn* $p\beta$ satisfies the condition 31:

$$
(p\beta)_-(k) = (p\beta)_-(pE(k/p)),
$$

from where:

$$
\begin{aligned}
|(p\beta)_-(k) - \alpha_-(k)| &= |(p\beta)_-(pE(k/p)) - \alpha_-(k)| \\
&= \alpha_-(pE(k/p)) - \alpha_-(k) \\
\leq \alpha_-(pE(k/p)) &- \alpha_-(p(E(p/k)+1))
\end{aligned}
\tag{41}
$$

Analogously, for the inequality 39.

3.5 Remark:

By definition 2.6, if we define:

$$
\begin{aligned}
1 - s = \bigvee &\{[\alpha_-(rp) - \alpha_-((r+1)p)] \\
\vee &[\alpha_+((r+1)p) - \alpha_+(rp)]; \\
&r = 0, 1, 2, \cdots\}
\end{aligned}
\tag{42}
$$

then:

$$
\alpha \sim^s (p\beta) \tag{43}
$$

For instance: if we take the example of the definition 2.19, where $p = 2, \alpha_-(i) = [A]_i, \alpha_+(i) = 1 - [A]_{i+1}, \beta_-(i) = \alpha_-(pi), \beta_+(i) = \alpha_+(pi)$:

i	$\alpha_-(i)$	$\alpha_+(i)$	$\beta_-(i)$	$\beta_+(i)$	$p_-\beta_-(i)$	$p_+\beta_+(i)$
0	1	0	1	0	1	0
1	1	0.1	0.9	0.2	0.9	0
2	0.9	0.2	0.8	0.4	0.9	0.2
3	0.8	0.2	0.6	0.6	0.8	0.2
4	0.8	0.4	0.3	0.7	0.8	0.4
5	0.6	0.4	0.2	0.9	0.6	0.4
6	0.6	0.6	0	1	0.6	0.6
7	0.4	0.7	0	1	0.3	0.6
8	0.3	0.7	0	1	0.3	0.7
9	0.3	0.8	0	1	0.2	0.7
10	0.2	0.9	0	1	0.2	0.9
11	0.1	1	0	1	0	0.9
12	0	1	0	1	0	1

and we have:

$\bigvee\{\alpha_-(i)-(p\beta)_-(i)\} = (1-1)\vee(1-0.9)\vee(0.9-0.9)\vee(0.8-0.8)\vee(0.8-0.8)\vee(0.6-0.6)\vee(0.6-0.6)\vee(0.4-0.3)\vee(0.3-0.3)\vee(0.3-0.2)\vee(0.2-0.2)\vee(0.1-0) = 0.1$

In the same way:$\bigvee\{\alpha_+(i)-(p\beta)_+(i)\} = 0.1$, and

$$\alpha \sim^s (p\beta) \tag{44}$$

with $1-s=0.1$

3.6 Proposition

If a *gnn* describes the cardinality of a *ffs* A, the conditions 31 and 33 are satisfied if and only if:

$$[A]_1 = [A]_2 = \cdots = [A]_p \geq [A]_{p+1} = \ldots [A]_{2p} \geq [A]_{2p+1} = \cdots \tag{45}$$

Proof

After the definition 2.9, the conditions 31 and 33 are satisfied if and only if the following equalities are fulfilled:

a)

$$[A]_1 = [A]_2 = \cdots = [A]_p \geq [A]_{p+1} = \cdots = [A]_{2p} \geq [A]_{2p+1} = \cdots \tag{46}$$

b)

$$\begin{aligned} 1-[A]_1 = 1-[A]_2 = \cdots &= 1-[A]_p \leq 1-[A]_{p+1} = \\ 1-[A]_{p+2} = \cdots &= 1-[A]_{2p} \leq 1-[A]_{2p+1} = \cdots \end{aligned} \tag{47}$$

but the above two conditions are really the same condition that we want to prove

3.7 Proposition

If a *gnn* α describes the cardinality of a *ffs* and the division of the *gnn* α and p has a solution, then the quotient is a *gnn* β, that describes the cardinality of some *ffs*.

Proof

The solution is defined in 32 and 34:

$$\beta_-(i) = \alpha_-(pi); \beta_+(i) = \alpha_+(pi) \tag{48}$$

Therefore,

$$\beta_-(i) = [A]_{pi}; \beta_+(i) = 1 - [A]_{pi+1} \tag{49}$$

If β can represent the cardinality of a *ffs* B, then:

$$\beta_-(i) = [B]_i; \beta_+(i) = 1 - [B]_{i+1} \tag{50}$$

So, we would have:

$$\beta_-(i) = [B]_i = [A]_{pi}; \beta_+(i) = 1 - [B]_{i+1} = 1 - [A]_{pi+1} \tag{51}$$

but the relationship $[B]_{i+1} = [A]_{p(i+1)}$ must hold as well. Hence, is necessary and sufficient that:$[A]_{pi+1} = [A]_{pi+p)}$ and this equation holds true by 3.6.

3.8 Proposition

The subset of [0 1] of all the values achieved by $\{[A]_i; i = 1, 2, \cdots, n = |\text{SuppA}|\}$, including its possible repetitions is the same subset of [0 1] of the values $\{A(x), x \in SuppA\}$, including its possible repetitions

Proof

Let $A(x_1) \geq A(x_2) \geq \cdots \geq A(x_{n-1}) \geq A(x_n)$ be the set of values achieved by the ffs A over its support decreasingly ordered.

Let $r = A(x_j)$ and $A(x_{j-k-1}) > A(x_{j-k}) = \cdots = A(x_j) = \cdots = A(x_{j+h}) > A(x_{j+h+1})$ for some numbers k and h.

Then;

$$j + h + 1 > |A_r| = j + h > j - h > j - h - 1 \tag{52}$$

Moreover, if $s > r$, then $s > A(x_{j-k})$ and, consequently, $|A_s| \leq j - k + 1$

So, $r = \max\{t||A_t| \geq i; i = j - k, \cdots, j + h\} = [A]_i; i = j - k, \cdots, j + h$

Conversely, if $r = [A]_i$ for some natural number i, then there exists an element x_j, such that $A(x_j) = r$ because, otherwise, let x_j such that $j = \max\{i|A(x_i \geq r\}|$ we would have $A(x_j > r$, and $|\{x \in \text{suppA}|A(x) \geq A(x_j)\}| = |\{x \in \text{suppA}|A(x) \geq r\}| \geq i$, which is a contradiction with the definition of r.

3.9 Proposition

If β is the quotient of a gnn α that describes the cardinality of a ffs A by a natural number p, then:

$$sc(A) = psc(B) \tag{53}$$

where B is any ffs whose cardinality is described by β and "sc" is any scalar cardinality.

Proof By the proposition 2.18: $\mathrm{sc}(A) = \Sigma_{x\in \mathrm{SuppA}}(f(A(x))$ By the proposition3.8:

$$\Sigma_{x\in \mathrm{SuppA}}(f(A(x)) = \Sigma_{i=1,\cdots,n}(f([A]_i)$$

but,

$$\begin{aligned} \Sigma_{i=1,\cdots,n}(f([A]_i) &= \\ f([A]_1) + f([A]_2) + \cdots + f([A]_p) + f([A]_{p+1}) + \cdots + f([A]_{2p}) + \cdots &= \\ f([A]_p) + f([A]_p) + \cdots + f([A]_p) + f([A]_{2p}) + \cdots + f([A]_{2p}) + \cdots &= \\ p(f([A]_p)) + p(f([A]_{2p})) + \cdots & \end{aligned} \tag{54}$$

Finally, if Propositions 3.8 and 2.18 are applied :

$$\begin{aligned} p(f([A]_p)) + p(f([A]_{2p})) + \cdots &= \\ p(f([B]_1)) + p(f([B]_2)) + \cdots = p\Sigma_{i=1,\cdots,|\mathrm{SuppB}|}(f([B]_i) &= \\ p\Sigma_{x\in \mathrm{SuppB}}(f(B(x)) = \mathrm{p\ sc(B)} & \end{aligned} \tag{55}$$

4 Examples

Let A be a *ffs* defined on SuppA= $x_1, \ldots, x_8$ with the values: $(0.5, 1, 0.3, 0.5, 0.8, 1, 0.8, 0.3)$ and we suppose that $p = 2$

i	$[A]_i$	$1-[A]_{i+1}$	$\lvert A\rvert(i)$	$\beta_-(i)$	$\beta_+(i)$
0	1	0	0	1	0
1	1	0	0	1	0.2
2	1	0.2	0.2	0.8	0.5
3	0.8	0.2	0.2	0.5	0.7
4	0.8	0.5	0.5	0.3	1
5	0.5	0.5	0.5	0	1
6	0.5	0.7	0.5	0	1
7	0.3	0.7	0.3	0	1
8	0.3	1	0.3	0	1

and the *gnn* $\beta(i) = min\{\beta_-(i), \beta_+(i)\}$ describes the cardinality $|B|(i)$ of a *ffs* whose values which are diferent of 0 are: $(0.3, 0.5, 0.8, 1)$

On the other hand, the set Supp(B) will be a subset of Supp(A)= $x_1, \ldots, x_8$ of cardinal equal to 4

1) Let f be the mapping defined as follows:

$$f(a) = 0 \text{ if } a < 1, f(1) = 1$$

Then we can set:

$$\mathrm{sc}(A) = \Sigma_{x \in \mathrm{SuppA}} f(A(x)) = |\{x \in \mathrm{SuppA}; A(x) = 1\}| = 2$$

$$\mathrm{sc}(B) = \Sigma_{x \in \mathrm{SuppA}} f(B(x)) = |\{x \in \mathrm{SuppA}; B(x) = 1\}| = 1$$

2) If $f(a) = 1$ if $a > 0, f(0) = 0$, then:

$$\mathrm{sc}(A) = |\mathrm{SuppA}| = 8$$

$$\mathrm{sc}(B) = |\mathrm{SuppB}| = 4$$

3) If $f(a) = a^k; a \in [0\ 1], k \in N$, we have:
If $k = 1$:

$$\mathrm{sc}(A) = \Sigma_{x \in \mathrm{SuppA}} A(x) = 5.2$$

$$\mathrm{sc}(B) = \Sigma_{x \in \mathrm{SuppB}} B(x) = 2.6$$

If $k = 2$:

$$\mathrm{sc}(A) = \Sigma_{x \in \mathrm{SuppA}} (A(x))^2 = 3.96$$

$$\mathrm{sc}(B) = \Sigma_{x \in \mathrm{SuppB}} (B(x))^2 = 1.98$$

In the following cases we make use of functions that satisfy the conditions of the proposition() and, besides, they are defined over a partition of the interval [0 1] given by any value $t \in [0\ 1]$. For each value, t, we would have a function f_t, from where we can calculate $\mathrm{sc}_t(A)$ and $\mathrm{sc}_t(B)$. In each case we will find $f_t, \mathrm{sc}_t(A), \mathrm{sc}_t(B)$ when t ranges the values $\{0.3, 0.5, 0.8, 1\}$ that are the range of the fuzzy set A. We will express this result by:

$$\mathrm{sc}(A) = \mathrm{sc}_{0.3}(A), \mathrm{sc}_{0.5}(A), \mathrm{sc}_{0.8}(A), \mathrm{sc}_1(A))$$

$$\mathrm{sc}(B) = \mathrm{sc}_{0.3}(B), \mathrm{sc}_{0.5}(B), \mathrm{sc}_{0.8}(B), \mathrm{sc}_1(B))$$

4) For the function $f(a) = 1$ if $a \geq t, f(a) = 0$, if $a < t$, we have:

$$\mathrm{sc}(A) = |A_t| = (8, 6, 4, 2)$$

$$\mathrm{sc}(B) = |B_t| = (4, 3, 2, 1)$$

5) If $f(a)=1$ if $a>t, f(a)=0$, if $a \leq t$, we have:

$$\mathrm{sc}(A)=|A_{>t}|=(6,4,2,0)$$

$$\mathrm{sc}(B)=|B_{>t}|=(3,2,1,0)$$

6) If $f(a)=1$ if $a \geq t, f(a)=a^k$, if $a<t$, we have:
for $k=1$:

$$\mathrm{sc}(A)=|A_t|+\sum_{A(x)<t} A(x)=(8,6.6,5.6,5.2)$$

$$\mathrm{sc}(B)=|B_t|+\sum_{B(x)<t} B(x)=(4,3.3,2.8,2.6)$$

for $k=2$:

$$\mathrm{sc}(A)=|A_t|+\sum_{A(x)<t} (A(x))^2=(8,6.18,4.66,3.96)$$

$$\mathrm{sc}(B)=|B_t|+\sum_{B(x)<t} (B(x))^2==(4,3.09,2.33,1.98)$$

7) If $f(a)=0$ if $a \leq t, f(a)=a^k$, if $a>t$, we have:
for $k=1$:

$$\mathrm{sc}(A)=\sum_{A(x)>t} A(x)=(4.6,3.6,2,0)$$

$$\mathrm{sc}(B)=\sum_{B(x)>t} B(x)=(2.3,1.8,1,0)$$

for $k=2$:

$$\mathrm{sc}(A)=\sum_{A(x)>t} (A(x))^2=(3.78,3.28,2,0)$$

$$\mathrm{sc}(B)=\sum_{B(x)<t} (B(x))^2=(1.89,1.64,1,0)$$

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