Mathware & Soft Computing 5 (1998) 201-211

ε-Partitions and α-Equivalences

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Abstract

The aim of this paper is to study a special type of fuzzy relations, the α-equivalences, as well as to consider the relation that connects these with the family of ε-partitions of the referential. Some classic equivalences between set, partitions and fuzzy relations are also studied.
Keywords: ε-partition, Hamming's distance, α-equivalence and ε-equality.

1 Introduction

Since Zadeh [22] introduced the definition of fuzzy set, numerous authors [6], [16], [19], ..., have studied the fuzzy relations, having awoken special interest the study of the fuzzy binary relations.

A very important type of crisp binary relations are the relations of equivalence R, that is, the subsets of the cartesian product of the crisp set X with himself, formed by the elements

$$\{(x, y) \in X \times X / xRy\}$$

that verify the following properties:
- Reflexive: $xRx; \forall x \in X$.
- Symmetrical: $xRy \Longleftrightarrow yRx; \forall x, y \in X$.
- Transitive: If xRy and yRz then $xRz; \forall x, y, z \in X$.

Various extensions of these properties [9] have been given in the theory of fuzzy sets, and consequently there are different extensions of the relations of equivalence [7], [19], [23], etc.

As in the crisp case, every equivalence relation remains thoroughly characterized by the partition that generates, and conversely, in the fuzzy case it has also been attempted to match the binary relations of "equivalence" and the fuzzy partitions [1], [3], [4], [5], [11], [13], [16], [21], etc. In most cases the resulting partition is given by Ruspini definition[18], or some modification of it [2].

The present work defines and studies the α-equivalences, a type of fuzzy binary relations that will generate and characterize in certain sense the ε-partitions [14], fuzzy partitions that extend the crisp partitions and collect the idea of "covering" of the total by "two by two disjoint" subsets , and the fuzziness that exists in the family of sets that form the partition.

2 Preliminary

2.1 fuzzy ε-partitions

Definition 2.1 *Let $\widetilde{A}$ be a fuzzy set on a referential X . It is said that the family $\{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\}$ of fuzzy subsets of $\widetilde{A}$ is a fuzzy ε-partition of $\widetilde{A}$, with ε in$[0,1]$, if it verifies that:*

$$(\widetilde{A}_i \cap \widetilde{A}_j)_\alpha = \emptyset, \forall i, j = 1, 2, \ldots, n, \text{ con } i \neq j, \forall \alpha > \varepsilon$$
$$(\bigcup_{i=1}^{n} \widetilde{A}_i)_\alpha = (\widetilde{A})_\alpha, \forall \alpha \in (\varepsilon, 1-\varepsilon)$$

where $(\widetilde{D})_\alpha$ is the α-cut of the set $\widetilde{D}$ according to the traditional definition of this concept [8].

Therefore, the previous conditions can be written as:

$$\{x \in X / \widetilde{A}_i \cap \widetilde{A}_j(x) \geq \alpha\} = \emptyset \qquad \forall i, j = 1, 2, \ldots, n, \text{ con } i \neq j, \forall \alpha > \varepsilon$$
$$\{x \in X / \cup_{i=1}^{n} \widetilde{A}_i(x) \geq \alpha\} = \{x \in X / \widetilde{A}(x) \geq \alpha\} \qquad \forall \alpha \in (\varepsilon, 1-\varepsilon)$$

When we considered the classic operations of union and intersection [12] (that is maximum and minimum of the membership functions), it is immediate that such conditions lead the following, more illustrative of the importance of the value of ε in this definition of fuzzy partition:

$$(\widetilde{A}_i \cap \widetilde{A}_j)(x) \leq \varepsilon, \forall i, j = 1, 2, \ldots, n, i \neq j \quad \forall x \in X$$
$$\begin{cases} \cup_{i=1}^{n} \widetilde{A}_i(x) \geq 1-\varepsilon & \forall x \in X / \widetilde{A}(x) \geq 1-\varepsilon \\ \cup_{i=1}^{n} \widetilde{A}_i(x) = \widetilde{A}(x) & \forall x \in X / \varepsilon < \widetilde{A}(x) < 1-\varepsilon \\ \cup_{i=1}^{n} \widetilde{A}_i(x) \leq \varepsilon & \forall x \in X / \widetilde{A}(x) \leq \varepsilon \end{cases}$$

Throughout this paper, it will be used the classic union and intersection operations previously considered, as well as the classic inclusion of fuzzy sets defined by:

$$\widetilde{A} \subset \widetilde{B} \Longleftrightarrow \mu_{\widetilde{A}}(x) \leq \mu_{\widetilde{B}}(x), \forall x \in X$$

In general it will be also considered that ε is a number lower than 0.5, to avoid problems as those of the definition of Ruspini, in which a set repeated n times can be a partition of other set totally different from it.

On working with finite referentials, the proximity between two partitions can be measured by means of the distance of Hamming [10].

Definition 2.2 *[14] Let* $\Pi_{\widetilde{A}} = \{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\}$ *and* $\Pi_{\widetilde{B}} = \{\widetilde{B}_1, \widetilde{B}_2, \ldots, \widetilde{B}_n\}$ *be two partitions formed by the same number of elements, within a finite referential* X*. The distance between* $\Pi_{\widetilde{A}}$ *and* $\Pi_{\widetilde{B}}$ *is defined as:*

$$d(\Pi_{\widetilde{A}}, \Pi_{\widetilde{B}}) = \min_{\sigma \in S_n} \sum_{i=1}^{n} d_H(\widetilde{A}_i, \widetilde{B}_{\sigma(i)}) = \min_{\sigma \in S_n} \sum_{i=1}^{n} \sum_{x \in X} |\mu_{\widetilde{A}_i}(x) - \mu_{\widetilde{B}_{\sigma(i)}}(x)|$$

where S_n *denotes the set of all permutations of the set* $\{1, 2, \ldots, n\}$ *and* d_H *the distance of Hamming between two fuzzy subsets.*

It can be proved in a simple way that the application d defined in 2.2 verifies the three axioms that define a distance, therefore it receives correctly this name.

2.2 Binary fuzzy relations

As it has already been commented in the introduction, the classic properties (reflexive, symmetrical and transitive) are studied in the fuzzy case leading to different extensions. Some of these extensions are formulated below:
A relation binary $\widetilde{R} \subset X \times X$ is said:
- Reflexive: If $x\widetilde{R}x = 1, \forall x \in X$
- Reflexive of order δ: If $x\widetilde{R}x = \delta, \forall x \in X$
- α-Reflexive: If $x\widetilde{R}x \geq \alpha, \forall x \in X$
- Strongly reflexive: If $x\widetilde{R}x \geq \max_y\{x\widetilde{R}y\}, \forall x \in X$
- Absolutely reflexive: If $x\widetilde{R}x > 0, \forall x \in X$
- Symmetrical: $x\widetilde{R}y = y\widetilde{R}x, \forall x, y \in X$
- Transitive: $x\widetilde{R}z \geq \max_y\{\min\{x\widetilde{R}y, y\widetilde{R}z\}\}$
- MP-Transitive: $x\widetilde{R}z \geq \max_y\{x\widetilde{R}y \cdot y\widetilde{R}z\}$
- Luk-Transitive: $x\widetilde{R}z \geq \max\{x\widetilde{R}y + y\widetilde{R}z - 1, 0\}, \forall y \in X$
- T-Transitive: $x\widetilde{R}z \geq \max\{x\widetilde{R}y + y\widetilde{R}z - 1, 0\}, \forall y \in X$

where $x\widetilde{R}y$ denotes the membership function of $\widetilde{R}$ in the point (x, y), that is,

$$x\widetilde{R}y \equiv \widetilde{R}(x, y), \forall x, y \in X$$

One of the most common forms of extending the relations of crisp equivalence is through the similarity relations.

Definition 2.3 *Let* X *be a referential and* $\widetilde{R}$ *a fuzzy subset of* $X \times X$*.* $\widetilde{R}$ *is a relation of similarity if* $\widetilde{R}$ *is reflexive, symmetrical and transitive.*

Definition 2.4 *Let* $\widetilde{R}$ *be a fuzzy relation, the* λ*-cut of* $\widetilde{R}$ *is defined as the crisp relation* $\widetilde{R}_\lambda$ *given by*

$$\widetilde{R}_\lambda = \{(x, y) \in X \times Y / x\widetilde{R}y \geq \lambda\}$$

As consequence of the following proposition, the relations of similarity remain characterized by theirs α-cuts.

Proposition 2.5 *Let* X *be a referential and let* $\widetilde{R}$ *be a fuzzy subset of* $X \times X$*.* $\widetilde{R}$ *is a relation of similarity if and only if* $\widetilde{R}_\lambda$ *is a crisp equivalence for all* λ *in* $[0, 1]$*.*

3 α-Equivalences

Definition 3.1 *Let X be a referential, and let $\widetilde{R}$ be a fuzzy subset of the cartesian product of $X \times X$. $\widetilde{R}$ is a fuzzy relation of α-equivalence, with $\alpha \in [0,1]$, if*

$$\widetilde{R} \text{ is } \alpha\text{-reflexive:[21] } x\widetilde{R}x \geq \alpha, \forall x \in X \tag{1}$$

$$\widetilde{R} \text{ is symmetrical: } x\widetilde{R}y = y\widetilde{R}x, \forall x, y \in X \tag{2}$$

$$\widetilde{R} \text{ is } \alpha\text{-transitive: If } x\widetilde{R}y \geq \alpha \text{ and } y\widetilde{R}z \geq \alpha \Longrightarrow x\widetilde{R}z \geq \alpha \text{ with } x, y, z \in X \tag{3}$$

Proposition 3.2 *If $\widetilde{R}$ is a relation of fuzzy α-equivalence, then its α-cut is a relation of crisp equivalence.*

The proof is immediate from the definition of α-equivalence. However, the reciprocal with this definition of α-equivalence is not certain, but, it would be verified modifying the condition 2, by the following

$$\widetilde{R} \text{ is } \alpha\text{-symmetrical: } x\widetilde{R}y \geq \alpha \Longleftrightarrow y\widetilde{R}x \geq \alpha, \forall x, y \in X \tag{4}$$

Definition 3.3 *Let X be a referential, and $\widetilde{R}$ a fuzzy subset of the cartesian product of $X \times X$. $\widetilde{R}$ is a fuzzy relation of weak α-equivalence, with $\alpha \in [0,1]$, if $\widetilde{R}$ is α-reflexive, α-symmetrical and α-transitive.*

In all this paper α-equivalences have been considered instead of weak α-equivalences, but all the development would be made in a totally analogous way for the latter.

In the following proposition, these two definitions are related to the usually used for similarity.

Proposition 3.4 *Let X be a referential, and $\widetilde{R}$ a fuzzy subset of the cartesian product of $X \times X$. Then:*
$\widetilde{R}$ *similarity* $\Longrightarrow$ $\widetilde{R}$ *α-equivalence* $\Longrightarrow$ $\widetilde{R}$ *weak α-equivalence*

The prove of this proposition is immediate, considering the characterizations of these relations through their α-cuts and the properties of these.

The concept of "class of equivalence" that is going to be studied below, it will be very important in the paragraph 5, in which the classes of α-equivalence will be related to the ε-partitions.

Definition 3.5 *Let X be a referential and $\widetilde{R}$ a fuzzy relation of α-equivalence on it. The class of α-equivalence of any element $a \in X$ is defined as the fuzzy set $\widetilde{[a]}$ defined by*

$$\widetilde{[a]}(x) = \max_{\{b \in X / a\widetilde{R}b \geq \alpha\}} (b\widetilde{R}x)$$

4 ε-Equalities

Below we define "almost" equal subsets, partitions and fuzzy relations. These definitions are necessary for the following paragraph.

Definition 4.1 *Let X be the referential and $\widetilde{A}$ and $\widetilde{B}$ fuzzy subsets in X. It is said that:*
1) $\widetilde{A}$ is ε-equal to $\widetilde{B}$, and it will be denoted by $\widetilde{A} \equiv_\varepsilon \widetilde{B}$, if and only if

$$(\widetilde{A})_\delta = (\widetilde{B})_\delta, \forall \delta > \varepsilon \tag{5}$$

2) $\widetilde{A}$ is weak ε-equal to $\widetilde{B}$, and it will be denoted by $\widetilde{A} \equiv_{\underline{\varepsilon}} \widetilde{B}$, if and only if

$$(\widetilde{A})_\delta = (\widetilde{B})_\delta, \forall \delta \in (\varepsilon, 1-\varepsilon) \tag{6}$$

Therefore it will be said that two fuzzy subsets are ε-equal if they are differentiated only in the points that "almost do not belong" to them. In the case in which a certain freedom (ε) is allowed in the points that "almost belong" to the sets as well as in those which "almost do not belong" , then it will be said to exit a weak ε-equality.

Definition 4.2 *Let $\widetilde{A}$ and $\widetilde{B}$ be two fuzzy subsets of the referential X. They are ε-different (weak ε-different), and is denoted $\not\equiv_\varepsilon$ ($\not\equiv_{\underline{\varepsilon}}$), if they do not verify the condition 5 (6).*

Proposition 4.3 *Let X be a referential and let $\widetilde{P}(X)$ be the set formed by all the fuzzy subsets of X. Then both $\equiv_\varepsilon$ and $\equiv_{\underline{\varepsilon}}$ are relations of classic equivalence on $\widetilde{P}(X)$, that is, verify the reflexive, symmetrical and transitive properties.*

The proof is because the equality between crisp sets is a equivalence relation.

From the equality between sets, it can be defined an equality between partitions.

Definition 4.4 *Let X be the referential and let Π and Π' be fuzzy partitions in X. It is said that:*
1) Π is ε-content in Π', and it will be denoted by $\Pi \subseteq_\varepsilon \Pi'$, if and only if

$$\forall \widetilde{A}_i \in P \text{ with } \widetilde{A}_i \not\equiv_\varepsilon \emptyset, \exists \widetilde{B}_j \in \Pi' / \widetilde{A}_i \equiv_\varepsilon \widetilde{B}_j$$

2) Π is weak ε-content in Π', and it will be denoted by $\Pi \subseteq_{\underline{\varepsilon}} \Pi'$ if and only if

$$\forall \widetilde{A}_i \in P \text{ with } \widetilde{A}_i \not\equiv_{\underline{\varepsilon}} \emptyset, \exists \widetilde{B}_j \in \Pi' / \widetilde{A}_i \equiv_{\underline{\varepsilon}} \widetilde{B}_j$$

Definition 4.5 *Let X be the referential and let Π and Π' be fuzzy partitions of it. It is said that:*
1) Π is ε-equal to Π', and it will be denoted by $\Pi \simeq_\varepsilon \Pi'$, if and only if

$$\Pi \subseteq_\varepsilon \Pi' \text{ y } \Pi' \subseteq_\varepsilon \Pi$$

2) Π is weak ε-equal to Π', and it will be denoted by $\Pi \simeq_{\underline{\varepsilon}} \Pi'$, if and only if

$$\Pi \subseteq_{\underline{\varepsilon}} \Pi' \text{ y } \Pi' \subseteq_{\underline{\varepsilon}} \Pi$$

From the proposition 4.3 formulated for sets, a similar result for the partitions can be established.

Corollary 4.6 *The relations $\simeq_\varepsilon$ and $\simeq_{\underline{\varepsilon}}$ defined in 4.5 verify the reflexive, symmetrical and transitive properties, therefore they are a true relation of equivalence in the classic sense, on the set of fuzzy partitions of the referential.*

Finally, since a binary relation is a fuzzy set, from definition 4.1 it follows:

Definition 4.7 *Let X be the referential and let $\widetilde{R}$ and $\widetilde{R}'$ be fuzzy binary relations on X. It is said that:*
1) $\widetilde{R}$ is ε-equal to $\widetilde{R}'$, and it will be denoted by $\widetilde{R} \equiv_\varepsilon \widetilde{R}'$, if and only if

$$(\widetilde{R})_\delta = (\widetilde{R}')_\delta, \forall \delta > \varepsilon$$

2) $\widetilde{R}$ is weak ε-equal to $\widetilde{R}'$, and it will be denoted by $\widetilde{R} \equiv_{\underline{\varepsilon}} \widetilde{R}'$, if and only if

$$(\widetilde{R})_\delta = (\widetilde{R}')_\delta, \forall \delta \in (\varepsilon, 1-\varepsilon)$$

that is, that $\{a, b \in X / a\widetilde{R}b \geq \alpha\} = \{a, b \in X / a\widetilde{R}'b \geq \alpha\}$ for any $\delta > \varepsilon$ and $\delta \in (\varepsilon, 1-\varepsilon)$ respectively.

As it happened with sets and fuzzy partitions:

Corollary 4.8 *The relations $\equiv_\varepsilon$ and $\equiv_{\underline{\varepsilon}}$ defined in 4.7 are relations of equivalence in the classic sense, on the set of fuzzy binary relations of the referential.*

5 α-Equivalences and ε-Partitions

In the following result the α-equivalences are related with the ε-partitions of referential.

Theorem 5.1 *Let X be a referential, and let $\{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\}$ be a ε-partition of this set. The fuzzy subset $\widetilde{R} \subseteq X \times X$ defined by*

$$x\widetilde{R}y = \max_{\widetilde{A}_i}\{\min\{\widetilde{A}_i(x), \widetilde{A}_i(y)\}\}$$

is a $(1-\varepsilon)$-equivalence.

Proof Let x, y and z be three elements in X, then:
i) $(1-\varepsilon)$-reflexive: $x\widetilde{R}x = \max_{\widetilde{A}_i}\{\widetilde{A}_i(x)\} = \cup_{i=1}^n \widetilde{A}_i(x) \geq 1-\varepsilon$.
ii) Symmetrical: As $\min\{\widetilde{A}_i(x), \widetilde{A}_i(y)\} = \min\{\widetilde{A}_i(y), \widetilde{A}_i(x)\}$ then $x\widetilde{R}y = y\widetilde{R}x$.
iii) $(1-\varepsilon)$-transitive: If $x\widetilde{R}y$ and $y\widetilde{R}z \geq 1-\varepsilon$ then $\exists \widetilde{A}_{i_0}$ and $\widetilde{A}_{i_1}$ such that $\min\{\widetilde{A}_{i_0}(x), \widetilde{A}_{i_0}(y)\} \geq 1-\varepsilon$ and $\min\{\widetilde{A}_{i_1}(y), \widetilde{A}_{i_1}(z)\} \geq 1-\varepsilon \Longrightarrow \widetilde{A}_{i_0}(x), \widetilde{A}_{i_0}(y)$, $\widetilde{A}_{i_1}(x), \widetilde{A}_{i_1}(z) \geq 1-\varepsilon$. But $i_0 = i_1$ because if $i_0 \neq i_1$ then $\widetilde{A}_{i_0} \cap \widetilde{A}_{i_1}(y) \geq 1-\varepsilon$ and

this is not possible. So $\widetilde{A}_{i_0}(x), \widetilde{A}_{i_0}(z) \geq 1-\varepsilon \Longrightarrow$
$x\widetilde{R}z = \max_{\widetilde{A}_i}\{\min\{\widetilde{A}_i(x), \widetilde{A}_i(z)\}\ \} \geq 1-\varepsilon.$ ■

Since this theorem relates the $(1-\varepsilon)$-equivalences with the ε-partitions, as it happened with the partitions and crisp relations of equivalence, it can be wondered that will happen to the crisp relation that it generates $\widetilde{R}$ through its $(1-\varepsilon)$-cut. The solution to this question is given in the following proposition:

Proposition 5.2 *Let X be a referential finite, and let $\Pi = \{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\}$ be a ε-partition of this set. If it is considered the relation of $(1-\varepsilon)$-equivalence obtained from Π in the previous theorem, and it is obtained the relation from crisp equivalence $\widetilde{R}_{1-\varepsilon}$, the crisp partition formed by the classes of equivalence of $\widetilde{R}_{1-\varepsilon}$ is the crisp partition closest to Π , measuring this proximity through the distance of Hamming 2.2.*

In the proof we have to prove that $d(\Pi, \Pi_{\widetilde{R}_{1-\varepsilon}})$ is lower or equal than $n|X|\varepsilon$ and lower or equal than $d(\Pi, \Pi^{'})$, being $\Pi^{'}$ any other crisp partition of X.

If an element $\widetilde{A}_{i_0}$ that take the value ε in all element x of the referential is added to the fuzzy partition, the new family continues being a ε-partition, but the resulting relation is more restrictive.

Proposition 5.3 *Let X be a referential, and let $\Pi = \{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\}$ be a ε-partition of this set that verifies that*

$$\exists i_0 \in \{1, 2, \ldots, n\} / \widetilde{A}_{i_0}(x) = \varepsilon, \forall x \in X \tag{7}$$

Then the relation of $(1-\varepsilon)$-equivalence defined in 5.1 is furthermore transitive.

When besides the condition of the proposition 5.3 it is verified that $\bigcup_{i=1}^{n} \widetilde{A}_i = X$, then the relation of $(1-\varepsilon)$-equivalence that generates is a similarity.

In this paragraph it has been studied as a ε-partition generates a relation of $(1-\varepsilon)$-equivalence. Reciprocally, if we have a α-equivalence relation, a fuzzy partition can be generated .

Proposition 5.4 *Let X be a referential and $\widetilde{R}$ a relation of α-equivalence defined on it, verifying the condition*

$$x\widetilde{R}y \geq \alpha \ \text{ó}\ x\widetilde{R}y \leq 1-\alpha, \forall x, y \in X, \alpha > 0.5 \tag{8}$$

then the subsets family of X formed by the different classes of α-equivalence is a $(1-\alpha)$-partition of X .

Proof
i) $\left(\cup_{i=1}^{m} \widetilde{[a_i]}\right)_\delta = (X)_\delta = X$ for all $\delta \in (1-\alpha, \alpha)$.

An inclusion is evident, so we are going to see the other: let a_i be an element in $X \Longrightarrow \widetilde{[a_i]}(a_i) \geq \alpha \Longrightarrow \cup_{i=1}^m \widetilde{[a_i]}(a_i) \geq \widetilde{[a_i]}(a_i) \geq \alpha$.
ii) $(\widetilde{[a_i]} \cap \widetilde{[a_j]})_{1-\alpha} = \emptyset \Longleftrightarrow \widetilde{[a_i]} \cap \widetilde{[a_j]}(x) \leq 1-\alpha, \forall x \in X$, we are going to study this intersection set:

$\widetilde{[a_i]} \cap \widetilde{[a_j]}(x) = \min\{\widetilde{[a_i]}(x), \widetilde{[a_j]}(x)\} = \min\{b_0 \widetilde{R} x, b_1 \widetilde{R} x\}$ with $b_0 \widetilde{R} a_i, b_1 \widetilde{R} a_j \geq \alpha$. If $\widetilde{[a_i]} \cap \widetilde{[a_j]}(x) > 1-\alpha \Longrightarrow b_0 \widetilde{R} x, b_1 \widetilde{R} x > 1-\alpha \Longrightarrow b_0 \widetilde{R} x, b_1 \widetilde{R} x \geq \alpha$, as $\widetilde{R}$ is α-transitive and symmetrical $\Longrightarrow b_0 \widetilde{R} b_1 \geq \alpha$. With this, we have that $\{b \in X / b\widetilde{R} a_i \geq \alpha\} = \{b \in X / b\widetilde{R} a_j \geq \alpha\}$, and so, $\widetilde{[a_i]}(x) = \max_{\{b \in X / b\widetilde{R} a_i \geq \alpha\}}(b\widetilde{R}x) = \max_{\{b \in X / b\widetilde{R} a_j \geq \alpha\}}(b\widetilde{R}x) = \widetilde{[a_j]}(x) \Longrightarrow \widetilde{[a_i]} = \widetilde{[a_j]}$ and this is a contradiction, so $b_0 \widetilde{R} x \leq 1-\alpha$ or $b_1 \widetilde{R} x \leq 1-\alpha \Longrightarrow \min\{b_0 \widetilde{R} x, b_1 \widetilde{R} x\} = \widetilde{[a_i]} \cap \widetilde{[a_j]}(x) \leq 1-\alpha$. ■

As consequence of the theorem 5.1 and the proposition 5.4, together with the definitions of ε-equal partitions and relations given in the paragraph 4, the following theorem is obtained.

Theorem 5.5 *Let X be a referential, the set $P = \{$ set of the ε-partitions of $X\}$ and the set $R = \{$ set of the α-equivalences on X that verify the condition 8$\}$. Under these hypothesis, the application*

$$\begin{array}{rcl} f : P & \longrightarrow & R \\ \Pi & \longrightarrow & \widetilde{R}_\Pi \end{array}$$

where $\widetilde{R}_\Pi$ is the fuzzy binary relation defined in the theorem 5.1 is inyective.

Proof The first is to prove that f is actually an application and after to prove that furthermore it is inyective.
• Application: If $\Pi = \{\widetilde{A}_1, \widetilde{A}_2, \ldots, \widetilde{A}_n\} \simeq_\varepsilon \Pi' = \{\widetilde{B}_1, \widetilde{B}_2, \ldots, \widetilde{B}_m\} \Longrightarrow f(\Pi) = \widetilde{R}_\Pi \equiv_\varepsilon \widetilde{R}_{\Pi'} = f(\Pi')$?, that is, $(\widetilde{R}_\Pi)_\delta = (\widetilde{R}_{\Pi'})_\delta, \forall \delta > \varepsilon$?

Let δ be greater than ε, then if $(x,y) \in (\widetilde{R}_\Pi)_\delta \Longrightarrow x\widetilde{R}_\Pi y = \max_{\widetilde{A}_i}\{\min\{\widetilde{A}_i(x), \widetilde{A}_i(y)\}\} \geq \delta \Longrightarrow \exists i_o / \max_{\widetilde{A}_i}\{\min\{\widetilde{A}_i(x), \widetilde{A}_i(y)\}\} = \min\{\widetilde{A}_{i_o}(x), \widetilde{A}_{i_o}(y)\} \geq \delta > \varepsilon$, as $(\widetilde{A}_i)_\delta = (\widetilde{B}_j)_\delta$ because $\Pi \simeq_\varepsilon \Pi'$, then $\exists j_o / B_{j_o}(x), B_{j_o}(y) \geq \delta \Longrightarrow x\widetilde{R}_{\Pi'} y = \max_{\widetilde{B}_i}\{\min\{\widetilde{B}_i(x), \widetilde{B}_i(y)\}\} \geq \min\{\widetilde{B}_{j_o}(x), \widetilde{B}_{j_o}(y)\} \geq \delta \Longrightarrow x, y \in (\widetilde{R}_{\Pi'})_\delta \Longrightarrow (\widetilde{R}_\Pi)_\delta \subset (\widetilde{R}_{\Pi'})_\delta$. The other content is proved symmetrically.
• Inyective: If $f(\Pi) = \widetilde{R}_\Pi \equiv_\varepsilon \widetilde{R}_{\Pi'} = f(\Pi')$ then $\Pi \simeq_\varepsilon \Pi'$?

Let $\widetilde{A}_i \in \Pi$ be with $\widetilde{A}_i \not\equiv_\varepsilon \emptyset$ then $\forall x \in (\widetilde{A}_i)_\delta$ with $\delta > \varepsilon$, as $x\widetilde{R}_\Pi x = \max_{\widetilde{A}_k}\{\widetilde{A}_k(x)\} = \widetilde{A}_i(x)$ because Π is a ε-partition, so, there are only a subset with membership function greater than ε in x (because the intersection is "empty"). As $\widetilde{R}_\Pi \equiv_\varepsilon \widetilde{R}_{\Pi'} \Longrightarrow x\widetilde{R}_\Pi x = x\widetilde{R}_{\Pi'} x = \widetilde{B}_j(x) > \varepsilon \Longrightarrow x \in (\widetilde{B}_j)_\delta$, so $(\widetilde{A}_i)_\delta \subset (\widetilde{B}_j)_\delta$. If we suppose that $(\widetilde{A}_i)_\delta \neq (\widetilde{B}_j)_\delta$ then $\exists x_o \in X/$

(a) $x_o \in (\widetilde{A}_i)_\delta$ and $x_o \notin (\widetilde{B}_j)_\delta$, or

(b) $x_o \notin (\widetilde{A}_i)_\delta$ and $x_o \in (\widetilde{B}_j)_\delta$.

We are going to prove the case (a), the other is identical:

$x\widetilde{R}_{\Pi}x_o \geq \min\{\widetilde{A}_i(x), \widetilde{A}_i(x_o)\} \geq \delta > \varepsilon$ but $x\widetilde{R}_{\Pi'}x_o = \max_{B_k}\{\min\{\widetilde{B}_k(x), \widetilde{B}_k(x_o)\}\}$; as $B_j(x) \geq \delta > \varepsilon$ and Π' is ε-partition, then $\forall k \neq j, \widetilde{B}_k(x) \leq \varepsilon < \delta$ and as $x_o \notin (\widetilde{B}_j)_\delta \Longrightarrow \widetilde{B}_j(x_o) < \delta$, then $x\widetilde{R}_{\Pi'}x_o < \delta$, but $(\widetilde{R}_\Pi)_\delta = (\widetilde{R}_{\Pi'})_\delta$, so it is not possible, and it is necessary that $(\widetilde{A}_i)_\delta = (\widetilde{B}_j)_\delta$. In the same form, if we fix a subset $\widetilde{B}_i \in \Pi'$ with $\widetilde{B}_i \not\equiv_\varepsilon \emptyset \Longrightarrow \exists \widetilde{A}_j \in \Pi/\widetilde{B}_j \equiv_\varepsilon \widetilde{A}_j$. So $\Pi = \Pi'$ because we have the two contents and f is a inyective application. ∎

Proposition 5.6 *If a relation of weak ε-equality is considered in the theorem 5.5 instead of a ε-equality, then the application f is a biyection.*

Proof The prove that f is application and inyective is totally analogous to the previous theorem. We are now to prove that in this case f is also surjective, that is, $\forall \widetilde{R} \in R, \exists \Pi \in P/f(\Pi) \equiv_{\underline{\varepsilon}} \widetilde{R}$?

The family $\Pi = \{\widetilde{[a_i]}/a_i \in X\}$ is a ε-partition as we can see in the proposition 5.4, but $f(\Pi) \equiv_{\underline{\varepsilon}} \widetilde{R}$?. If we denote $f(\Pi) = \widetilde{R}'$, then we are going to see that $\widetilde{R} \equiv_{\underline{\varepsilon}} \widetilde{R}'$. Let x and y be two elements in X, we have three cases:
i) If $x\widetilde{R}y \geq 1-\varepsilon$, then $x\widetilde{R}'y = \max_{\widetilde{[a_i]}}\{\min\{\widetilde{[a_i]}(x), \widetilde{[a_i]}(y)\}\} \geq \min\{\widetilde{[x]}(x), \widetilde{[x]}(y)\} \geq 1-\varepsilon$, because, as $\widetilde{R}$ is a $1-\varepsilon$-equivalence then:

$$\begin{cases} \widetilde{[x]}(x) = \max_{b \in X/b\widetilde{R}x \geq 1-\varepsilon}(b\widetilde{R}x) \geq x\widetilde{R}x \geq 1-\varepsilon; \\ \widetilde{[x]}(y) = \max_{b \in X/b\widetilde{R}x \geq 1-\varepsilon}(b\widetilde{R}y) \geq x\widetilde{R}y \geq 1-\varepsilon. \end{cases}$$

So, if $x\widetilde{R}y \geq 1-\varepsilon \Longrightarrow x\widetilde{R}'y \geq 1-\varepsilon$.

To see the other implication, if $x\widetilde{R}'y \geq 1-\varepsilon \Longrightarrow \exists i/\min\{\widetilde{[a_i]}(x), \widetilde{[a_i]}(y)\} = \{b\widetilde{R}x, c\widetilde{R}y\} \geq 1-\varepsilon$ with b and c such that $b\widetilde{R}a_i, c\widetilde{R}a_i \geq 1-\varepsilon \Longrightarrow x\widetilde{R}y \geq 1-\varepsilon$ because $\widetilde{R}$ is ε-transitive. So $\forall x, y \in X, x\widetilde{R}y \geq 1-\varepsilon \Longleftrightarrow x\widetilde{R}'y \geq 1-\varepsilon$.
ii) If $x\widetilde{R}y \leq \varepsilon$, then $x\widetilde{R}'y = \min\{\widetilde{[a_i]}(x), \widetilde{[a_i]}(y)\}$. If $\widetilde{[a_i]}(x) \geq 1-\varepsilon \Longrightarrow \widetilde{[a_i]}(y) \leq \varepsilon$, because they are not in the same class of $(1-\varepsilon)$-equivalence. In other case $\widetilde{[a_i]}(x) \leq \varepsilon$ because Π is a ε-partition of X; so if $x\widetilde{R}y \leq \varepsilon \Longrightarrow x\widetilde{R}'y \leq \varepsilon$.

To see the other implication, we can have a similar way to the previous point.

We have proof than $(\widetilde{R})_\alpha = (\widetilde{R}')_\alpha$, for all $\alpha \in (\varepsilon, 1-\varepsilon)$, that is, $\widetilde{R}$ and $\widetilde{R}'$ are weak ε-equals, and so f is a biyection. ∎

6 Concluding remarks

In this work it has been attempted to find the class of fuzzy binary relations that generate the ε-partitions, that is, the “equivalence” relations as well as to obtain an identification between “equivalence relations” and “partitions” analogous to that of crisp case. Evidently, this is only a first approach to the problem, in future studies we will attempt to analyze the relations that generate partitions of any subset of the referential. It is also between our purposes to study the resulting relation of a refinement, of product partitions, as well as the relation that connects the "equivalences" generated by independent partitions [15], etc. In inverse sense, we would also like to study the resulting partition of the composition of two relations, of the union, of the intersection, etc.

Acknowledgements

The research in it has been supported in part by DGICYT Grant No. PB94-1328; its financial support is gratefully acknowledged. We wish also to thank Prof. Antonia Salas for her support.

References

[1] J.C. Bezdek and J.D. Harris (1978). Fuzzy partitions and relations: An axiomatic basis for clustering. *Fuzzy Sets and Systems* **1**:111-127.

[2] J. Bezdek (1981). Pattern Recognition with Fuzzy Objective Functions Algorithms. *Plenum Press, New York.*

[3] S.K. Bhakat and P. Das (1992). q-Similitudes and q-fuzzy partitions. *Fuzzy Sets and Systems* **51**: 195-202.

[4] S. Bodjanova (1993). Some binary relations and operations on the set of fuzzy partitions. *Fuzzy Sets and Systems* **53**:305-317.

[5] B. Bouchon and G. Cohen (1983). On fuzzy relations and partitions. *Avances in Fuzzy Sets, Possibility Theory and Applications*:97-105.

[6] M.K. Chakraborty and M. Das (1983). Studies in fuzzy relations over fuzzy subsets. *Fuzzy Sets and Systems* **9**:79-89.

[7] M.K. Chakraborty and M. Das (1983). On fuzzy equivalence I. *Fuzzy Sets and Systems* **11**:185-193.

[8] D. Dubois and H. Prade (1980). Fuzzy Sets and Systems. Theory and Applications. *Academic Press, Inc.*

[9] F. Esteva (1991). Cálculo con relaciones borrosas. *Lógica borrosa y sus aplicaciones. Curso de verano de la Universidad Complutense.*

[10] A. Kaufmann (1975). Introduction to the theory of Fuzzy Subsets. *Academic Press, New York,* **Vol 1**.

[11] J. Jacas and L. Valverde (1989). On Fuzzy relations, Metrics and Cluster Analisys. *en J.L. Verdegay et al. Aproximate Reasoning Tools for Artificial Intelligence, Verlag TUV, Rheinland*:21-38.

[12] G.J. Klir and T.A. Folger (1988). Fuzzy Sets, Uncertainty, and Information. *Prentice-Hall International (UK) Limited*

[13] R. López de Mántaras and L. Valverde (1984). New results in fuzzy clustering based on the concept of indistinguishability relations. *Proc. 7th. Intern. Conf. on Pattern Recognition, Montreal.*

[14] S. Montes, M.J. López-Palomo and P. Gil (1995). Las ε-particiones. *V Congreso Estylf'95.* **1**:96-99.

[15] S. Montes, A. Bodini and P. Gil (1996). Fuzzy partitions and classifications. *IPMU'96*:en proceso.

[16] S. Ovchinnikov (1981). Structure of fuzzy binary relations. *Fuzzy Sets and Systems* **6**:169-195.

[17] S. Ovchinnikov (1991). Similarity relations, fuzzy partitions, and fuzzy orderings. *Fuzzy Sets and Systems* **40**:107-126.

[18] E.H. Ruspini (1969). A new approach to clustering. *Inf. Control* **15**:22-32.

[19] E. Trillas and J.L. Castro (1989). Logic and Fuzzy relations. *en J.L. Verdegay et al. Aproximate Reasoning Tools for Artificial Intelligence, Verlag TUV, Rheinland.*

[20] E. Trillas (1982). Assaig sobre les relacions d'indistingibilitat. *Actes del 1er. Congres Català de Lógica Matemática*:51-59.

[21] R.T. Yeh and S.Y. Bang (1975). Fuzzy relations, fuzzy graphs, and their applications to clustering analysis. *Fuzzy Sets and their Applications to Cognitive and Decision Processes. Academic Press*:125-149.

[22] L.A. Zadeh (1965). Fuzzy sets. *Information and Control* **8**:338-353.

[23] L.A. Zadeh (1971). Similarity relations and fuzzy orderings. *Inform. Sci.* **3**:177-200.