Some Theoretical Remarks of Octonionic Analysis

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Abstract. In this article we first review the classical results of octonions and octonionic analysis. Then we consider some theoretical properties of the theory and compare it to quaternionic analysis and Clifford analysis.

INTRODUCTION

Octonionic analysis is a function theory related to the functions taking their values in the famous non-associative and non-commutative division algebra of octonions. Our point of view to the theory is to consider the functions which are in the kernel of the Cauchy-Riemann operator ∂_x. We started our study in our paper [9] and this paper is a continuation of it. One can say that in [9] we gave all the necessary formulas, and that the nature of this small paper is more philosophical, i.e., here we study some similarities and differences of the similar type of function theories, namely quaternionic analysis and Clifford analysis. Comparing this text with the standard text of quaternionic analysis one may see a lot of similarities, but the theories have also crucial differences. These differences are located in the places where associativity is needed. This is of course a priori a trivial observation, but here we try to give some explicit points where this happens, e.g., the lack of Leibniz rule.

PRELIMINARIES

In this section we recall the preliminaries of octonions and analysis on it. Details and more information may be found in our paper [9] and standard literature, e.g., [1, 3, 8, 10, 11]. We also assume that a reader is familiar with the classical theory of quaternions and quaternionic analysis, e.g., [3, 6, 7, 8, 10, 11].

Octonions

The algebra of octonions $\mathbb{O}$ is a well known non-commutative and non-associative algebra, generated by the basis elements $\{1, e_1, \ldots, e_7\}$ satisfying the multiplication table:

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

In the table, 1 is the identity element of $\mathbb{O}$. An arbitrary octonion $x \in \mathbb{O}$ may be written in the form $x = x_0 + \underline{x}$, where x_0 is the *scalar part* and

$$\underline{x} = x_1 e_1 + \cdots + x_7 e_7$$

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the *vector part* of the quaternion, where $x_0, x_1, \ldots, x_7 \in \mathbb{R}$. The *conjugation* is defined by $\overline{x} = x_0 - \underline{x}$, satisfying $\overline{xy} = \overline{y}\,\overline{x}$. In the multiplication table, we see that the elements $\{1, e_1, e_2, e_3\}$ generate the quaternion algebra $\mathbb{H}$ as a natural subalgebra of $\mathbb{O}$. Every octonion $x \in \mathbb{O}$ admits also the *quaternionic form* $x = a + be_4$, where

$$a = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \text{ and } b = x_4 + x_5e_1 + x_6e_2 + x_7e_3$$

are quaternions. The *quaternionic parts* a, b of x satisfy the algebraic rules

(a) $e_4a = \overline{a}e_4$,
(b) $e_4(ae_4) = -\overline{a}$,
(c) $(ae_4)e_4 = -a$,
(d) $a(be_4) = (b\underline{a})e_4$,
(e) $(ae_4)b = (a\overline{b})e_4$,
(f) $(ae_4)(be_4) = -\overline{b}a$,

which gives us explicit rules to make computations by quaternionic forms. The conjugate of a quaternionic form $x = a + be_4$ is $\overline{x} = \overline{a} - be_4$. Considering only the vector parts $\underline{x}$ and $\underline{y}$ we obtain the decomposition

$$\underline{x}\underline{y} = \frac{1}{2}(\underline{x}\underline{y} + \underline{y}\underline{x}) + \frac{1}{2}(\underline{x}\underline{y} - \underline{y}\underline{x}), \tag{1}$$

where $\underline{x} \cdot \underline{y} := -\frac{1}{2}(\underline{x}\underline{y} + \underline{y}\underline{x}) = x_1y_1 + \cdots + x_7y_7$ is the *scalar product* and $\underline{x} \times \underline{y} := \frac{1}{2}(\underline{x}\underline{y} - \underline{y}\underline{x})$ the (7-dimensional) *cross product* of vectors $\underline{x}$ and $\underline{y}$.

Octonionic Analysis

Octonionic analysis is dealing with the functions $f\colon \Omega \subset \mathbb{R}^8 \to \mathbb{O}$, where we usually make the identification $\mathbb{R}^8 \cong \mathbb{O}$ assuming that our functions are of an octonion variable. Our point of view is based on the *Cauchy-Riemann operator*

$$\partial_x = \partial_{x_0} + e_1\partial_{x_1} + \cdots + e_7\partial_{x_7},$$

and on its vector part, the *Dirac operator*

$$\partial_{\underline{x}} = e_1\partial_{x_1} + \cdots + e_7\partial_{x_7},$$

both acting on componentwise differentiable octonion valued functions. The conjugate of the Cauchy-Riemann operator $\partial_x = \partial_{x_0} + \partial_{\underline{x}}$ is defined as $\partial_{\overline{x}} = \partial_{x_0} - \partial_{\underline{x}}$. Both of the operators factorize the Laplacian, i.e., $\partial_{\overline{x}}\partial_x = \partial_x\partial_{\overline{x}} = \Delta_x$ and $\partial_{\underline{x}}^2 = -\Delta_{\underline{x}}$, where $\Delta_x := \partial_{x_0}^2 + \Delta_{\underline{x}}$ and $\Delta_{\underline{x}} := \sum_{j=1}^7 \partial_{x_j}^2$. As in the quaternionic analysis, we make the following crucial definition.

Definition 1 *Let $\Omega \subset \mathbb{O}$ be open and $f\colon \Omega \to \mathbb{O}$ a componentwise differentiable function. If*

$$\partial_x f = 0 \quad (\textit{resp. } f\partial_x = 0)$$

in Ω, then f is called left *(resp.* right*)* monogenic *in Ω.*

In our point of view the octonionic analysis means to study monogenic octonion valued functions and their function theoretic properties. The Cauchy-Riemann operator admit also its *quaternionic form* $\partial_x = \partial_u + \partial_v e_4$, where quaternionic Cauchy-Riemann operators are

$$\partial_u = \partial_{x_0} + e_1\partial_{x_1} + e_2\partial_{x_2} + e_3\partial_{x_3} \text{ and } \partial_v = \partial_{x_4} + e_1\partial_{x_5} + e_2\partial_{x_6} + e_3\partial_{x_7},$$

acting on octonionic valued differentiable functions $f = f(u, v)$, where we represent the variable x in the quaternionic form $x = u + ve_4$ and consider functions as functions of two quaternion variables. We may represent an octionion valued function in a quaternionic form $f(u, v) = g(u, v) + h(u, v)e_4$, where g and h are quaternion valued. For quaternion valued functions we have the algebraic rules

(a) $\partial_u(ge_4) = (g\partial_u)e_4$,
(b) $(\partial_u e_4)g = (\partial_u\overline{g})e_4$,

(c) $(\partial_u e_4)(g e_4) = -\overline{g}\partial_u$.
(d) $(g e_4)\partial_u = (g\partial_{\overline{u}})e_4$,
(e) $g(\partial_u e_4) = (\partial_u g)e_4$,
(f) $(g e_4)(\partial_u e_4) = -\partial_{\overline{u}} g$.

These rules allow us to make computations using quaternion formed objects in octonionic analysis. The following result allows us to transforms a problem of octonionic monogenic function completely to a problem of the quaternionic analysis.

Proposition 1 (Quaternionic Cauchy-Riemann systems [9]) *Assume that the quaternionic form of a differentiable function f is $f = g + he_4$. Then*
(a) $\partial_x f = 0$ *if and only if*

$$\partial_u g = \overline{h}\partial_v,$$

$$h\partial_u = -\partial_v \overline{g},$$

(b) $f\partial_x = 0$ *if and only if*

$$g\partial_u = \partial_{\overline{v}} h,$$

$$h\partial_{\overline{u}} = -\partial_v g.$$

Let us consider still the vector valued functions $\underline{f} = f_1 e_1 + \cdots + f_7 e_7$ and operators. The scalar and cross product allow us to define the *divergence operator* $\partial_{\underline{x}} \cdot \underline{f} := -\frac{1}{2}(\partial_{\underline{x}}\underline{f} + \underline{f}\partial_{\underline{x}}) = \partial_{x_1} f_1 + \cdots + \partial_{x_7} f_7$ and the *rotor*

$$\partial_{\underline{x}} \times \underline{f} := \frac{1}{2}(\partial_{\underline{x}}\underline{f} - \underline{f}\partial_{\underline{x}}) = \sum_{\substack{i,j=1 \\ i\neq j}}^{7} \partial_{x_i} f_j e_i e_j.$$

Then decomposition (1) gives

$$\partial_{\underline{x}}\underline{f} = -\partial_{\underline{x}} \cdot \underline{f} + \partial_{\underline{x}} \times \underline{f},$$
$$\underline{f}\partial_{\underline{x}} = -\partial_{\underline{x}} \cdot \underline{f} - \partial_{\underline{x}} \times \underline{f}.$$

For a function $f = f_0 + \underline{f}$ with real and vector parts f_0 and $\underline{f}$ we have

$$\partial_x f = \partial_{x_0} f_0 + \partial_{x_0}\underline{f} + \partial_{\underline{x}} f_0 - \partial_{\underline{x}} \cdot \underline{f} + \partial_{\underline{x}} \times \underline{f}, \quad (2)$$
$$f\partial_x = \partial_{x_0} f_0 + \partial_{x_0}\underline{f} + \partial_{\underline{x}} f_0 - \partial_{\underline{x}} \cdot \underline{f} - \partial_{\underline{x}} \times \underline{f}. \quad (3)$$

Comparing the real and vector parts of equation (2) we deduce that a function is left monogenic if and only if its real and vector parts f_0 and $\underline{f}$ satisfy the generalized octonionic *Moisil-Teodorescu system*

$$\partial_{x_0} f_0 - \partial_{\underline{x}} \cdot \underline{f} = 0,$$
$$\partial_{x_0}\underline{f} + \partial_{\underline{x}} f_0 + \partial_{\underline{x}} \times \underline{f} = 0.$$

ON RIESZ SYSTEM IN OCTONIONIC ANALYSIS

In [13] Stein and Weiß represented the so called *Riesz system* in a component form. In $\mathbb{R}^8$ this is

$$\partial_{x_0} f_0 - \partial_{x_1} f_1 - \ldots - \partial_{x_7} f_7 = 0,$$
$$\partial_{x_0} f_i + \partial_{x_i} f_0 = 0, \quad (i = 1, \ldots, 7),$$
$$\partial_{x_i} f_j - \partial_{x_j} f_i = 0, \quad (i, j = 1, \ldots, 7,\ i \neq j).$$

In this section our aim is to study how this system is connected to octonionic analysis. First we observe that we may write the Riesz system in the operator form

$$\partial_{x_0} f_0 - \partial_{\underline{x}} \cdot \underline{f} = 0,$$
$$\partial_{x_0} \underline{f} + \partial_{\underline{x}} f_0 = 0,$$
$$\partial_{\underline{x}} \times \underline{f} = 0.$$

Comparing the Moisil-Teodorescu system with the Riesz system we see that left monogenic functions do not satisfy the Riesz system. The Riesz system may be characterized as follows.

Theorem 1 *Let $f\colon \Omega \to \mathbb{O}$ be a differentiable octonion valued function. Then f satisfies the Riesz system if and only if $\partial_x f = f \partial_x = 0$.*

Proof. If $\partial_x f = f\partial_x = 0$, adding and subtracting equations (2) and (3) and comparing the real and vector parts, we obtain the Riesz system. The other direction is trivial. □

The preceding observations tell us that the classical theory of the Clifford analysis (see, e.g., [2, 5]) differs from the theory of octonionic analysis. In Clifford analysis it is enough to use one of the Cauchy-Riemann operators to get Riesz system by restricting to the so called paravector valued functions.

In the quaternionic analysis, the similar result to the preceding theorem holds in four dimensions. It is natural that the theories of quaternionic and octonionic analysis are similar in this sense, because we haven't have any case yet where associativity would have been needed.

RADIAL ALGEBRA IDENTITIES FOR CAUCHY-RIEMANN OPERATOR

Behind Clifford analysis stays the more general abstract structure, called the *radial algebra*, and analysis on it; see [4, 12]. In the case of octonionic analysis, it is not possible to use the radial algebra because of the lack of associativity. But we may still take a look at the operator identities, and check which of those hold. At the same time we will see the connections and differences between the theories.

In our case the "vector variables" are octonions $x \in \mathbb{O}$ and the "vector derivative" is the Cauchy-Riemann operator ∂_x. The axioms for the vector derivative are given at the page 298 in [12]. In the next proposition, we obtain the following modifications of the axioms.

Proposition 2 *Assume that f is a real valued and F an octonion valued differentiable function. Let $a \in \mathbb{O}$. Then*
(D1) $\partial_x(fF) = (\partial_x f)F + f(\partial_x F)$ and $(fF)\partial_x = F(\partial_x f) + f(F\partial_x)$,
(D2) $\partial_x(Fa) \neq (\partial_x F)a$ and $(aF)\partial_x \neq a(F\partial_x)$,
(D3) $\partial_x(F\partial_x) = (\partial_x F)\partial_x$,
(D4) $\partial_x(x\overline{x}) = \partial_x(\overline{x}x) = 2x$ and $\partial_x(x_0 y_0 + \underline{x}\cdot\underline{y}) = y$.

Proof. The proof for (D1) is obvious. To prove (D2), one needs to find a simple counterexample, e.g., $F = x_1 e_2$ and $a = e_4$. The proof for (D3) follows from the fact $x(yx) = (xy)x$, see [9]. Also the proof of (D4) is trivial. □

The property (D2) is different than in the quaternionic analysis, where the equality holds. The biggest consequence is that the similar Leibniz product formula than in the quaternionic analysis ("$\partial_x(fg) = (\partial_x f)g + \dot{\partial}_x f \dot{g}$", see e.g. [7]) does not exist in octonionic analysis. The lack of the Leibniz rule have many theoretical implications. For example to find polynomial solutions for the equation $\partial_x f = 0$ is much more complicated than in the quaternionic and Clifford analysis.

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REFERENCES

[1] J. C. Baez, Bull. Amer. Math. Soc. (N.S.) **39**, 145–205 (2002).
[2] F. Brackx, R. Delanghe, and F. Sommen, *Clifford Analysis* (Pitman, Boston, MA, 1982).
[3] J. H. Conway and D. A. Smith, *On quaternions and octonions: their geometry, arithmetic, and symmetry* (A K Peters, Ltd., Natick, MA, 2003).
[4] H. De Schepper, A. G. Adán, and F. Sommen, Complex Anal. Oper. Theory **11**, 1139–1172 (2017).
[5] R. Delanghe, F. Sommen, and V. Souček, *Clifford algebra and spinor-valued functions* (Kluwer Academic Publishers Group, Dordrecht, 1992).
[6] K. Gürlebeck, K. Habetha, and W. Sprößig, *Funktionentheorie in der Ebene und im Raum* (Birkhäuser Verlag, Basel, 2006).
[7] K. Gürlebeck and W. Sprößig, *Quaternionic and Clifford calculus for physicists and engineers* (Wiley, Chichester, 1997).
[8] R. Hermann, *Spinors, Clifford and Cayley algebras* (Rutgers University, New Brunswick, N.J., 1974).
[9] J. Kauhanen and H. Orelma, *Cauchy-Riemann operators in octonionic analysis*, arXiv:1701.08698, 2017.
[10] P. Lounesto, *Clifford algebras and spinors.* (Cambridge University Press, Cambridge, 2001).
[11] I. Porteous, *Clifford algebras and the classical groups.* (Cambridge University Press, Cambridge, 1995).
[12] F. Sommen, Port. Math. **54**, 287–310 (1997).
[13] E. Stein, G. Weiß, Amer. J. Math. **90**, 163–196 (1968).