International Global Navigation Satellite Systems Society
IGNSS Symposium 2013

Outrigger Gold Coast, Qld Australia
16-18 July, 2013

Estimation of initial state and model parameters for autonomous GNSS orbit prediction

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ABSTRACT

In self-assisted GNSS the orbit of a satellite is predicted by solving the differential equation that models its motion. Our motion model includes the most important forces: Earth's gravity, lunar and solar gravity and solar radiation pressure. Unmodeled forces are taken into account by using Gaussian white noise term with covariance matrix estimated offline from historical orbital data. The estimation of model parameters (solar radiation pressure and Earth orientation parameters) and initial state for the prediction includes both offline and online stages. In the offline stage, priors for the solar radiation pressure parameters are estimated using precise orbits issued by the International GNSS service (IGS). In the online stage, the satellite's broadcast ephemeris is used to estimate the initial state and model parameters. The estimation of the initial state is formulated as non-linear continuous-time filtering problem with discrete-time measurements. The filtering equations are solved numerically and the performance of different numerical methods (Extended, Cubature and Unscented Kalman filters) is compared. Using the estimated initial state and model parameters, the satellite orbits are predicted 5 days into the future. The accuracy and consistency of the predicted orbits is analysed by comparing with the IGS precise ephemerides. In this paper only GPS satellites are considered, but the method can be extended to other satellite systems.

KEYWORDS: Satellite orbit prediction; Gaussian filtering; Estimation

1. INTRODUCTION

In autonomous or “self-assisted” GNSS orbit prediction the orbit is predicted in the positioning device using only information from the satellite’s broadcast ephemeris. The predicted orbit can be used for example to reduce the time to first fix (TTFF) of a standalone GNSS receiver (Mattos, 2008), (Zhang *et al*., 2008), (Lytvyn *et al*., 2012).

Like most GNSS prediction algorithms, our algorithm is based on integrating the satellite’s equation of motion several days forward using initial conditions computed from the satellite’s broadcast ephemeris. Our equation of motion includes the four most significant forces acting on the satellite: the gravitations of the Earth, the Sun and the Moon, and the solar radiation pressure. The models for the gravitational terms are covered in (Seppänen *et al*. 2011), (Seppänen *et al*., 2012). Our two-parameter solar radiation pressure model is presented in (Ala-Luhtala *et al*., 2012). The uncertainty caused by modelling errors and unmodeled forces are taken into account by adding a Gaussian white noise acceleration term to the satellite’s equation of motion. The covariance matrix for the Gaussian white noise term is estimated using historical precise ephemeris data.

One of the main problems in the autonomous orbit prediction is obtaining the initial conditions needed to start the integration. The broadcast position and velocity are given in the Earth-centered, Earth-fixed reference frame and must be transformed to an inertial reference frame. However, the Earth orientation parameters (EOP) that are needed in the transformation are unknown. We have also noticed that the velocity computed directly from the broadcast ephemeris is not accurate enough for prediction. In our previous work, we solved the initial condition determination problem by fitting the motion model to the broadcast data using an iterative least-squares minimization algorithm (Seppänen *et al*., 2011), (Seppänen *et al*., 2012).

In this paper we propose a Bayesian filtering algorithm for the determination of the initial state. The filtering solution has the advantage that it requires only one iteration and enables a probability-based interpretation of the problem. The filtering problem we are considering in this paper is nonlinear with continuous-time process model and discrete-time measurements. The exact solution is analytically intractable and numerical approximations are used. We consider the approximate solution obtained using the Extended Kalman Filter (Jazwinski, 1970), which is based on linearization using a first-order Taylor polynomial. We also consider the Unscented Kalman Filter (UKF) (Julier *et al*., 1995) and the Cubature Kalman Filter (CKF) (Arasaratnam and Haykin, 2009), which are based on sigma-point approximations for the statistical moments needed in the filtering algorithm.

The paper is organized as follows. The satellite’s equation of motion and the reference frames are covered in Sections 2.1 and 2.2. In Section 2.3 a method for estimating the process noise covariance matrix using precise ephemeris data is presented. The method for estimating the initial state using the broadcast ephemeris is presented in Section 2.4. Section 2.5 summarizes the proposed orbit prediction algorithm. The algorithms prediction accuracy is assessed in Chapter 3. The paper closes with conclusion and discussion in Chapter 4.

2. MODEL

2.1 Equation of motion

The four most significant forces affecting the satellite are the gravitational attractions of the Earth, the Sun and the Moon, and the solar radiation pressure (srp). The acceleration of the satellite is

$$\boldsymbol{a}_{\mathrm{sat}} = \boldsymbol{a}_{\mathrm{earth}} + \boldsymbol{a}_{\mathrm{sun}} + \boldsymbol{a}_{\mathrm{moon}} + \boldsymbol{a}_{\mathrm{srp}}, \tag{1}$$

where all the accelerations are in an inertial reference frame.

The acceleration caused by the Earth is computed by taking the gradient of the gravity potential *U*. To account for the uneven mass distribution of the Earth, the gravity potential is written using a spherical harmonic series (Montenbruck and Gill, 2005). Seppänen (2010) found that at GNSS satellite altitudes, terms up to the order of at least 4 should be used. In our implementation, we have used terms up to the degree and order 8.

The gravitational acceleration caused by any celestial body can be computed using equation

$$\mathbf{a}_{\mathrm{cb}} = GM\left(\frac{\mathbf{r}_{\mathrm{cb}} - \mathbf{r}}{||\mathbf{r}_{\mathrm{cb}} - \mathbf{r}||^3} - \frac{\mathbf{r}_{\mathrm{cb}}}{||\mathbf{r}_{\mathrm{cb}}||^3}\right), \tag{2}$$

where *M* is the mass of the body, $\mathbf{r_{cb}}$ is its position in Earth centered inertial reference frame, and $\mathbf{r}$ is the position of the satellite in the same reference frame. The orbits of the sun and the moon are computed using simple models presented by Montenbruck and Gill (2005. See references (Seppänen et al, 2011, Seppänen et al, 2012) for more details about the computation of the gravity terms.

For the acceleration caused by the solar radiation pressure, we use a two-parameter empirical model (Ala-Luhtala *et al*., 2012)

$$\mathbf{a}_{\mathrm{srp}} = \lambda\left(-\frac{\alpha_1 C}{r_{\mathrm{sun}}^2}\mathbf{e}_s + \alpha_2 \mathbf{e}_y\right). \tag{3}$$

The first term inside the parenthesis describes the effect of direct solar radiation pressure, which is directed along the line joining the satellite and the sun. The magnitude of the direct solar radiation pressure depends on the satellite's distance to the sun r_{sun}. The term α_1 is a satellite specific parameter that scales the direct radiation pressure, and *C* is a known constant common for all the satellites. The second term inside the parenthesis models the so called y-bias acceleration, which is directed along the satellites solar panel axis (Springer *et al*., 1999), (Froideval, 2009). The y-bias parameter α_2 is also satellite specific. To account for the shadowing of the Earth, we use a time varying term λ that is based on the conical shadow model described by Montenbruck and Gill (2005). The times when the satellite enters Earth's shadow are called eclipse seasons.

In addition to the previously described four forces, there are numerous smaller forces acting on the satellite. These include for example the gravitation of other celestial bodies, the radiation pressure of the proportion of the incident sunlight that is reflected by the Earth (albedo), and Earth tides (Montenbruck and Gill, 2005). We do not attempt to model the forces accurately, but instead take them into account by adding a stochastic acceleration term to Eq. (1). In this paper, we use a Gaussian white noise model. Formally, we can write the new acceleration equation as

$$\mathbf{a} = \mathbf{a}_{\text{sat}} + \mathbf{L}(\mathbf{r}, \mathbf{v})\mathbf{w}, \tag{4}$$

where $\mathbf{a}_{\text{sat}}$ is computed using Eq. (1) and $\mathbf{w}$ is a Gaussian white noise stochastic process with zero mean and covariance matrix $\mathbf{Q}_{\mathbf{a}}$. We have chosen to use the satellite centered RTN-coordinate system (Radial, Tangential, Normal) for the white noise term. The transformation matrix $\mathbf{L}(\mathbf{r}, \mathbf{v})$ transforms the white noise into the inertial reference frame. The transformation from Earth centered inertial (ECI) reference system to the RTN system is given by (Tapley *et al*., 2004)

$$\mathbf{r}_{\mathbf{RTN}} = [\mathbf{e}_R \quad \mathbf{e}_T \quad \mathbf{e}_N]\mathbf{r}_{\text{ECI}}, \tag{5}$$

where the unit vectors are

$$\mathbf{e}_R = \frac{\mathbf{r}_{\text{ECI}}}{||\mathbf{r}_{\text{ECI}}||}, \tag{6}$$

$$\mathbf{e}_T = \mathbf{e}_N \times \mathbf{e}_{R,} \tag{7}$$

$$\mathbf{e}_N = \frac{\mathbf{r}_{\text{ECI}} \times \mathbf{v}_{\text{ECI}}}{||\mathbf{r}_{\text{ECI}} \times \mathbf{v}_{\text{ECI}}||}. \tag{8}$$

Eq. (4) should be interpreted as a first order Itô stochastic differential equation

$$d\mathbf{x} = \mathbf{f}(\mathbf{x})dt + \mathbf{D}(\mathbf{x})d\boldsymbol{\beta}_{\mathbf{a}.}, \tag{9}$$

where

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{r}, \mathbf{v}) = \begin{bmatrix} \mathbf{v} \\ \mathbf{a}(\mathbf{r}) \end{bmatrix}, \tag{10}$$

$$\mathbf{D}(\mathbf{x}) = \mathbf{D}(\mathbf{r}, \mathbf{v}) = \begin{bmatrix} \mathbf{0}_{3\times 3} \\ \mathbf{L}(\mathbf{r}, \mathbf{v}) \end{bmatrix} \tag{11}$$

and $\boldsymbol{\beta}_{\mathbf{a}}$ is a 3-dimensional Brownian motion with diffusion matrix $\mathbf{Q}_{\mathbf{a}}$. The state $\mathbf{x}$ includes the position and velocity of the satellite in the inertial reference frame. See for example references (Øksendal, 2003) and (Jazwinski, 1970) for more information about stochastic differential equations.

2.2 Reference frames

An Earth-fixed, Earth-centered (ECEF) system has its origin at the mass center of the Earth and its axes are fixed with respect to the Earth's surface. In GPS, the reference frame is WGS84, which, for our purposes, can be considered equal to the Terrestrial Reference System (TRS) maintained by the International Earth Rotation and Reference Systems Service (IERS). The origin of the TRS system is the Earth's centre of mass and its z-axis is the mean rotational axis of the Earth.

An inertial reference system maintained by the IERS is the Celestial Reference System (CRS). CRS is a reference system whose coordinate axes maintain their orientation with respect to distant stars. The origin of this reference frame is the center of the Earth and Earth is in an accelerated motion while orbiting around the sun. Therefore CRS is not precisely inertial, but is an adequate approximation of an inertial reference frame for our purposes. The transformation from the TRS system at epoch *t* to the CRS system is

$$\mathbf{r}_{\mathrm{TRS}}(t) = \mathbf{W}(t)\mathbf{G}(t)\mathbf{N}(t)\mathbf{P}(t)\mathbf{r}_{\mathrm{CRS}}, \tag{12}$$

where the matrices **W**, **G**, **N** and **P** describe polar motion, Earth rotation, nutation, and precession, respectively. See references (Seppänen *et al*., 2011) and (Seppänen *et al*, 2012) for details on the computation of matrices **G**, **N** and **P**. The polar motion matrix **W** is described using equation

$$\mathbf{W}(t) = \mathbf{R}_y\big(-x_p(t)\big)\mathbf{R}_x\big(-y_p(t)\big), \tag{13}$$

where x_p and y_p are the polar motion parameters and $\mathbf{R}_x$ and $\mathbf{R}_y$ are rotation matrices about the x- and y-axes. Together with $\mathrm{dUT1}$, x_p and y_p are also called Earth orientation parameters (EOP). $\mathrm{dUT1}$ is the difference between Universal Time (UT1) and Coordinated Universal Time (UTC). This difference is small and in our implementation we use $\mathrm{dUT1} = 0$. This approximation leads to a median error of 4.2 m in the satellite's position for a 4 day long prediction (Seppänen *et al*., 2012). The daily values for these parameters can be found from the homepage of IERS (IERS, 2013).

Instead of the CRS frame, we choose the inertial reference frame to be an intermediate reference system, denoted by $\mathrm{TIRS}(t_0)$, at time t_0. The transformation from $\mathrm{TIRS}(t_0)$ to TRS is given by

$$\mathbf{r}_{\mathrm{TRS}} = \mathbf{W}(t)\mathbf{G}(t)\mathbf{N}(t)\mathbf{P}(t)\mathbf{P}^T(t_0)\mathbf{N}^T(t_0)\mathbf{G}^T(t_0)\mathbf{r}_{\mathrm{TIRS}(t_0)}. \tag{14}$$

For a prediction of a few days, the nutation and precession matrices remain almost unchanged. That is, we can make the approximations $\mathbf{P}(t)\mathbf{P}^{\mathbf{T}}(t_0) \approx \mathbf{I}$ and $\mathbf{N}(t)\mathbf{N}^{\mathbf{T}}(t_0) \approx \mathbf{I}$. Using these approximations Eq. (14) is reduced to

$$\mathbf{r}_{\mathrm{TRS}} = \mathbf{W}(t)\mathbf{R}_z\big((t - t_0)\omega\big)\mathbf{r}_{\mathrm{TIRS}(t_0)}. \tag{15}$$

We used also the result $\mathbf{G}(t)\mathbf{G}^T(t_0) = \mathbf{R}_z\big((t - t_0)\omega\big)$, where ω is the angular velocity of the Earth's rotation (Seppänen *et al*., 2012). We use the notation $\mathbf{T}(t) = \mathbf{W}(t)\mathbf{R}_z\big((t - t_0)\omega\big)$ for the transformation matrix from $\mathrm{TIRS}(t_0)$ to TRS.

The transformation for the velocity can be derived by differentiating Eq. (15) with respect to time. This gives

$$\mathbf{v}_{\mathrm{TRS}} = \mathbf{T}(t)\mathbf{v}_{\mathrm{TIRS}(t_0)} - \boldsymbol{\omega}\times\big(\mathbf{T}(t)\mathbf{r}_{\mathrm{TIRS}(t_0)}\big), \tag{16}$$

where $\boldsymbol{\omega} = [0 \quad 0 \quad \omega]^T$ is the angular velocity vector of the Earth's rotation.

2.3 Offline estimation of the process noise covariance

For the process noise covariance we use a diagonal matrix

$$\mathbf{Q_a} = \begin{bmatrix} e^{q_R} & 0 & 0 \\ 0 & e^{q_T} & 0 \\ 0 & 0 & e^{q_N} \end{bmatrix}, \tag{17}$$

where the diagonal elements are the variances in the radial, tangential and normal directions. The exponential parametrization is used for scaling to avoid numerical errors caused by very small values, and also to convert the problem of estimating the parameter values to an unconstrained optimization problem. The variances are estimated using precise ephemeris data from the National Geospatial-intelligence Agency (NGA, 2013). The reason for using NGA instead of IGS precise ephemerides is that from NGA we get also the velocity of the satellite.

Consider that we have the precise ephemerides $\mathbf{x_0}, \mathbf{x_1}, \dots, \mathbf{x_N}$ (i.e. the position and velocity) at times $t_0, t_1, \dots, t_N$ in an inertial reference frame. The transformation to inertial reference frame can be done using the daily EOP values provided by IERS (IERS, 2013). The posterior distribution for the process noise covariance parameters $\mathbf{q} = [q_R \quad q_T \quad q_N]$ is

$$p(\mathbf{q} \,|\mathbf{x}_0, \dots, \mathbf{x}_N) \propto p(\mathbf{x}_0, \dots, \mathbf{x}_N |\, \mathbf{q}) p(\mathbf{q}). \tag{18}$$

The likelihood can be written by

$$p(\mathbf{x}_0, \dots, \mathbf{x}_N \mid \mathbf{q}) = \prod_{i=1}^{N} p(\mathbf{x}_i \mid \mathbf{x}_{i-1}, \mathbf{q}) p(\mathbf{x}_0) \tag{19}$$

The conditional probability densities in Eq. (19) could in principle be obtained by discretizing the stochastic differential equation (9) using time step $\Delta t = t_i - t_{i-1}$. However, due to the nonlinear model and state dependent noise term, the conditional densities are difficult to compute. For this reason, we approximate the conditional densities in Eq. (19) with a multivariate normal density

$$p(\mathbf{x}_i | \mathbf{x}_{i-1}, \mathbf{q}) = \mathrm{Normal}(\mathbf{x}_i | \mathbf{m}_i, \mathbf{\Sigma}_i(\mathbf{q})) \tag{20}$$

where $\mathbf{m}_i$ and $\mathbf{\Sigma}_i(\mathbf{q})$ are solutions to the differential equations

$$\frac{d\mathbf{m}_i(t)}{dt} = \mathbf{f}(\mathbf{m}(t)) \tag{21}$$

$$\frac{d\mathbf{\Sigma}_i(t, \mathbf{q})}{dt} = \mathbf{\Sigma}(t, \mathbf{q}) \mathbf{F}_\mathbf{x}^T(\mathbf{m}(t)) + \mathbf{F}_\mathbf{x}(\mathbf{m}(t)) \mathbf{\Sigma}(t, \mathbf{q}) + \mathbf{L}(\mathbf{m}(t)) \mathbf{Q_a}(\mathbf{q}) \mathbf{L}^T(\mathbf{m}(t)) \tag{22}$$

with initial conditions $\mathbf{m}(0) = \mathbf{x}_{i-1}$ and $\mathbf{\Sigma}(0, \mathbf{q}) = \mathbf{0}$. The matrix $\mathbf{F_x}$ is the Jacobian matrix of $\mathbf{f}$. Note that this is the same approximation made in the prediction step of the Extended Kalman filter (Jazwinski, 1970), (Särkkä and Sarmavuori, 2013). The solar radiation pressure parameters needed to compute **f** are fixed to the values given in (Ala-Luhtala *et al*., 2012). Satellite PRN 1 was replaced since results in (Ala-Luhtala *et al*., 2012), so new values $\alpha_1 = 1.5464$ and $\alpha_2 = 0.0033$ were estimated.

We seek an estimate for the parameters $\mathbf{q}$ by maximizing the posterior distribution in Eq. (18). The prior is taken to be uniform $p(\mathbf{q}) \propto 1$. With the uniform prior, the maximum can be found by minimizing the negative log-likelihood $-\log p(\mathbf{x}_0, \dots, \mathbf{x}_N|\mathbf{q})$. Using the multivariate normal probability densities for the conditional distributions in the likelihood, the negative log-likelihood is given by

$$\begin{aligned} -\log p(\mathbf{x}_0, \dots, \mathbf{x}_N) \\ = \frac{1}{2}\sum_{i=1}^{N} [\log|\boldsymbol{\Sigma}_i(\mathbf{q})| + (\mathbf{x}_i - \mathbf{m}_i)^T \boldsymbol{\Sigma}_i^{-1}(\mathbf{q})(\mathbf{x}_i - \mathbf{m}_i)] \\ + \text{const.} \end{aligned} \tag{23}$$

All the terms that do not depend on the parameter vector $\mathbf{q}$ are absorbed into the constant term. A conjugate gradient method (Luenberger and Ye, 2008) can be used for the minimization of Eq. (23). The expressions for the Jacobian and Hessian are computed analytically.

We estimate the parameters using precise ephemeris data from GPS weeks 1670-1686. For each day during the time period, we estimate a value for the $\mathbf{q}$. This provides a time series of parameter estimates. We want a single time independent value for $\mathbf{q}$, so we take the median of the resulting time series. We exclude estimates made during eclipse seasons. As an example, the median values for satellite PRN 3 are $\mathbf{q} = [-32.3 \quad -29.5 \quad -29.2]$. The values for the other satellites in the GPS constellation are $q_R \in$ [-30.4, -32.5], $q_T \in$ [-27.7, -29.9], $q_N \in$ [-28.9, -29.6].

2.4 Online estimation of initial state

Our previous studies have shown that the velocity computed directly from the satellite's broadcast ephemeris is much too inaccurate for prediction of several days (Seppänen *et al*., 2011), (Seppänen *et al*., 2012). Also, the broadcast position and velocity must be transformed from the ECEF coordinate system to the inertial coordinate system using Eq. (11). This transformation depends on the EOP values, which are not currently part of the navigation message. In addition, the satellite’s solar radiation pressure parameters must also be estimated. This leads to the problem of estimating the satellite's initial state (i.e. the positon and velocity in an inertial reference frame, the EOP values and the solar radiation pressure parameters) using the data available in the navigation message. In the following, a Bayesian filtering based solution for the problem is presented.

The state-space model for the problem is given by

$$d\mathbf{x} = \mathbf{f}(\mathbf{x})dt + \mathbf{D}(\mathbf{x})d\boldsymbol{\beta}, \tag{24}$$

$$\mathbf{y}_k = \mathbf{h}(\mathbf{x}) + \boldsymbol{\epsilon}_k, \tag{25}$$

where

$$\mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{r}, \mathbf{v}, \mathbf{p}) = \begin{bmatrix} \mathbf{v} \\ \mathbf{a}(\mathbf{r}, \mathbf{p}) \\ \mathbf{0}_{d\times 1} \end{bmatrix}, \tag{26}$$

$$\mathbf{D}(\mathbf{x}) = \mathbf{D}(\mathbf{r}, \mathbf{v}, \mathbf{p}) = \begin{bmatrix} \mathbf{0}_{3\times 3} & \mathbf{0}_{3\times d} \\ \mathbf{L}(\mathbf{r}, \mathbf{v}) & \mathbf{0}_{3\times d} \\ \mathbf{0}_{d\times 3} & \mathbf{I}_{d\times d} \end{bmatrix}, \quad d\boldsymbol{\beta} = \begin{bmatrix} d\boldsymbol{\beta}_{\mathbf{a}} \\ d\boldsymbol{\beta}_{\mathbf{p}} \end{bmatrix}, \tag{27}$$

$$\mathbf{h}(\mathbf{x}) = \mathbf{h}(\mathbf{r}, \mathbf{v}, \mathbf{p}) = [\mathbf{T}(\mathbf{p}) \quad \mathbf{0}_{3\times 3} \quad \mathbf{0}_{3\times d}] \begin{bmatrix} \mathbf{r} \\ \mathbf{v} \\ \mathbf{p} \end{bmatrix}. \tag{28}$$

The state of the satellite **x** consists of the position and velocity of the satellite in the inertial reference frame, and the d-dimensional ($d = 4$) vector **p**, that contains the EO-parameters x_p and y_p, and the solar radiation parameters α_1 and α_2. We note, that the acceleration term $\mathbf{a}(\mathbf{r}, \mathbf{p})$ depends not only on the solar radiation pressure parameters, but also on the EO-parameters, since coordinate transformation to ECEF reference frame is needed to compute the gravitational acceleration caused by the Earth (Seppänen *et al.*, 2011), (Seppänen *et al.*, 2012). For the parameter vector **p** we assume a simple model

$$d\mathbf{p} = d\boldsymbol{\beta}_{\mathbf{p}}, \tag{29}$$

where $\boldsymbol{\beta}_{\mathbf{p}}$ is a d-dimensional Brownian motion stochastic process with a diagonal diffusion matrix $\mathbf{Q}_{\mathbf{p}}$. That is, we are assuming that the parameters stay approximately constant over the broadcast ephemeris's time interval. Equation (20) gives now the augmented system of equations Eq. (9) and Eq. (22). The Brownian motions $\boldsymbol{\beta}_{\mathbf{a}}$ and $\boldsymbol{\beta}_{\mathbf{p}}$ are assumed independent, so that the diffusion matrix for the joint process is

$$\mathbf{Q} = \begin{bmatrix} \mathbf{Q}_{\mathbf{a}} & \mathbf{0} \\ \mathbf{0} & \mathbf{Q}_{\mathbf{p}} \end{bmatrix}. \tag{30}$$

The measurements $\mathbf{y}_k$ are the ECEF positions computed from the broadcast ephemeris and the matrix **T(p)** is a transformation matrix from the inertial reference frame to the ECEF reference frame. For GPS, the navigation message is valid for a 4-hour time interval from $t_{\text{toe}} - 2h$ to $t_{\text{toe}} + 2h$, where t_{toe} is the time of ephemeris. Using the 16 Kepler-like parameters included in the navigation message, we can compute the satellite's ECEF position at any time during the 4-hour time interval. Note that the measurement model is nonlinear, since the parameters **p** appear nonlinearly in the matrix **T(p)**. The measurement noise $\boldsymbol{\epsilon}_k$ is assumed to be zero mean Gaussian white noise with covariance matrix **R**. The covariance matrix **R** is chosen to be diagonal, with equal variances $r = (1\text{ m})^2$ for each coordinate axis. This is the square of the reference accuracy of GPS broadcast position (IGS, 2013).

We want to estimate the state at time t_k given all the measurements up to that time. The solution for this Bayesian filtering problem is the posterior distribution $p(\mathbf{x}(t_k)|\,\mathbf{y}_{1:k})$. The filtering algorithm recursively computes the posterior distribution, starting from a prior distribution $p(\mathbf{x}(t_0))$. In the prediction step of the filter, we compute the distribution $p(\mathbf{x}(t_k)|\mathbf{y}_{1:k-1})$. In the update step, the information from the newest measurement is used to get the distribution $p(\mathbf{x}(t_k)|\,\mathbf{y}_{1:k})$.

In the nonlinear problem considered in this paper, the computations are intractable and approximations must be used. We consider here the Gaussian filtering approximation (Ito and Xiong, 2000), (Särkkä and Sarmavuori, 2013), where we approximate the posterior distribution with a normal distribution

$$p(\mathbf{x}(t_k) \mid \mathbf{y}_{1:k}) \approx \mathrm{Normal}(\mathbf{x}(t_k) \mid \mathbf{m}(t_k), \mathbf{P}(t_k)). \tag{31}$$

Using this approximation, the prediction step of the filter reduces to solving the ordinary differential equations

$$\frac{d\mathbf{m}}{dt} = \mathbb{E}[\mathbf{f}(\mathbf{x})], \tag{32}$$

$$\frac{d\mathbf{P}}{dt} = \mathbb{E}[(\mathbf{x}-\mathbf{m})\mathbf{f}^{\mathbf{T}}(\mathbf{x})] + \mathbb{E}[\mathbf{f}(\mathbf{x})(\mathbf{x}-\mathbf{m})^{\mathbf{T}}] + \mathbb{E}[\boldsymbol{\Sigma}(\mathbf{x})], \tag{33}$$

where

$$\boldsymbol{\Sigma}(\mathbf{x}) = \mathbf{D}(\mathbf{x})\mathbf{Q}\mathbf{D}^{\mathbf{T}}(\mathbf{x}). \tag{34}$$

The Eqs. (32) and (33) are integrated from t_{k-1} to t_k using initial conditions $\mathbf{m}(t_{k-1})$ and $\mathbf{P}(t_{k-1})$ from the previous filtering step.

Let $\mathbf{m}^-(t_k)$ and $\mathbf{P}^-(t_k)$ be the solutions to the differential equations (32) and (33) at the end point t_k. The update step for the filter is given by

$$\boldsymbol{\mu}_k = \mathbb{E}[\mathbf{h}(\mathbf{x})] \tag{35}$$

$$\mathbf{S}_k = \mathbb{E}[(\mathbf{h}(\mathbf{x})-\boldsymbol{\mu}_k)(\mathbf{h}(\mathbf{x})-\boldsymbol{\mu}_k)^T] + \mathbf{R} \tag{36}$$

$$\mathbf{D}_k = \mathbb{E}[(\mathbf{x}-\mathbf{m}_k^-)(\mathbf{h}(\mathbf{x})-\boldsymbol{\mu}_k)^T] \tag{37}$$

$$\mathbf{K}_k = \mathbf{D}_k\mathbf{S}_k^{-1} \tag{38}$$

$$\mathbf{m}(t_k) = \mathbf{m}^-(t_k) + \mathbf{K}_k(\mathbf{y}_k - \boldsymbol{\mu}_k) \tag{39}$$

$$\mathbf{P}(t_k) = \mathbf{P}^-(t_k) - \mathbf{K}_k\mathbf{S}_k\mathbf{K}_k^T \tag{40}$$

The expectations are now taken with respect to the distribution $\mathrm{Normal}(\mathbf{m}^-(t_k), \mathbf{P}^-(t_k))$. The Gaussian expectations in Eqs. (32), (33) and (35)-(37) are computed using numerical approximations. Using different approximations, different filters are obtained, as follows.

2.3.1 The extended Kalman filter

The extended Kalman filter can be derived by linearizing the nonlinear process and measurement functions using first order Taylor polynomial. The prediction step of the extended Kalman filter is given by (Jazwinski, 1970), (Särkkä and Sarmavuori, 2013)

$$\frac{d\mathbf{m}}{dt} = \mathbf{f}(\mathbf{m}) \tag{41}$$

$$\frac{d\mathbf{P}}{dt} = \mathbf{P}\mathbf{F}_{\mathbf{x}}^{\mathbf{T}}(\mathbf{m}) + \mathbf{F}_{\mathbf{x}}(\mathbf{m})\mathbf{P} + \mathbf{\Sigma}(\mathbf{m}) \tag{42}$$

where $\mathbf{F}_{\mathbf{x}}$ is the Jacobian matrix of **f.** The update step equations (35)-(37) are given by

$$\boldsymbol{\mu}_k = \mathbf{h}(\mathbf{m}^-(t_k)) \tag{43}$$

$$\mathbf{S}_k = \mathbf{H}_{\mathbf{x}}\mathbf{P}_k^-\mathbf{H}_{\mathbf{x}}^T + \mathbf{R} \tag{44}$$

$$\mathbf{D}_k = \mathbf{P}_k^-\mathbf{H}_{\mathbf{x}}^T \tag{45}$$

where $\mathbf{H}_{\mathbf{x}}$ is the Jacobian matrix of $\mathbf{h}$, evaluated at $\mathbf{m}^-(t_k)$.

2.3.2 The unscented Kalman filter and the cubature Kalman filter

The prediction step of the UKF in the present continuous-discrete case is given by (Särkkä and Sarmavuori, 2013)

$$\frac{d\mathbf{m}}{dt} = \sum_{i=0}^{2n} W_m^{(i)}\mathbf{f}(\mathbf{x}^{(i)}) \tag{46}$$

$$\frac{d\mathbf{P}}{dt} = \sum_{i=0}^{2n} W_P^{(i)}\left[\mathbf{f}\left(\mathbf{x}^{(i)}\right)\boldsymbol{\xi}_i^T\sqrt{\mathbf{P}}^T + \sqrt{\mathbf{P}}\boldsymbol{\xi}_i\mathbf{f}^T\left(\mathbf{x}^{(i)}\right) + \mathbf{\Sigma}(\mathbf{x}^{(i)})\right] \tag{47}$$

$$\mathbf{x}^{(i)} = \mathbf{m} + \sqrt{\mathbf{P}}\boldsymbol{\xi}_i, \tag{48}$$

$$\xi_0 = \mathbf{0}, \qquad \xi_i = \begin{cases} \sqrt{\lambda + n}\boldsymbol{e}_i & i = 1, \dots, n \\ -\sqrt{\lambda + n}\boldsymbol{e}_i & i = n+1, \dots, 2n \end{cases} \tag{49}$$

$$W_m^{(0)} = \frac{\lambda}{n+\lambda}, W_P^{(0)} = \frac{\lambda}{n+\lambda}(1 - \alpha^2 + \beta) \tag{50}$$

$$W_m^{(i)} = W_P^{(i)} = \frac{1}{2(n+\lambda)}, \qquad i = 1, \dots, 2n \tag{51}$$

where the matrix square root is defined to be the lower triangular matrix of the Cholesky decomposition $\mathbf{P} = \sqrt{\mathbf{P}}\sqrt{\mathbf{P}}^{\boldsymbol{T}}$ and $\lambda = \alpha^2(n + \kappa) - n$. Also, α, κ and β are parameters of the UKF. The differential equations are solved from time t_{k-1} to t_k.

Let $\mathbf{m}^-(t_k)$ and $\mathbf{P}^-(t_k)$ be the solutions at the end point. For the measurement update step, we form first the sigma-points

$$\mathcal{X}^{(i)} = \mathbf{m}^-(t_k) + \sqrt{\mathbf{P}^-(t_k)}\boldsymbol{\xi}_i, \qquad \mathcal{Y}^{(i)} = \mathbf{h}\left(\mathcal{X}^{(i)}\right), \tag{52}$$

where $\boldsymbol{\xi}_i$ are the same as in Eq. (43). Now we can approximate the expectations in Eqs. (35)-(37) with

$$\boldsymbol{\mu}_k = \sum_{i=0}^{2n} W_m^{(i)} \mathcal{Y}^{(i)} \tag{53}$$

$$\mathbf{S}_k = \sum_{i=0}^{2n} W_P^{(i)} (\mathcal{Y}^{(i)} - \boldsymbol{\mu}_k)(\mathcal{Y}^{(i)} - \boldsymbol{\mu}_k)^T \tag{54}$$

$$\mathbf{D}_k = \sum_{i=0}^{2n} W_P^{(i)} \left(\mathcal{X}^{(i)} - \mathbf{m}^-(t_k)\right)\left(\mathcal{Y}^{(i)} - \boldsymbol{\mu}_k\right)^T \tag{55}$$

Choosing $\alpha = 1$, $\beta = \kappa = 0$ we get the Cubature Kalman Filter (Särkkä and Sarmavuori, 2013).

2.5 Predicting the satellite's orbit

The orbit prediction algorithm is summarized in this section. We assume that the process noise covariance matrix has been estimated for the satellite in question. First we compute the ECEF positions, which are used as measurements, from the broadcast ephemeris. We use the time interval $t_{\mathrm{toe}} - 1.5\mathrm{h}$ to $t_{\mathrm{toe}} + 1.5\mathrm{h}$, with 5 minutes time step. The antenna offset is corrected using values provided by the NGA (NGA, 2013).

To start the filtering algorithm, we need a prior $p(\mathbf{x}(t_0))$ for the state at the initial time of $t_0 = t_{\mathrm{toe}} - 1.5\mathrm{h}$. We use a normal prior $p\big(\mathbf{x}(t_0)\big) = \mathrm{Normal}(\mathbf{x}(t_0) \mid \mathbf{m}(t_0), \mathbf{P}(t_0))$, where the mean and covariance are set as follows. For the position, we take the prior mean in ECEF coordinates from the broadcast message. The variance is taken to be $(1\ \mathrm{m})^2$, which is the square of the reference accuracy for GPS broadcast position (IGS, 2013). For velocity, the prior mean is the ECEF velocity computed from the broadcast message and the variance is taken to be $(10^{-4}\ \mathrm{m/s})^2$. The formulas for computing velocity from the broadcast ephemeris are given for example in (Korvenoja and Piché, 2000). The prior mean and variance for the EO-parameters are taken to be the mean and variance of the daily precise values provided by the IERS over the years 2008--2011. The prior means for the solar radiation pressure parameters are given in (Ala-Luhtala *et al.*, 2012). Satellite PRN 1 was replaced since results in (Ala-Luhtala *et al.*, 2012), so new values $\alpha_1 = 1.5464$ and $\alpha_2 = 0.0033$ were estimated. A prior variance of $(10^{-6})^2$ is used for both solar radiation pressure parameters.

For the prediction algorithm, we need the position and velocity prior in the inertial reference frame. The transformation to the inertial reference frame is a nonlinear function of the position, velocity and the parameters. To get the mean and variance in the inertial reference frame, we need to compute expectations of the form

$$\mathbf{m}_0^{\mathrm{IN}} = \mathbb{E}[\mathbf{g}(\mathbf{x}_0^{\mathrm{ECEF}})] \tag{56}$$

$$\mathbf{P}_0^{\mathrm{IN}} = \mathbb{E}\left[\left(\mathbf{g}(\mathbf{x}_0^{\mathrm{ECEF}}) - \mathbf{m}_0^{\mathrm{IN}}\right)\left(\mathbf{g}(\mathbf{x}_0^{\mathrm{ECEF}}) - \mathbf{m}_0^{\mathrm{IN}}\right)^T\right], \tag{57}$$

where $\mathbf{g}$ is the function that transforms the position and velocity into the inertial reference frame. The expectations in Eqs. (56) and (57) can be computed using the sigma point approximations in Eqs. (53) and (54).

After the estimation of the initial state, we can start the prediction from the time $t = t_{\mathrm{toe}} + 1.5\mathrm{h}$. The prediction can be computed by using the same filtering equations, but omitting the update step.

3 EVALUATING TEST RESULTS

Tests are done using broadcast ephemerides from GPS weeks 1679 to 1710. Each test consists of estimating the initial state from one broadcast ephemeris, and then predicting the orbit for a 5- day interval. For the UKF we use parameter values $\alpha = 0.001$, $\kappa = 0$ and $\beta = 0$.

The prediction errors for 5 day prediction are presented in Figure 1. The results show the combined orbit prediction errors for the whole GPS satellite constellation. For each GPS satellite 40 predictions were made using different initial times. With unhealthy satellites removed, the total number of predictions made was 1215. We note that satellite PRN 24 was unavailable for the time period used in this paper. We see that all the filtering methods give very similar means. The 95% interval for the total error is about 65m. Looking at the individual RTN error components, we see that most of the error is in the tangential direction. The values for the 95% intervals of RTN errors at day 5 of prediction are approximately 4m, 62m and 7m for the R, T and N coordinates respectively. The small radial error is a favourable result, since this component has the largest effect on the pseudorange error (Seppänen *et al*., 2012). Comparing the results to the errors using our earlier implementation (Ala-Luhtala *et al*., 2012), we can conclude that the method proposed in this paper seems to have about the same accuracy in terms of RTN errors.

The consistency of the orbit prediction is assessed by determining the proportion of cases the precise position $\mathbf{r}_{\mathrm{PE}}$ is inside the 95% probability ellipsoid defined by equation

$$(\mathbf{r} - \mathbf{r}_{\mathrm{PE}})^T \mathbf{P}^{-1} (\mathbf{r} - \mathbf{r}_{\mathrm{PE}}) \leq \beta,$$

where $\mathbf{r}$ and $\mathbf{P}$ are the predicted position and corresponding covariance matrix, and β is the value of the chi-squared inverse cumulative distribution function at point 0.95, with degrees of freedom 3. The consistency of the prediction measures how well the variance of the prediction corresponds to the realised error. The results are listed in Table 1. All methods have consistencies close to the ideal value of 0.95. This is a clear improvement over our earlier prediction algorithms**,** where consistencies of 0.25-0.40 were observed for predictions of over 3 days.

	EKF	UKF	CKF
$toe + 1.5h$	0.90	0.94	0.94
Day 1	0.97	0.99	0.98
Day 2	0.96	0.98	0.97
Day 3	0.94	0.98	0.95
Day 4	0.92	0.97	0.94
Day 5	0.91	0.97	0.93

Table 1: 95% consistencies of the predicted orbits

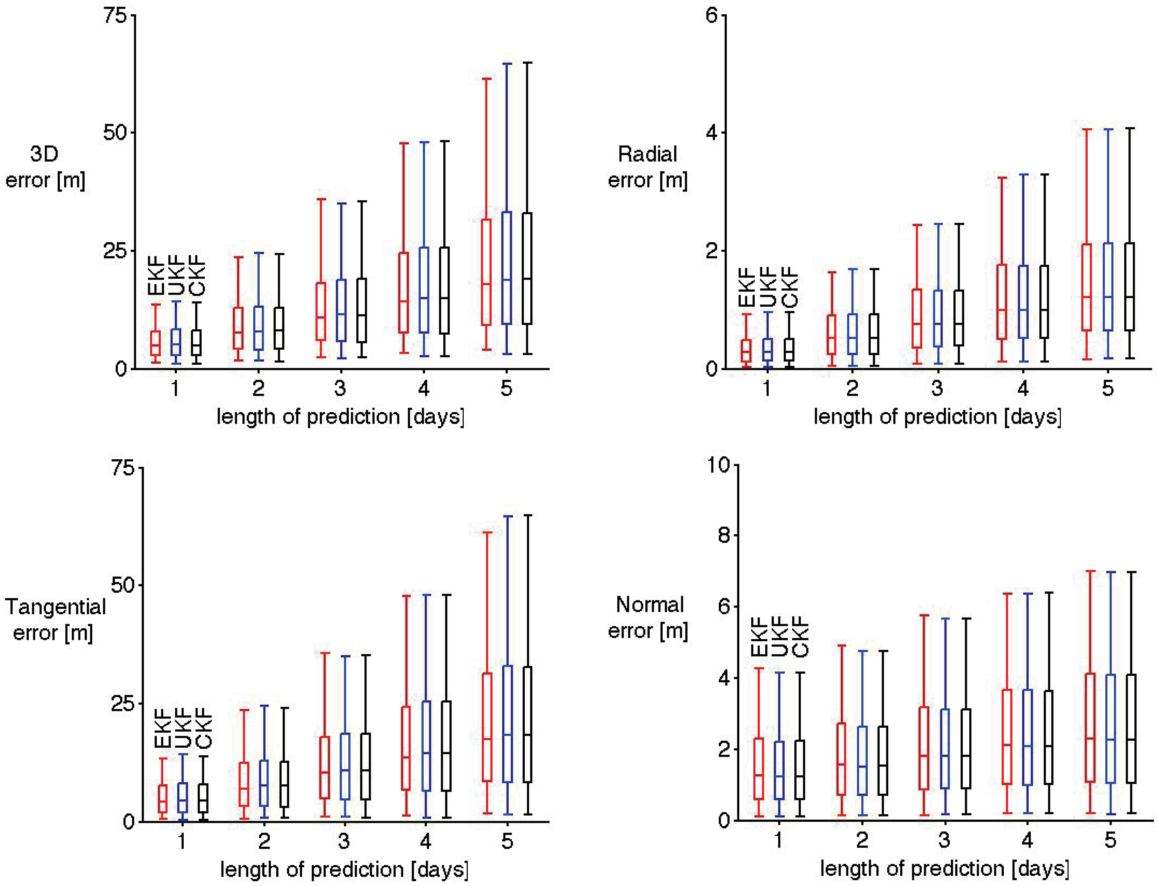

Figure 1: Box plots of the 3D and RTN errors for the different filters. The boxes present the 25%, 50% and 75% quantiles and the whiskers extending from the boxes show the 5% and 95% quantiles.

4 CONCLUSION AND DISCUSSION

This paper considers the prediction of GPS satellite orbits using information from the satellite's broadcast ephemeris. The model for the satellite's equation of motion includes the four major forces affecting the satellite: gravitational forces of Earth, Moon, and Sun, and the solar radiation pressure. The uncertainty caused by modelling errors and unmodeled forces are taken into account by including a Gaussian white noise term in the equation of motion. The covariance for the process noise is estimated using precise ephemeris data.

To start the prediction, we need to determine the satellite's initial position and velocity in an inertial reference frame, values of the EO-parameters and the solar radiation pressure parameters. We have shown how these parameters can be found using a Bayesian filtering algorithm. Three different filters were considered in this paper: the Extended Kalman filter, the Unscented Kalman filter and the Cubature Kalman filter. After the estimation of the initial state, the prediction can be carried out by computing only the prediction step of the filtering algorithm, using numerical integration to propagate the mean and variance.

The proposed method is assessed by computing the orbit prediction error in the RTN reference frame, using precise ephemerides from the IGS as reference. All the methods give almost identical errors for the predicted orbit. Errors are largest in the tangential direction, where the 95% interval of the error is about 62m for a prediction of 5 days. The 95% intervals of the errors in the radial and normal directions are about 4 m and 7 m respectively for the 5 day prediction.

Using filtering algorithms for prediction provides an estimate for the variance of the position. We analyse the predicted variance by checking if the true position of the satellite is inside the 95% probability ellipsoid for the predicted position. The results show that all the filtering methods provide good consistencies. The UKF and CKF tend to have slightly larger values for the predicted variance and overall slightly better consistency results than the EKF.

From the results we conclude that the UKF and CKF do not seem to offer any clear improvement over the EKF. The consistency results are slightly better for the UKF and CKF, but the computational cost in our implementation is about 6 times larger than for EKF. The method proposed in this paper seems to have about the same accuracy as our previous method, where a deterministic algorithm was used to solve the initial state (Ala-Luhtala *et al*., 2012). A downside in predicting also the variance is that the differential equations are no longer independent of the velocity, and we cannot use the efficient Runge-Kutta-Nyström numerical integration method. The results for this paper were produced using Runge-Kutta method of order 4 with 15 second time step. The relatively small time step means that we need a large number of force model evaluations in the numerical integration. The computations can be made more efficient by using a more sophisticated numerical integration method, e.g. the Gauss-Jackson method (Jackson, 1924), (Berry and Healy, 2004). With the Gauss-Jackson method, we could use much larger time step and hence reduce the number of force model evaluations.

The method described here could be easily implemented also for the European Galileo and Chinese Compass satellite systems, since their broadcast ephemeris format is similar to GPS. For GLONASS the implementation is more difficult, since each broadcast ephemeris is valid only for a 30-minute time interval. This interval may be too short for accurately estimating the parameters of the model. A possible solution may be to use more than one broadcast ephemeris.

ACKNOWLEDGEMENTS

This research was funded by Nokia Inc. and Tampere Doctoral Programme in Information Science and Engineering (TISE).

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