Cauchy-Riemann Operators in Octonionic Analysis

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Abstract. In this paper we first recall the definition of the octonion algebra and its algebraic properties. We derive the so called e_4-calculus and using it we obtain the list of generalized Cauchy-Riemann systems for octonionic monogenic functions.

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1. Introduction

The algebra of octonions is a well known non-associative division algebra. The second not so well known feature is that we may define a function theory in spirit of classical theory of complex holomorphic functions, and study its properties. This theory has its limitations, since the multiplication is neither commutative nor associative.

The octonions or Cayley numbers were first defined in 1843 by John T. Graves. Nowadays the systematic way to define octonions is the so called Cayley-Dickson construction, which we will also use in this paper. See historical remarks on ways to define the octonions in [1]. The first literary source of octonionic analysis is the article [4] published by Paolo Dentoni and Michele Sce in 1973. In the article authors introduced the basic operators and functions, and studied some of their function theoretic properties. We will recall all of their definitions that we will need in this paper. After Dentoni and Sce octonionic analysis has been studied, and some function theoretic properties, as well as solutions of the fundamental system, have been obtained, see for example [7, 8, 12] and their references.

In this paper we will aggregate existing results, unify their notations, and then study their new features. The first part of this paper is a survey of known results, where we give a detailed definition for the octonions. To make our practical calculations easier, we derive the so called "e_4-calculus" on it. Then we recall the notion of the Cauchy-Riemann operator. A function in its kernel is called monogenic. To find explicit monogenic functions directly from the definitions is too complicated, because the algebraic properties give too many limitations. To give an explicit characterization of monogenic functions, we separate variables, or represent the target space as a direct sum of subspaces. Using this trick we obtain a list of real, complex, and quaternionic partial differential equation systems, which are all generalizations of the complex Cauchy-Riemann system. These systems allow us to study explicit monogenic functions. We compute an example, assuming that the functions are biaxially symmetric.

Authors like to emphasize, that this work is the starting point for our future works on this fascinating field of mathematics. A reader should notice, that although the algebraic calculation rules look really complicated, one may still derive practical formulas to analyse the properties of the quantities of the theory. It seems that there are two possible ways to study the octonionic analysis in our sense. In the first one, one just takes results from classical complex or quaternionic analysis and tries to prove them. The second one is to concentrate to algebraic properties and features of the theory, and to try to find something totally new, in the framework of the algebra. We believe that the latter gives us deeper intuition of the theory, albeit the steps forward are not always so big.

2. On Octonion Algebra

In this section we recall the definition of the octonions and study their algebraic properties. We develop the so called e_4-calculus, which we will use during the rest of the paper to simplify practical computations.

2.1. Definition of Octonions

Let us denote the field of complex numbers by $\mathbb{C}$ and the skew field of quaternions by $\mathbb{H}$. We assume that the complex numbers are generated by the basis elements $\{1, i\}$ and the quaternions by $\{1, i, j, k\}$ with the well known defining relations

$$i^2 = j^2 = k^2 = ijk = -1.$$

We expect that the reader is familiar with the complex numbers and the quaternions. We give [1, 3, 9, 13] as a basic reference.

The systematic way to define octonions is the so called Cayley-Dickson construction. The construction produces a sequence of algebras over the field of real numbers, each with twice the dimension of the previous one. The previous algebra of a Cayley-Dickson step is assumed to be an algebra with a conjugation. Starting from the algebra of real numbers $\mathbb{R}$ with the trivial conjugation $x \mapsto x$, the Cayley-Dickson construction produces the algebra of complex numbers $\mathbb{C}$ with the conjugation $x + iy \mapsto x - iy$. Then applying Cayley-Dickson construction to the complex numbers produces quaternions

$\mathbb{H}$ with the conjugation. The quaternion conjugation is given as follows. An arbitrary $x \in \mathbb{H}$ is of the form

$$x = x_0 + \underline{x}$$

where $x_0 \in \mathbb{R}$ is the real part and $\underline{x} = x_1 i + x_2 j + x_3 k$ is the vector part of the quaternion x. Vector parts are isomorphic to the three dimensional Euclidean vector space $\mathbb{R}^3$. Then the conjugate of x obtained from the Cayley-Dickson construction is denoted by $\overline{x}$, and defined by

$$\overline{x} = x_0 - \underline{x}.$$

Now the Cayley-Dickson construction proceeds as follows. Consider pairs of quaternions, i.e., the space $\mathbb{H} \oplus \mathbb{H}$. We define the multiplication for the pairs as

$$(a, b)(c, d) = (ac - \overline{d}b, da + b\overline{c})$$

where $a, b, c, d \in \mathbb{H}$. With this multiplication the pairs of quaternions $\mathbb{H} \oplus \mathbb{H}$ is an eight dimensional algebra generated by the elements

$$e_0 := (1, 0),\ e_1 := (i, 0),\ e_2 := (j, 0),\ e_3 := (k, 0),$$
$$e_4 := (0, 1),\ e_5 := (0, i),\ e_6 := (0, j),\ e_7 := (0, k).$$

Denoting $1 := e_0$ and using the definition of the product, we may write the following table.

	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
1	1	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	e_1	-1	e_3	$-e_2$	e_5	$-e_4$	$-e_7$	e_6
e_2	e_2	$-e_3$	-1	e_1	e_6	e_7	$-e_4$	$-e_5$
e_3	e_3	e_2	$-e_1$	-1	e_7	$-e_6$	e_5	$-e_4$
e_4	e_4	$-e_5$	$-e_6$	$-e_7$	-1	e_1	e_2	e_3
e_5	e_5	e_4	$-e_7$	e_6	$-e_1$	-1	$-e_3$	e_2
e_6	e_6	e_7	e_4	$-e_5$	$-e_2$	e_3	-1	$-e_1$
e_7	e_7	$-e_6$	e_5	e_4	$-e_3$	$-e_2$	e_1	-1

We see that $e_0 = 1$ is the unit element of the algebra. Using the table, it is an easy task to see that the algebra is not associative nor commutative. We also see that the elements $\{1, e_1, e_2, e_3\}$ generate the quaternion algebra, i.e., $\mathbb{H}$ is a subalgebra.

The preceding algebra is called the *algebra of octonions* and it is denoted by $\mathbb{O}$. An arbitrary $x \in \mathbb{O}$ may be represented in the form

$$x = x_0 + \underline{x}$$

where $x_0 \in \mathbb{R}$ is the real part of the octonion x, and

$$\underline{x} = x_1 e_1 + x_2 e_2 + x_3 e_3 + x_4 e_4 + x_5 e_5 + x_6 e_6 + x_7 e_7$$

is the vector part, where $x_1, ..., x_7 \in \mathbb{R}$. Vector parts are isomorphic to the seven dimensional Euclidean vector space $\mathbb{R}^7$. The whole algebra of octonions is naturally identified with the vector space $\mathbb{R}^8$. The Cayley-Dickson construction produces naturally a conjugation $(a, b)^* := (\overline{a}, -b)$ for octonions, where

$\overline{a}$ is the quaternion conjugate. Because there is no risk of confusion, we will denote the conjugate of $x \in \mathbb{O}$ by $\overline{x}$. Using the real and vector parts, we have

$$\overline{x} = x_0 - \underline{x}.$$

We refer [1, 3, 5] for more detailed description of this construction.

2.2. Algebraic Properties

In this subsection we collect some algebraic properties and results of the octonions to better understand its algebraic structure.

The next result shows that $\mathbb{O}$ is an alternative division algebra.

Proposition 2.1 (cf. [13]**).** *If $x, y \in \mathbb{O}$ then*

$$x(xy) = x^2y, \ (xy)y = xy^2, \ (xy)x = x(yx),$$

and each non-zero $x \in \mathbb{O}$ has an inverse.

We see that the associativity holds in the case $(xy)x = x(yx)$. Unfortunately, this is almost the only non-trivial case when the associativity holds:

Proposition 2.2 (cf. [3]**).** *If*

$$x(ry) = (xr)y$$

for all $x, y \in \mathbb{O}$, then r is real.

Hence, the use of parentheses is something we need to keep in mind, when we compute using the octonions. The alternative properties given in Proposition 2.1 implies the following identities.

Proposition 2.3 (Moufang Laws, [3, 11]**).** *For each $x, y, z \in \mathbb{O}$*

$$(xy)(zx) = (x(yz))x = x((yz)x).$$

The inverse element x^{-1} of non-zero $x \in \mathbb{O}$ may be computed as follows. We define the norm by $|x| = \sqrt{x\overline{x}} = \sqrt{\overline{x}x}$. A straightforward computation shows that the norm is well defined, and

$$|x|^2 = \sum_{j=0}^{7} x_j^2.$$

In addition,

$$x^{-1} = \frac{\overline{x}}{|x|^2}.$$

By the following result $\mathbb{O}$ is a composition algebra. We will say that the octonions have a *multiplicative norm.*

Proposition 2.4 (cf. [3, 5, 13]**).** *The norm of $\mathbb{O}$ satisfies the* composition law

$$|xy| = |x||y|$$

for all $x, y \in \mathbb{O}$.

The composition law has algebraic implications for conjugation, since the conjugate may be written using the norm in the form $\overline{x} = |x+1|^2 - |x|^2 - 1 - x$. The following formulas are easy to prove by brute force computations. But the reader should notice, that actually they are consequences of the composition laws, not directly related only to octonions.

Proposition 2.5 (cf. [3]**).** *If $x, y \in \mathbb{O}$, then*

$$\overline{\overline{x}} = x \quad \text{and} \quad \overline{xy} = \overline{y}\,\overline{x}.$$

In general we say that an algebra A is a composition algebra, if it has a norm $N\colon A \to \mathbb{R}$ such that $N(ab) = N(a)N(b)$ for all $a, b \in A$. We know that $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are composition algebras. It is an interesting algebraic task to prove that actually this list is complete.

Theorem 2.6 (Hurwitz, [3]**).** *$\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ are the only composition algebras.*

2.3. e_4–Calculus

In principle it is possible to carry out all of the computations with the octonions just using the multiplication table. However, this often leads to very impractical calculations. In this subsection we study how to compute with the octonions in practise. Our starting point is the observation that every octonion $x \in \mathbb{O}$ can be written in the form

$$x = a + be_4,$$

where $a, b \in \mathbb{H}$. This form is called the *quaternionic form* of an octonion. If

$$x = x_0 + x_1e_1 + \cdots + x_7e_7,$$

then

$$a = x_0 + x_1e_1 + x_2e_2 + x_3e_3 \quad \text{and} \quad b = x_4 + x_5e_1 + x_6e_2 + x_7e_3.$$

Using the multiplication table we can prove the following.

Lemma 2.7. *Let e_i and e_j, where $i, j \in \{1, 2, 3\}$, be the basis elements for the vector part of the quaternions $\mathbb{H}$. Then*

(a) $e_i(e_je_4) = (e_je_i)e_4$,
(b) $(e_ie_4)e_j = -(e_ie_j)e_4$,
(c) $(e_ie_4)(e_je_4) = e_je_i$.

These rules imply the following rules for the vectors.

Lemma 2.8. *Let $\underline{a} = a_1e_1 + a_2e_2 + a_3e_3$ and $\underline{b} = b_1e_1 + b_2e_2 + b_3e_3 \in \mathbb{H}$ be vectors ($a_i, b_j \in \mathbb{R}$). Then*

(a) $e_4\underline{a} = -\underline{a}e_4$
(b) $e_4(\underline{a}e_4) = \underline{a}$
(c) $(\underline{a}e_4)e_4 = -\underline{a}$
(d) $\underline{a}(\underline{b}e_4) = (\underline{b}\,\underline{a})e_4$
(e) $(\underline{a}e_4)\underline{b} = -(\underline{a}\,\underline{b})e_4$
(f) $(\underline{a}e_4)(\underline{b}e_4) = \underline{b}\,\underline{a}$

Using preceding formulae, it is easy to obtain similar formulas for all quaternions $\underline{a} = a_0 + a_1e_1 + a_2e_2 + a_3e_3$ and $\underline{b} = b_0 + b_1e_1 + b_2e_2 + b_3e_3$.

Lemma 2.9. *Let $a, b \in \mathbb{H}$. Then*

(a) $e_4a = \overline{a}e_4$
(b) $e_4(ae_4) = -\overline{a}$
(c) $(ae_4)e_4 = -a$
(d) $a(be_4) = (ba)e_4$
(e) $(ae_4)b = (a\overline{b})e_4$
(f) $(ae_4)(be_4) = -\overline{b}a$

The previous relations are called the *rules of e_4-calculus for the octonions.* The idea is that when we multiply octonions, we overcome the lack of associativity of octonions by writing octonions in the quaternionic form and using the rules of Lemma 2.9 to modify the products into the quaternionic form. The situation is similar to computing with complex numbers, where we usually write $a + bi$ $(a, b \in \mathbb{R})$ and multiply using the relation $i^2 = -1$. As an example, we compute the following lemmata.

Lemma 2.10. *Let $x = a_1 + b_1e_4$ and $y = a_2 + b_2e_4$ $(a_i, b_j \in \mathbb{H})$ be octonions in the quaternionic form. Then their product in quaternionic form is*

$$xy = (a_1a_2 - \overline{b}_2b_1) + (b_1\overline{a}_2 + b_2a_1)e_4.$$

Proof. Apply Lemma 2.9:

$$\begin{aligned} xy &= (a_1 + b_1e_4)(a_2 + b_2e_4) \\ &= a_1a_2 + (b_1e_4)a_2 + a_1(b_2e_4) + (b_1e_4)(b_2e_4) \\ &= a_1a_2 + (b_1\overline{a}_2)e_4 + (b_2a_1)e_4 - \overline{b}_2b_1. \end{aligned}$$ □

Lemma 2.11. *For an octonion $a + be_4$ $(a, b \in \mathbb{H})$ in the quaternionic form we have*

$$\begin{aligned} \overline{a + be_4} &= \overline{a} - be_4, \\ |a + be_4|^2 &= |a|^2 + |b|^2. \end{aligned}$$

3. Cauchy-Riemann Operators

In this section we begin to study the basic analytical properties of the octonion valued functions. First we recall some basic properties and after that we express some equivalent systems related to the decomposition of octonions. Using these equivalent systems we may avoid difficulties caused by non-associativity.

3.1. Definitions and Basic Properties

In the octonionic analysis we consider functions defined on a set $\Omega \subset \mathbb{R}^8 \cong \mathbb{O}$ and taking values in $\mathbb{O}$. Similarly than in the case of quaternionic analysis, we may consider octonionic analyticity, and see that the generalization of Cauchy-Riemann equations is the only way to get a nice function class (see [8]). We begin by connecting to an octonion

$$x = x_0 + x_1 e_1 + \cdots + x_7 e_7$$

the derivative operator

$$\partial_x = \partial_{x_0} + e_1 \partial_{x_1} + \cdots + e_7 \partial_{x_7}. \tag{3.1}$$

This derivative operator is called the *Cauchy-Riemann operator*. The vector part of it

$$\partial_{\underline{x}} = e_1 \partial_{x_1} + \cdots + e_7 \partial_{x_7} \tag{3.2}$$

is called the *Dirac operator*. Now it is easy to represent the Cauchy-Riemann operator and its conjugate as

$$\partial_x = \partial_{x_0} + \partial_{\underline{x}} \quad \text{and} \quad \partial_{\overline{x}} = \partial_{x_0} - \partial_{\underline{x}}.$$

Remark 3.1. These operators were defined by Dentoni and Sce in [4]. They called the operator ∂_x the *operator of Fueter and Moisil*. In this paper we will follow the notation used in Clifford analysis (cf. [2]) hoping that the reader will get a better understanding of the octonionic analysis by comparing them to each other. In [6] we study the similarities and differences between Clifford and octonionic analyses.

The function $f\colon \Omega \subset \mathbb{O} \to \mathbb{O}$ is of the form

$$f = \sum_{j=0}^{7} e_j f_j$$

where $f_j\colon \Omega \subset \mathbb{O} \to \mathbb{R}$. If the components of f have partial derivatives, then ∂_x operates from the left as

$$\partial_x f = \sum_{i=0}^{7} e_i \partial_{x_i} f = \sum_{i,j=0}^{7} e_i e_j \partial_{x_i} f_j$$

and from the right as

$$f \partial_x = \sum_{i=0}^{7} f e_i \partial_{x_i} = \sum_{i,j=0}^{7} e_j e_i \partial_{x_i} f_j.$$

Definition 3.2. Let $\Omega \subset \mathbb{O}$ be open and assume that the components of $f\colon \Omega \to \mathbb{O}$ have partial derivatives. If $\partial_x f = 0$ (resp. $f \partial_x = 0$) in Ω, then f is called *left (resp. right) monogenic in* Ω.

Remark 3.3. These functions were defined in [4], where the authors called them *left- and right regular*.

We define the *Laplace operator* as

$$\Delta_x = \partial_{x_0}^2 + \partial_{x_1}^2 + \cdots + \partial_{x_7}^2.$$

Because $\overline{x}x = x\overline{x}$, it follows that in $C^2(\Omega, \mathbb{O})$, where $\Omega \subset \mathbb{O}$ is open,

$$\partial_{\overline{x}}\partial_x = \partial_x\partial_{\overline{x}} = \Delta_x. \tag{3.3}$$

From Proposition 2.1 it follows $(\overline{x}x)y = \overline{x}(xy)$, and therefore for $f \in C^2(\Omega, \mathbb{O})$

$$(\partial_{\overline{x}}\partial_x)f = \partial_{\overline{x}}(\partial_x f) = \partial_x(\partial_{\overline{x}} f).$$

Similarly

$$f(\partial_{\overline{x}}\partial_x) = (f\partial_{\overline{x}})\partial_x = (f\partial_x)\partial_{\overline{x}}.$$

These properties give us, like in the quaternionic case a relation between monogenicity and harmonicity.

Proposition 3.4 (cf. [4]**).** *If a function $f \in C^2(\Omega, \mathbb{O})$ is left or right monogenic, then f is harmonic.*

Some basic function theoretical results have already been studied in octonionic analysis, e.g., the following classical integral formula holds.

Theorem 3.5 (cf. [7]**).** *Let M be an 8-dimensional compact, oriented, smooth manifold with boundary ∂M contained in some open connected subset $\Omega \subset \mathbb{R}^8$. If the function $f\colon \Omega \to \mathbb{O}$ is left monogenic, then for each $x \in M$*

$$f(x) = \frac{1}{\omega_8} \int_{\partial M} \frac{\overline{x-y}}{|x-y|^8} (n(y)f(y))\, dS(y),$$

where ω_8 is the volume of the sphere S^7, n is the outward pointing unit normal on ∂M, and dS is the scalar surface element on the boundary.

Using this theorem, similarly than in the quaternionic analysis case, we may prove many function theoretic results, for example the mean value theorem, maximum modulus theorem, and Weierstrass type approximation theorems, see [7].

3.2. Equivalent Systems for Monogenic Functions

In this paper our aim is to gain a better understanding of the monogenic functions in the octonionic analysis. In this subsection we consider equivalent real, complex, and quaternionic formulations of the equation $\partial_x f = 0$. The idea is that by separating the variables we obtain equivalent systems, which allow us to avoid problems caused by non-associativity of the octonions. These systems are motivated by the use of the subalgebras in the Cayley-Dickson process.

3.2.1. A Real Decomposition. We start from the most trivial case, which is already well known (see [4]). We observe that the octonion algebra may be represented as a direct sum of 1-dimensional real subspaces

$$\mathbb{O} = \bigoplus_{j=0}^{7} e_j \mathbb{R}.$$

Now we can separate the variables and also split the target space and the Cauchy-Riemann operator due to this decomposition: we write the variables, the functions, and the Cauchy-Riemann operator in the form

$$x = \sum_{j=0}^{7} x_j e_j, \quad f = \sum_{j=0}^{7} f_j e_j, \quad \partial_x = \sum_{j=0}^{7} e_j \partial_{x_j}.$$

A straightforward computation yields that f is left monogenic if and only if its component functions $f_0, f_1, ..., f_7$ satisfy the 8×8 real partial differential equation system

$$\begin{cases} \partial_{x_0} f_0 - \partial_{x_1} f_1 - \ldots - \partial_{x_7} f_7 = 0, \\ \partial_{x_0} f_1 + \partial_{x_1} f_0 + \partial_{x_2} f_3 - \partial_{x_3} f_2 + \partial_{x_4} f_5 - \partial_{x_5} f_4 - \partial_{x_6} f_7 + \partial_{x_7} f_6 = 0, \\ \partial_{x_0} f_2 + \partial_{x_2} f_0 - \partial_{x_1} f_3 + \partial_{x_3} f_1 + \partial_{x_4} f_6 - \partial_{x_6} f_4 + \partial_{x_5} f_7 - \partial_{x_7} f_5 = 0, \\ \partial_{x_0} f_3 + \partial_{x_3} f_0 + \partial_{x_1} f_2 - \partial_{x_2} f_1 + \partial_{x_4} f_7 - \partial_{x_7} f_4 - \partial_{x_5} f_6 + \partial_{x_6} f_5 = 0, \\ \partial_{x_0} f_4 + \partial_{x_4} f_0 - \partial_{x_1} f_5 + \partial_{x_5} f_1 - \partial_{x_2} f_6 + \partial_{x_6} f_2 - \partial_{x_3} f_7 + \partial_{x_7} f_3 = 0, \\ \partial_{x_0} f_5 + \partial_{x_5} f_0 + \partial_{x_1} f_4 - \partial_{x_4} f_1 - \partial_{x_2} f_7 + \partial_{x_7} f_2 + \partial_{x_3} f_6 - \partial_{x_6} f_3 = 0, \\ \partial_{x_0} f_6 + \partial_{x_6} f_0 + \partial_{x_1} f_7 - \partial_{x_7} f_1 + \partial_{x_2} f_4 - \partial_{x_4} f_2 - \partial_{x_3} f_5 + \partial_{x_5} f_3 = 0, \\ \partial_{x_0} f_7 + \partial_{x_7} f_0 - \partial_{x_1} f_6 + \partial_{x_6} f_1 + \partial_{x_2} f_5 - \partial_{x_5} f_2 + \partial_{x_3} f_4 - \partial_{x_4} f_3 = 0. \end{cases}$$

This explicit characterization of monogenic functions by the real partial differential equation system has its advantages. For example, in [2] the system is used to study monogenic functions using computer algebra.

Example. Let $h\colon \Omega \to \mathbb{R}$ be a harmonic function on an open set $\Omega \subset \mathbb{R}^8$. Then we may construct a solution f by setting

$$f_0 = \partial_{x_0} h, \quad f_j = -\partial_{x_j} h, \quad j = 1, ..., 7.$$

Remark 3.6. The 8×8 system above is different to the classical Riesz system of Stein and Weiss can be expressed

$$\begin{cases} \partial_{x_0} f_0 - \partial_{x_1} f_1 - \ldots - \partial_{x_7} f_7 = 0, \\ \partial_{x_0} f_i + \partial_{x_i} f_0 = 0, \quad i = 1, \ldots, 7, \\ \partial_{x_i} f_j - \partial_{x_j} f_i = 0, \quad i, j = 1, \ldots, 7, \ i \neq j, \end{cases}$$

see [15]. We discuss the connection between these systems in more detail in [6] and deduce that solutions of the Riesz system are equivalent with both sided monogenic functions.

3.2.2. A Complex Decomposition. The preceding subsection motivates us to proceed further using similar techniques. Now we observe that the octonions may be express as a direct sum of complex numbers

$$\mathbb{O} = \mathbb{C} \oplus \mathbb{C}e_2 \oplus (\mathbb{C} \oplus \mathbb{C}e_2)e_4,$$

where a basis of $\mathbb{C}$ is $\{1, e_1\}$. We may write an octonion with respect to this decomposition as

$$x = z_1 + z_2e_2 + (z_3 + z_4e_2)e_4,$$

where we denote

$$z_1 = x_0 + x_1e_1, \quad z_2 = x_2 + x_3e_1,$$
$$z_3 = x_4 + x_5e_1, \quad z_4 = x_6 + x_7e_1.$$

Similarly we express a function f as a sum of complex valued functions $f_j = f_j(z_1, z_2, z_3, z_4)$ in the form

$$f = f_1 + f_2e_2 + (f_3 + f_4e_2)e_4. \tag{3.4}$$

If we define complex Cauchy-Riemann operators as

$$\partial_{z_1} = \partial_{x_0} + e_1\partial_{x_1}, \quad \partial_{z_2} = \partial_{x_2} + e_1\partial_{x_3},$$
$$\partial_{z_3} = \partial_{x_4} + e_1\partial_{x_5}, \quad \partial_{z_4} = \partial_{x_6} + e_1\partial_{x_7},$$

we may split the Cauchy-Riemann operator as

$$\partial_x = \partial_{z_1} + \partial_{z_2}e_2 + (\partial_{z_3} + \partial_{z_4}e_2)e_4.$$

Again, after straightforward computations, one have that $\partial_x f = 0$ is equivalent to the complex 4×4 equation system

$$\begin{cases} \partial_{z_1} f_1 - \partial_{z_2}\overline{f}_2 - \partial_{z_3}\overline{f}_3 - \partial_{\overline{z}_4} f_4 = 0, \\ \partial_{z_1} f_2 + \partial_{z_2}\overline{f}_1 + \partial_{\overline{z}_3} f_4 - \partial_{z_4}\overline{f}_3 = 0, \\ \partial_{z_1} f_3 - \partial_{\overline{z}_2} f_4 + \partial_{z_3}\overline{f}_1 + \partial_{z_4}\overline{f}_2 = 0, \\ \partial_{\overline{z}_1} f_4 + \partial_{z_2} f_3 - \partial_{z_3} f_2 + \partial_{z_4} f_1 = 0. \end{cases}$$

Similarly than in the case of real decomposition, one may construct solutions to this system using complex harmonic functions. One can also prove that the component functions are harmonic in the sense of several complex variables. We leave the details for the reader. We do not discuss this decomposition detailed here, but our aim is to study it more in future.

3.3. Quaternionic Cauchy-Riemann Equations

In this section we extend our procedure to the next level. We express the octonion algebra as a direct sum

$$\mathbb{O} = \mathbb{H} \oplus \mathbb{H}e_4$$

of quaternions. This decomposition corresponds to the quaternionic forms of octonions. Every function $f\colon \Omega \subset \mathbb{O} \to \mathbb{O}$ can be written in the form $f = g + he_4$, where g and h are quaternionic valued. If we also write the variable in the quaternionic form $x = u+ve_4$, we observe that $g, h\colon \mathbb{H}\times\mathbb{H} \to \mathbb{H}$ are functions of two quaternionic variables. Similarly we split

$$\partial_x = \partial_u + \partial_v e_4,$$

where ∂_u and ∂_v are quaternionic Cauchy-Riemann operators. The rules of e_4-calculus in Lemma 2.9 give us immediately:

Lemma 3.7. *Suppose that the components of $f\colon \Omega \subset \mathbb{H} \to \mathbb{H}$ have partial derivatives, and let $\partial_u = \partial_{u_0} + e_1\partial_{u_1} + e_2\partial_{u_2} + e_3\partial_{u_3}$ be the quaternionic Cauchy-Riemann operator. Then we have*

(a) $\partial_u(fe_4) = (f\partial_u)e_4$,
(b) $(\partial_u e_4)f = (\partial_u\overline{f})e_4$,
(c) $(\partial_u e_4)(fe_4) = -\overline{f}\partial_u$,
(d) $(fe_4)\partial_u = (f\partial_{\overline{u}})e_4$,
(e) $f(\partial_u e_4) = (\partial_u f)e_4$,
(f) $(fe_4)(\partial_u e_4) = -\partial_{\overline{u}}f$.

Using these rules, we obtain the following equivalent systems:

Proposition 3.8 (Quaternionic Cauchy-Riemann systems). *Assume that the components of $f\colon \Omega \subset \mathbb{H}\times\mathbb{H} \to \mathbb{O}$ have partial derivatives, and write f in the quaternionic form $f = g + he_4$, where $g, h\colon \Omega \subset \mathbb{H}\times\mathbb{H} \to \mathbb{H}$. Then*

(a) $\partial_x f = 0$ *if and only if*

$$\begin{cases} \partial_u g = \overline{h}\partial_v, \\ h\partial_u = -\partial_v\overline{g}. \end{cases} \tag{3.5}$$

(b) $f\partial_x = 0$ *if and only if*

$$\begin{cases} g\partial_u = \partial_{\overline{v}}h, \\ h\partial_{\overline{u}} = -\partial_v g. \end{cases} \tag{3.6}$$

Proof. Let $f = g + he_4$ and $\partial_x = \partial_u + \partial_v e_4$. Then we have

$$\begin{aligned} \partial_x f &= (\partial_u + \partial_v e_4)(g + he_4) \\ &= \partial_u g + \partial_u(he_4) + (\partial_v e_4)g + (\partial_v e_4)(he_4) \\ &= \partial_u g + (h\partial_u)e_4 + (\partial_v\overline{g})e_4 - \overline{h}\partial_v, \end{aligned}$$

which gives us (a). Computations for (b) are similar. □

As a special case:

Corollary 3.9. (a) *If $\underline{g} = \underline{h} = 0$, then $\partial_x f = 0$ if and only if*

$$\begin{cases} \partial_u g_0 = \partial_v h_0, \\ \partial_u h_0 = -\partial_v g_0. \end{cases}$$

(b) *If $g_0 = h_0 = 0$, then $\partial_x f = 0$ if and only if*

$$\begin{cases} \partial_u \underline{g} = -\underline{h}\partial_v, \\ \underline{h}\partial_u = \partial_v \underline{g}. \end{cases}$$

Proposition 3.4 implies:

Proposition 3.10. *If g and $h\colon \Omega \subset \mathbb{H}\times\mathbb{H} \to \mathbb{H}$ satisfy Cauchy-Riemann system* (3.5) *or* (3.6), *then g and h are harmonic.*

In this point of view, octonionic analysis is actually two variable quaternionic analysis and there is a natural biaxial behaviour. Since $\mathbb{O}$ is an alternating algebra, i.e., $x(yx) = (xy)x$, we may also define *inframonogenic functions* as in the classical case (see [10]) as functions which satisfy the system $\partial_x f \partial_x = 0$. For inframonogenic functions we obtain the following equivalent decomposition.

Proposition 3.11. *A function $f = g + he_4 \in C^2(\mathbb{O}, \mathbb{O})$ is inframonogenic if and only if*

$$\Delta_v \overline{g} - \partial_u g \partial_u + \overline{h} \partial_v \partial_u + \partial_{\overline{v}} h \partial_u = 0,$$
$$\Delta_u h + \partial_v \overline{g} \partial_{\overline{u}} + \partial_v \partial_u g - \partial_v \overline{h} \partial_v = 0.$$

Proof. As above,

$$\partial_x f = q_1 + q_2 e_4,$$

where $q_1 = \partial_u g - \overline{h} \partial_v$ and $q_2 = \partial_v \overline{g} + h \partial_u$. Using the differentiation rules of Lemma 3.7 we compute

$$\begin{aligned} \partial_x f \partial_x &= (q_1 + q_2 e_4)(\partial_u + \partial_v e_4) \\ &= q_1 \partial_u + (q_2 e_4)\partial_u + q_1(\partial_v e_4) + (q_2 e_4)(\partial_v e_4) \\ &= q_1 \partial_u + (q_2 \partial_{\overline{u}}) e_4 + (\partial_v q_1) e_4 - \partial_{\overline{v}} q_2 \\ &= \partial_u g \partial_u - \overline{h} \partial_v \partial_u - \Delta_v \overline{g} - \partial_{\overline{v}} h \partial_u \\ &\quad + (\partial_v \overline{g} \partial_{\overline{u}} + \Delta_u h + \partial_v \partial_u g - \partial_v \overline{h} \partial_v) e_4. \end{aligned}$$ □

3.4. Real Biaxially Radial Solutions – a Connection to Holomorphic Functions

In this last subsection we present the following example. Let us consider real valued functions g and $h\colon \mathbb{O} \to \mathbb{R}$ which are axially symmetric (invariant under the action of the spingroup) in the following sense: for all $q \in S^3$

$$g(u_0, \underline{u}, v_0, \underline{v}) = g(u_0, \overline{q}\underline{u}q, v_0, \overline{q}\underline{v}q). \tag{3.7}$$

Then the functions g and h depend only on u_0, v_0, $a = |\underline{u}|^2$, $b = |\underline{v}|^2$ and $c = \langle \underline{u}, \underline{v} \rangle$, see [14, Section 4]. Using the definition of the Dirac operator (3.2) and the Chain rule we get

$$\partial_{\underline{u}} g = 2\underline{u}\partial_a g + \underline{v}\partial_c g, \quad \partial_{\underline{v}} g = 2\underline{v}\partial_b g + \underline{u}\partial_c g. \tag{3.8}$$

By Corollary 3.9 (a) the function $f = g + he_4$ is monogenic if g and h satisfy the system

$$\begin{cases} \partial_u g = \partial_v h, \\ \partial_u h = -\partial_v g. \end{cases} \tag{3.9}$$

Substituting (3.8) and similar equations for h into (3.9), we obtain the system

$$\begin{cases} 2\partial_a g - \partial_c h = 0, \\ \partial_c g - 2\partial_b h = 0, \\ 2\partial_a h + \partial_c g = 0, \\ \partial_c h + 2\partial_b g = 0, \\ \partial_{u_0} g - \partial_{v_0} h = 0, \\ \partial_{u_0} h + \partial_{v_0} g = 0. \end{cases} \tag{3.10}$$

The first four equations of (3.10) give us

$$\begin{cases} \partial_a g + \partial_b g = 0, \\ \partial_a h + \partial_b h = 0. \end{cases} \tag{3.11}$$

Let us consider solutions of (3.11) of the form

$$g = G(u_0, v_0, a-b, c) \quad \text{and} \quad h = H(u_0, v_0, a-b, c). \tag{3.12}$$

The first four equations of (3.10) yield

$$\begin{cases} 2\partial_d G - \partial_c H = 0, \\ 2\partial_d H + \partial_c G = 0, \end{cases} \tag{3.13}$$

where $d = a - b$. Let us look for a solution of the form $H = e^d p(c, u_0, v_0)$ and $G = e^d q(c, u_0, v_0)$. The system (3.13) then reads

$$\begin{cases} 2q - \partial_c p = 0, \\ 2p + \partial_c q = 0. \end{cases} \tag{3.14}$$

This system has a solution

$$\begin{cases} p(c, u_0, v_0) = -\alpha(u_0, v_0)\cos(2c) + \beta(u_0, v_0)\sin(2c), \\ q(c, u_0, v_0) = \alpha(u_0, v_0)\sin(2c) + \beta(u_0, v_0)\cos(2c), \end{cases} \tag{3.15}$$

i.e.,

$$\begin{cases} g = \alpha(u_0, v_0)e^d \sin(2c) + \beta(u_0, v_0)e^d \cos(2c), \\ h = -\alpha(u_0, v_0)e^d \cos(2c) + \beta(u_0, v_0)e^d \sin(2c). \end{cases} \tag{3.16}$$

Substituting these into the last two equations of (3.10) we obtain

$$\begin{cases} \partial_{u_0}\alpha = \partial_{v_0}\beta, \\ \partial_{v_0}\alpha = -\partial_{u_0}\beta. \end{cases} \tag{3.17}$$

This is the Cauchy-Riemann system. Hence, any holomorphic function $\alpha + i\beta$ gives us a monogenic function $f\colon \mathbb{H} \times \mathbb{H} \to \mathbb{O}$,

$$\begin{aligned} &f(u_0 + \underline{u} + (v_0 + \underline{v})e_4) = \\ &e^{|\underline{u}|^2 - |\underline{v}|^2}\Big(\alpha(u_0, v_0)\sin(2\langle \underline{u}, \underline{v}\rangle) + \beta(u_0, v_0)\cos(2\langle \underline{u}, \underline{v}\rangle) \\ &\qquad + \big(-\alpha(u_0, v_0)\cos(2\langle \underline{u}, \underline{v}\rangle) + \beta(u_0, v_0)\sin(2\langle \underline{u}, \underline{v}\rangle\big)e_4\Big). \end{aligned}$$

This method allows us to construct biaxially rotation invariant monogenic functions. In the above example the function takes values in $\mathbb{C}$, generated by $\{1, e_4\}$. An interesting problem in the future is to find general biaxially rotation invariant functions. That kind of explicit functions would help us to better understand monogenic functions in octonionic analysis.

References

[1] Baez, J. C., *The octonions.* Bull. Amer. Math. Soc. (N.S.) 39 (2002), no. 2, 145–205.

[2] Colombo F., Sabadini I., Sommen F., Struppa D., *Analysis of Dirac systems and computational algebra*, Progress in Mathematical Physics, Vol. 39, Birkhäuser, Boston, 2004.

[3] Conway, J. H., Smith, D. A., *On quaternions and octonions: their geometry, arithmetic, and symmetry.* A K Peters, Ltd., Natick, MA, 2003.

[4] Dentoni P., Sce M., *Funzioni regolari nellalgebra di Cayley*, Rend. Sem. Mat. Univ. Padova 50(1973), 251–267

[5] Hermann, R., *Spinors, Clifford and Cayley algebras*, Interdisciplinary mathematics, Vol. VII. Department of Mathematics, Rutgers University, New Brunswick, N.J., 1974.

[6] Kauhanen J., Orelma H., *Some Theoretical Remarks of Octonionic Analysis*, AIP Conf. Proc. 1907, 030056 (2017)

[7] Li, X., Peng, L., *The Cauchy integral formulas on the octonions,* Bull. Belg. Math. Soc. Simon Stevin 9 (2002), no. 1, 47–64.

[8] Li, X., Kai, Z., Peng, L., *Characterization of octonionic analytic functions*, Complex Var. Theory Appl. 50 (2005), no. 13, 1031–1040.

[9] Lounesto, P., *Clifford algebras and spinors. Second edition*, London Mathematical Society Lecture Note Series, 286. Cambridge University Press, Cambridge, 2001.

[10] Malonek, H., Peña Peña, D., Sommen, F., *Fischer decomposition by inframonogenic functions,* Cubo 12 (2010), no. 2, 189–197.

[11] Moufang, R., *Alternativkörper und der Satz vom vollständigen Vierseit* (D_9), Abh. Math. Sem. Univ. Hamburg 9 (1933), no. 1, 207–222.

[12] Nôno K.,*On the octonionic linearization of Laplacian and octonionic function theory*, Bull. Fukuoka Univ. Ed., Part III, 37(1988), 1–15.

[13] Porteous, I., *Clifford algebras and the classical groups,* Cambridge Studies in Advanced Mathematics, 50. Cambridge University Press, Cambridge, 1995.

[14] Sommen, F., *Clifford analysis in two and several vector variables,* Appl. Anal. 73 (1999), no. 1-2, 225–253.

[15] Stein E., Weiss G., *Generalization of the Cauchy-Riemann equations and representation of the rotation group*, Amer. J. Math. 90 (1968), 163–199.

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