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A CRITICAL COMPARISON OF TWO DISCONTINUOUS GALERKIN METHODS FOR THE NAVIER-STOKES EQUATIONS USING SOLENOIDAL APPROXIMATIONS

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Abstract. *This paper compares two methods to solve incompressible problems, in particular the Navier-Stokes equations, using a discontinuous polynomial interpolation that is exactly divergence-free in each element. The first method is an Interior Penalty Method, whereas the second method follows the Compact Discontinuous Galerkin [1] approach for the diffusive part of the problem. In both cases the Navier-Stokes equations are then solved using a fractional-step method, using an implicit method for the diffusion part and a semi-implicit method for the convection. Numerical examples compare the efficiency and the accuracy of the two proposed methods.*

1 INTRODUCTION

Since its introduction the Discontinuous Galerkin (DG) method has become popular for its high-order accuracy. DG methods are actually particulary well suited for the construction of high-order accurate space discretization of problems on general unstructured grids, by the mean of high-order polynomial approximations within elements. Another advantage of the DG discretization is its compactness since the coupling is strictly restricted to the elements sharing a face.

Recently, the DG method has been successfully used to treat convection-diffusion problems such as the solution of the Navier-Stokes equations. For example Cockburn et al. developed an accurate and stable scheme, the Local Discontinuous Galerkin (LDG) method for the Stokes, Oseen and Navier-Stokes equations [2, 3]. One drawback of this scheme is

the loss of the compactness of the discretization form, due to the fact that the degrees of freedom in one element are connected not only to those in the neighboring elements but also to those in the elements neighboring the immediate neighbors. In [1] a Compact Discontinuous Galerkin (CDG) method is applied to elliptic problems. The CDG method is very similar to the LDG method but it eliminates the coupled degrees of freedom between the non-neighboring elements.

Bassi and Rebay also proposed methods to solve the Navier-Stokes equations using high order DG methods [4] but especially apply it to compressible problems. To solve incompressible problems, an interesting possibility is to use explicit divergence-free bases. In [5] a DG method with solenoidal approximation has been applied to the Stokes equations and an Interior Penalty Method (IPM) employed to construct the weak form.

The aim of this paper is to formulate the Compact Discontinuous Galerkin method and the Interior Penalty Method for the incompressible Navier-Stokes equations using solenoidal approximations as seen in [5] and then to present a critical comparison between them.

2 TWO FORMULATIONS OF THE NAVIER-STOKES PROBLEM

Let $\Omega \subset \mathbb{R}^{\mathtt{n_{sd}}}$ be an open, bounded domain with piecewise linear boundary $\partial\Omega$ and $\mathtt{n_{sd}}$ the number of spatial dimensions. Suppose that Ω is partitioned in $\mathtt{n_{el}}$ disjoint subdomains K, which for example correspond to different materials, with also piecewise linear boundaries ∂K that define an internal interphase Γ. The strong form for the homogeneous steady Navier-Stokes problem can be written as:

$$-2\nu\boldsymbol{\nabla}\cdot\boldsymbol{\nabla}^{\mathtt{S}}\boldsymbol{u} + \boldsymbol{\nabla}p + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) = \boldsymbol{f} \qquad \text{in } \Omega \tag{1a}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \qquad \text{in } \Omega \tag{1b}$$

$$\boldsymbol{u} = \boldsymbol{u}_D \qquad \text{on } \Gamma_D \tag{1c}$$

$$-p\boldsymbol{n} + 2\nu(\boldsymbol{n}\cdot\boldsymbol{\nabla}^{\mathtt{S}})\boldsymbol{u} = \boldsymbol{t} \qquad \text{on } \Gamma_N \tag{1d}$$

where $\boldsymbol{f} \in \mathcal{L}_2(\Omega)$ is a source term, $\boldsymbol{u}$ the flux velocity, p its pressure, ν the kinematic viscosity and $\boldsymbol{\nabla}^{\mathtt{S}} = \frac{1}{2}(\boldsymbol{\nabla} + \boldsymbol{\nabla}^T)$.

2.1 The Compact Discontinuous Galerkin method

In the CDG approach [1], the problem is first transformed into a first order in derivative problem, introducing the gradient of the velocity as an additional variable. This additional variable is then eliminated at an elemental level, using the so-called lifting operators.

Introducing the velocity gradient $\boldsymbol{\sigma} = 2\nu\boldsymbol{\nabla}^{\mathtt{S}}\boldsymbol{u}$, the Navier-Stokes equations can be

rewritten as the following system of first order in derivative equations

$$\boldsymbol{\sigma} = 2\nu\boldsymbol{\nabla}^{\mathtt{S}}\boldsymbol{u} \qquad \text{in } \Omega, \tag{2a}$$
$$-\boldsymbol{\nabla}\cdot\boldsymbol{\sigma} + \boldsymbol{\nabla}p + \boldsymbol{\nabla}\cdot(\boldsymbol{u}\otimes\boldsymbol{u}) = \boldsymbol{f} \qquad \text{in } \Omega, \tag{2b}$$
$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0 \qquad \text{in } \Omega, \tag{2c}$$
$$\boldsymbol{u} = \boldsymbol{u}_D \qquad \text{on } \Gamma_D, \tag{2d}$$
$$\boldsymbol{n}\cdot\boldsymbol{\sigma} - p\boldsymbol{n} = \boldsymbol{t} \qquad \text{on } \Gamma_N. \tag{2e}$$

2.1.1 DG formulation of the Navier-Stokes equations

Multiplying the equations (2) by smooth test functions $\boldsymbol{\tau}$, $\boldsymbol{v}$ and q respectively and integrating by parts over an arbitrary subset $K \subset \Omega$ with outward normal unit vector $\boldsymbol{n}$ the weak problem is

$$\int_K \boldsymbol{\sigma} : \boldsymbol{\tau} d\boldsymbol{x} = -2\nu \int_K \boldsymbol{u}\cdot\boldsymbol{\nabla}\cdot\boldsymbol{\tau} d\boldsymbol{x} + 2\nu \int_{\partial K} \boldsymbol{u}\cdot\boldsymbol{n} ds, \tag{3a}$$

$$\int_K \left[\boldsymbol{\sigma} : \boldsymbol{\nabla}^{\mathtt{S}}\boldsymbol{v} - p\boldsymbol{\nabla}\cdot\boldsymbol{v}\right] d\boldsymbol{x} - \int_{\partial K} \left[\boldsymbol{\sigma} : (\boldsymbol{v}\otimes\boldsymbol{n}) - p\boldsymbol{v}\cdot\boldsymbol{n}\right] ds$$
$$- \int_K (\boldsymbol{u}\otimes\boldsymbol{u})\cdot\boldsymbol{\nabla}\boldsymbol{v} d\boldsymbol{x} + \int_{\partial K} \boldsymbol{u}\cdot\boldsymbol{n}\cdot\boldsymbol{u}\cdot\boldsymbol{v} ds = \int_K \boldsymbol{f}\cdot\boldsymbol{v} d\boldsymbol{x} \tag{3b}$$

$$-\int_K \boldsymbol{u}\cdot\nabla q + \int_{\partial K} \boldsymbol{u}\cdot\boldsymbol{n} q ds = 0 \tag{3c}$$

Note that the above equations are well defined for functions $(\boldsymbol{\sigma}, \boldsymbol{u}, p)$ and $(\boldsymbol{\tau}, \boldsymbol{v}, q)$ in $\boldsymbol{\Sigma} \times \boldsymbol{V} \times Q$ where the spaces

$$\boldsymbol{\Sigma} = \{\boldsymbol{\tau} \in [\mathcal{L}_2(\Omega)]^{\mathtt{n}_{\mathtt{sd}}^2} | \boldsymbol{\tau}|_K \in [\mathcal{H}^1(K)]^{\mathtt{n}_{\mathtt{sd}}^2} \quad \forall K \in \mathcal{T}_h\}, \tag{4a}$$
$$\boldsymbol{V} = \{\boldsymbol{v} \in [\mathcal{L}_2(\Omega)]^{\mathtt{n}_{\mathtt{sd}}} | v|_K \in [\mathcal{H}^1(K)]^{\mathtt{n}_{\mathtt{sd}}} \quad \forall K \in \mathcal{T}_h\}, \tag{4b}$$
$$Q = \{q \in \mathcal{L}_2(\Omega) | \int_\Omega q d\boldsymbol{x} = 0, q|_K \in \mathcal{H}^1(K) \quad \forall K \in \mathcal{T}_h\}, \tag{4c}$$

are associated with the triangulation $\mathcal{T}_h = \{K\}$ of Ω. In addition the finite element subspaces $\boldsymbol{\Sigma}_h \subset \boldsymbol{\Sigma}$, $\boldsymbol{V}_h \subset \boldsymbol{V}$ and $Q_h \subset Q$ are introduced as

$$\boldsymbol{\Sigma}_h = \{\boldsymbol{\tau} \in [\mathcal{L}_2(\Omega)]^{\mathtt{n}_{\mathtt{sd}}^2} | \boldsymbol{\tau}|_K \in [\mathcal{P}_k(K)]^{\mathtt{n}_{\mathtt{sd}}^2} \quad \forall K \in \mathcal{T}_h\}, \tag{5a}$$
$$\boldsymbol{V}_h = \{\boldsymbol{v} \in [\mathcal{L}_2(\Omega)]^{\mathtt{n}_{\mathtt{sd}}} | v|_K \in [\mathcal{P}_k(K)]^{\mathtt{n}_{\mathtt{sd}}} \quad \forall K \in \mathcal{T}_h\}, \tag{5b}$$
$$Q_h = \{q \in \mathcal{L}_2(\Omega) | \int_\Omega q d\boldsymbol{x} = 0, q|_K \in \mathcal{P}_k(K) \quad \forall K \in \mathcal{T}_h\}, \tag{5c}$$

where $\mathcal{P}_k(K)$ is the space of polynomial functions of degree at most $k \geq 1$ on K.

To write expressions applicable over the whole domain, the jump and average operators need to be defined. Two adjacent elements K^+ and K^- of the triangulation $\mathcal{T}_h$ are

considered and $e = \partial K^+ \cap \partial K^-$ is their common edge. Assume that $\boldsymbol{n}^\pm$ denote the unit vectors normal to $\partial K^\pm$ at any point on the edge e. The average operator is defined as

$$\{\circledcirc\} = \begin{cases} \kappa^+ \circledcirc^+ + \kappa^- \circledcirc^- & \text{on } \Gamma, \\ \circledcirc & \text{on } \partial\Omega. \end{cases}$$

and the jump operator as

$$[\![p\,\boldsymbol{n}]\!] = \begin{cases} p^+\,\boldsymbol{n}^+ + p^-\,\boldsymbol{n}^- & \text{on } \Gamma \\ p\,\boldsymbol{n} & \text{on } \partial\Omega \end{cases} \qquad \text{for scalars,}$$

$$[\![\boldsymbol{n} \otimes \boldsymbol{v}]\!] = \begin{cases} \boldsymbol{n}^+ \otimes \boldsymbol{v}^+ + \boldsymbol{n}^- \otimes \boldsymbol{v}^- & \text{on } \Gamma \\ \boldsymbol{n} \otimes \boldsymbol{v} & \text{on } \partial\Omega \end{cases} \qquad \text{or}$$

$$[\![\boldsymbol{n} \cdot \boldsymbol{v}]\!] = \begin{cases} \boldsymbol{n}^+ \cdot \boldsymbol{v}^+ + \boldsymbol{n}^- \cdot \boldsymbol{v}^- & \text{on } \Gamma \\ \boldsymbol{n} \cdot \boldsymbol{v} & \text{on } \partial\Omega \end{cases} \qquad \text{for vectors,}$$

$$[\![\boldsymbol{n} \cdot \boldsymbol{\sigma}]\!] = \begin{cases} \boldsymbol{n}^+ \cdot \boldsymbol{\sigma}^+ + \boldsymbol{n}^- \cdot \boldsymbol{\sigma}^- & \text{on } \Gamma \\ \boldsymbol{n} \cdot \boldsymbol{\sigma} & \text{on } \partial\Omega \end{cases} \qquad \text{for second order tensors.}$$

Using these definitions, the DG formulation of the weak form (3) is: find $\boldsymbol{\sigma}_h \in \boldsymbol{\Sigma}_h$, $u_h \in \boldsymbol{V}_h$, $p_h \in Q_h$ such that for all $K \in \mathcal{T}_h$

$$\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\tau} = -2\nu \int_\Omega \boldsymbol{u}_h \cdot \boldsymbol{\nabla}_h \cdot \boldsymbol{\tau} + 2\nu \int_{\Gamma \cup \partial\Omega} \hat{\boldsymbol{u}}_h^\sigma \cdot [\![\boldsymbol{\tau} \cdot \boldsymbol{n}]\!] ds, \tag{6a}$$

$$\begin{aligned} &\int_\Omega [\boldsymbol{\sigma}_h : \boldsymbol{\nabla}_h^{\mathtt{S}} \boldsymbol{v} - p_h \boldsymbol{\nabla}_h \cdot \boldsymbol{v}] d\boldsymbol{x} - \int_{\Gamma \cup \partial\Omega} [\hat{\boldsymbol{\sigma}}_h : [\![\boldsymbol{v} \otimes \boldsymbol{n}]\!] - \hat{p}_h [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!]] ds \\ &\quad - \int_\Omega (\boldsymbol{u}_h \otimes \boldsymbol{u}_h) \cdot \boldsymbol{\nabla}_h \boldsymbol{v} d\boldsymbol{x} + \int_{\Gamma \cup \partial\Omega} \hat{\boldsymbol{f}}(\boldsymbol{u}_h, \boldsymbol{u}_h) \cdot \boldsymbol{n} \cdot \boldsymbol{v} ds = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} d\boldsymbol{x} \end{aligned} \tag{6b}$$

$$- \int_\Omega \boldsymbol{u}_h \cdot \nabla_h q + \int_{\Gamma \cup \partial\Omega} \hat{\boldsymbol{u}}_h^p \cdot [\![\boldsymbol{n} q]\!] ds = 0 \tag{6c}$$

where $\hat{\boldsymbol{u}}_h^\sigma$, $\hat{\boldsymbol{\sigma}}_h$, $\hat{p}_h$, $\hat{\boldsymbol{f}}(\boldsymbol{u}_h, \boldsymbol{u}_h)$ and $\hat{\boldsymbol{u}}_h^p$ are numerical fluxes to be defined.

Note that if we use the integration by parts formula

$$- \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla} \cdot \boldsymbol{\tau} d\boldsymbol{x} = \int_\Omega \boldsymbol{\tau} \cdot \boldsymbol{\nabla}^{\mathtt{S}} \boldsymbol{v} d\boldsymbol{x} - \int_\Gamma ([\![\boldsymbol{n} \otimes \boldsymbol{v}]\!] : \{\boldsymbol{\tau}\} + \{\boldsymbol{v}\} \cdot [\![\boldsymbol{n} \cdot \boldsymbol{\tau}]\!]) ds - \int_{\partial\Omega} \boldsymbol{v} \cdot \boldsymbol{\tau} \cdot \boldsymbol{n} ds \tag{7}$$

valid for all $\boldsymbol{\tau} \in [\mathcal{H}^1(\mathcal{T}_h)]^{\mathtt{n}_{\mathtt{sd}}^2}$ and $\boldsymbol{u} \in [\mathcal{H}^1(\mathcal{T}_h)]^{\mathtt{n}_{\mathtt{sd}}}$, equation (6a) can be written as

$$\begin{aligned} \int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\tau} d\boldsymbol{x} = 2\nu \int_\Omega \boldsymbol{\tau} \cdot \boldsymbol{\nabla}_h^{\mathtt{S}} \boldsymbol{u}_h d\boldsymbol{x} - 2\nu \int_\Gamma ([\![\boldsymbol{n} \otimes \boldsymbol{u}_h]\!] : \{\boldsymbol{\tau}\} - \{\hat{\boldsymbol{u}}_h^\sigma - \boldsymbol{u}_h\} \cdot [\![\boldsymbol{n} \cdot \boldsymbol{\tau}]\!]) ds \\ + 2\nu \int_{\partial\Omega} (\hat{\boldsymbol{u}}_h^\sigma - \boldsymbol{u}_h) \cdot \boldsymbol{\tau} \cdot \boldsymbol{n} ds \end{aligned} \tag{8}$$

Let us now describe the different numerical fluxes of equation (6).

The convective numerical flux. The convective flux $\hat{f}(\boldsymbol{u}_h, \boldsymbol{u}_h)$ is defined in a similar way as Lomtev and Karniadakis do in [6]. A standard Roe-splitting flux is used

$$\hat{\boldsymbol{f}}(\boldsymbol{u}, \boldsymbol{u}) \cdot \boldsymbol{n} = \frac{1}{2}\left[(\boldsymbol{F}_x(\boldsymbol{u}^e), \boldsymbol{F}_y(\boldsymbol{u}^e)) + \left(\boldsymbol{F}_x(\boldsymbol{u}^i), \boldsymbol{F}_y(\boldsymbol{u}^i)\right)\right] \cdot \boldsymbol{n} - \frac{1}{2} R\,|\Lambda|\,L(\boldsymbol{u}^e - \boldsymbol{u}^i)$$

where $\boldsymbol{u}^i$ and $\boldsymbol{u}^e$ are respectively the velocities in the interior and in the exterior of the element under consideration,

$$\text{where} \quad \boldsymbol{F}_x(\boldsymbol{u}) = \begin{pmatrix} u^2 \\ uv \end{pmatrix} \quad \text{and} \quad \boldsymbol{F}_y(\boldsymbol{u}) = \begin{pmatrix} uv \\ v^2 \end{pmatrix},$$

and A, jacobian matrix of $\boldsymbol{F}$, is written in terms of the left and right eigenvectors: $A = R\Lambda L$, where Λ contains the corresponding eigenvalues in the diagonal.

The diffusive numerical fluxes. Following the definition of the diffusive fluxes by Cockburn et al. for their LDG method [2], if a face e lies inside the domain Ω

$$\hat{\boldsymbol{\sigma}}_h = \{\boldsymbol{\sigma}_h\} - C_{11}[\![\boldsymbol{n} \otimes \boldsymbol{u}_h]\!] + \boldsymbol{C}_{12} \otimes [\![\boldsymbol{n} \cdot \boldsymbol{\sigma}_h]\!], \quad \hat{\boldsymbol{u}}_h^\sigma = \{\boldsymbol{u}_h\} - \boldsymbol{C}_{12} \cdot [\![\boldsymbol{n} \otimes \boldsymbol{u}_h]\!] \quad \text{on } \Gamma,$$

and if e lies on the boundary,

$$\hat{\boldsymbol{\sigma}}_h = \boldsymbol{\sigma}_h - C_{11}(\boldsymbol{u}_h - \boldsymbol{u}_D) \otimes \boldsymbol{n}, \quad \hat{\boldsymbol{u}}_h^\sigma = \boldsymbol{u}_D \quad \text{on } \Gamma_D.$$

C_{11} is a positive constant and $\boldsymbol{C}_{12}$ is a vector that has to be determined for each interior edge of the domain according to

$$\boldsymbol{C}_{12} = \frac{1}{2}(S_{K^+}^{K^-}\boldsymbol{n}^+ + S_{K^-}^{K^+}\boldsymbol{n}^-)$$

where $S_{K^+}^{K^-} \in \{0, 1\}$ is a switch defined for each element edge. The switches always satisfy that

$$S_{K^+}^{K^-} + S_{K^-}^{K^+} = 1.$$

There is several possible choices of the switches, see for example [2]. One possibility is the *natural switch*, which takes into account the element numbering to set $S_{K^\mp}^{K^\pm}$. Another alternative is to use a *consistent* switch that satisfies

$$0 < \sum_{e \in \partial K} S_{K^\mp}^{K^\pm} < \mathtt{n_{sd}} + 1. \tag{9}$$

Calculating $\boldsymbol{C}_{12}$ depending on the numbering of the nodes for each element K is the option chosen here and is a consistent switch.

The numerical fluxes related to the incompressibility constraint. If a face e lies inside the domain Ω

$$\hat{\boldsymbol{u}}_h^p = \{\boldsymbol{u}_h\}, \quad \hat{p}_h = \{p_h\} \quad \text{on } \Gamma,$$

whereas on the boundary

$$\hat{\boldsymbol{u}}_h^p = \boldsymbol{u}_D, \quad \hat{p}_h = p_h \quad \text{on } \Gamma_D.$$

Finally in order to account for the natural condition on Γ_N, we must choose $\hat{\boldsymbol{\sigma}}_h$ and $\hat{p}_h$ such that

$$\boldsymbol{n} \cdot \hat{\boldsymbol{\sigma}}_h - \hat{p}_h \boldsymbol{n} = \boldsymbol{t} \quad \text{on } \Gamma_N$$

and set

$$\hat{\boldsymbol{u}}_h^p = \boldsymbol{u} \quad \text{and} \quad \hat{\boldsymbol{u}}_h^\sigma = \boldsymbol{u} \quad \text{on } \Gamma_N$$

2.1.2 CDG formulation of the steady Navier-Stokes equations

Using the expression of the fluxes previously defined, equation (8) can now be written

$$\int_\Omega \boldsymbol{\sigma}_h : \boldsymbol{\tau} d\boldsymbol{x} = 2\nu \int_\Omega \boldsymbol{\tau} \cdot \boldsymbol{\nabla}_h^{\mathtt{S}} \boldsymbol{u}_h d\boldsymbol{x} - 2\nu \int_\Gamma (\llbracket \boldsymbol{n} \otimes \boldsymbol{u}_h \rrbracket : \{\boldsymbol{\tau}\} + \boldsymbol{C}_{12} \cdot \llbracket \boldsymbol{n} \otimes \boldsymbol{u} \rrbracket \cdot \llbracket \boldsymbol{n} \cdot \boldsymbol{\tau} \rrbracket) ds + 2\nu \int_{\Gamma_D} (\boldsymbol{u}_D - \boldsymbol{u}_h) \cdot \boldsymbol{\tau} \cdot \boldsymbol{n} ds \quad (10)$$

To obtain an expression for $\boldsymbol{\sigma}_h$ as a function of $\boldsymbol{u}_h$, lifting operators are introduced following the method introduced by Arnold et al. [7]. For all $e \in \Gamma, r^e : [\mathcal{L}_2(e)]^{\mathtt{n}_{\mathtt{sd}}^2} \to \boldsymbol{\Sigma}_h, l^e : [\mathcal{L}_2(e)]^{\mathtt{n}_{\mathtt{sd}}} \to \boldsymbol{\Sigma}_h$ and for each $e \in \Gamma_D, r_D^e : [\mathcal{L}_2(e)]^{\mathtt{n}_{\mathtt{sd}}} \to \boldsymbol{\Sigma}_h$, are defined as

$$\begin{aligned}
\int_\Omega r^e(\boldsymbol{v}) : \boldsymbol{\tau} d\boldsymbol{x} &= \int_e \boldsymbol{v} : \{\boldsymbol{\tau}\} ds, && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\
\int_\Omega l^e(\boldsymbol{v}) : \boldsymbol{\tau} d\boldsymbol{x} &= \int_e \boldsymbol{v} \cdot \llbracket \boldsymbol{n} \cdot \boldsymbol{\tau} \rrbracket ds, && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\
\int_\Omega r_D^e(\boldsymbol{v}) : \boldsymbol{\tau} d\boldsymbol{x} &= \int_e \boldsymbol{v} \cdot \boldsymbol{n} \cdot \boldsymbol{\tau} ds, && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h.
\end{aligned} \quad (11)$$

Note that here the lifting operators are defined edgewise, which is where CDG differs from LDG, see [1]. The lifting operators using by the LDG methods are defined as: $r : [\mathcal{L}_2(\Gamma)]^{\mathtt{n}_{\mathtt{sd}}^2} \to \boldsymbol{\Sigma}_h, l : [\mathcal{L}_2(\Gamma)]^{\mathtt{n}_{\mathtt{sd}}} \to \boldsymbol{\Sigma}_h$ and for each $e \in \Gamma_D, r_D : [\mathcal{L}_2(\Gamma_D)]^{\mathtt{n}_{\mathtt{sd}}} \to \boldsymbol{\Sigma}_h$, are defined as

$$\begin{aligned}
\int_\Omega r(\boldsymbol{v}) : \boldsymbol{\tau} d\boldsymbol{x} &= \int_\Gamma \boldsymbol{v} : \{\boldsymbol{\tau}\} ds, && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\
\int_\Omega l(\boldsymbol{v}) : \boldsymbol{\tau} d\boldsymbol{x} &= \int_\Gamma \boldsymbol{v} \cdot \llbracket \boldsymbol{n} \cdot \boldsymbol{\tau} \rrbracket ds, && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h, \\
\int_\Omega r_D(\boldsymbol{v}) : \boldsymbol{\tau} d\boldsymbol{x} &= \int_{\Gamma_D} \boldsymbol{v} \cdot \boldsymbol{n} \cdot \boldsymbol{\tau} ds, && \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h.
\end{aligned} \quad (12)$$

From these definitions the correspondence between the CDG and LDG lifting operators is clear: for all $\boldsymbol{v} \in [\mathcal{L}_2(\Gamma)]^{\mathtt{n}_{\mathtt{sd}}^2}$

$$r(\boldsymbol{v}) = \sum_{e\in\Gamma} r^e(\boldsymbol{v}), \quad l(\boldsymbol{v}) = \sum_{e\in\Gamma} l^e(\boldsymbol{v}), \quad r_D(\boldsymbol{v}) = \sum_{e\in\Gamma} r_D^e(\boldsymbol{v}). \tag{13}$$

Using this notation, the equation (6a) defining $\boldsymbol{\sigma}_h$ in terms of $\boldsymbol{u}_h$ can be rewritten as

$$\boldsymbol{\sigma}_h = 2\nu \boldsymbol{\nabla}_h^{\mathtt{S}} \boldsymbol{u}_h - \boldsymbol{\sigma}_e \tag{14}$$

where

$$\boldsymbol{\sigma}_e = 2\nu(r^e(\llbracket \boldsymbol{n} \otimes \boldsymbol{u}_h \rrbracket) + l^e(\boldsymbol{C}_{12} \cdot \llbracket \boldsymbol{n} \otimes \boldsymbol{u}_h \rrbracket) - r_D^e(\boldsymbol{u}_D) + r_D^e(\boldsymbol{u}_h)). \tag{15}$$

Following the rationale detailed in [1] for an elliptic problem, eliminating the additional variable $\boldsymbol{\sigma}_h$, the CDG scheme for the steady Navier-Stokes equations is: find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $p_h \in V_h$ such that

$$\mathcal{A}_h^{CDG}(\boldsymbol{u}_h, \boldsymbol{v}) + \mathcal{O}_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}) + \mathcal{B}_h(\boldsymbol{v}, p_h) = \mathcal{F}_h^{CDG}(\boldsymbol{v}), \forall \boldsymbol{v} \in \boldsymbol{V}_h \tag{16}$$

where

$$\begin{aligned}
\mathcal{A}_h^{CDG}(\boldsymbol{u}, \boldsymbol{v}) &= \int_\Omega 2\nu \boldsymbol{\nabla}^{\mathtt{S}} \boldsymbol{u} : \boldsymbol{\nabla}^{\mathtt{S}} \boldsymbol{v} d\boldsymbol{x} - \int_{\Gamma\cup\Gamma_D} 2\nu(\llbracket \boldsymbol{n} \otimes \boldsymbol{u} \rrbracket : \{\boldsymbol{\nabla}^{\mathtt{S}} \boldsymbol{v}\} + \{\boldsymbol{\nabla}^{\mathtt{S}} \boldsymbol{u}\} : \llbracket \boldsymbol{n} \otimes \boldsymbol{v} \rrbracket) ds \\
&\quad - \int_\Gamma 2\nu(\boldsymbol{C}_{12} \otimes \llbracket \boldsymbol{n} \otimes \boldsymbol{u} \rrbracket : \llbracket \boldsymbol{n} \cdot \boldsymbol{\nabla}^{\mathtt{S}} \boldsymbol{v} \rrbracket + \boldsymbol{C}_{12} \otimes \llbracket \boldsymbol{n} \cdot \boldsymbol{\nabla}^{\mathtt{S}} \boldsymbol{u} \rrbracket : \llbracket \boldsymbol{n} \otimes \boldsymbol{v} \rrbracket) ds \\
&+ \sum_{e\in\Gamma} \int_\Omega 2\nu(r^e(\llbracket \boldsymbol{n} \otimes \boldsymbol{u} \rrbracket) + l^e(\boldsymbol{C}_{12} \otimes \llbracket \boldsymbol{n} \otimes \boldsymbol{u} \rrbracket) + r_D^e(\boldsymbol{u})) : (r^e(\llbracket \boldsymbol{n} \otimes \boldsymbol{v} \rrbracket) + l^e(\boldsymbol{C}_{12} \otimes \llbracket \boldsymbol{n} \otimes \boldsymbol{v} \rrbracket) + r_D^e(\boldsymbol{v})) d\boldsymbol{x} \\
&\quad + \int_{\Gamma\cup\Gamma_D} C_{11} \llbracket \boldsymbol{n} \otimes \boldsymbol{u} \rrbracket \cdot \llbracket \boldsymbol{n} \otimes \boldsymbol{v} \rrbracket ds \\
\mathcal{O}_h(\boldsymbol{u}, \boldsymbol{u}; \boldsymbol{v}) &= -\int_\Omega (\boldsymbol{u} \otimes \boldsymbol{u}) \cdot \nabla_h \boldsymbol{v} d\boldsymbol{x} + \int_{\Gamma\cup\Gamma_D} \hat{\boldsymbol{f}}(\boldsymbol{u}, \boldsymbol{u}) \cdot \boldsymbol{n} \cdot \boldsymbol{v} ds \\
\mathcal{B}_h(\boldsymbol{v}, p) &= -\int_\Omega p \boldsymbol{\nabla}_h \cdot \boldsymbol{v} d\boldsymbol{x} + \int_{\Gamma\cup\Gamma_D} \{p\} \llbracket \boldsymbol{v} \cdot \boldsymbol{n} \rrbracket ds \\
\mathcal{F}_h^{CDG}(\boldsymbol{v}) &= \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v} d\boldsymbol{x} - \int_{\Gamma_D} 2\nu \boldsymbol{n} \otimes \boldsymbol{u}_D : \boldsymbol{\nabla}^{\mathtt{S}} \boldsymbol{v} ds \\
&\quad - \sum_{e\in\Gamma} \int_\Omega 2\nu(\boldsymbol{n} \otimes \boldsymbol{u}_D) : (r^e(\llbracket \boldsymbol{n} \otimes \boldsymbol{v} \rrbracket) + l^e(\boldsymbol{C}_{12} \otimes \llbracket \boldsymbol{n} \cdot \boldsymbol{v} \rrbracket)) d\boldsymbol{x} \\
&\quad - \sum_{e\in\Gamma} \int_\Omega 2\nu \boldsymbol{n} \otimes \boldsymbol{v} : r_D^e(\boldsymbol{u}_D) d\boldsymbol{x} + \int_{\Gamma_D} C_{11}(\boldsymbol{n} \otimes \boldsymbol{u}_D) \cdot (\boldsymbol{n} \otimes \boldsymbol{v}) ds + \int_{\Gamma_N} \boldsymbol{v} \cdot \boldsymbol{t} ds
\end{aligned}$$

Using the following identity for any $\boldsymbol{v} \in [\mathcal{H}^1(\widehat{\Omega})]^{\mathtt{n}_{\mathtt{sd}}}$ and any $q \in \mathcal{L}_2(\Omega)$

$$-\int_\Omega q \boldsymbol{\nabla} \cdot \boldsymbol{v} d\boldsymbol{x} + \int_{\Gamma\cup\partial\Omega} \{q\} \llbracket \boldsymbol{n} \cdot \boldsymbol{v} \rrbracket ds = \int_\Omega \boldsymbol{v} \cdot \boldsymbol{\nabla} q d\boldsymbol{x} - \int_{\Gamma\cup\partial\Omega} \llbracket q\boldsymbol{n} \rrbracket \{\boldsymbol{v}\},$$

the incompressibility equation (6c) can be rewritten as

$$\mathcal{B}_h(\boldsymbol{u}, q) = \mathcal{G}_h(q), \tag{17}$$

where

$$\mathcal{G}_h(q) = \int_{\Gamma_D} q\boldsymbol{u}_D \cdot \boldsymbol{n} ds. \tag{18}$$

Note that the compactness of the CDG scheme compared to the LDG comes from the product of the lifting operators (in $\mathcal{A}_h^{CDG}(\boldsymbol{u}, \boldsymbol{v})$). Indeed in the CDG scheme the lifting operators are associated to individual edges and there are no connectivities between non-neighbor elements, whereas in the LDG scheme, the product of the lifting operator implies connectivities between non-neighbor elements.

2.2 The Interior Penalty Method

Following the approach presented in [5] for the Stokes equations, an Interior Penalty Method formulation is proposed for the Navier-Stokes equations. The convective numerical flux and the numerical flux related to the incompressibility constraint are the same as the ones used in the CDG approach. Concerning the diffusive numerical fluxes, a penalty parameter C_{11} is introduced to ensure the coercivity of the bilinear form

$$\mathcal{A}_h^{IPM}(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega 2\nu \boldsymbol{\nabla}^{\mathtt{S}}\boldsymbol{u} : \boldsymbol{\nabla}^{\mathtt{S}}\boldsymbol{v} d\boldsymbol{x} - \int_{\Gamma\cup\Gamma_D} 2\nu(\llbracket \boldsymbol{n}\otimes\boldsymbol{u}\rrbracket : \{\boldsymbol{\nabla}^{\mathtt{S}}\boldsymbol{v}\} + \{\boldsymbol{\nabla}^{\mathtt{S}}\boldsymbol{u}\} : \llbracket \boldsymbol{n}\otimes\boldsymbol{v}\rrbracket) ds$$
$$+ \int_{\Gamma\cup\Gamma_D} C_{11} \llbracket \boldsymbol{n}\otimes\boldsymbol{u}\rrbracket \cdot \llbracket \boldsymbol{n}\otimes\boldsymbol{v}\rrbracket ds$$

Using the same discrete spaces as the ones previously defined, the IPM scheme for the steady Navier-Stokes equations is: find $\boldsymbol{u}_h \in \boldsymbol{V}_h$ and $p_h \in P_h$ such that

$$\mathcal{A}_h^{IPM}(\boldsymbol{u}_h, \boldsymbol{v}) + \mathcal{O}_h(\boldsymbol{u}_h, \boldsymbol{u}_h, \boldsymbol{v}) + \mathcal{B}_h(\boldsymbol{v}, p_h) = \mathcal{F}_h^{IPM}(\boldsymbol{v}), \forall \boldsymbol{v} \in \boldsymbol{V}_h \tag{19}$$

where $\mathcal{O}_h(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v})$ and $\mathcal{B}_h(\boldsymbol{v}, p)$ are defined as for the CDG method and

$$\mathcal{F}_h^{IPM}(\boldsymbol{v}) = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} d\boldsymbol{x} - \int_{\Gamma_D} 2\nu \boldsymbol{n}\otimes\boldsymbol{u}_D : \boldsymbol{\nabla}^{\mathtt{S}}\boldsymbol{v} ds + \int_{\Gamma_D} C_{11}(\boldsymbol{n}\otimes\boldsymbol{u}_D)\cdot(\boldsymbol{n}\otimes\boldsymbol{v}) ds + \int_{\Gamma_N} \boldsymbol{v}\cdot\boldsymbol{t} ds$$

See that this formulation is compact in the sense that the degrees of freedom of one element are only connected to those of the immediate neighbors.

3 SOLENOIDAL INTERPOLATION SPACES

Following approaches such as the ones from Crouzeix and Raviart [8] or Griffiths [9], who were the first ones to introduce solenoidal spaces, and the approach proposed in

[10, 11], which uses for the interpolation of the velocity piecewise polynomials *(exactly) divergence-free*, a similar rationale is pursued following the one presented in [5].

Knowing that any function in $[\mathcal{H}^1(K)]^{\mathtt{n_{sd}}}$, can be written as the sum of a solenoidal part and an irrotational one, the functional space for the velocity can be split in the direct sum: $[\mathcal{H}^1(\Omega)]^{\mathtt{n_{sd}}} = \boldsymbol{\mathcal{S}} \oplus \boldsymbol{\mathcal{I}}$ where

$$\boldsymbol{\mathcal{S}} := \left\{ \boldsymbol{v} \in [\mathcal{H}^1(\Omega)]^{\mathtt{n_{sd}}} \mid \boldsymbol{\nabla} \cdot \boldsymbol{v}|_K = 0 \text{ for } i = 1, \dots, \mathtt{n_{el}} \right\},$$
$$\boldsymbol{\mathcal{I}} \subset \left\{ \boldsymbol{v} \in [\mathcal{H}^1(\Omega)]^{\mathtt{n_{sd}}} \mid \boldsymbol{\nabla} \times \boldsymbol{v}|_K = \boldsymbol{0} \text{ for } i = 1, \dots, \mathtt{n_{el}} \right\}.$$

Note also that the solution of the original problem (1) or (16) belongs to $\boldsymbol{\mathcal{S}}$.

The finite counterparts of $\boldsymbol{\mathcal{S}}$ and $\boldsymbol{\mathcal{I}}$ are

$$\boldsymbol{\mathcal{S}}_h = \left\{ \boldsymbol{v} \in [\mathcal{H}^1(\Omega)]^{\mathtt{n_{sd}}} \mid \boldsymbol{v}|_K \in [\mathcal{P}^k(K)]^{\mathtt{n_{sd}}},\ \boldsymbol{\nabla} \cdot \boldsymbol{v}|_{\Omega_i} = 0 \text{ for } i = 1, \dots, \mathtt{n_{el}} \right\},$$
$$\boldsymbol{\mathcal{I}}_h \subset \left\{ \boldsymbol{v} \in [\mathcal{H}^1(\Omega)]^{\mathtt{n_{sd}}} \mid \boldsymbol{v}|_K \in [\mathcal{P}^k(K)]^{\mathtt{n_{sd}}},\ \boldsymbol{\nabla} \times \boldsymbol{v}|_{\Omega_i} = \boldsymbol{0} \text{ for } i = 1, \dots, \mathtt{n_{el}} \right\},$$

such that $\boldsymbol{\mathcal{I}}_h \subset \boldsymbol{\mathcal{I}}$. Note that the following relations and inclusions are verified: $\boldsymbol{V}_h = \boldsymbol{\mathcal{S}}_h \oplus \boldsymbol{\mathcal{I}}_h$, $\boldsymbol{V}_h \subset [\mathcal{H}^1(\Omega)]^{\mathtt{n_{sd}}}$, $Q_h \subset \mathcal{L}_2(\Omega)$ and $\boldsymbol{\mathcal{S}}_h \subset \boldsymbol{\mathcal{S}}$.

Under these circumstances, problem (16) can be split in two *uncoupled* problems. The first one solves for the *divergence-free* velocities and *hybrid pressures*. Following [12] the space of hybrid pressures (pressures along the edges in 2D or faces in 3D) is simply:

$$\boldsymbol{P} := \{ \tilde{p} \mid \tilde{p} : \Gamma \cup \Gamma_D \longrightarrow \mathbb{R} \ \text{ and } \ \tilde{p} = [\![\boldsymbol{n} \cdot \boldsymbol{v}]\!] \ \text{ for some } \ \boldsymbol{v} \in \boldsymbol{\mathcal{S}} \}. \tag{20}$$

Note that the finite dimensional subspace associated with the hybrid pressures, $\boldsymbol{P}^h \subset \boldsymbol{P}$, can be defined directly from (20) restricting velocities to $\boldsymbol{\mathcal{S}}_h$.

The approximations obtained in this case are $\boldsymbol{u}_h \in \boldsymbol{\mathcal{S}}_h$ and $\tilde{p}_h \in \boldsymbol{P}_h$ solution of

$$\begin{cases} \mathcal{A}_h(\boldsymbol{u}_h, \boldsymbol{v}) + \mathcal{B}_h^S(\boldsymbol{v}, \tilde{p}_h) + \mathcal{O}_h(\boldsymbol{u}_h, \boldsymbol{u}_h; \boldsymbol{v}) = \mathcal{F}_h(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{\mathcal{S}}_h, \\ \mathcal{B}_h^S(\boldsymbol{u}_h, \tilde{q}) = \mathcal{G}_h(q) & \forall \tilde{q} \in \boldsymbol{P}_h, \end{cases} \tag{21a}$$

where

$$\mathcal{B}_h^S(\boldsymbol{v}, \tilde{p}) = \int_{\Gamma \cup \Gamma_D} \tilde{p}_h [\![\boldsymbol{v} \cdot \boldsymbol{n}]\!] ds$$

and $\mathcal{A}_h(\boldsymbol{u}_h, \boldsymbol{v}) = \mathcal{A}_h^{CDG}(\boldsymbol{u}_h, \boldsymbol{v})$ or $\mathcal{A}_h(\boldsymbol{u}_h, \boldsymbol{v}) = \mathcal{A}_h^{IPM}(\boldsymbol{u}_h, \boldsymbol{v})$, and $\mathcal{F}_h(\boldsymbol{v}) = \mathcal{F}_h^{CDG}(\boldsymbol{v})$ or $\mathcal{F}_h(\boldsymbol{v}) = \mathcal{F}_h^{IPM}(\boldsymbol{v})$, depending on which method is used.

The second problem, which requires the solution of the previous one, evaluates the "interior" pressures: find $p_h \in Q_h$

$$\mathcal{B}_h^I(\boldsymbol{v}, p_h) = \mathcal{F}_h(\boldsymbol{v}) - \mathcal{A}_h(\boldsymbol{u}_h, \boldsymbol{v}) - \mathcal{B}_h^S(\boldsymbol{v}, \tilde{p}_h) - \mathcal{O}_h(\boldsymbol{u}_h, \boldsymbol{u}_h; \boldsymbol{v}) \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{I}}_h, \tag{21b}$$

where

$$\mathcal{B}_h^I(\boldsymbol{v}, p) = -\int_\Omega p \boldsymbol{\nabla}_h \cdot \boldsymbol{v} d\boldsymbol{x}. \tag{22}$$

It is important to note that equation (21b) can be solved element by element and pressure is the only unknown.

4 SOLUTION OF THE NAVIER-STOKES EQUATIONS BY A FRACTIONAL STEP METHOD

To solve the non-linear Navier-Stokes equations an iterative method in time is used. Starting with the weak form obtained for the *steady* case (21a), the weak form associated to the *unsteady* problem is now: given $\boldsymbol{u}_h^0 \in \boldsymbol{\mathcal{S}}_h$ find $\boldsymbol{u}_h \in \boldsymbol{\mathcal{S}}_h$ and $\tilde{p}_h \in \boldsymbol{P}_h$ solution of

$$\begin{cases} \Big(\dfrac{\partial \boldsymbol{u}_h}{\partial t}, \boldsymbol{v}\Big) + \mathcal{A}_h(\boldsymbol{u}_h, \boldsymbol{v}) + \mathcal{B}_h^S(\boldsymbol{v}, \tilde{p}_h) + \mathcal{O}_h(\boldsymbol{u}_h, \boldsymbol{u}_h; \boldsymbol{v}) = \mathcal{F}_h(\boldsymbol{v}) & \forall \boldsymbol{v} \in \boldsymbol{\mathcal{S}}_h, \\ \mathcal{B}_h^S(\boldsymbol{u}_h, \tilde{q}) = \mathcal{G}_h(q) & \forall \tilde{q} \in \boldsymbol{P}_h \end{cases} \tag{23}$$

Once again the expressions of $\mathcal{A}_h(\boldsymbol{u}_h, \boldsymbol{v})$ and $\mathcal{F}_h(\boldsymbol{v})$ depend on the choice of the method, CDG or IPM.

Let us denote by $\boldsymbol{u}_h^n$ and $\tilde{p}_h^n$ the velocity and pressure at time t^n, where $t^n = t^{n-1} + \Delta t$. Following the method developed in [13], the fields $\boldsymbol{u}_h^{n+1}$ and $\tilde{p}_h^{n+1}$ are calculated from $\boldsymbol{u}_h^n$ but here an implicit Euler method with a pressure increment is used to derive $\boldsymbol{u}_h^*$ for the diffusive part (Stokes problem). Indeed, it has been observed, by Goda in [14], that adding an old value of the pressure gradient in the first sub-step and then accordingly correcting the velocity in the second sub-step increases the accuracy of the scheme and improves the convergence properties. The idea was then made popular by Van Kan [15] who proposed a second-order incremental pressure-correction scheme. Using the Backward Difference Formula of second order to approximate the time derivative, and a semi-implicit scheme for the nonlinear convective term of the Navier-Stokes equation, the incremental pressure-correction reads as follows. The first step is: from $\boldsymbol{u}_h^n$, $\boldsymbol{u}_h^{n-1}$ and $\tilde{p}_h^n$, find $\boldsymbol{u}_h^* \in [\mathcal{H}^1(\Omega)]^{\mathtt{n_{sd}}}$ such that

$$\Big(\frac{3\boldsymbol{u}_h^* - 4\boldsymbol{u}_h^n + \boldsymbol{u}_h^{n-1}}{2\Delta t}, \boldsymbol{v}\Big) + \mathcal{A}_h(\boldsymbol{u}_h^*, \boldsymbol{v}) + \mathcal{O}_h(2\boldsymbol{u}_h^n - \boldsymbol{u}_h^{n-1}, \boldsymbol{u}_h^n; \boldsymbol{v}) = \mathcal{F}_h^*(\boldsymbol{v}) - \mathcal{B}_h^S(\boldsymbol{v}, \tilde{p}_h^n) \qquad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{S}}_h, \tag{24a}$$

then the second step is: from $\boldsymbol{u}_*^h$ and $\tilde{p}_h^n$, find $\boldsymbol{u}_h^{n+1} \in \boldsymbol{\mathcal{S}}_h$ and $\tilde{p}_h^n \in \boldsymbol{P}^h$ such that

$$\begin{cases} \Big(\dfrac{3\boldsymbol{u}_h^{n+1} - 3\boldsymbol{u}_h^*}{2\Delta t}, \boldsymbol{v}\Big) + \mathcal{B}_h^S(\boldsymbol{v}, \tilde{p}_h^{n+1} - \tilde{p}_h^n) = \boldsymbol{0} & \forall \boldsymbol{v} \in \boldsymbol{\mathcal{S}}_h, \\ \mathcal{B}_h^S(\boldsymbol{u}_h^{n+1}, \tilde{q}) = \mathcal{G}_h(q) & \forall \tilde{q} \in \boldsymbol{P}^h. \end{cases} \tag{24b}$$

Denoting by $\mathbf{u}$ and $\mathbf{p}$ the vectors of nodal values of the velocity $\boldsymbol{u}$ and hybrid pressure $\tilde{p}$, the following matrix form is obtained: the first sub-step is

$$\mathbf{u}^* = (3\mathbf{M} + 2\Delta t(\mathbf{A} + \mathbf{O}(2\mathbf{u}^n - \mathbf{u}^{n-1}))^{-1}(\mathbf{M}(4\mathbf{u}^n - \mathbf{u}^{n-1}) + 2\Delta t(\mathbf{F}^* - \mathbf{C}^T\mathbf{p}^n)) \tag{25a}$$

and the second substep

$$\begin{cases} \mathbf{p}^{n+1} = \mathbf{p}^n + \dfrac{3}{2}(\mathbf{C}\mathbf{M}^{-1}\mathbf{C}^T)^{-1}\dfrac{\mathbf{C}\mathbf{u}^* - \mathbf{G}}{\Delta t}, \\ \mathbf{u}^{n+1} = \mathbf{u}^* - \dfrac{2}{3}\Delta t\mathbf{M}^{-1}\mathbf{C}^T(\mathbf{p}^{n+1} - \mathbf{p}^n). \end{cases} \tag{25b}$$

where $\mathbf{M}$ is the mass matrix, $\mathbf{A}$ the diffusion matrix, $\mathbf{O}$ the convection matrix, $\mathbf{C}^T$ the edge pressure matrix and $\mathbf{F}$ and $\mathbf{G}$ vectors taking into account the force term and the boundary conditions.

After the convergence of the iterative process, the interior pressure p is retrieved by a direct method solving equation (21b).

5 NUMERICAL EXAMPLES

5.1 Driven cavity example

A standard benchmark test for the Navier-Stokes equations is first considered. A plane flow of an isothermal fluid in a lid-driven cavity is modelled in a 2D square domain $\Omega =]0,1[\times]0,1[$, with zero body force and one moving wall. A continuous velocity ($\boldsymbol{u} = (1,0)^T$ for $0.1 \leq x \leq 0.9$ $\boldsymbol{u} = (10x,0)^T$ for $0 \leq x \leq 0.1$ and $0.9 \leq x \leq 1$) is imposed on the exterior upper boundary $\{y = 1\}$, and a zero velocity $\boldsymbol{u} = (0,0)^T$ is enforced on the three other sides. Calculations are made for $Re = 1, 100, 400$. Figure 1 shows the results, which fit to the expected solution. As the Reynolds number is increased, the position of the main vortex moves towards the center of the cavity, both CDG method and IPM giving very similar results. In order to further compare them, the velocity profiles at the vertical centerline are shown in figure 2 for different values for Re. First, it can be noticed that as the Reynolds number increases, the boundary layers are more obvious and the variations in the velocity are sharper. Second, results for CDG method and IPM are almost identical. To compare more precisely the results of both methods an analytical example is taken in the next section.

5.2 General results on a Stokes example

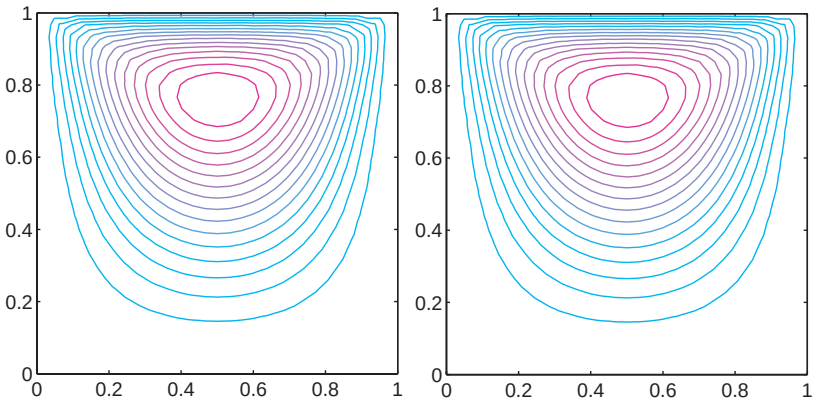
A Stokes example with analytical solution is now considered to compare the two proposed methodologies and in particular to study the properties of the diffusion matrix and the error of both methods. The Stokes equations are solved in a 2D square domain $\Omega =]0,1[\times]0,1[$ with Dirichlet boundary conditions on three edges, and a Neumann boundary condition on the fourth edge $\{y = 0\}$. A body force

$$\boldsymbol{f} = \begin{pmatrix} 12(1-2y)x^4 + 24(-1+2y)x^3 + 12(-4y+6y^2-4y^3+1)x^2 \\ +(-2+24(y-3y^2+2y^3))x + 1 - 4y + 12y^2 - 8y^3 \\ 8(1-6y+6y^2)x^3 + 12(-1+6y-6y^2)x^2 \\ +(4+48(y^2-y^3)+24(y^4-y))x - 12y^2 + 24y^3 - 12y^4 \end{pmatrix}$$

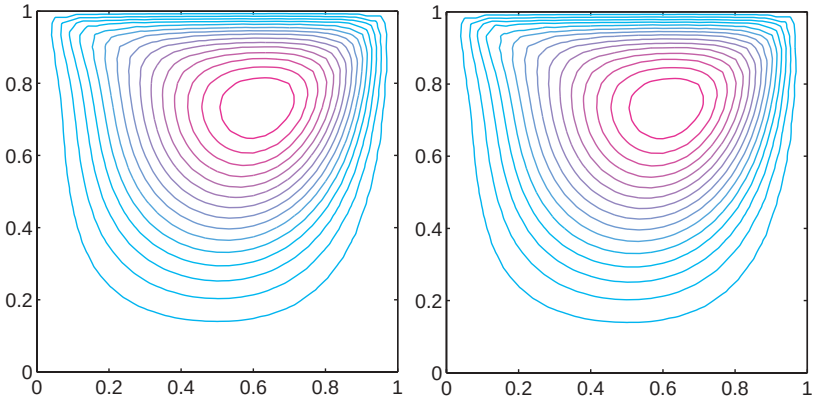
is imposed in order to have the polynomial exact solution

$$\boldsymbol{u} = \begin{pmatrix} x^2(1-x)^2(2y-6y^2+4y^3) \\ -y^2(1-y)^2(2x-6x^2+4x^3) \end{pmatrix},$$

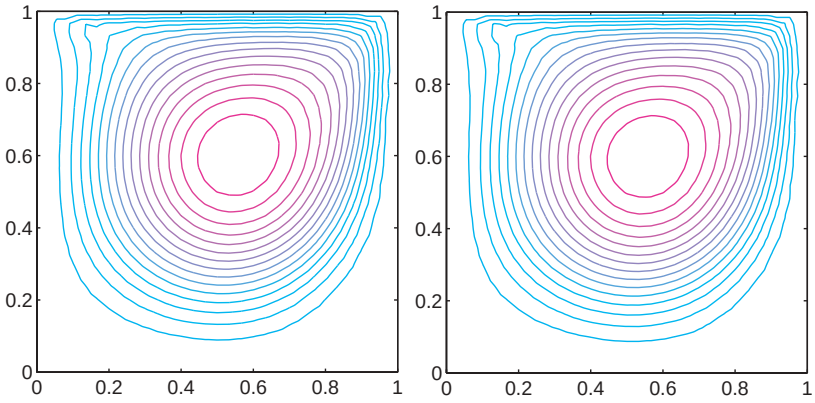
$$p = x(1-x).$$



(a) $Re = 1$

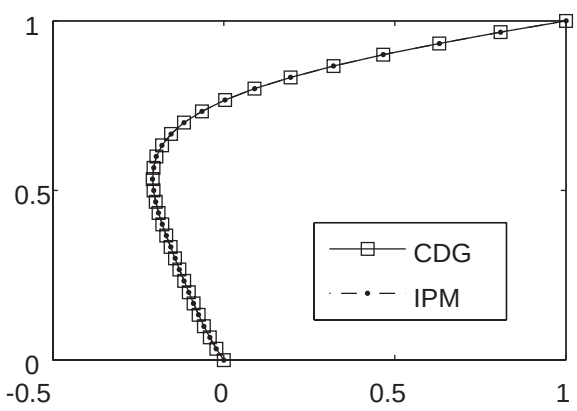


(b) $Re = 100$

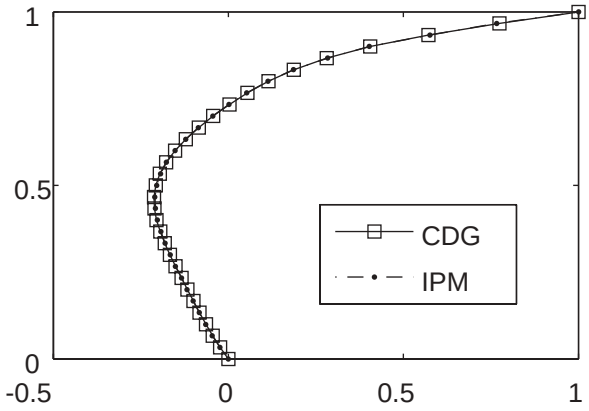


(c) $Re = 400$

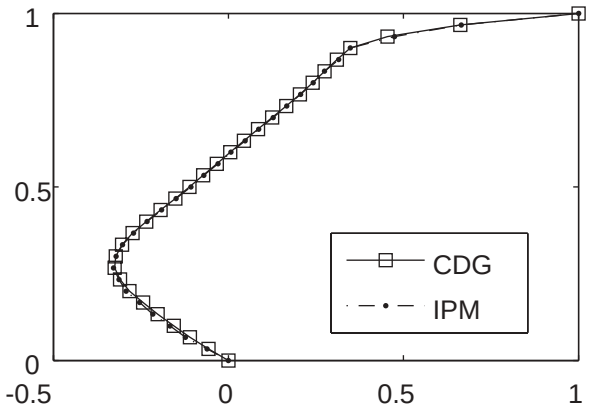
Figura 1: Velocity streamlines for the CDG method (left) and the IPM (right), for different Reynolds numbers $Re = 1, 100, 400$, $k = 2$, 450 elements, $C_{11} = 1$ (CDG), $C_{11} = 10$ (IPM).



(a) $Re = 1$

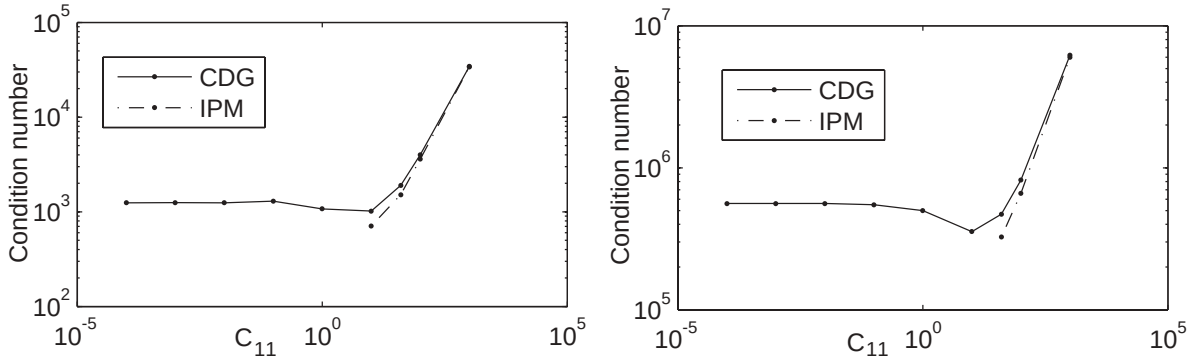


(b) $Re = 100$



(c) $Re = 400$

Figura 2: Velocity profiles at the vertical centerline for the CDG and IPM methods, for different values of Re



(a) $k = 2$ (b) $k = 4$

Figura 3: Condition number of the diffusion matrix function of the stabilization parameter C_{11} for the CDG and IPM methods, for a quadratic approximation of the velocity (left) and a fourth order of approximation of the velocity (right)

First the influence of the stabilization parameter C_{11} on the conditioning of the diffusion matrix is studied. Figure 3 shows the values of the condition number of the diffusion matrix for two different orders of interpolation and for the CDG and IPM methods. Note that unlike the Interior Penalty Method, the CDG methods (and it would also be the case for the LDG method) allow to choose a value of C_{11} as small as wanted for the interior edges, see [1]. It gives thus more freedom to chose this parameter and the value of the condition number is rather constant for values of C_{11} less or equal than 10. On the other hand, the value of C_{11} for the IPM has to be big enough to ensure the coercivity of the bilinear form, for example $C_{11} \geq 10$ for a quadratic velocity and $C_{11} \geq 40$ for a fourth order interpolation of the velocity. Then the IPM gives slightly better results than the CDG method. Then, increasing the value of C_{11} increases dramatically the condition number, equally for both methods.

Since there is a larger range of valid values for the stabilization parameter C_{11} for the CDG method let us study how its choice affects the accuracy of the solution of the Stokes equation. Figure 4 shows the results for the velocity and pressure errors for three different values of $C_{11} = 0, 1, 10$. It can be seen that though a value of 0 or 1 for C_{11} gives very similar results, the value of $C_{11} = 10$ give worse results, especially for a coarse mesh in the velocity case and for any size of the mesh in the pressure case. Note that the accuracy is only weakly dependent on the value of C_{11} since optimal convergence rate are obtained in any case.

These results lead us to choose a small value for C_{11} for the CDG method $C_{11} = 1$ for example, which will be optimal from both the condition numbers and the error estimates. Note that if $C_{11} = 0$ is chosen, a non-zero C_{11} has to be taken on the boundaries in order to treat properly the boundary conditions. Concerning the IPM, given that any value

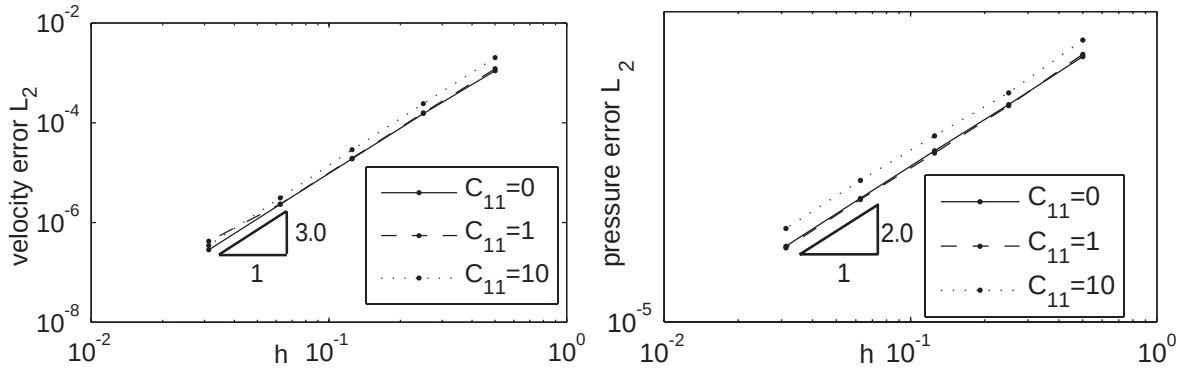


Figura 4: Comparison of the errors obtained with the CDG method for different values of C_{11}, for a quadratic approximation of the velocity (left) and a linear interpolation of the pressure (right).

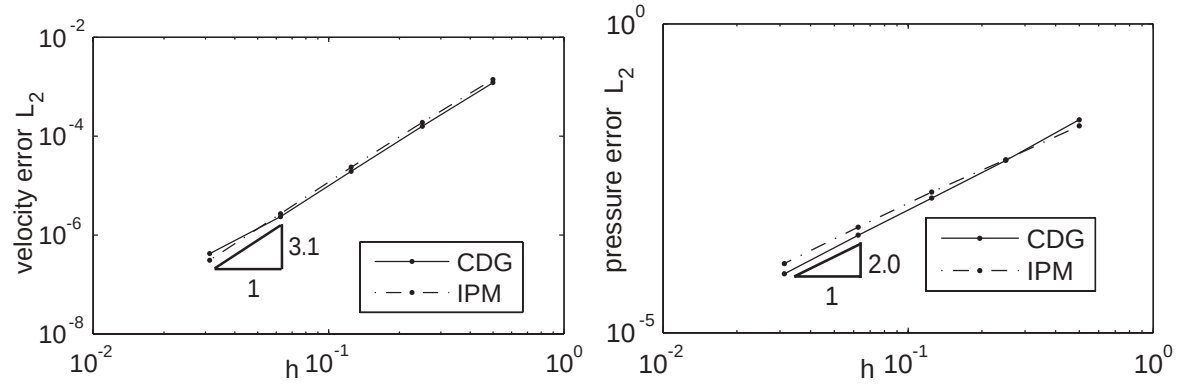


Figura 5: Comparison of the errors obtained with the CDG and the IPM methods, for a quadratic approximation of the velocity (left) and a linear interpolation of the pressure (right).

of C_{11} bigger than the minimum value necessary to ensure the coercivity of the bilinear form will deteriorate the conditioning of the diffusion matrix, it is preferable to chose the smallest value possible of C_{11}.

Both CDG and IPM are now compared from the accuracy point of view, for a quadratic approximation of the velocity and a linear interpolation of the pressure and choosing $C_{11} = 1$ for CDG and $C_{11} = 10$ for IPM. As it can be seen in figure 5 the results are very similar. Depending on the mesh size, one or the other method gives a better result but in any case both methods reach optimal convergence rates for velocity and pressure errors.

6 CONCLUSIONS

A critical comparison of the Compact Discontinuous Galerkin method and the Interior Penalty Method is presented here. The two methods are derived using different rationales. The CDG method first transforms the Navier-Stokes problem into a first order in derivative problem and then eliminate the additional variable using the so-called lifting

operators. The IPM rationale is more straight-forward and deals directly with the second order in derivative problem. But both problems present the major advantage -relative to the LDG method for example- that they are compact formulations: the degrees of freedom of one element are only connected to those of the immediate neighbors.

The CDG presents the other advantage that the stabilization parameter can take any value, whereas the penalty parameter in the IPM has to be taken big enough to ensure the coercivity of the bilinear form. Apart from this difference in the choice of the characteristic parameters both methods present very similar results, from the conditioning and the accuracy points of view, both methods reach optimal convergence rates for velocity and pressure errors.

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