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Concept Lattices Associated with L-Fuzzy W-contexts

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Abstract

We generalize in this paper the L-Fuzzy concept theory we developed in a previous paper ([1]), using the composition of L-Fuzzy relations. This theory models knowledge acquisition and classification and takes as departure point Wille's idea ([5]).

We begin the work defining L-Fuzzy W-contexts as the tuples (L,W,X,Y,$\underset{\smile}{R}$) where W, X and Y are the sets of labels, objects and attributes respectively, and $\underset{\smile}{R} \in L^{X\times Y}$ is an L-Fuzzy relation.

From these contexts, we will give the operators needed to define the L-Fuzzy W-concepts. These concepts will be pairs of relations $(\underset{\smile}{P}, \underset{\smile}{Q})$ where $\underset{\smile}{P} \in L^{W\times X}$, $\underset{\smile}{Q} \in L^{W\times Y}$ satisfying $\underset{\smile}{P}_1 = \underset{\smile}{Q}$ and $\underset{\smile}{Q}_2 = \underset{\smile}{P}$ with the operator 1 and 2 definitions given.

After proving the lattice structure of the L-Fuzzy W-concepts set, we analyse a practical example where we interpret the new concept definition

Key words: L-Fuzzy concepts, L-Fuzzy W-concepts, L-Fuzzy W-contexts, Conceptual knowledge.

1 Introduction

Using the L-Fuzzy concept theory ([1]), we analyse the contexts (L,X,Y,$\underset{\smile}{R}$) where X and Y are the object and attribute sets respectively, and $\underset{\smile}{R} \in L^{X\times Y}$. We make this study using the $L-Fuzzy$ concepts, which are pairs $(\underset{\smile}{A}, \underset{\smile}{B})$ where $\underset{\smile}{A} \in L^X$, $\underset{\smile}{B} \in L^Y$ satisfying $\underset{\smile}{A}_1 = \underset{\smile}{B}$ and $\underset{\smile}{B}_2 = \underset{\smile}{A}$.

We consider important, as we show in example 1, to define a new context that allows us define concepts as pairs of L-Fuzzy relations. To do this, we will change the L-Fuzzy context definition to apply it to the new situation and we will define the L-Fuzzy W-contexts so as to obtain the L-Fuzzy W-concepts.

2 The L-Fuzzy W-context

Let $(L, \leq)$ be a complete lattice, with $'$ and $\top$ a negation operator and a t-conorm in L. Moreover, let W, X, Y be non empty sets, and let $\underset{\sim}{R} \in L^{X \times Y}$ be an L-Fuzzy relation.

Definition 1. An <u>L-Fuzzy W-context</u> is a tuple (L,W,X,Y,$\underset{\sim}{R}$) where W, X and Y are the sets of labels, objects and attributes respectively.

Example 1. Suppose that an academy offers training courses for the unemployed.

To simulate this situation, we will take the W-context $(L, W, X, Y, \underset{\sim}{R})$ where L={0,0.1,0.2,0.3,...0.9,1} is the lattice, W={interesting, very specific, at a suitable level for his knowledge } contains personal opinions about the courses, X={computer science, accounting, mechanics } is the set of courses, Y={domestic helper, waiter, secretary, car salesman} is formed by the jobs, and the table 1 represents the relation between the courses and the jobs

$\underset{\sim}{R}$	domestic helper	waiter	accounting	car salesman
computer science	0	0.2	0.9	0.5
accounting	0	0.4	1	0.7
mechanics	0.2	0	0	0.5

Table 1

3 Derivation operators and constructor operadors

Definition 2. Let $\underset{\sim}{P} \in L^{W \times X}$. We define the associated relation $\underset{\sim}{P_1}$ in $L^{W \times Y}$

$$\underset{\sim}{P_1}(w, y) = \inf_{x \in X} (\underset{\sim}{P'}(w, x) \top \underset{\sim}{R}(x, y)) \qquad (1)$$

In the same way, given $\underset{\sim}{Q} \in L^{W \times Y}$ we associate to it $\underset{\sim}{Q_2} \in L^{W \times X}$ satisfying

$$\underset{\sim}{Q_2}(w, x) = \inf_{y \in Y} (\underset{\sim}{Q'}(w, y) \top \underset{\sim}{R}(x, y)) \qquad (2)$$

We can show, in an easy way, the relationship between these definitions and the composition of relations of the Fuzzy theory. If we take the $sup - \star$ definition (Dubois and Prade [3]), and we generalize it to a complete lattice L, we have the following expression:

$$(\underset{\sim}{P} \star \underset{\sim}{R})(w, y) = \sup_{x \in X} (\underset{\sim}{P}(w, x) \star \underset{\sim}{R}(x, y)), \quad \forall \underset{\sim}{P} \in L^{W \times X}, \forall \underset{\sim}{R} \in L^{X \times Y}$$

Thus, we can prove the following

Proposition 1. For every t-conorm $\top$ in L and for every negation operator $'$, the following equalities are true:

$$\underset{\sim}{P}_1 = (\underset{\sim}{P} \star \underset{\sim}{R}')',\ \forall \underset{\sim}{P} \in L^{W\times X}$$
$$\underset{\sim}{Q}_2 = (\underset{\sim}{Q} \star (\underset{\sim}{R}^{op})')',\ \forall \underset{\sim}{Q} \in L^{W\times Y}$$

where $\star$ is the t-norm in L associated to $(\top,')$ and $\underset{\sim}{R}^{op}$ is the opposite relation to $\underset{\sim}{R}$ defined by $\underset{\sim}{R}^{op}(y,x) = \underset{\sim}{R}(x,y), \forall x \in X, \forall y \in Y$.

We can also define the <u>constructor operators</u>([1]) φ and ψ, using the above equalities, and denote them:

$$\varphi : L^{W\times X} \to L^{W\times X} \,/\, \varphi(\underset{\sim}{P}) = \underset{\sim}{P}_{12} \qquad (3)$$

$$\psi : L^{W\times Y} \to L^{W\times Y} \,/\, \psi(\underset{\sim}{Q}) = \underset{\sim}{Q}_{21} \qquad (4)$$

We prove that to obtain $\varphi(\underset{\sim}{P})$ and $\psi(\underset{\sim}{Q})$ it is sufficient to apply to each row of $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ the constructor operators defined in the L-Fuzzy theory.

These operators satisfy the properties proved in [1], and since φ and ψ preserve the usual order in L-Fuzzy relations, $\Omega = fix(\varphi)$ and $\Sigma = fix(\psi)$ are complete lattices (Tarski [4]). We will use this result to give the L-Fuzzy W-context definition ([1]).

However, since φ and ψ are monotonous maps, we can use the P. and R. Cousot theory ([2]), which gives a constructive version of Tarski's theorem ([4]), to calculate the supremum and infimum elements of the lattices $fix(\varphi)$ and $fix(\psi)$, and the supremum and infimum of a family:

$$\underset{\sim}{0}_{\Omega} = luis(\varphi)(\underset{\sim}{0}) \qquad \underset{\sim}{0}_{\Sigma} = luis(\psi)(\underset{\sim}{0}) \qquad (5)$$
$$\underset{\sim}{1}_{\Omega} = llis(\varphi)(\underset{\sim}{1}) \qquad \underset{\sim}{1}_{\Sigma} = llis(\psi)(\underset{\sim}{1}) \qquad (6)$$

$\forall\{\underset{\sim}{P}_i, i \in I\} \in \Omega$, $\forall\{\underset{\sim}{Q}_i, i \in I\} \in \Sigma$

$$\bigvee_{\Omega} \underset{\sim}{P}_i = luis(\varphi)\left(\bigvee \underset{\sim}{P}_i\right) \qquad \bigvee_{\Sigma} \underset{\sim}{Q}_i = luis(\psi)\left(\bigvee \underset{\sim}{Q}_i\right) \qquad (7)$$
$$\bigwedge_{\Omega} \underset{\sim}{P}_i = llis(\varphi)\left(\bigwedge \underset{\sim}{P}_i\right) \qquad \bigwedge_{\Sigma} \underset{\sim}{Q}_i = llis(\psi)\left(\bigwedge \underset{\sim}{Q}_i\right) \qquad (8)$$

where, given a function f that preserves the order, $luis(f)(\underset{\sim}{P})$ is the limit of a stationary upper iteration sequence starting with $\underset{\sim}{P}$ and $llis(f)(\underset{\sim}{Q})$ is the limit of a stationary lower iteration sequence starting with $\underset{\sim}{Q}$.

4 L-Fuzzy W-concepts lattice

Let (L,W,X,Y,$\underset{\sim}{R}$) be an L-Fuzzy W-context and let $fix(\varphi)$ and $fix(\psi)$ be the fixed point sets of the constructor operators.

Definition 3. For every $\underset{\sim}{P} \in fix(\varphi)$, the pair $(\underset{\sim}{P}, \underset{\sim}{P_1})$ with is said to be an L-Fuzzy W-concept of $(L, W, X, Y, \underset{\sim}{R})$.

We can give this definition taking $\underset{\sim}{Q} \in fix(\psi)$. In this case, $(\underset{\sim}{Q_2}, \underset{\sim}{Q})$ will be the L-Fuzzy W-concept.

Moreover, as in [1], we prove the following:

Theorem 1. The L-Fuzzy W-concept set

$$\underset{\sim}{\mathcal{L}}_W = \{(\underset{\sim}{P}, \underset{\sim}{P_1}) \,/\, \underset{\sim}{P} \in fix(\varphi)\} = \{(\underset{\sim}{Q_2}, \underset{\sim}{Q}) \,/\, \underset{\sim}{Q} \in fix(\psi)\}$$

with the order relation $\preceq$ defined by

$$\forall (\underset{\sim}{P}, \underset{\sim}{P_1}), (\underset{\sim}{T}, \underset{\sim}{T_1}) \in \underset{\sim}{\mathcal{L}}_W, \qquad (\underset{\sim}{P}, \underset{\sim}{P_1}) \preceq (\underset{\sim}{T}, \underset{\sim}{T_1}) \Longleftrightarrow \underset{\sim}{P} \leq \underset{\sim}{T}$$

is a complete lattice.

We will denote the L-Fuzzy W-concept lattice $(\underset{\sim}{\mathcal{L}}_W, \preceq)$.

We can prove, as in [1], that the maximum and minimum elements of the lattice $(\underset{\sim}{\mathcal{L}}_W, \preceq)$ are respectively:

$$\underset{\sim}{0}_{\underset{\sim}{\mathcal{L}}_W} = (\underset{\sim}{0_\Omega}, \underset{\sim}{1_\Sigma}) \quad \text{and} \quad \underset{\sim}{1}_{\underset{\sim}{\mathcal{L}}_W} = (\underset{\sim}{1_\Omega}, \underset{\sim}{0_\Sigma})$$

Moreover, if we use an upper semicontinuous t-conorm $\top$ to define φ and ψ, then we have the following

Proposition 2. For every family

$$\mathcal{F} = \{(\underset{\sim}{P_i}, (\underset{\sim}{P_i})_1),\ \underset{\sim}{P_i} \in \Omega\} = \{((\underset{\sim}{Q_i})_2, \underset{\sim}{Q_i}),\ \underset{\sim}{Q_i} \in \Sigma\} \subseteq \underset{\sim}{\mathcal{L}}_W$$

we can express the supremum and the infimum of $\mathcal{F}$ in this way:

$$\bigvee_{\underset{\sim}{\mathcal{L}}_W}\left(\underset{\sim}{P_i},(\underset{\sim}{P_i})_1\right) = \left(\bigvee_{\Omega}\underset{\sim}{P_i},\bigwedge_{\Sigma}(\underset{\sim}{P_i})_1\right)$$

$$\bigwedge_{\underset{\sim}{\mathcal{L}}_W}\left((\underset{\sim}{Q_i})_2,\underset{\sim}{Q_i}\right) = \left(\bigwedge_{\Omega}(\underset{\sim}{Q_i})_2,\bigvee_{\Sigma}\underset{\sim}{Q_i}\right)$$

where the supremum and infimum in Σ and Ω are calculated using (7) and (8).

5 Relationship between $\underset{\sim}{\mathcal{L}}$ and $\underset{\sim}{\mathcal{L}}_W$.

If we take an element $(\underset{\sim}{P},\underset{\sim}{P}_1) \in \underset{\sim}{\mathcal{L}}_W$, and we take into account the definitions given in [1], then we can observe that it is formed by a pair of relations where each row of $\underset{\sim}{P}$, labeled by an element of W, is the *Fuzzy*-extension of an L-Fuzzy concept of $(L, X, Y, \underset{\sim}{R})$, and each row of $\underset{\sim}{P}_1$ its correspondent *Fuzzy*-comprehension. This allows us to calculate the L-Fuzzy W-concept lattice throught the L-Fuzzy concept lattice.

Moreover, we can obtain its cardinality in this way:

Proposition 3. Let $(L, W, X, Y, \underset{\sim}{R})$ be an L-Fuzzy W-context with $Card(W)$=k. If $\underset{\sim}{\mathcal{L}}$ is the L-Fuzzy concept lattice of $(L, X, Y, \underset{\sim}{R})$ and if $Card(\underset{\sim}{\mathcal{L}})$=n finite, then $Card(\underset{\sim}{\mathcal{L}}_W) = n^k$.

<u>Proof:</u> We have to take into account that for every $\underset{\sim}{P} \in L^{W\times X}$, to calculate $\varphi(\underset{\sim}{P}) = \underset{\sim}{P}_{12}$ we apply the operator φ, defined in the L-Fuzzy concept theory, to each row of $\underset{\sim}{P}$. So, $Card(\underset{\sim}{\mathcal{L}}_W) = n^k$.■

We can say that the W-concept $(\underset{\sim}{P},\underset{\sim}{P}_1)$ is formed by the L-Fuzzy concepts derived from L-Fuzzy sets that form the rows of the original relation.

On the other hand, if we have a single label, then there is a strong relationship between the W-concepts derived from $(L, W, X, Y, \underset{\sim}{R})$ and those obtained from $(L, X, Y, \underset{\sim}{R})$, i.e., the latter will be pairs of L-Fuzzy sets, as we see in the following:

Proposition 4. If $Card(W) = 1$, then $\underset{\sim}{\mathcal{L}}_W(L, W, X, Y, \underset{\sim}{R}) \backsimeq \underset{\sim}{\mathcal{L}}(L, X, Y, \underset{\sim}{R})$.

<u>Proof:</u> If $Card(W) = 1$, then $L^{W\times X} \backsimeq L^X$ and $L^{W\times Y} \backsimeq L^Y$, and taking into account these isomorphisms, the definitions (1) and (2) and their correpondent in the L-Fuzzy case ([1]), it is obvious that $\underset{\sim}{\mathcal{L}}_W(L, W, X, Y, \underset{\sim}{R}) \backsimeq \underset{\sim}{\mathcal{L}}(L, X, Y, \underset{\sim}{R})$.■

6 Interpretation of the L-Fuzzy W-concepts througth an example.

We are going to came back to example 1 and suppose that Table 2 represents the opinion of a student about the courses (objects), taking into account particular characteristics (labels).

$\underset{\backsim}{Q}$	computer science	accounting	mechanics
interesting	0.5	0.5	0.2
very specific	0.7	0.8	0.1
at a suit. level for his know.	0.8	0.3	0

Table 2

Applying the P. and R. Cousot([2]) theory to our work and taking into account that φ preserves the order, we can calculate the fixed point $\underset{\backsim}{Q}^*$ from $\underset{\backsim}{Q}$ and the L-Fuzzy W-concept $(\underset{\backsim}{Q}^*, \underset{\backsim 1}{Q}^*)$

$\underset{\backsim}{Q}^*$	computer science	accounting	mechanics
interesting	0.5	0.5	0.2
very specific	0.5	0.6	0.5
at a suit. level for his know.	0.5	0.6	0.1

Table 3

$\underset{\backsim 1}{Q}^*$	domestic helper	waiter	secretary	car salesman
interesting	0.5	0.5	0.8	0.5
very specific	0.4	0.4	0.5	0.5
at a suit. level for his know.	0.4	0.4	0.9	0.5

Table 4

From this concept we can conclude:

• If the most important thing is to suit the courses and jobs to the student's knowledge (last label), then he must study a course of accounting and another of computer science; and he will be able to work as secretary.

• If we are going to take into account his training in specific areas, we will choose the accounting course and the more suitable jobs will be secretary or car salesman.

• If we want that person to study and work in what interests him, he must study computer science and accounting; and finally work as a secretary.

• If we look at the opinion of this student on some aspects of the courses, and the relation between these courses and the jobs, then the jobs of domestic helper or waiter are not suitable for him. Moreover, there is a strong relationship between what is both interesting and at a suitable level for the knowledge of this person.

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