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Extensions of Set Functions*

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Abstract

We establish a necessary and sufficient condition for a function defined on a subset of an algebra of sets to be extendable to a positive additive function on the algebra. It is also shown that this condition is necessary and sufficient for a regular function defined on a regular subset of the Borel algebra of subsets of a given compact Hausdorff space to be extendable to a measure.
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1 Introduction

A standard method of constructing a measure in a given set X is to define first an additive function on an algebra $\mathcal{A}$ of subsets of X and then extend this function to a measure on the σ–algebra generated by $\mathcal{A}$. This 'extension problem' is an important part of the classical measure theory. Standard examples include Hahn's extension theorem and the Borel measure in $[0,1]$ (cf. [3, III.5]).

In the paper, we are concerned with the following problem: Let X be a set and $\mathcal{A}$ be an algebra of subsets of X. Given a subset $\mathcal{S}$

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of $\mathcal{A}$ and a real valued function α on $\mathcal{S}$, find necessary and sufficient conditions for α to be extendable to a positive additive function μ on $\mathcal{A}$.

The following condition is instrumental in our treatment of the extension problem:

$$\sum_{A\in\mathcal{F}} n(A)\chi_{_A}(s) \geq 0,\ \forall s \in X \quad \Rightarrow \quad \sum_{A\in\mathcal{F}} n(A)\alpha(A) \geq 0, \qquad \text{[R]}$$

for any finite family $\mathcal{F} \subseteq \mathcal{S}$, where coefficients $n(A)$'s are arbitrary integers and $\chi_{_A}$ stands for the characteristic function of a set $A \subseteq X$.

We show that condition [R] is necessary (Section 2) and sufficient (Section 3) for α to be extendable to a positive additive set function. In the case when X is a finite set, a stronger result is also established in Section 3. To obtain these results, we only assume that X is a finite union of elements of $\mathcal{S}$ (this assumption is dropped in the case of a finite set X).

We make additional assumptions about the quadruple $(X, \mathcal{A}, \mathcal{S}, \alpha)$ when treating the extension problem for measures in sections 4 and 5. In both sections, X is a compact Hausdorff space. In Section 4, $\mathcal{A}$ is the Borel algebra $\mathcal{B}$ of subsets of X, whereas in Section 5, $\mathcal{A}$ is the σ–algebra generated by $\mathcal{S}$. Assuming, in addition, that $\mathcal{S}$ and α satisfy some 'regularity' conditions, we show that [R] is a necessary and sufficient condition for α to be extendable to a positive regular measure on $\mathcal{A}$.

Our approach to the extension problem comes close to that of Bruno de Finetti in his "Probability Theory" [4] (Sections 9 and 10). In particular, his "convexity condition" (Section 15 in Appendix) is equivalent to condition [R], although de Finetti formulates it in rather different terms.

2 Condition [R]

The following lemma establishes a useful equivalent form of condition [R].

Lemma 1. [R] *is equivalent to the following condition*

$$\sum_{A\in\mathcal{F}} c(A)\chi_{_A}(s) \geq 0,\ \forall s \in X \quad \Rightarrow \quad \sum_{A\in\mathcal{F}} c(A)\alpha(A) \geq 0, \qquad (1)$$

for any finite family $\mathcal{F} \subseteq \mathcal{S}$, where coefficients $c(A)$'s are arbitrary real numbers.

Proof. It suffices to show that [R] implies (1). Suppose that for some real coefficients $c(A)$'s such that $\sum_{A\in\mathcal{F}} c(A)\,\chi_A \geq 0$ we have $\sum_{A\in\mathcal{F}} c(A)\alpha(A) < 0$. There are rational numbers $p(A)$'s such that $\sum_{A\in\mathcal{F}} p(A)\alpha(A) < 0$ and $p(A) \geq c(A)$ for all $A \in \mathcal{F}$. Clearly,

$$\sum_{A\in\mathcal{F}} p(A)\chi_A \geq \sum_{A\in\mathcal{F}} c(A)\chi_A \geq 0.$$

Multiplying both inequalities $\sum_{A\in\mathcal{F}} p(A)\chi_A \geq 0$ and $\sum_{A\in\mathcal{F}} p(A)\alpha(A) < 0$ by a common multiple of the denominators of nonzero coefficients $p(A)$'s, we obtain a contradiction to [R].

□

Suppose that α is a restriction of a positive additive set function μ on $\mathcal{A}$. Note that the first sum in (1) is, by definition, a simple function on X. Then condition (1) states that the integral of a positive simple function is positive ([3, III.2.14]). Thus we have the following proposition.

Proposition 1. [R] *is a necessary condition for a function α on $\mathcal{S}$ to be extendable to a positive additive set function on $\mathcal{A}$.*

3 Extensions to positive additive set functions

We denote by B_0 the vector space of all simple functions (with respect to $\mathcal{A}$) on X and denote by $B_0^\#$ – the algebraic dual space. The space $B_0^\#$ is isomorphic to the vector space of all additive set functions μ on $\mathcal{A}$. The isomorphism is given by

$$\mu \mapsto f_\mu \quad \text{where} \quad f_\mu(x) = \int x(s)\,\mu(ds). \tag{2}$$

The set C of all positive simple functions on X is a convex cone in B_0. Thus B_0 is an ordered vector space. A functional $f \in B_0^\#$ is *monotone* if $x \geq y$ implies $f(x) \geq f(y)$. A functional f is monotone if and only if it is *positive*, i.e., $x \geq 0$ implies $f(x) \geq 0$.

We shall use the following general fact about monotone linear extensions of linear functionals on ordered vector spaces ([1, Theorem 1, §6, ch. 2]).

Theorem 1. *Let L be a vector space with a cone C. Let L_0 be a subspace of L such that for each x in L, $x + L_0$ meets C if and only if $-x + L_0$ meets C. Let f_0 in $L_0^\#$ be monotone. Then there exists an extension f of f_0 which is monotone and in $L^\#$.*

Now we prove the main theorem of this section.

Theorem 2. *Let $\mathcal{S}$ be a subset of $\mathcal{A}$ such that X is a finite union of sets in $\mathcal{S}$ and let α be a function on $\mathcal{S}$. Then α can be extended to a positive additive function μ on $\mathcal{A}$ if and only if it satisfies condition* [R].

Proof. Necessity was established in Proposition 1.

Sufficiency. Let L_0 be the subspace of B_0 generated by the characteristic functions of sets in $\mathcal{S}$. For $x = \sum_{A \in \mathcal{F}} c(A)\chi_A \in L_0$ where $\mathcal{F}$ is a finite subset of $\mathcal{S}$, we define

$$f_0(x) = \sum_{A \in \mathcal{F}} c(A)\alpha(A).$$

It follows immediately from (1) that f_0 is well–defined and is a positive linear functional on L_0.

Note that for any $x \in B_0$ the set $x + L_0$ meets the cone C of positive functions in B_0. Indeed, let $X = \cup_{i=1}^n A_i$, $A_i \in \mathcal{S}$ and define $x_0 = \sum_{i=1}^n \chi_{A_i} \in L_0$. Then, for $m = \sup_{s \in X} |x(s)|$, $x + mx_0 \in C$.

By Theorem 1, f_0 admits an extension to a positive linear functional f on B_0. By defining $\mu(A) = f(\chi_A)$ for $A \in \mathcal{A}$, we obtain an extension of α to a positive additive function on $\mathcal{A}$.

□

Note that the assumption that X is a finite union of sets in $\mathcal{S}$ is essential in the theorem. Indeed, let X be an infinite set, $\mathcal{A} = 2^X$, and let $\mathcal{S}$ be the family of all singletons in $\mathcal{A}$. Let us define $\alpha(\{s\}) = 1$, $\forall s \in X$. Thus defined α satisfies condition [R] but cannot be extended to a monotone additive function on $\mathcal{A}$.

On the other hand, in the case of a finite set X we have a stronger result.

Theorem 3. *Let X be a finite set, $\mathcal{S} \subseteq \mathcal{A}$, and α be a function on $\mathcal{S}$. Then α can be extended to a positive additive function μ on $\mathcal{A}$ if and only if it satisfies condition* [R] *with coefficients from a finite set of integers.*

Proof. Again, we need to prove sufficiency only. Let $X' = \cup\mathcal{S}$ and $\mathcal{A}'$ be the algebra of subsets of X' consisting of sets in $\mathcal{A}$ that are subsets of X'. By Theorem 2, α can be extended to a positive additive set function μ' on $\mathcal{A}'$. For an $A \in \mathcal{A}$, we define $\mu(A) = \mu'(A \cap X')$. Clearly, μ is a positive additive set function on $\mathcal{A}$.

Let us consider characteristic functions of sets in $\mathcal{S}$ as integral vectors in $\mathbb{R}^{|X|}$ and let C be the intersection of the subspace generated by these vectors with the positive cone in $\mathbb{R}^{|X|}$. The cone C is a rational polyhedral cone and therefore has an integral Hilbert basis (Theorem 16.4 in [5]). Thus we can use only vectors from this basis in the right side of the implication in [R]. It follows that in the case of finite set X coefficients in [R] can be taken from a finite set of integers.

□

Remark. It was noted by Jean–Paul Doignon (personal communication) that sufficiency of condition [R] in the finite case is a direct consequence of Farkas' lemma [5, Corollary 7.1d].

4 Extensions to measures I

The following example shows that, in general, condition [R] is not sufficient for a function α to be extendable to a positive measure (σ–additive set–function) on a σ–algebra $\mathcal{A}$.

Example 1. Let $X = [0, 1]$ and $\mathcal{S} = \{[0, t) : t \in (0, 1]\} \cup \{[0, t] : t \in [0, 1]\}$. Note that $X \in \mathcal{S}$. We define $\alpha(\{0\}) = 0$ and $\alpha(A) = 1$ if A is $[0, t)$ or $[0, t]$ for $0 < t \cdot\ 1$. It is easy to verify that thus defined α satisfies condition [R].

Let μ be a σ–additive extension of α to the σ–algebra $\mathcal{B}$ of Borel subsets of $[0,1]$. We have

$$\begin{aligned}
&\mu((t,1]) = 1 - \alpha([0,t]) = 0, \quad \text{for } t > 0,\\
&\mu((0,1]) = 1 - \alpha(\{0\}) = 1,\\
&\mu((s,t]) = 1 - \alpha([0,s]) - \mu((t,1]) = 0, \quad \text{for } 0 < s < t.
\end{aligned}$$

By σ–additivity of μ,

$$1 = \mu((0,1]) = \mu\left(\bigcup_1^\infty \left(\frac{1}{k+1}, \frac{1}{k}\right]\right) = \sum_1^\infty \mu\left(\left(\frac{1}{k+1}, \frac{1}{k}\right]\right) = 0,$$

a contradiction. On the other hand, by condition [R], there is an additive extension of α to $\mathcal{B}$.

This example suggests that in order to keep [R] as a necessary and sufficient condition for extendibility of a set function to a measure, some constrains should be imposed on the quadruple $(X, \mathcal{A}, \mathcal{S}, \alpha)$. Namely, we assume that X is a compact Hausdorff space and introduce the following 'regularity' conditions on $\mathcal{S}$ and α.

Definition 1. (i) A family $\mathcal{S}$ of subsets of X is said to be *regular* if

(a) For each $E \in \mathcal{S}$ and a closed set $F \subseteq E$ there is $E' \in \mathcal{S}$ such that
$$F \subseteq E' \subseteq \mathrm{cl}E' \subseteq E.$$

(b) For each $E \in \mathcal{S}$ and an open set $G \supseteq E$ there is $E'' \in \mathcal{S}$ such that
$$E \subseteq \mathrm{int}E'' \subseteq E'' \subseteq G.$$

(ii) A function α on a family $\mathcal{S}$ is said to be *regular* if for each $E \in \mathcal{S}$ and $\varepsilon > 0$ there is a set F in $\mathcal{S}$ whose closure is contained in E and a set G whose interior contains E such that $|\alpha(G) - \alpha(F)| < \varepsilon$.

In this section, $\mathcal{A}$ is the Borel algebra $\mathcal{B}$ of subsets of X.

Example 2. Since X is a normal space, the families of all open sets and of all closed sets in X are examples of regular families of Borel sets (cf. [2, VII.3.2(2)]).

Example 3. Let $X = [0,1]$ and S be the family of all intervals in the form $[a, b)$. Clearly, S is a regular family of Borel sets.

Example 4. Let $\mathcal{S} = \mathcal{B}$ and let $\alpha = \mu$ – a regular positive additive set function on $\mathcal{B}$ in the usual sense (cf. [3, III.5.11]). Then α is a regular function in the sense of Definition 1.

Lemma 2. *Let μ be a regular positive measure on the Borel algebra $\mathcal{B}$ and let $\mathcal{S}$ be a regular family of Borel sets. Then the restriction of μ to $\mathcal{S}$ is a regular function on $\mathcal{S}$.*

Proof. Let $E \in \mathcal{S}$ and $\varepsilon > 0$. Since μ is regular and positive, there is a closed set $F \subseteq E$ and an open set $G \supseteq E$ such that $\mu(G) - \mu(F) < \varepsilon$. Since $\mathcal{S}$ is regular, there are $E', E'' \in \mathcal{S}$ such that $F \subseteq E' \subseteq \mathrm{cl}E' \subseteq E \subseteq \mathrm{int}E'' \subseteq E'' \subseteq G$. Since μ is positive, $\mu(E'') - \mu(E') < \varepsilon$. Therefore the restriction of μ to $\mathcal{S}$ is a regular set function on $\mathcal{S}$.

□

Lemma 3. *Let S be a regular family of Borel sets such that X is a finite union of sets in $\mathcal{S}$ and let α be a regular function on $\mathcal{S}$ satisfying condition* [R]. *Then α is extendable to a regular positive measure on $\mathcal{B}$.*

Proof. By Theorem 2, α admits an extension to a positive additive set function μ on $\mathcal{B}$. Since μ is bounded, it defines a bounded positive linear functional f on the Banach space B of all uniform limits of functions in B_0 endowed with the norm $\|\cdot\|_\infty$. This functional is given by [3, IV.5.1]

$$f(x) = \int x(s)\,\mu(ds), \quad x \in B_0.$$

By the Riesz representation theorem [3, IV.6.3] the restriction of this functional (which we denote by the same symbol f) to the space $C(X)$ of continuous functions on S is given by

$$f(x) = \int x(s)\,\mu^*(ds), \quad x \in C(X),$$

where μ^* is a regular positive measure on $\mathcal{B}$.

Now it suffices to show that $\mu^*(E) = \mu(E)$ on $\mathcal{S}$. Let $E \in \mathcal{S}$ and $\varepsilon > 0$. Since μ^* is positive and regular there is a closed set F and an open set G such that

$$F \subseteq E \subseteq G, \quad \mu^*(F) \cdot \ \mu^*(E) \cdot \ \mu^*(G), \quad \text{and} \quad \mu^*(G) - \mu^*(F) < \varepsilon.$$

Since $\mathcal{S}$ is regular, there are $E', E'' \in \mathcal{S}$ such that

$$F \subseteq E' \subseteq \mathrm{cl}E' \subseteq E \subseteq \mathrm{int}E'' \subseteq E'' \subseteq G \quad \text{and} \quad \mu(E'') - \mu(E') < \varepsilon.$$

We denote $F' = \mathrm{cl}E'$ and $G' = \mathrm{int}E''$. Since μ and μ^* are positive,

$$\mu(G') - \mu(F') < \varepsilon \quad \text{and} \quad \mu^*(G') - \mu^*(F') < \varepsilon. \tag{3}$$

Since X is a normal space, by Urysohn's lemma, there is a continuous function x such that

$$\begin{array}{ll} 0 \cdot \ x(s) \cdot \ 1, & \text{for all } s \in X, \\ x(s) = 1, & \text{for all } s \in F', \\ x(s) = 0, & \text{for all } s \notin G'. \end{array}$$

For a natural number n, we define a family of $n+1$ intervals in $[0,1]$ by

$$I_k = \begin{cases} \left[\frac{k-1}{n}, \frac{k}{n}\right), & \text{for } 1 \cdot \ k \cdot \ n; \\ \{1\}, & \text{for } k = n+1. \end{cases}$$

The family of Borel sets $E_k = x^{-1}(I_k),\ 1 \cdot \ k \cdot \ n+1$, forms a partition of X. Clearly, $\bigcup_{k=2}^n E_k \subseteq G' \setminus F'$. Therefore, by the first inequality in (3),

$$\sum_{k=2}^{n} \mu(E_k) \cdot \ \mu(G') - \mu(F') < \varepsilon. \tag{4}$$

Let x_n be a function defined by $x_n(s) = \frac{k-1}{n}$ for $s \in E_k,\ 1 \cdot \ k \cdot \ n+1$. Thus

$$|f(x) - f(x_n)| \cdot \ \|f\| \cdot \|x - x_n\| < \frac{1}{n}\,\|f\| \tag{5}$$

Further,

$$x_n = \sum_{k=1}^{n+1} \tfrac{k-1}{n} \chi_{E_k} = \sum_{k=2}^{n} \tfrac{k-1}{n} \chi_{E_k} + \chi_{E_{k+1}}.$$

Thus

$$f(x_n) = \sum_{k=2}^{n} \tfrac{k-1}{n} \mu(E_k) + \mu(E_{k+1}),$$

which implies, by (4),

$$f(x_n) - \mu(E_{k+1}) = \sum_{k=2}^{n} \tfrac{k-1}{n} \mu(E_k) < \varepsilon.$$

This inequality together with one in (5) imply

$$|f(x) - \mu(E_{n+1})| < \varepsilon + \frac{1}{n}\|f\|. \tag{6}$$

Clearly, $F' \subseteq E_{n+1} \subseteq G'$, and $F' \subseteq E \subseteq G'$. Thus, by (3),

$$|\mu(E_{n+1}) - \mu(E)| < \varepsilon. \tag{7}$$

Since $f(x) = \int x(s)\, \mu^*(ds)$, we have $\mu^*(F') \cdot \ f(x) \cdot \ \mu^*(G')$. On the other hand, $\mu^*(F') \cdot \ \mu^*(E) \cdot \ \mu^*(G')$. By the second inequality in (3),

$$|\mu^*(E) - f(x)| < \varepsilon. \tag{8}$$

Combining inequalities (6), (7), and (8), we have

$$|\mu^*(E) - \mu(E)| < 3\varepsilon + \frac{1}{n}\|f\|.$$

Hence, $\mu^*(E) = \mu(E) = \alpha(E)$.

□

Combining the results of Lemma 2 and Lemma 3, we have the following theorem.

Theorem 4. *Let $\mathcal{S}$ be a regular family of Borel sets such that X is a finite union of sets in $\mathcal{S}$. A function α on $\mathcal{S}$ is extendable to a regular positive measure on $\mathcal{B}$ if and only if it is regular and satisfies condition* [R].

5 Extensions to measures II

In this section we make a different assumption about components of the quadruple $(X, \mathcal{A}, \mathcal{S}, \alpha)$. Namely, let X again be a compact Hausdorff space, $\mathcal{S}$ be a family of subsets of X, and let $\mathcal{A}$ be the σ–algebra generated by $\mathcal{S}$.

Lemma 4. *Let $\mathcal{S}$ be a regular family of subsets of X. The restriction of a regular positive measure μ on $\mathcal{A}$ to $\mathcal{S}$ is a regular function on $\mathcal{S}$.*

Proof. Let $E \in \mathcal{S}$ and $\varepsilon > 0$. Since μ is regular and positive, there is $F \in \mathcal{A}$ such that $\mathrm{cl}F \subseteq E$ and $G \in \mathcal{A}$ such that $\mathrm{int}G \supseteq E$ such that $\mu(G) - \mu(F) < \varepsilon$. Since $\mathcal{S}$ is regular, there are $E', E'' \in \mathcal{S}$ such that

$$F \subseteq \mathrm{cl}F \subseteq E' \subseteq \mathrm{cl}E' \subseteq E \subseteq \mathrm{int}E'' \subseteq E'' \subseteq \mathrm{int}G \subseteq G.$$

Since μ is positive, $\mu(E'') - \mu(E') < \varepsilon$. Therefore the restriction of μ to $\mathcal{S}$ is a regular set function on $\mathcal{S}$.

□

Lemma 5. *Let S be a regular family of subsets of X such that X is a finite union of sets in $\mathcal{S}$ and let α be a regular function on $\mathcal{S}$ satisfying condition* [R]*. Then α is extendable to a regular positive measure μ on the σ–algebra $\mathcal{A}$ generated by $\mathcal{S}$.*

Proof. Let $\mathcal{A}_0$ be the algebra generated by $\mathcal{S}$. By Theorem 2, α admits an extension to a positive additive set function μ on $\mathcal{A}_0$. It suffices to show that μ is a regular function on $\mathcal{A}_0$. Indeed, by Theorem 14 in [3, III.5], a regular function on $\mathcal{A}_0$ admits an extension to a positive measure on $\mathcal{A}$.

Let $\varepsilon > 0$ and A and B be two sets in $\mathcal{S}$. Since $\mathcal{S}$ is a regular family and α is a regular function, there are $A_1, A_2 \in \mathcal{S}$ and $B_1, B_2 \in \mathcal{S}$ such that

$$A_1 \subseteq \mathrm{cl}A_1 \subseteq A \subseteq \mathrm{int}A_2 \subseteq A_2, \quad \alpha(A_2) - \alpha(A_1) < \varepsilon/2,$$

and

$$B_1 \subseteq \mathrm{cl}B_1 \subseteq B \subseteq \mathrm{int}B_2 \subseteq B_2, \quad \alpha(B_2) - \alpha(B_1) < \varepsilon/2,$$

We have

$$\mu(A_1 \cup B_1) + \mu(A_1 \cap B_1) = \mu(A_1) + \mu(B_1) = \alpha(A_1) + \alpha(B_1)$$

and

$$\mu(A_2 \cup B_2) + \mu(A_2 \cap B_2) = \mu(A_2) + \mu(B_2) = \alpha(A_2) + \alpha(B_2).$$

Hence,

$$[\mu(A_2 \cup B_2) - \mu(A_1 \cup B_1)] + [\mu(A_2 \cap B_2) - \mu(A_1 \cap B_1)] = \\ = [\alpha(A_2) - \alpha(A_1)] + [\alpha(B_2) - \alpha(B_1)] < \varepsilon,$$

implying

$$\mu(A_2 \cup B_2) - \mu(A_1 \cup B_1) < \varepsilon \quad \text{and} \quad \mu(A_2 \cap B_2) - \mu(A_1 \cap B_1) < \varepsilon.$$

Clearly,

$$\mathrm{cl}(A_1 \cup B_1) \subseteq A \cup B \subseteq \mathrm{int}(A_1 \cup B_1) \quad \text{and} \\ \mathrm{cl}(A_1 \cap B_1) \subseteq A \cap B \subseteq \mathrm{int}(A_1 \cap B_1).$$

Thus the regularity condition for μ is satisfied for unions and intersections of sets in $\mathcal{S}$. Hence, μ is a regular function on $\mathcal{A}_0$.

□

Combining the results of Lemma 4 and Lemma 5, we have the following theorem.

Theorem 5. *Let $\mathcal{A}$ be the σ–algebra generated by a regular family $\mathcal{S}$ of subsets of X such that X is a finite union of sets in $\mathcal{S}$. A function α on $\mathcal{S}$ is extendable to a regular positive measure on $\mathcal{A}$ if and only if it is regular and satisfies condition* [R].

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