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On the Central Limit Theorem on IFS-events

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Abstract

A probability theory on IFS-events has been constructed in [3], and axiomatically characterized in [4]. Here using a general system of axioms it is shown that any probability on IFS-events can be decomposed onto two probabilities on a Lukasiewicz tribe, hence some known results from [5], [6] can be used also for IFS-sets. As an application of the approach a variant of Central limit theorem is presented.

Keywords. Probability theory, IFS-events

1 Introduction

An IFS-set A on a space Ω as a couple (μ_A, ν_A) is understood, $\mu_A : \Omega \to \langle 0, 1\rangle$, $\nu_A : \Omega \to \langle 0, 1\rangle$ such that $\mu_A(\omega) + \nu_A(\omega) \leq 1$ for any $\omega \in \Omega$ (see[1]). The function μ_A is called the membership function, the function ν_A is called the non membership function. An IFS-set $A = (\mu_A, \nu_A)$ is called IFS-event if μ_A, ν_A are $\mathcal{S}$-measurable with respect to a given σ-algebra of subsets of Ω.

In [3] P. Grzegorzewski and E. Mrowka considered a classical probability space $(\Omega, \mathcal{S}, \mathcal{P})$ and they suggested to define a probability measure on the set $\mathcal{G}$ of all IFS events as an interval valued function $\mathcal{P}$ by the following way. Probability $\mathcal{P}(A)$ of an event $A = (\mu_A, \nu_A)$ is the interval

$$\mathcal{P}(A) = [\textstyle\int_\Omega \mu_A \, dP, 1 - \int_\Omega \nu_A \, dP]. \qquad (*)$$

If $\nu_A = 1 - \mu_A$, then the interval is the singleton $\int_\Omega \mu_A \, dP$, hence the Grzegorzewski and Mrowka definition is an extension of the Zadeh definition. The probability $\mathcal{P}$

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is the function $\mathcal{P} : \mathcal{G} \to \mathcal{J}$, where $\mathcal{J}$ is the family of all compact subintervals of the unit interval $I = [0, 1]$. In [3] many properties of the mapping $\mathcal{P}$ were discovered. Then in [4] it was proved that any function $\mathcal{P} : \mathcal{G} \to \mathcal{J}$ satisfying some properties (as continuity, some kind of additivity etc.) has the form $(*)$.

Special attention should by devoted to the notion of additivity of $\mathcal{P}$. Namely in fuzzy sets theory there are many possibilities how the define the intersection and the union of fuzzy sets. Recall that the representation theorem works with the Lukasiewicz connectives $\oplus$, $\odot$, hence the additivity has the form

$$A \odot B = (0, 1) \Longrightarrow \mathcal{P}(A \oplus B) = \mathcal{P}(A) + \mathcal{P}(B)$$

In the paper we shall use a more general situation. Instead of the set of all measurable functions with values in $\langle 0, 1\rangle$ we shall consider any Lukasiewicz tribe $\mathcal{T}$. Instead of IFS-events we consider the family $\mathcal{F}$ of all couples (f, g) of elements of $\mathcal{T}$ such that $f + g \leq 1$. Then we define axiomatically the notion of a probability as a function from $\mathcal{F}$ to the family $\mathcal{J}$ of all compact subintervals of the unit interval. Moreover, we define the notion of an observable, that is an analogue of the notion of a random variable in the Kolmogorov theory. This notion is introduced here for the first time with regard to IFS-events. The main result of the paper are the representation theorems representing probabilities and observables in $\mathcal{F}$ by the corresponding notions in $\mathcal{T}$. Consequently it is possible to transpone some known theorems from the probability theory on tribes to the more general case of IFS-events. As an illustration of the developed method the central limit theorem is presented.

In Section 2 we give the definitions of basic notions and some examples. Section 3 contains the representation theorems. In Section 4 and Section 5 a version of the central limit theorems is presented.

2 Probabilities and observables

Recall that a tribe is a non-empty family $\mathcal{T}$ of functions $f : \Omega \to \langle 0, 1\rangle$ satisfying the following conditions:

(i) $f \in \mathcal{T} \Longrightarrow 1 - f \in \mathcal{T}$;

(ii) $f, g \in \mathcal{T} \Longrightarrow f \oplus g = \min(f + g, 1) \in \mathcal{T}$;

(iii) $f_n \in \mathcal{T}$ $(n = 1, 2, ...)$, $f_n \nearrow f \Longrightarrow f \in \mathcal{T}$.

Of course, a tribe is a special case of a σ-MV-algebra.

In the preceding definition (instead of $\max(a, b)$) we have used the first Lukasiewicz operation $\oplus$: $\langle 0, 1\rangle \times \langle 0, 1\rangle \to \langle 0, 1\rangle$, $a \oplus b = \min(a + b, 1)$. The second binary operation $\odot$ is defined by the equality $a \odot b = \max(a + b - 1, 0)$. It is easy to see that $\chi_A \oplus \chi_B = \chi_{A\cup B}$, $\chi_A \odot \chi_B = \chi_{A\cap B}$. Recall [5,6] that probability (= a state) on a Lukasiewicz tribe $\mathcal{T}$ is any mapping $p : \mathcal{T} \to \langle 0, 1\rangle$ satisfying the following conditions:

(i) $p(1_\Omega) = 1$;

(ii) if $f \odot g = 0_\Omega$, then $p(f \oplus g) = p(f) + p(g)$;

(iii) if $f_n \nearrow f$, then $p(f_n) \nearrow p(f)$.

Example 1. Let $\mathcal{S}$ be a σ-algebra of subsets of a set Ω, $P : \mathcal{S} \to \langle 0, 1\rangle$ be a probability measure. χ_A be the characteristic function of a set $A \in \mathcal{S}$. Put $\mathcal{T} = \{\chi_A; A \in \mathcal{S}\}$, $p(\chi_A) = P(A)$. Then $\mathcal{T}$ is a tribe and p is a probability on $\mathcal{T}$.

Example 2. Again let $(\Omega, \mathcal{S}, \mathcal{P})$ be a probability space, $\mathcal{T}$ be the set of all $\mathcal{S}$-measurable function $f : \Omega \to \langle 0, 1\rangle$, $p(f) = \int_\Omega f\, dP$. Then $\mathcal{T}$ is a tribe and p is a probability on $\mathcal{T}$ defined by Zadeh [7].

During the whole text we fix the tribe $\mathcal{T}$ and the generated family $\mathcal{F}$.

Definition 1. By an IFS-event we understand any element of the family

$$\mathcal{F} = \{(\mu_A, \nu_A); \mu_A, \nu_A \in \mathcal{T}, \mu_A + \nu_A \leq 1\}$$

To define the notion of probability on IFS-events we need to introduce operations on $\mathcal{F}$. Let $A = (\mu_A, \nu_A)$, $B = (\mu_B, \nu_B)$. Then we define

$$A \oplus B = (\mu_A \oplus \mu_B, \nu_A \odot \nu_B),$$
$$A \odot B = (\mu_A \odot \mu_B, \nu_A \oplus \nu_B).$$

If $A_n = (\mu_{A_n}, \nu_{A_n})$, then we write

$$A_n \nearrow A \Longleftrightarrow \mu_{A_n} \nearrow \mu_A, \quad \nu_{A_n} \searrow \nu_A.$$

If $\nu_A = 1 - \mu_A$, $\nu_B = 1 - \mu_B$, then

$$A \oplus B = (\mu_A \oplus \mu_B, (1 - \mu_A) \odot (1 - \mu_B)) = (\mu_A \oplus \mu_B, 1 - \mu_A \oplus \mu_B),$$

and similarly $A \odot B = (\mu_A \odot \mu_B, 1 - \mu_A \odot \mu_B)$.

A probability $\mathcal{P}$ on $\mathcal{F}$ is a mapping from $\mathcal{F}$ to the family $\mathcal{J}$ of all closed intervals $\langle a, b\rangle$ such that $0 \leq a \leq b \leq 1$. Here we define

$$\langle a, b\rangle + \langle c, d\rangle = \langle a + c, b + d\rangle,$$
$$\langle a_n, b_n\rangle \nearrow \langle a, b\rangle \Longleftrightarrow a_n \nearrow a,\ b_n \nearrow b.$$

Definition 2. By an IFS-probability on $\mathcal{F}$ we understand any function $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ satisfying the following properties :

(i) $\mathcal{P}((1_\Omega, 0_\Omega)) = \langle 1, 1\rangle = \{1\}$; $\mathcal{P}((0_\Omega, 1_\Omega)) = \langle 0, 0\rangle = \{0\}$;

(ii) $\mathcal{P}(A \oplus B) + \mathcal{P}(A \odot B) = \mathcal{P}(A) + \mathcal{P}(B)$ for any $A, B \in \mathcal{F}$;

(iii) if $A_n \nearrow A$, then $\mathcal{P}(A_n) \nearrow \mathcal{P}(A)$.

$\mathcal{P}$ is called separating, if $\mathcal{P}((f, g)) = \langle p(f), 1 - q(g)\rangle$ for some $p, q : \mathcal{T} \to \langle 0, 1\rangle$.

Example 3. ([3]). Let $(\Omega, \mathcal{S}, P)$ be a probability space $\mathcal{T} = \{f; f : \Omega \to \langle 0,1\rangle, f$ is $\mathcal{S}$ measurable$\}$, and for $A \in \mathcal{F}$, $A = (\mu_A, \nu_A)$, put

$$\mathcal{P}(A) = \left\langle \int_\Omega \mu_A \, dP, \quad 1 - \int_\Omega \nu_A \, dP \right\rangle.$$

Then $\mathcal{P}$ is probability with respect to Definition 2. Indeed,

$$\mathcal{P}(1_\Omega, 0_\Omega) = \left\langle \int_\Omega 1_\Omega \, dP, \quad 1 - \int_\Omega 0_\Omega \, dP \right\rangle = \langle 1, 1\rangle,$$
$$\mathcal{P}(0_\Omega, 1_\Omega) = \left\langle \int_\Omega 0_\Omega \, dP, \quad 1 - \int_\Omega 1_\Omega \, dP \right\rangle = \langle 0, 0\rangle.$$

The property (iii) has been proved in [3], we shall prove (ii). We have

$$\begin{aligned}
&\mathcal{P}(A \oplus B) + \mathcal{P}(A \odot B) = \\
&= \mathcal{P}((\mu_A \oplus \mu_B, \nu_A \odot \nu_B)) + \mathcal{P}(\mu_A \odot \mu_B, \nu_A \oplus \nu_B) \\
&= \left\langle \int (\mu_A \oplus \mu_B) \, dP, 1 - \int (\nu_A \odot \nu_B) \, dP \right\rangle \\
&+ \left\langle \int (\mu_A \odot \mu_B) \, dP, 1 - \int (\nu_A \oplus \nu_B) \, dP \right\rangle \\
&= \left\langle \int (\mu_A \oplus \mu_B + \mu_A \odot \mu_B) \, dP, 2 - \int (\nu_A \odot \nu_B + \nu_A \oplus \nu_B) \, dP \right\rangle \\
&= \left\langle \int \mu_A \, dP + \int \mu_B \, dP, 1 - \int \nu_A \, dP + 1 - \int \nu_B \, dP \right\rangle \\
&= \mathcal{P}(A) + \mathcal{P}(B).
\end{aligned}$$

Moreover, in [4] it has been proved that under two additional conditions any IFS-probability $\mathcal{P}$ on the family $\mathcal{F}$ generated by a σ-algebra $\mathcal{S}$, has the above form.

More generally, if $p, q : \mathcal{T} \to \langle 0,1\rangle$, $p \le q$ are probabilities, then $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ defined by $\mathcal{P}((f, g)) = \langle p(f), 1 - q(g)\rangle$, is a probability. In the special case $\mathcal{P}((f, 1 - f)) = \langle p(f), q(f)\rangle$.

The second important notion in the probability theory is the notion of a random variable. According to the terminology used in quantum structures we shall speak about observables instead of random variables. Recall that an observable with values in $\mathcal{T}$ is a mapping $x : \mathcal{B}(R) \to \mathcal{T}$ ($\mathcal{B}(R)$ being the σ-algebra of Borel subsets of R) satisfying the following properties:

(i) $x(R) = 1_\Omega$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = 0_\Omega$, and $x(A \cup B) = x(A) + x(B)$;

(iii) if $A_n \nearrow A$ then $x(A_n) \nearrow x(A)$.

Definition 3. A mapping $x : \mathcal{B}(R) \to \mathcal{F}$ is called an IFS-observable, if it satisfies the following conditions:

(i) $x(R) = (1_\Omega, 0_\Omega)$;

(ii) if $A \cap B = \emptyset$, then $x(A) \odot x(B) = (0_\Omega, 1_\Omega)$, and $x(A \cup B) = x(A) \oplus x(B)$;

(iii) if $A_n \nearrow A$, then $x(A_n) \nearrow x(A)$.

3 Representation theorems

Theorem 1. *$\mathcal{P} : \mathcal{F} \to \mathcal{J}$ is a separating IFS-probability if and only if $p, q : \mathcal{T} \to \langle 0, 1\rangle$ are probabilities.*

Proof. Let $A = (f_1, f_2)$, $B = (g_1, g_2) \in \mathcal{F}$. Then

$$\begin{aligned}
A \oplus B &= (f_1 \oplus g_1, f_2 \odot g_2),\\
A \odot B &= (f_1 \odot g_1, f_2 \oplus g_2),\\
\mathcal{P}(A \oplus B) &= \langle p(f_1 \oplus g_1), 1 - q(f_2 \odot g_2)\rangle,\\
\mathcal{P}(A \odot B) &= \langle p(f_1 \odot g_1), 1 - q(f_2 \oplus g_2)\rangle,\\
\mathcal{P}(A \oplus B) + \mathcal{P}(A \odot B) &= \langle p(f_1 \oplus g_1) + p(f_1 \odot g_1), 2 - q(f_2 \odot g_2) - q(f_2 \oplus g_2)\rangle,\\
\mathcal{P}(A) + \mathcal{P}(B) &= \langle p(f_1, 1 - q(g_1)\rangle + \langle p(g_1), 1 - q(g_2)\rangle\\
&= \langle p(f_1) + p(g_1), 2 - q(g_1) - q(g_2)\rangle,
\end{aligned}$$

hence

$$\begin{aligned}
p(f_1 \oplus g_1) + p(f_1 \odot g_1) &= p(f_1) + p(g_1),\\
q(f_2 \odot g_2) + q(f_2 \oplus g_2) &= q(f_2) + q(g_2),
\end{aligned}$$

for all $f_1, f_2, g_1, g_2 \in \mathcal{T}$.

By these two equalities the additivity of p and q follows. If $I = (1_\Omega, 0_\Omega)$, then

$$\langle p(1_\Omega), 1 - q(0_\Omega)\rangle = \mathcal{P}(I) = \{1\},$$

hence $p(1_\Omega) = 1$.

On the other hand, if $O = (0_\Omega, 1_\Omega)$, then

$$\langle p(0_\Omega), 1 - q(1_\Omega)\rangle = \mathcal{P}(O) = \{0\},$$

hence $1 - q(1_\Omega) = 0$, $q(1_\Omega) = 1$.

Now we prove the continuity of p and q. First let $f_n \in \mathcal{T}$, $(n = 1, 2, \dots)$, $f_n \nearrow f$. Put $F_n = (f_n, 1 - f_n)$. Then $F_n \in \mathcal{F}$, $F_n \nearrow F = (f, 1 - f)$. Therefore

$$\langle p(f_n), 1 - q(f_n)\rangle = \mathcal{P}(F_n) \nearrow \mathcal{P}(F) = \langle p(f), 1 - q(f)\rangle,$$

hence $p(f_n) \nearrow p(f)$, $1 - q(f_n) \searrow 1 - q(f)$, $q(f_n) \nearrow q(f)$. □

Theorem 2. *Let $x : \mathcal{B}(R) \to \mathcal{F}$. For any $A \in \mathcal{B}(R)$ denote $x(A) = (x^\flat(A), 1 - x^\sharp(A))$. Then x is IFS-observable if and only if $x^\flat : \mathcal{B}(R) \to \mathcal{T}$, $x^\sharp : \mathcal{B}(R) \to \mathcal{T}$ are observables.*

Proof. Since

$$(1_\Omega, 0_\Omega) = x(R) = (x^\flat(R), 1 - x^\sharp(R)),$$

we obtain

$$x^\flat(R) = 1_\Omega, x^\sharp(R) = 1_\Omega.$$

Let $A \cap B = \emptyset$. Then

$$\begin{aligned}(0_\Omega, 1_\Omega) &= x(A) \odot x(B) \\ &= (x^\flat(A), 1 - x^\sharp(A)) \odot (x^\flat(B), 1 - x^\sharp(B)) \\ &= (x^\flat(A) \odot x^\flat(B), (1 - x^\sharp(A)) \oplus (1 - x^\sharp(B))),\end{aligned}$$

hence $0_\Omega = x^\flat(A) \odot x^\flat(B)$. Further

$$1_\Omega = (1 - x^\sharp(A)) \oplus (1 - x^\sharp(B)) = (1 - x^\sharp(A) + 1 - x^\sharp(B)) \wedge 1,$$

hence

$$\begin{aligned}1 - x^\sharp(A) + 1 - x^\sharp(B) &\geq 1, \\ 1 &\geq x^\sharp(A) + x^\sharp(B), \\ x^\sharp(A) \odot x^\sharp(B) &= (x^\sharp(A) + x^\sharp(B) - 1) \vee 0 = 0.\end{aligned}$$

Moreover,

$$\begin{aligned}&(x^\flat(A \cup B), 1 - x^\sharp(A \cup B)) = \\ &= x(A \cup B) = x(A) \oplus x(B) \\ &= (x^\flat(A), 1 - x^\sharp(A)) \oplus (x^\flat(B), 1 - x^\sharp(B)) \\ &= (x^\flat(A) \oplus x^\flat(B), (1 - x^\sharp(A)) \odot (1 - x^\sharp(B))) \\ &= (x^\flat(A) + x^\flat(B), (1 - x^\sharp(A) + 1 - x^\sharp(B) - 1) \vee 0) \\ &= (x^\flat(A) + x^\flat(B), 1 - (x^\sharp(A) + x^\sharp(B))).\end{aligned}$$

Therefore

$$\begin{aligned}x^\flat(A \cup B) &= x^\flat(A) + x^\flat(B), \\ x^\sharp(A \cup B) &= x^\sharp(A) + x^\sharp(B).\end{aligned}$$

Finally, let $A_n \nearrow A$. Then

$$(x^\flat(A_n), 1 - x^\sharp(A_n)) = x(A_n) \nearrow x(A) = (x^\flat(A), 1 - x^\sharp(A)),$$

hence

$$x^\flat(A_n) \nearrow x^\flat(A), \quad 1 - x^\sharp(A_n) \searrow 1 - x^\sharp(A), \quad \text{i. e. } x^\sharp(A_n) \nearrow x^\sharp(A). \qquad \square$$

It is easy to see that the mappings

$$p_{x^\flat} = p \circ x^\flat : \mathcal{B}(R) \to \langle 0,1\rangle, \quad q_{x^\sharp} = q \circ x^\sharp : \quad \mathcal{B}(R) \to \langle 0,1\rangle$$

are probability measures. Therefore we define

$$E(x^\flat) = \int_R t\, dp_{x^\flat}(t), \quad E(x^\sharp) = \int_R t\, dq_{x^\sharp}(t)$$

if these integrals exist. In this case we say that x is integrable. Further we define

$$\sigma^2(x^\flat) = \int_R (t - E(x^\flat))^2\, dp_{x^\flat}(t), \sigma^2(x^\sharp) = \int_R (t - E(x^\sharp))^2\, dq_{x^\sharp}(t)$$

if these integral exists. In this case we say that x belongs to L^2.

Theorem 3. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be a separating IFS-probability given by $\mathcal{P}((f,g)) = \langle p(f), 1 - q(g)\rangle$, $x : \mathcal{B}(R) \to \mathcal{F}$ be an IFS-observable given by $x(A) = (x^\flat(A), 1 - x^\sharp(A))$. Then $\mathcal{P} \circ x : \mathcal{B}(R) \to \mathcal{J}$ is given by*

$$\mathcal{P} \circ x(A) = \langle p(x^\flat(A)), q(x^\sharp(A))\rangle.$$

Proof. Evidently

$$\begin{aligned}\mathcal{P} \circ x(A) &= \mathcal{P}(x(A)) = \mathcal{P}((x^\flat(A), 1 - x^\sharp(A))) \\ &= \langle p(x^\flat(A)), 1 - q(1 - x^\sharp(A))\rangle \\ &= \langle p(x^\flat(A)), q(x^\sharp(A))\rangle.\end{aligned}$$

□

4 Independence

Definition 4. An n-dimensional IFS-observable is a mapping $h : \mathcal{B}(R^n) \to \mathcal{F}$ satisfying the following conditions:

(i) $h(R^n) = (1_\Omega, 0_\Omega)$;

(ii) if $A \cap B = \emptyset$, then $h(A) \odot h(B) = (0_\Omega, 1_\Omega)$, and $h(A \cup B) = h(A) + h(B)$;

(iii) if $A_n \nearrow A$, then $h(A_n \nearrow h(A)$.

Recall that observables $(x_1, \dots, x_n) : \mathcal{B}(R) \to \mathcal{T}$ are called independent if there exists n-dimensional observable $h : \mathcal{B}(R^n) \to \mathcal{T}$ such that

$$p(h(A_1 \times A_2 \times \cdots \times A_n)) = p(x_1(A_1)) \cdot p(x_2(A_2)) \cdot \cdots \cdot p(x_n(A_n)))$$

for any $(A_1, \dots, A_n) \in \mathcal{B}(R)$.

Definition 5. IFS-observables $x_1, ..., x_n : \mathcal{B}(R) \to \mathcal{F}$ are called independent with respect to an IFS-probability $\mathcal{P}$, if there exists n-dimensional observable $h : \mathcal{B}(R) \to \mathcal{F}$ such that

$$\mathcal{P}(h(A_1 \times A_2 \times \cdots \times A_n)) = \mathcal{P}(x_1(A_1)) \otimes \mathcal{P}(x_2(A_2)) \otimes \cdots \otimes (\mathcal{P}(x_n(A_n))$$

for any $(A_1, A_2, \ldots, A_n) \in \mathcal{B}(R)$. Here

$$\langle a_1, b_1 \rangle \otimes \langle a_2, b_2 \rangle \otimes \cdots \otimes \langle a_n, b_n \rangle = \langle a_1 a_2 \ldots a_n, b_1 b_2 \ldots b_n \rangle$$

for any $\langle a_i, b_i \rangle \in \mathcal{J} (i = 1, 2, \ldots, n)$.

Theorem 4. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be a separating probability. Then IFS-observables $x_1, x_2, \ldots, x_n \in \mathcal{B}(R) \to \mathcal{F}$ are independent if and only if the corresponding observables $x_1^\flat, x_2^\flat, \ldots, x_n^\flat : \mathcal{B}(R) \to \mathcal{T}$ are independent as well as $x_1^\sharp, x_2^\sharp, \ldots, x_n^\sharp : \mathcal{B}(R) \to \mathcal{T}$.*

Proof. Let $A_1, A_2, \ldots, A_n \in \mathcal{B}(R)$. Then by Theorem 3

$$\begin{aligned}
&\langle p(h^\flat(A_1 \times \cdots \times A_n)), q(h^\sharp(A_1 \times \cdots \times A_n)) \rangle = \\
=&\mathcal{P}(h(A_1 \times \cdots \times A_n)) = \mathcal{P}(x_1(A_1)) \otimes \mathcal{P}(x_2(A_2)) \otimes \cdots \otimes \mathcal{P}(x_n(A_n)) \\
=&\langle p(x_1^\flat(A_1), q(x_1^\sharp(A_1)) \rangle \otimes \cdots \otimes \langle p(x_n^\flat(A_n)), q(x_n^\sharp(A_n)) \rangle \\
=&\langle p(x_1^\flat(A_1)) \cdot p(x_2^\flat(A_2) \cdot \cdots \cdot p(x_n^\flat(A_n)), q(x_1^\sharp(A_1)) \cdot q(x_2^\sharp(A_2)) \cdot \cdots \cdot q(x_n^\sharp(A_n)) \rangle
\end{aligned}$$

hence

$$\begin{aligned}
p(h^\flat(A_1 \times \cdots \times A_n)) &= p(x_1^\flat(A_1)) \cdot p(x_2^\flat(A_2)) \cdot \cdots \cdot p(x_n^\flat(A_n)), \\
q(h^\sharp(A_1 \times \cdots \times A_n)) &= q(x_1^\sharp(A_1)) \cdot q(x_2^\sharp(A_2)) \cdot \cdots \cdot q(x_n^\sharp(A_n)).
\end{aligned}$$

□

5 Central limit theorem

A sequence $(x_n)_{n=1}^\infty$ of IFS-observables is called independent, if $(x_1, x_2, \ldots, x_n)$ are independent for any n. They are equally distributed if

$$x_n((-\infty, t)) = x_1((-\infty, t))$$

for any $n \in N$ and $t \in R$.

If $k : R^n \to R$ is any Borel function and $x_1^\flat, \ldots, x_n^\flat : \mathcal{B}(R) \to \mathcal{T}$ are observable, and $h^\flat$ their joint observable we define $k(x_1^\flat, \ldots, x_n^\flat)$ by the formula

$$k(x_1^\flat, \ldots, x_n^\flat)(A) = h^\flat(k^{-1}(A))$$

for any $A \in \mathcal{B}(R)$. E.g.

$$\frac{\sqrt{n}}{\sigma_1} \left(\frac{1}{n} \sum_{i=1}^n x_i^\flat - a_1 \right) : \mathcal{B}(R) \to \mathcal{T}$$

is defined by the formula

$$\frac{\sqrt{n}}{\sigma_1} \left(\frac{1}{n} \sum_{i=1}^n x_i^\flat - a_1 \right) (A) = h^\flat(k^{-1}(A)),$$

where

$$k(u_1, u_2, \ldots, u_n) = \frac{\sqrt{n}}{\sigma_1} \left(\frac{1}{n} \sum_{i=1}^n u_i - a_1 \right).$$

Theorem 5. *Let $\mathcal{P} : \mathcal{F} \to \mathcal{J}$ be a separating probability. Let $(x_n)_1^\infty$ be a sequence of independent equally distributed IFS-observables from L^2, $E(x_1^\flat) = a_1$, $E(x_1^\sharp) = a_2$, $\sigma^2(x_1^\flat) = {\sigma_1}^2$, $\sigma^2(x_1^\sharp) = {\sigma_2}^2$, $y_n^\flat = \frac{\sqrt{n}}{\sigma_1}(\frac{1}{n}\sum_{i=1}^n x_i^\flat - a_1)$, $y_n^\sharp = \frac{\sqrt{n}}{\sigma_2}(\frac{1}{n}\sum_{i=1}^n x_i^\sharp - a_2)$, $y_n = (y_n^\flat, 1 - y_n^\sharp)$. Then*

$$\lim_{n\to\infty} \mathcal{P}(y_n((-\infty, t))) = \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp(-\frac{u^2}{2})\, du \right\}$$

for every $t \in R$.

Proof. By Theorem 4 $(x_n^\flat)_1^\infty$, $(x_n^\sharp)_1^\infty$ are independent and evidently equally distributed. Let $h_n^\flat$, be the joint observable of $x_1^\flat, \ldots, x_n^\flat$,

$$k_n(h_1, \ldots, h_n) = \frac{\sqrt{n}}{\sigma_1}\left(\frac{1}{n}\sum_{i=1}^{n} u_i - a_1\right), \quad y_n^\flat = h_n^\flat \circ k_n^{-1}.$$

Then by [5], Theorem 3.12

$$\lim_{n\to\infty} p(y_n^\flat((-\infty, t))) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(\frac{-u^2}{2}\right) du.$$

Similarly

$$\lim_{n\to\infty} q(y_n^\sharp((-\infty, t))) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{t} \exp\left(\frac{-u^2}{2}\right) du.$$ □

6 Conclusions

The paper is concerned in the probability theory on IFS-events. The main result of the paper is an original method of achieving new results of the probability theory on IFS-events by the corresponding results holding for fuzzy events. The method can be developed in two directions. First instead of a tribe of fuzzy sets one could try to consider any MV-algebra. Secondly, instead of independency of observables a kind of compatibility could be introduced and then conditional probabilities could be considered.

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