for distance-regular graphs

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Abstract

We analyze when the Moore–Penrose inverse of the combinatorial Laplacian of a distance–regular graph is a M–matrix; that is, it has non–positive off–diagonal elements or, equivalently when the Moore-Penrose inverse of the combinatorial Laplacian of a distance–regular graph is also the combinatorial Laplacian of another network. When this occurs we say that the distance–regular graph has the M–property. We prove that only distance–regular graphs with diameter up to three can have the M–property and we give a characterization of the graphs that satisfy the M-property in terms of their intersection array. Moreover we exhaustively analyze the strongly regular graphs having the M-property and we give some families of distance regular graphs with diameter three that satisfy the M-property.

1 Introduction

Very often problems in biological, physical and social sciences can be reduced to problems involving matrices which have some special structure. One of the most common situation is when the matrix has non–positive off–diagonal and non–negative diagonal entries; that is $L = kI - A$, $k > 0$ and $A \geq 0$, where the diagonal entries of A are less or equal than k. These matrices appear in relation to systems of equations or eigenvalue problems in a broad variety of areas including finite difference methods for solving partial differential equations, input–output production and growth models in economics or Markov processes in probability and statistics. Of course, the combinatorial community can recognize within this type of matrices, the combinatorial Laplacian of a k–regular graph where A is its adjacency matrix.

If k is at least the spectral radius of A, then L is called a *M–matrix*. We remark that M–matrices arise naturally in some discretizations of differential operators, particularly those with a minimum/maximum principle, such as the Laplacian, and as such are well–studied in scientific computing. In fact M–matrices satisfy monotonicity properties that are the discrete counterpart of the minimum principle, and it makes them suitable for the resolution of large sparse systems of linear equations by iterative methods.

As well as a symmetric, irreducible and non–singular M–matrix appears as the discrete counterpart of a Dirichlet problem for a self–adjoint elliptic operator, its inverse corresponds with the Green operator associated with the boundary value problem. On the other hand, when the M–matrix is singular, it can be seen as a discrete analogue of the Poisson equation for a self–adjoint elliptic operator on a manifold without boundary and then, its Moore–Penrose inverse corresponds with the Green operator too. A well–known property of an irreducible non–singular M–matrix is that its inverse is non–negative, [3]. However, the scenario changes dramatically when the matrix is an irreducible and singular M–matrix. In this case, it is known that the matrix has a generalized inverse which is non–negative, but this is not always true for any generalized inverse. For instance, it may happen that the Moore–Penrose inverse has some negative entries. We focus here in studying when the Moore–Penrose inverse of a symmetric, singular and irreducible M–matrix is itself an M–matrix. In particular, we study the case of distance–regular graphs and more specifically strongly regular graphs.

2 Preliminaries

The triple $\Gamma = (V, E, c)$ denotes a finite network; that is, a finite connected graph without loops nor multiple edges, with vertex set V, whose cardinality equals n, and edge set E, in which each edge $\{x, y\}$ has been assigned a *conductance* $c(x, y) > 0$. So, the conductance can be considered as a symmetric function $c\colon V \times V \longrightarrow [0, +\infty)$ such that $c(x, x) = 0$ for any $x \in V$ and moreover, $x \sim y$, that is vertex x is adjacent to vertex y, iff $c(x,y) > 0$. We define the *degree function* k as $k(x) = \sum_{y \in V} c(x,y)$, for each $x \in V$. The usual distance from vertex x to vertex y is denoted by $d(x,y)$ and $D = \max\{d(x,y) : x,y \in V\}$ stands for the *diameter* of Γ. We denote as $\Gamma_i(x)$ the set of vertices at distance i from vertex x, $\Gamma_i(x) = \{y : d(x,y) = i\}$ $0 \le i \le D$. The *complement* of Γ is defined as the graph $\overline{\Gamma}$ on the same vertices such that two vertices are adjacent iff they are not adjacent in Γ; that is $x \sim y$ in $\overline{\Gamma}$ iff $c(x,y) = 0$.

The set of real–valued functions on V is denoted by $\mathcal{C}(V)$. When necessary, we identify the functions in $\mathcal{C}(V)$ with vectors in $\mathbb{R}^{|V|}$ and the endomorphisms of $\mathcal{C}(V)$ with $|V|$–order square matrices.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the endomorphism of $\mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(u)(x) = \sum_{y \in V} c(x,y) \Big(u(x) - u(y) \Big) = k(x)u(x) - \sum_{y \in V} c(x,y)\, u(y), \quad x \in V.$$

It is well–known that $\mathcal{L}$ is a positive semi–definite self–adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. So, $\mathcal{L}$ can be interpreted as an irreducible, symmetric, diagonally dominant and singular M–matrix, L. Therefore, the Poisson equation $\mathcal{L}(u) = f$ on V has solution iff $\sum_{x \in V} f(x) = 0$ and, when this happens, there exists a unique solution $u \in \mathcal{C}(V)$ such that $\sum_{x \in V} u(x) = 0$, see [1].

The *Green operator* is the linear operator $\mathcal{G} : \mathcal{C}(V) \longrightarrow \mathcal{C}(V)$ that assigns to any $f \in \mathcal{C}(V)$ the unique solution of the Poisson equation with data $f - \frac{1}{n} \sum_{x \in V} f(x)$ such that $\sum_{x \in V} u(x) = 0$. It is easy to prove that $\mathcal{G}$ is a positive semi–definite self–adjoint operator and has 0 as its lowest eigenvalue whose associated eigenfunctions are constant. Moreover, if $\mathcal{P}$ denotes the projection on the subspace of constant functions then,

$$\mathcal{L} \circ \mathcal{G} = \mathcal{G} \circ \mathcal{L} = \mathcal{I} - \mathcal{P}.$$

In addition, we define the *Green function* as $G : V \times V \longrightarrow \mathbb{R}$ given by $G(x,y) = \mathcal{G}(\varepsilon_y)(x)$, where ε_y stands for the Dirac function at y. Therefore, interpreting $\mathcal{G}$, or G, as a matrix it is nothing else but $\mathsf{L}^\dagger$ the Moore–Penrose inverse of L, the matrix associated with $\mathcal{L}$. In consequence, $\mathsf{L}^\dagger$ is a M–matrix iff $G(x,y) \le 0$ for any $x,y \in V$ with $x \ne y$ and then $\mathcal{G}$ can be identified with the combinatorial Laplacian of a new connected network with the same vertex set, that we denote by $\Gamma^\dagger$.

From now on we will say that a network Γ has the *M–property* iff $\mathsf{L}^\dagger$ is a M–matrix.

Next we obtain a necessary and sufficient condition for a network to have the M–property. In [1] it was proved that for any $x \in V$, there exists a unique $\nu^x \in \mathcal{C}(V)$ such that $\nu^x(x) = 0$, $\nu^x(y) > 0$ for any $y \neq x$ and verifying

$$\mathcal{L}(\nu^x) = 1 - n\varepsilon_x \quad \text{on } V. \tag{1}$$

We call ν^x the *equilibrium measure of* $V \setminus \{x\}$ and then we define *capacity* as the function $\mathsf{cap} \in \mathcal{C}(V)$ given by $\mathsf{cap}(x) = \sum\limits_{y \in V} \nu^x(y)$.

Theorem 1 *The network Γ has the M–property iff for any $y \in V$*

$$\mathsf{cap}(y) \leq n\nu^y(x) \qquad \textit{for any } x \sim y.$$

In this case, $\overline{\Gamma}$ is a subgraph of the subjacent graph of $\Gamma^\dagger$.

Proof: The Green function is given by

$$G(x,y) = \frac{1}{n^2}\big(\mathsf{cap}(y) - n\,\nu^y(x)\big),$$

see [1]. Therefore, $\mathsf{L}^\dagger$ is a M–matrix iff

$$\mathsf{cap}(y) \leq n \min_{x \in V \setminus \{y\}} \big\{\nu^y(x)\big\}.$$

The results follow by keeping in mind that $\min\limits_{x \in V \setminus \{y\}} \big\{\nu^y(x)\big\} = \min\limits_{x \sim y} \big\{\nu^y(x)\big\}$, since if the minimum is attained at $z \not\sim y$, then

$$1 = \mathcal{L}(\nu^y)(z) = \sum_{x \in V} c(z,x)\big(\nu^y(z) - \nu^y(x)\big) \leq 0,$$

which is a contradiction. $\square$

3 Distance–regular graphs with the M–property

We aim here at characterizing when the Moore–Penrose inverse of the combinatorial Laplacian matrix of a distance–regular graph is a M–matrix.

A connected graph Γ is called *distance–regular* if there are integers b_i, c_i, $i = 0, \dots, D$ such that for any two vertices $x, y \in \Gamma$ at distance $i = d(x, y)$, there are exactly c_i neighbours of y in $\Gamma_{i-1}(x)$ and b_i neighbours of y in $\Gamma_{i+1}(x)$, where for any vertex $x \in \Gamma$ the set of vertices at distance i from it is denoted by $\Gamma_i(x)$. Moreover, $|\Gamma_i(x)|$ will be denoted by k_i. In particular, Γ is regular of degree $k = b_0$. The sequence

$$\iota(\Gamma) = \{b_0, b_1, \dots, b_{D-1}; c_1, \dots, c_D\},$$

is called the *intersection array* of Γ. In addition, $a_i = k - c_i - b_i$ is the number of neighbours of y in $\Gamma_i(x)$, for $d(x, y) = i$. Clearly, $b_D = c_0 = 0$, $c_1 = 1$ and the diameter of Γ is D. Usually, the parameters a_1 and c_2 are denoted by λ and μ, respectively. For all the properties related with distance–regular graphs we refer the reader to [4, 7].

The parameters of a distance–regular graph satisfy many relations, among them we will make an extensive use of the following:

(i) $k_0 = 1$ and $k_i = \dfrac{b_0 \cdots b_{i-1}}{c_1 \cdots c_i}, \quad i = 1, \dots, D.$

(ii) $n = 1 + k + k_2 + \cdots + k_D$.

(iii) $k > b_1 \geq \cdots \geq b_{D-1} \geq 1$.

(iv) $1 \leq c_2 \leq \cdots \leq c_D \leq k$.

(v) If $i + j \leq D$, then $c_i \leq b_j$ and $k_i \leq k_j$ when, in addition, $i \leq j$.

Additional relations between the parameters give more information about the structure of distance–regular graphs. For instance, Γ is *bipartite* iff $a_i = 0$, $i = 1, \dots, D$, whereas Γ is *antipodal* iff $b_i = c_{D-i}$, $i = 0, \dots, D$, $i \neq \lfloor \frac{D}{2} \rfloor$ and then $b_{\lfloor \frac{D}{2} \rfloor} = t c_{\lceil \frac{D}{2} \rceil}$, $t \geq 1$ and Γ is an antipodal $(t+1)$–cover of its folded graph, see [7, Prop. 4.2.2].

The following lemma shows that the equilibrium measures for a distance–regular graph, and hence the capacity function, can be expressed in terms of the parameters of its intersection array, see [1, Prop. 4.1] for the details.

Lemma 2 *Let Γ be a distance–regular graph. Then, for all $x, y \in V$*

$$\nu^x(y) = \sum_{j=0}^{d(x,y)-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big) \quad \textit{and} \quad \mathsf{cap}(x) = \sum_{j=0}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2.$$

Proposition 3 *A distance–regular graph Γ has the M–property iff*

$$\sum_{j=1}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2 \leq \frac{n-1}{k}.$$

Moreover, the subjacent graph of $\Gamma^\dagger$ is K_n when the above inequality is strict and $\overline{\Gamma}$ otherwise.

Proof: From Theorem 1 and Lemma 2, the Moore–Penrose inverse of L is a M–matrix iff

$$\sum_{j=0}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2 \leq \frac{n(n-1)}{k}$$

that is, iff

$$\frac{(n-1)^2}{k} + \sum_{j=1}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2 \leq \frac{n(n-1)}{k}.$$

Finally, the above inequality is an equality iff $G(x,y) = 0$ when $d(x,y) = 1$, since for any $y \in V$, $\nu^y(x)$ is constant on $\Gamma_1(y)$. Therefore, the subjacent graph of $\Gamma^\dagger$ is $\overline{\Gamma}$. $\square$

A distance–regular graph of order n has diameter $D = 1$ iff it is the complete graph K_n. In this case, the above inequality holds since the left side term vanishes. Therefore, any complete graph has the M–property. In fact, $\mathsf{L}^\dagger = \frac{1}{n^2}\,\mathsf{L}$, see [2], and hence, $\Gamma^\dagger$ is also a complete network.

Corollary 4 *If Γ has the M–property and $D \geq 2$, then*

$$\lambda \leq 3k - \frac{k^2}{n-1} - n.$$

and hence $n < 3k$.

Proof: When $D \geq 2$, from the inequality in Proposition 3 we get that

$$\frac{(n-k-1)^2}{kb_1} \leq \sum_{j=1}^{D-1} \frac{1}{k_j b_j}\Big(\sum_{i=j+1}^{D} k_i\Big)^2 \leq \frac{n-1}{k}.$$

Therefore, $(n-1-k)^2 \leq (n-1)b_1 = (n-1)(k-1-\lambda)$ and the upper bound for λ follows. In addition, this inequality implies that $0 \leq \lambda < 3k-n$ and then $3k > n$. □

The inequality $3k > n$ turns out to be a strong restriction for a distance–regular graph to have the M–property. For instance, if $n \geq 3$, the n–cycle, C_n, is a distance–regular graph with diameter $D = \left\lfloor \frac{n}{2} \right\rfloor$ whose intersection array is

$$\iota(C_n) = \{2, 1, \dots, 1; 1, \dots, 1, c_D\},$$

where $c_D = 1$ when n is odd and $c_D = 2$ when D is even, see [7]. So, if C_n has the M–property, necessarily $6 > n$ and this occurs iff either $D = 1$; that is $n = 3$, or $D = 2$; that is $n = 4, 5$. Moreover for $n = 4, 5$, C_n has the M–property since

$$(\mathsf{L}^\dagger)_{ij} = \frac{1}{12n}\Big(n^2 - 1 - 6|i-j|(n - |i-j|)\Big), \qquad i, j = 1, \dots, n,$$

see for instance, [2, 10].

In the following result we generalize the above observation, by showing that only distance–regular graph with small diameter can satisfy the M–property.

Proposition 5 *If Γ is a distance–regular graph with the M–property, then $D \leq 3$.*

Proof: If $D \geq 4$, then from property (v) of the parameters, $k = k_1 \leq k_i$, $i = 2, 3$ and hence,

$$3k < 1 + 3k \leq 1 + k + k_2 + k_3 \leq n,$$

and hence Γ has not the M–property. □

3.1 Strongly regular graphs

A distance–regular graph whose diameter equals 2 is called *strongly regular graph*. This kind of distance–regular graph is usually represented throughout the four parameters (n, k, λ, μ) instead its intersection array, see [7, 9]. Clearly the four parameters of a strongly regular graph are not independent, since

$$(n-1-k)\mu = k(k-1-\lambda). \tag{2}$$

For this reason some authors drop the parameter n in the above array, see for instance [4]. Moreover, Equality (3) implies that $2k - n \le \lambda < k - 1$, since $1 \le \mu \le k$ and $D = 2$.

Observe that the only n–cycles satisfying the M–property are precisely C_3, that is the complete graph with 3 vertices, and C_4 and C_5 that are strongly regular graphs.

In the following result we characterize those strongly regular graphs that have the M–property, in terms of their parameters.

Proposition 6 *A strongly regular graph with parameters* (n, k, λ, μ) *has the M–property iff*

$$\mu \ge k - \frac{k^2}{n-1}.$$

Proof: Clearly for $D = 2$ the inequality in Corollary 4 characterizes the strongly regular graphs satisfying the M–property. The result follows taking into account that from Equality (3)

$$\lambda \le 3k - \frac{k^2}{n-1} - n \iff k(n-1-k) \le \mu(n-1). \qquad \square$$

Kirkland et al. in [11, Theorem 2.4] gave another characterization of strongly regular graphs with the M–property in terms of the combinatorial Laplacian eigenvalues.

It is straightforward to verify that Petersen graph does not have the M–property. So, it is natural to ask if there exist many strongly regular graphs satisfying the above inequality. Prior to answer this question, we recall that if Γ is a strongly regular graph with parameters (n, k, λ, μ), then its complement graph is also a strongly regular graph with parameters $(n, n-k-1, n-2-2k+\mu, n-2k+\lambda)$, see for instance [5], which in particular implies that $\mu \geq 2(k+1)-n$. Strongly regular graphs with the same parameters as their complement are called *conference graphs* and then their parameters are $(4m+1, 2m, m-1, m)$ where $m \geq 2$. Moreover it is known that such a graph exits iff $m = p^2 + q(q+1)$, where $p, q \geq 1$, see [9].

Now we are ready to answer the raised question.

Corollary 7 *If Γ is strongly regular graph, then either Γ or $\overline{\Gamma}$ has the M–property. Moreover, both of them have the M–property iff Γ is a conference graph.*

Proof: If we define $\bar{k} = n-k-1$, $\bar{\lambda} = n-2-2k+\mu$ and $\bar{\mu} = n-2k+\lambda$, then

$$\bar{k} - \frac{\bar{k}^2}{n-1} = k - \frac{k^2}{n-1}$$

and hence

$$\bar{\mu} \geq \bar{k} - \frac{\bar{k}^2}{n-1} \iff \lambda \geq 3k - \frac{k^2}{n-1} - n \iff \mu \leq k - \frac{k^2}{n-1},$$

where the equality in the left side holds iff the equality in the right side holds. Moreover, any of the above inequalities is an equality iff $\bar{\mu} = \mu$ and $\bar{\lambda} = \lambda$; that is iff Γ is a conference graph. The remaining claims follow from Proposition 6. $\square$

Many strongly regular graphs appear associated with the so–called partial geometries. A *Partial Geometry with parameters* $s, t, \alpha \geq 1$, $pg(s,t,\alpha)$, is an incident structure of points and lines such that every line has $s+1$ points, every point is on $t+1$ lines, two distinct lines meet in at most one point and given a line and a point not in it, there are exactly α lines through the point which meet the line. Therefore, the parameters of a partial geometry satisfy the inequalities $1 \leq \alpha \leq \min\{t+1, s+1\}$. We refer the reader to the surveys [6, 8, 9] for the main properties of partial geometries and their relation with strongly regular graphs.

The number of points and lines in $pg(s,t,\alpha)$ are $n = \dfrac{1}{\alpha}(s+1)(st+\alpha)$ and $\ell = \dfrac{1}{\alpha}(t+1)(st+\alpha)$, respectively. The *point graph of* $pg(s,t,\alpha)$ has the points as vertices and two vertices are adjacent iff they are collinear. Therefore, it is a regular graph with degree $k = s(t+1)$. Moreover, when

$\alpha = s+1$, the partial geometry is called *Linear space* and its point graph is the complete graph K_n. When $\alpha \le s$, the point graph is a strongly regular graph with parameters $\big(n, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\big)$. A strongly regular graph is called *pseudo geometric graph* if its associated parameters are of the former form.

Corollary 8 *A pseudo geometric graph with parameters* $\big(n, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\big)$ *has the* M*–property iff*

$$\alpha(2ts+t+\alpha) \ge st(s+1).$$

Next we study when the point graphs associated with some well–known families of partial geometries, o more generally when some families of pseudo geometric graphs, verify the M–property.

1. *Dual Linear Spaces:* In this case $\alpha = t+1$ and hence the point graph has the M–property iff $s \le 2(t+1)$. When, $t=1$ and $s=m-2$ the corresponding pseudo geometric graph are the so–called *triangular graph* T_m whose parameters are $\big(\binom{m}{2}, 2(m-2), m-2, 4\big)$. So, T_m has the M–property iff $m = 4,5,6$. Notice, that T_m is also the line graph of the complete graph K_m.

2. *Transversal Designs:* In this case $\alpha = s$ and hence the corresponding pseudo geometric graph is the *complete multipartite graph* $K_{(s+1)\times(t+1)}$ whose parameters are $\big((s+1)(t+1), s(t+1), (s-1)(t+1), s(t+1)\big)$ and it has the M–property. Observe that these graphs are the complement of $s+1$ disjoint copies of K_{t+1} which are characterized as the unique graphs such that $\mu = 0$, see [9, Theorem 1.2]. Therefore, $K_{(s+1)\times(t+1)}$ are the unique strongly regular graphs such that $\mu = k$, that is the only antipodal strongly regular graphs. Finally, note that the graph $K_{(s+1)\times 2}$ is also know as *Cocktail party graph.*

3. *Dual Transversal Designs:* In this case $\alpha = t$, $t > 1$ and hence the corresponding pseudo geometric graph is the *Pseudo–Latin square graph* $PL_r(m)$ whose parameters are $(m^2, r(m-1), r^2-3r+m, r(r-1))$, where $r = t+1$ and $m = s+1$. It has the M–property iff $s \le 2t$. For $t=2$ it is the line graph of the complete bipartite graph $K_{m,m}$, also called *squared lattice graph.*

4. *Generalized quadrangles:* In this case $\alpha = 1$, $s > 1$ and hence the parameters of the corresponding pseudo geometric graph are $\big((s+$

$1)(st+1), s(t+1), s-1, t+1\big)$. Therefore, it has the M–property iff $ts+t+1 \geq s^2 t$ and hence iff $t=1$ and $s=2$. Note that when $t=1$ these graphs are the so–called *Hamming graph* $H(2, s+1)$ *or Lattice.* Observe that the complement of $H(2, s+1)$ is the pseudo–latin square graph $PL_s(s+1)$ that satisfies the M–property.

When $1 < \alpha < \min\{t, s\}$, the point graph of $pg(s,t,\alpha)$ is called *Proper pseudo–geometric.* An example of this structure are the so–called *Kneser graphs* $K(m,2)$, *where* $m \geq 6$ *is even*, in which case $s = \dfrac{m}{2} - 1$, $t = m-4$ and $\alpha = \dfrac{m}{2} - 2$. For arbitrary $m \geq 5$, the *Kneser graph* $K(m,2)$ is the graph whose vertices represent the 2–subsets of $\{1, \dots, m\}$, and where two vertices are connected if and only if they correspond to disjoint subsets. The parameters of the Kneser graph $K(m,2)$ are $\big(\binom{m}{2}, \binom{m-2}{2}, \binom{m-4}{2}, \binom{m-3}{2}\big)$, that coincide with the parameters of the complement of T_m. Therefore, it has the M–property iff $m \geq 7$ as expected. In addition, for m odd $K(m,2)$ is an example of strongly regular graph that is not a pseudo geometric graph, which also implies that the complement of a pseudo geometric graph is not necessarily a pseudo geometric graph.

3.2 Distance–regular graphs with diameter $D=3$

In this section we characterize those distance–regular graphs with diameter 3 that have the M–property. In this case, the intersection array is

$$\iota(\Gamma) = (k, b_1, b_2; 1, c_2, c_3).$$

Again the parameters are not independent, since

$$(n-1-k)c_2c_3 = kb_1(b_2+c_3). \tag{3}$$

The next result follows straightforwardly from Proposition 3.

Proposition 9 *A distance–regular graph with* $D=3$ *has the* M*–property iff*

$$k^2 b_1\big(b_2c_2 + (b_2+c_3)^2\big) \leq c_2^2 c_3^2 (n-1).$$

A simple example verifying the above condition is the 3–*cube,* Q_3. Next we study when bipartite or antipodal distance–regular graphs have the M–property. Recall that the first ones are the incidence graph of a symmetric 2–design, whereas the second ones are covers of a complete graph.

The intersection array of a bipartite distance–regular graph with diameter $D = 3$ is $\iota(\Gamma) = \big(k, k-1, k-\mu; 1, \mu, k\big)$, where $1 \le \mu \le k-1$. Then, $n - 1 = \dfrac{1}{\mu}(2k(k-1)+\mu)$ and hence Γ has the M–property iff

$$k(k-1)(4k-5\mu) \le \mu^2.$$

Notice that the inequality holds when $\mu \ge \dfrac{4k}{5}$. For instance, it is true for $\mu = k-1$ when $k \ge 5$, and it is true for $\mu = k-2$ when $k \ge 10$.

The intersection array of an antipodal distance–regular graph with diameter $D = 3$ is $\iota(\Gamma) = \big(k, t\mu, 1; 1, \mu, k\big)$, where $t \ge 1$ and $1 \le m < k$. These graphs are the $(t+1)$–cover of the complete graph K_{k+1}. Then, $n = (t+1)(k+1)$ and hence Γ has the M–property iff

$$t(k+1)^2 \le \mu k(t+1).$$

When $t = 1$, the antipodal distance–regular graphs are known as *Taylor graphs*, $T(k,\mu)$. Then $T(k,\mu)$ has the M–property iff $(k+1)^2 \le 2k\mu$. Moreover, if Γ is a Taylor graph with $1 \le m < k-1$, it is well-known that the graph $\Gamma_{(2)}$ on the same vertices and such that two vertices of $\Gamma_{(2)}$ are adjacent if and only if their distance in Γ is 2 is also a Taylor graph whose intersection array is

$$\iota(\Gamma_{(2)}) = \big\{k, k-1-m, 1; 1, k-1-m, k\big\}.$$

Then, $\Gamma_{(2)}$ has the M–property iff $2km \le (k-2)^2 - 5$.

Corollary 10 *If Γ is the Taylor graph $T(k,m)$, then either Γ or $\Gamma_{(2)}$ has the M–property, except when $m = \dfrac{k}{2}-2, \dfrac{k}{2}-1, \dfrac{k}{2}, \dfrac{k}{2}+1$ when k is even and $m = \left\lceil\dfrac{k}{2}\right\rceil - 2, \left\lceil\dfrac{k}{2}\right\rceil - 1, \left\lceil\dfrac{k}{2}\right\rceil$ when k is odd, in which case none of them has the M–property.*

Finally, the only bipartite and antipodal distance–regular graphs with $D = 3$ have intersection array

$$\iota(\Gamma) = \big\{k, k-1, 1; 1, k-1, k\big\}$$

and they are called *k–crown graphs.* Therefore, they are Taylor graphs with $\mu = k-1$ and hence they have the M–property for any $k \ge 5$.

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