and restricted edge-connectivity of 3-arc graphs

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Abstract

A $3-arc$ of a graph G is a 4-tuple (y,a,b,x) of vertices such that both (y,a,b) and (a,b,x) are paths of length two in G. Let $\overleftrightarrow{G}$ denote the symmetric digraph of a graph G. The *3-arc graph* $X(G)$ of a given graph G is defined to have vertices the arcs of $\overleftrightarrow{G}$. Two vertices (ay), (bx) are adjacent in $X(G)$ if and only if (y,a,b,x) is a 3-arc of G. The purpose of this work is to study the edge-connectivity and restricted edge-connectivity of 3-arc graphs. We prove that the 3-arc graph $X(G)$ of every connected graph G of minimum degree $\delta(G) \geq 3$ has edge-connectivity $\lambda(X(G)) \geq (\delta(G)-1)^2$; and restricted edge- connectivity $\lambda_{(2)}(X(G)) \geq 2(\delta(G)-1)^2 - 2$ if $\kappa(G) \geq 2$. We also provide examples showing that all these bounds are sharp.

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [10] for terminology and definitions.

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For every $v \in V(G)$, $N_G(v)$ denotes the neighborhood of v, that is, the set of all vertices adjacent to v. The *degree* of a vertex v is $d(v) = |N_G(v)|$ and the minimum degree $\delta = \delta(G)$ of the graph G is the minimum degree over all vertices of G.

A graph G is called *connected* if every pair of vertices is joined by a path. If $S \subset V(G)$ and $G - S$ is not connected, then S is said to be a *cutset*. A *component* of a graph G is a maximal connected subgraph of G. A (noncomplete) connected graph is called *k-connected* if every cutset has cardinality at least k. The *connectivity* $\kappa(G)$ of a (noncomplete) connected graph G is defined as the maximum integer k such that G is k-connected. The *minimum* cutsets are those having cardinality $\kappa(G)$. The *connectivity* of a complete graph $K_{\delta+1}$ on $\delta + 1$ vertices is defined as $\kappa(K_{\delta+1}) = \delta$. Analogously, for edge connectivity an *edge-cut* in a graph G is a set W of edges of G such that $G - W$ is nonconnected. If W is a minimum edge-cut of a connected graph G, then $G - W$ contains exactly two components. Every connected graph on at least two vertices has an edge-cut. The *edge-connectivity* $\lambda(G)$ of a graph G is the minimum cardinality of an edge-cut of G. A classic result due to Whitney is that for every graph G, $\kappa(G) \leq \lambda(G) \leq \delta(G)$. A graph is *maximally connected* if $\kappa(G) = \delta(G)$, and *maximally edge-connected* if $\lambda(G) = \delta(G)$.

Though the parameters κ, λ of connectivities give the minimum cost to disrupt the network, they do not take into account what remains after deletion. Even two graphs with the same connectivity κ, λ may be considered to have different reliabilities, since the number of minimum cutsets or edge-cuts is different. Superconnectivity is a stronger measure of connectivity, introduced by Boesch and Tindell in [8], whose study has deserved some attention in the last years, see for instance, [1, 6, 7, 19, 20]. A maximally connected [edge-connected] graph is called *super-*κ [*super-*λ] if for every cutset [edge-cut] W of cardinality $\delta(G)$ there exists a component C of $G - W$ of cardinality $|V(C)| = 1$. The study of super-κ [super-λ] graphs has a particular significance in the design of reliable networks, mainly due to the fact that attaining superconnectivity implies minimizing the number of minimum cutsets [edge-cuts] (see [7, 20]).

In order to measure the super edge-connectivity we use the following parameter introduced by Esfahanian and Hakimi [11]. The *restricted edge-connectivity* $\lambda_{(2)} = \lambda_{(2)}(G)$ is the minimum cardinality over all *restricted edge-cuts* W, i.e., those such that there are no isolated vertices in $G - W$. A restricted edge-cut W is called a $\lambda_{(2)}$*-cut* if $|W| = \lambda_{(2)}$. Obviously for any $\lambda_{(2)}$-cut W, the graph $G - W$ consists of exactly two components

$C, \overline{C}$ and clearly $|V(C)| \geq 2, |V(\overline{C})| \geq 2$. A connected graph G is called $\lambda_{(2)}$-*connected* if $\lambda_{(2)}$ exists. Esfahanian and Hakimi [11] showed that each connected graph G of order $n(G) \geq 4$ except a star, is $\lambda_{(2)}$-connected and satisfies $\lambda_{(2)} \leq \xi$, where $\xi = \xi(G)$ denotes the *minimum edge-degree* of G defined as $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$. Furthermore, a $\lambda_{(2)}$-connected graph is said to be $\lambda_{(2)}$-*optimal* if $\lambda_{(2)} = \xi$. Recent results on this property are obtained in [2, 5, 12, 13, 18, 21, 23]. Notice that if $\lambda_{(2)} \leq \delta$, then $\lambda_{(2)} = \lambda$. When $\lambda_{(2)} > \delta$ (that is to say, when every edge cut of order δ isolates a vertex) the graph must be super-λ. Therefore, by means of this parameter we can say that a graph G is super-λ if and only if $\lambda_{(2)} > \delta$. Thus, we can measure the super edge-connectivity of the graph as the value of the restricted edge-connectivity $\lambda_{(2)}$.

Let $\overleftrightarrow{G}$ denote the symmetric digraph of a graph G. For adjacent vertices u, v of $V(G)$ we use (u, v) to denote the arc from u to v, and $(v, u)(\neq (u, v))$ to denote the arc from v to u. A *3-arc* is a 4-tuple (y, a, b, x) of vertices such that both (y, a, b) and (a, b, x) are paths of length two in G. The *3-arc graph* $X(G)$ of a given graph G is defined to have vertices the arcs of $\overleftrightarrow{G}$ and they are denoted as (uv). Two vertices $(ay), (bx)$ are adjacent in $X(G)$ if and only if (y, a, b, x) is a 3-arc of G, see [17, 22]. Equivalently, two vertices $(ax), (by)$ are adjacent in $X(G)$ if and only if $d_G(a, b) = 1$; that is, the tails a, b of the arcs $(a, x), (b, y) \in A(\overleftrightarrow{G})$ are at distance one in G. Thus the number of edges of $X(G)$ is $\sum_{uv \in E(G)}(d(u) - 1)(d(v) - 1)$ so that the minimum degree of $X(G)$ is $(\delta(G) - 1)^2$. There is a bijection between the edges of $X(G)$ and those of the 2-path graph $P_2(G)$, which is defined to have vertices the paths of length two in G such that two vertices are adjacent if and only if the union of the corresponding paths is a path or a cycle of length three, see [9]. Since $P_2(G)$ is a spanning subgraph of the second iterated line graph $L_2(G) = L(L(G))$ (see e.g. [14]), we have a relation between 3-arc graphs and line graphs. Some results on the connectivity of P_2-path graphs are studied e.g. in [3, 4, 15].

The purpose of this paper is to study the edge-connectivity, the restricted edge-connectivity and vertex-connectivity of the 3-arc graph $X(G)$ of a given graph G. The following theorem gather together the results on connectivity of 3-arc-graph $X(G)$ obtained by Knor and Zhou [16].

Theorem 1 [16] *Let G be a graph with minimum degree $\delta(G)$.*

(i) *$X(G)$ is connected if G is connected and $\delta(G) \geq 3$.*

(ii) $\kappa(X(G)) \geq (\kappa(G)-1)^2$ *if* $\kappa(G) \geq 3$.

The main results contained in this paper are the following:
Let G be a connected graph with minimum degree $\delta(G) \geq 3$.

(i) $\lambda(X(G)) \geq (\delta(G)-1)^2$.

(ii) $\lambda_{(2)}(X(G)) \geq 2(\delta(G)-1)^2 - 2$ *if* $\kappa(G) \geq 2$.

(iii) $\kappa(X(G)) \geq \min\{\kappa(G)(\delta(G)-1), (\delta(G)-1)^2\}$.

(iv) $X(G)$ *is super-*κ *if* $\kappa(G) = \delta(G)$ *and* $\delta(X(G)) = (\delta(G)-1)^2$.

2 Results on the edge-connectivity and restricted edge-connectivity of 3-arc graphs

Let $X(G)$ be the 3-arc graph of a graph G. If (ay) and (bx) are adjacent in $X(G)$ then the edge $(ay)(bx)$ will be called an *ab-edge* (or *ba*-edge). Observe that $(ay)(bx) = (bx)(ay)$ but $(ay) \neq (ya)$ and $(bx) \neq (xb)$. For any edge $ab \in E(G)$ let $\mathcal{V}^a_{ab} = \{(ay) \in V(X(G)) : y \in N_G(a) - b\}$. Observe that the induced subgraph of $X(G)$ by the set $\mathcal{V}^a_{ab} \cup \mathcal{V}^b_{ba}$ is the complete bipartite graph $K_{|\mathcal{V}^a_{ab}|,|\mathcal{V}^b_{ba}|} = K_{d(a)-1,d(b)-1}$.

If W is a minimal edge cut of a connected graph G, then, $G - W$ necessarily contains exactly two components C and $\overline{C}$, so it is usual to denote an edge cut W as $[C, \overline{C}]$ where $[C, \overline{C}]$ denotes the set of edges between C and its complement $\overline{C}$.

Lemma 2 *Let G be a graph and* $[C, \overline{C}]$ *an edge-cut of* $X(G)$*. Let* $ab \in E(G)$*, if* $[C, \overline{C}]$ *contains ab-edges, then it contains at least* $\min\{d(a)-1, d(b)-1\}$ *ab-edges.*

Proof: Suppose that $(ay)(bx)$ is an edge of $[C, \overline{C}]$ such that $(ay) \in V(C)$ and $(bx) \in V(\overline{C})$. Then $\mathcal{V}^a_{ab} \cap V(C) \neq \emptyset$ and $\mathcal{V}^b_{ba} \cap V(\overline{C}) \neq \emptyset$. Let denote by $|\mathcal{V}^a_{ab} \cap V(C)| = r_a \geq 1$, $|\mathcal{V}^b_{ba} \cap V(C)| = r_b \geq 0$, $|\mathcal{V}^a_{ab} \cap V(\overline{C})| = \overline{r}_a \geq 0$ and $|\mathcal{V}^b_{ba} \cap V(\overline{C})| = \overline{r}_b \geq 1$. Moreover, these numbers must satisfy $r_a + \overline{r}_a = d(a) - 1$ and $r_b + \overline{r}_b = d(b) - 1$. Furthermore, the number of *ab*-edges contained in $[C, \overline{C}]$ is $r_a\overline{r}_b + r_b\overline{r}_a$, that is,

$$|[C, \overline{C}]| \geq r_a\overline{r}_b + r_b\overline{r}_a. \quad (1)$$

If $r_b = 0$, then $\overline{r}_b = d(b) - 1$. As $r_a \geq 1$, (1) implies $|[C, \overline{C}]| \geq d(b) - 1$ and the lemma follows. Similarly, if $\overline{r}_a = 0$, the result is also true. Therefore, we can assume that $r_a, r_b, \overline{r}_a, \overline{r}_b \geq 1$. In this case $r_a\overline{r}_b + r_b\overline{r}_a \geq r_a + \overline{r}_a = d(a)-1$, and $r_a\overline{r}_b + r_b\overline{r}_a \geq r_b + \overline{r}_b = d(b) - 1$, and the result holds. □

Suppose that $[C, \overline{C}]$ is an edge-cut of $X(G)$. Let denote by $\omega(\alpha) = \{e \in E(G) : e = \alpha\beta\}$ and define $\mathcal{A} = \{\alpha\beta \in E(G) : (\alpha y)(\beta x) \in [C, \overline{C}]\}$. Then, as a consequence of the above lemma, we have $|[C, \overline{C}]| \geq |\mathcal{A}|(\delta(G) - 1)$. Next we prove that $|[C, \overline{C}]| \geq (\delta(G) - 1)^2$.

Lemma 3 *Let G be a graph and $[C, \overline{C}]$ an edge-cut of $X(G)$. Let $ab \in E(G)$ and suppose that $ab \in \mathcal{A}$. Then $|(\omega(a) \cup \omega(b)) \cap \mathcal{A}| \geq (\delta - 1)^2$.*

Proof: Suppose that for all $y \in N(a) - b$, $ay \in \mathcal{A}$. Then there are at least δ different ay-edges in $[C, \overline{C}]$, and by Lemma 2 the number of ay-edges in $[C, \overline{C}]$ is at least $\delta(\delta - 1) > (\delta - 1)^2$. The same occurs if for every $x \in N(b) - a$, $bx \in \mathcal{A}$. Therefore we may assume that there exists $y_0 \in N_G(a) - b$ such that $ay_0 \notin \mathcal{A}$ and there exists $x_0 \in N_G(b) - a$ such that $bx_0 \notin \mathcal{A}$.

As $ab \in \mathcal{A}$, $(ay')(bx') \in [C, \overline{C}]$ for some $y' \in N(a) - b$ and $x' \in N(b) - a$, and without loss of generality we may suppose that $(ay') \in V(C)$, $(bx') \in V(\overline{C})$. Suppose that $(ay_0)(bx_0) \notin [C, \overline{C}]$. Without loss of generality we may assume that $(ay_0), (bx_0) \in V(\overline{C})$ in which case $(ay')(bx_0) \in [C, \overline{C}]$ because $(ay') \in V(C)$. Then we can continue the proof assuming that there is an edge $(ay)(bx) \in [C, \overline{C}]$ such that $bx \notin \mathcal{A}$, i.e., there are no bx-edges in $[C, \overline{C}]$.

First suppose that $\mathcal{V}_{xb}^x \cap V(C) \neq \emptyset$. Let $B = \{x' \in N_G(b) \setminus \{x, a\} : (x'z) \in V(C)\}$ and $\overline{B} = \{x' \in N_G(b) \setminus \{x, a\} : (x'z) \in V(\overline{C})\}$. Observe that for all $x' \in B \cup \overline{B}$, $(x'z)$ is adjacent to $(bx) \in V(\overline{C})$, and $(x'z)$ is adjacent to (ba). Hence the edge-cut $[C, \overline{C}]$ must contain $|B|$ different bx'-edges. Moreover, since (ba) is adjacent to every $(xb') \in \mathcal{V}_{xb}^x$ and $bx \notin \mathcal{A}$, then $(ba) \in V(C)$ because our assumption $\mathcal{V}_{xb}^x \cap V(C) \neq \emptyset$. Hence $[C, \overline{C}]$ also contains $|\overline{B}|$ different bx'-edges yielding that $[C, \overline{C}]$ contains at least $|B| + |\overline{B}| + |\{ab\}| = d(b) - 1 \geq \delta - 1$ different bv-edges with $v \in N(b)$ and by Lemma 2, the result holds.

Second suppose that $\mathcal{V}_{xb}^x \subset V(\overline{C})$. Hence $\mathcal{V}_{ba}^b \subset V(\overline{C})$ because every $(bx') \in \mathcal{V}_{ba}^b$ is adjacent to every $(xb') \in \mathcal{V}_{xb}^x$ and $[C, \overline{C}]$ does not contain bx-edges. If $ay \notin \mathcal{A}$, reasoning for ay in the same way as for bx we get that $\mathcal{V}_{ab}^a \subset V(C)$. Thus as $\mathcal{V}_{ba}^b \subset V(\overline{C})$ it follows that $[C, \overline{C}]$ contains at least

$(d(a)-1)(d(b)-1) \geq (\delta-1)^2$ ab-edges and the lemma holds. Therefore, suppose that $ay \in \mathcal{A}$.

We know that there exists $v \in N_G(a) - y$ such that $av \notin \mathcal{A}$. As (va') is adjacent to (ay) for all $(va') \in \mathcal{V}^v_{va}$ it follows that $\mathcal{V}^v_{va} \subset V(C)$ (because $(ay) \in V(C)$ and $av \notin \mathcal{A}$). Hence $\mathcal{V}^a_{av} \subset V(C)$ because every $(ay') \in \mathcal{V}^a_{av}$ is adjacent to $(va') \in \mathcal{V}^v_{va}$. As $\mathcal{V}^b_{ba} \subset V(\overline{C})$ it follows that $[C, \overline{C}]$ contains at least $(d(a)-2)(d(b)-1)$ ab-edges. Further, as $ay \in \mathcal{A}$, by Lemma 2, $[C, \overline{C}]$ also contains at least $\delta - 1$ ay-edges, yielding that the number of au-edges contained $|[C, \overline{C}]|$ is at least $(\delta-2)(\delta-1) + (\delta-1) = (\delta-1)^2$, and the lemma holds. □

Theorem 4 *Let G be a connected graph with minimum degree $\delta \geq 3$. Then*

$$\lambda(X(G)) \geq (\delta-1)^2.$$

Proof: Let $[C, \overline{C}]$ be a minimum edge-cut of $X(G)$ and $\mathcal{A} = \{ab \in E(G) : (ay)(bx) \in [C, \overline{C}]\}$. As G is connected and $\delta \geq 3$, then $X(G)$ is connected yielding that $|\mathcal{A}| \geq 1$. So considering $ab \in \mathcal{A}$, and using Lemma 3 we get $|[C, \overline{C}]| \geq (\delta-1)^2$, following the theorem. □

The following corollary is an immediate consequence from Theorem 4, and from the fact that if G is a graph of minimum degree δ having an edge xy such that $d(x) = \delta$ and $d(y') = \delta$ for all $y' \in N_G(x) - y$, then the minimum degree of $X(G)$ is $\delta(X(G)) = (\delta-1)^2$.

Corollary 5 *Let G be a connected graph of minimum degree $\delta \geq 3$ having an edge xy such that $d(x) = \delta$ and $d(y') = \delta$ for all $y' \in N_G(x) - y$. Then the 3-arc graph $X(G)$ of G is maximally edge-connected.*

Figure 1 shows a 3-regular graph G with $\lambda(G) = 1$ and its 3-arc graph $X(G)$ which has $\lambda(X(G)) = 4 = \delta(X(G))$. However $X(G)$ is not super-λ and hence is not $\lambda_{(2)}$-optimal. And Figure 2 shows a 3-regular graph G with $\lambda(G) = \kappa(G) = 2$, and its 3-arc graph $X(G)$ which has $\lambda(X(G)) = 4$ and $\lambda_{(2)}(X(G)) = 6 = \xi(X(G))$, i.e., this graph is $\lambda_{(2)}$-optimal. In what follows we give a lower bound on the restricted edge-connectivity $\lambda_{(2)}(X(G))$ where G is a graph having connectivity $\kappa(G) \geq 2$.

Two edges which are incident with a common vertex are *adjacent*.

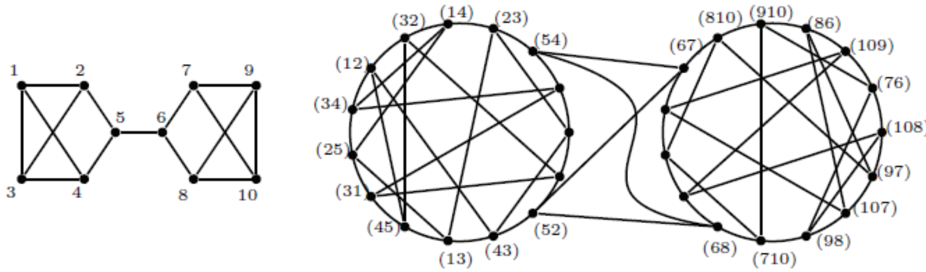

Figure 1: A 3-regular graph with $\lambda = 1$ and its 3-arc graph.

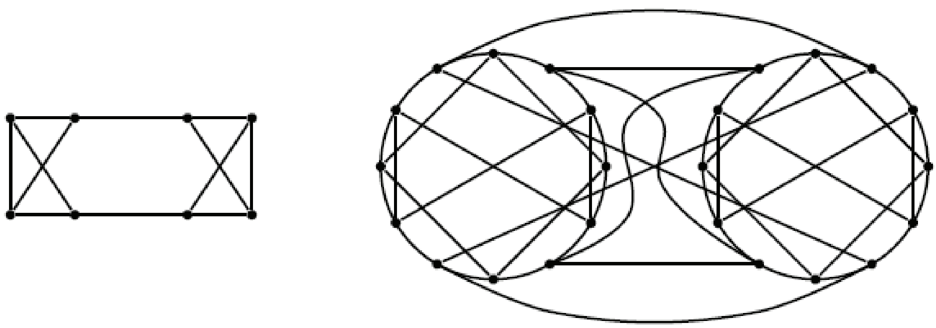

Figure 2: A 3-regular graph with $\lambda = 2$ ($\kappa = 2$) and its 3-arc graph.

Lemma 6 *Let G be a graph with minimum degree $\delta \geq 3$ and vertex connectivity $\kappa \geq 2$. Let $[C, \overline{C}]$ be a restricted edge-cut of $X(G)$ and consider the set $\mathcal{A} = \{ab \in E(G) : (ay)(bx) \in [C, \overline{C}]\}$. Then there are at least two nonadjacent edges in $\mathcal{A}$.*

Proof: Clearly $\mathcal{A} \neq \emptyset$, because $X(G)$ is connected. Thus let $(ay) \in V(C)$ and $(bx) \in V(\overline{C})$ be two adjacent vertices in $X(G)$, which implies that $ab \in \mathcal{A}$. Since $[C, \overline{C}]$ is a restricted edge-cut, then there exist $(uy') \in V(C)$ and $(wx') \in V(\overline{C})$ adjacent to (ay) and (bx) in $X(G)$, respectively. Observe that we may assume that $u \neq w$ because $\delta \geq 3$. Since G is 2-connected we can find a path $R : u = r_0, r_1, \ldots, r_k = w$ from u to w in $G - a$. As $\delta \geq 3$, there exists $v_i \in N(r_i) \setminus \{r_{i-1}, r_{i+1}\}$ for each $i = 1, \ldots, k-1$. Moreover we may choose $v_0 = y'$ and $v_k = x'$. Then the path R induces in $X(G)$ the path $R^* : (uy'), (r_1v_1), \ldots, (r_{k-1}v_{k-1}), (wx')$ (observe that if $k = 1$ then $R^* : (uy'), (wx')$). Since $(uy') \in V(C)$ and $(wx') \in V(\overline{C})$, it follows that $[C, \overline{C}] \cap E(R^*) \neq \emptyset$, hence $r_ir_{i+1} \in \mathcal{A}$ for some $i \in \{0, \ldots, k\}$. Since $a \notin V(R)$ then $a \notin \{r_i, r_{i+1}\}$.

Now reasoning analogously, we can find a path $S : u = s_0, s_1, \ldots, s_\ell = w$ from u to w in $G - b$ that induces a path S^* from $(uy') \in V(C)$ to $(wx') \in V(\overline{C})$. This implies that $[C, \overline{C}] \cap E(S^*) \neq \emptyset$, hence $s_js_{j+1} \in \mathcal{A}$ for some $j \in \{0, \ldots, \ell\}$. Since $b \notin V(S)$ then $b \notin \{s_j, s_{j+1}\}$.

As $ab, r_ir_{i+1}, s_js_{j+1} \in \mathcal{A}$, $a \notin \{r_i, r_{i+1}\}$ and $b \notin \{s_j, s_{j+1}\}$, it follow that al least two of the edges of $\{ab, r_ir_{i+1}, s_js_{j+1}\}$ are nonadjacent. □

Theorem 7 *Let G be a graph with minimum degree $\delta \geq 3$ and vertex connectivity $\kappa \geq 2$. Then $X(G)$ has restricted edge-connectivity $\lambda_{(2)}(X(G)) \geq 2(\delta - 1)^2 - 2$.*

Proof: Let $[C, \overline{C}]$ be a restricted edge-cut of $X(G)$ and consider the set $\mathcal{A} = \{ab \in E(G) : (ay)(bx) \in [C, \overline{C}]\}$. From Lemma 6, $\mathcal{A}$ contains two nonadjacent edges ab and cd. By Lemma 3, the number of au-edges and bv-edges, $u, v \in N(a) \cup N(b)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$, and the number of cu-edges and dv-edges, $u, v \in N(c) \cup N(d)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$. If $|[\{a, b\}, \{c, d\}] \cap \mathcal{A}| \leq 2$ then $|[C, \overline{C}]| \geq 2(\delta - 1)^2 - |[\{a, b\}, \{c, d\}] \cap \mathcal{A}| \geq 2(\delta - 1)^2 - 2$. If $3 \leq |[\{a, b\}, \{c, d\}] \cap \mathcal{A}| \leq 4$ then we may assume without loss of generality that $ac, bd \in \mathcal{A}$, hence, by applying Lemma 3, the number of au-edges and cv-edges, $u, v \in N(a) \cup N(c)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$, and the number of bu-edges and dv-edges, $u, v \in N(b) \cup N(d)$ contained in $[C, \overline{C}]$ is at least $(\delta - 1)^2$. Thus,

$$\begin{aligned} |[C, \overline{C}]| &\geq 2(\delta - 1)^2 - |[\{a, b\}, \{c, d\}] \cap \mathcal{A}| + 2(\delta - 1)^2 - |[\{a, c\}, \{b, d\}] \cap \mathcal{A}| \\ &\geq 4(\delta - 1)^2 - 8 \\ &\geq 2(\delta - 1)^2 - 2, \end{aligned}$$

since $\delta \geq 3$. Hence the theorem is valid. □

Figure 3 shows that $\lambda(G) \geq 2$ is not enough to guarantee that $\lambda_{(2)}(X(G)) \geq 2(\delta - 1)^2 - 2$. In this example G is a 4-regular graph with $\lambda = 2$ and $\kappa = 1$, but $\lambda_{(2)}(X(G)) = 12 < 16$.

The following corollary is an immediate consequence from Theorem 7, and from the fact that if G is graph of minimum degree δ having an edge

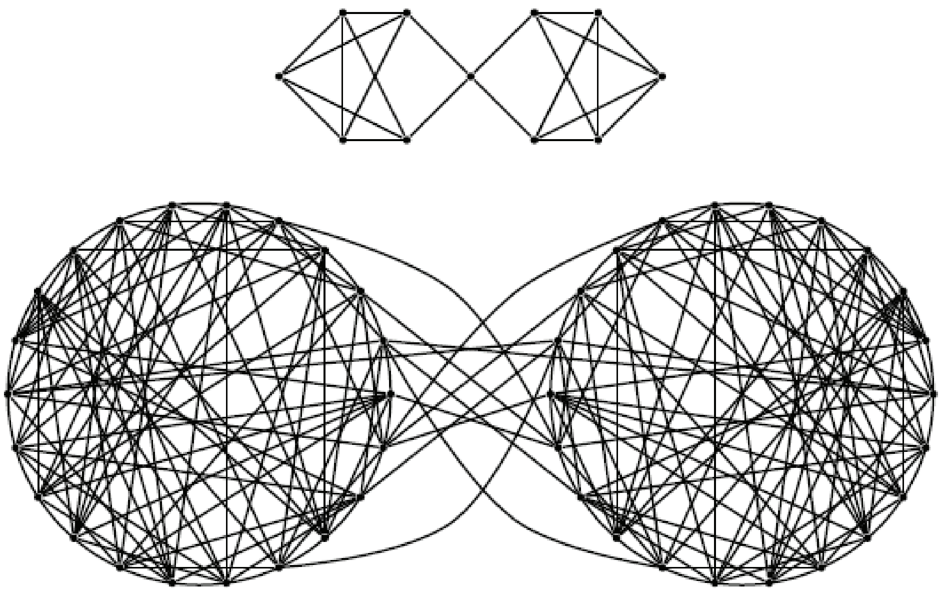

Figure 3: The 3-arc graph of a 4-regular graph with $\kappa = 1$ and $\lambda = 2$ with $\lambda_{(2)}(X(G)) = 12$.

xy such that $d(x) = \delta$, $d(y) = \delta$ and such that every $w \in (N_G(x) - y) \cup (N_G(y) - x)$ also has degree δ, then the minimum edge degree of $X(G)$ is $\xi(X(G)) = 2(\delta - 1)^2 - 2$.

Corollary 8 *Let G be a graph of minimum degree $\delta \geq 3$ and vertex connectivity $\kappa \geq 2$ having an edge xy such that $d(x) = \delta$, $d(y) = \delta$ and such that every $w \in (N_G(x) - y) \cup (N_G(y) - x)$ also has degree δ. Then the 3-arc graph $X(G)$ has restricted edge connectivity $\lambda_{(2)}(X(G)) = \xi(X(G)) = 2(\delta - 1)^2 - 2$.*

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