# of semiregular cages with odd girth

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Abstract

A graph is said to be edge-superconnected if each minimum edge-cut consists of all the edges incident with some vertex of minimum degree. A graph G is said to be a $\{d, d+1\}$-semiregular graph if all its vertices have degree either d or $d+1$. A smallest $\{d, d+1\}$-semiregular graph G with girth g is said to be a $(\{d, d+1\}; g)$-cage. We show that every $(\{d, d+1\}; g)$-cage with odd girth g is edge-superconnected.

1 Introduction

We only consider undirected simple graphs without loops or multiple edges. Unless otherwise stated, we follow [9] for basic terminology and definitions. Let G stand for a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *distance* $d_G(u, v) = d(u, v)$ between two vertices of the graph G is the length of a shortest path between u and v, and the *diameter* of G denoted by $diam(G)$ is the maximum distance between any pair of vertices; when G is not connected, then $diam(G) = +\infty$. For $w \in V$ and $S \subset V$, $d(w, S) = d_G(w, S) = \min\{d(w, s) : s \in S\}$ denotes the *distance* between w and S. For every $S \subset V$ and every nonnegative integer $r \geq 0$, $N_r(S) = \{w \in V : d(w, S) = r\}$ denotes the *neighborhood of S at distance r*. Thus the set of vertices adjacent to a vertex v is $N(v) = N_1(\{v\})$, and the *degree* of a vertex v in G is $deg_G(v) = deg(v) = |N(v)|$, whereas the *minimum degree* $\delta = \delta(G)$ is the minimum degree over all vertices of G. A graph is called *r-regular* if every vertex of the graph has degree r.

A graph G is called *connected* if every pair of vertices is joined by a path. An *edge-cut* in a graph G is a set W of edges of G such that $G - W$ is disconnected. A graph is *k-edge-connected* if every edge-cut contains at least k edges. If W is a minimal edge-cut of a connected graph G, then necessarily, $G - W$ contains exactly two components. The *edge-connectivity* $\lambda = \lambda(G)$ of a graph G is the minimum cardinality of an edge-cut of G. A classic result is $\lambda \leq \delta$ for every graph G. A graph is *maximally edge-connected* if $\lambda = \delta$.

One might be interested in more refined indices of reliability. Even two graphs with the same edge-connectivity λ may be considered to have different reliabilities. As a more refined index than the edge-connectivity, edge-superconnectivity is proposed in [6, 7]. A subset of edges W is called *trivial* if it contains the set of edges incident with some vertex of the graph. Clearly, if $|W| \leq \delta - 1$, then W is nontrivial. A graph is said to be *edge-superconnected* if $\lambda = \delta$ and every minimum edge-cut is trivial.

The *degree set* D of a graph G is the set of distinct degrees of the vertices of G. The *girth* $g(G)$ is the length of a shortest cycle in G. A *$(D; g)$-graph* is a graph having degree set D and girth g. Let $n(D; g)$ denote the least order of a $(D; g)$-graph. Then a $(D; g)$-graph with order $n(D; g)$ is called a *$(D; g)$-cage*. If $D = \{r\}$ then a $(D; g)$-cage is a $(r; g)$-cage. When $D = \{r, r + 1\}$, we refer to $(D; g)$-cages as *semiregular cages*.

The existence of $(r; g)$-cages was proved by Erdös and Sachs [10] in the decade of the 60's, and using this result Chartrand et al. [8] proved the existence of $(D; g)$-cages. Some of the structural properties of $(r; g)$-cages that have been studied are the vertex and the edge connectivity; concerning this problem Fu, Huang and Rodger [11] conjectured that every $(r; g)$-cage is r-connected, and they proved the statement for $r = 3$. Other contributions supporting this conjecture can be seen in [15, 16, 17, 20]. Moreover, some structural properties of $(r; g)$-cages have been extended for $(D; g)$-cages, for example the monotonicity of the order with respect to the girth (see Theorem 1) and the upper bound for the diameter (see Theorem 2). The edge-superconnectivity of cages was established in [18, 19]. For semiregular cages, it has been proved in [3] that they are maximally edge connected. The main objective of this work is to prove that every $(\{d, d +$

$1\}; g)$-cage with odd girth $g \geq 5$ is edge-superconnected. With this aim we need the following two results.

Theorem 1 [4] *Let g_1, g_2 be two integers such that $3 \leq g_1 < g_2$. Then $n(\{d, d+1\}; g_1) < n(\{d, d+1\}; g_2)$.*

Theorem 2 [5] *The diameter of a $(\{d, d+1\}; g)$-cage is at most g.*

2 Main theorem

In order to study the edge-superconnectivity of a graph in terms of its diameter and its girth, the following results were established [1, 2, 13].

Proposition 3 *Let $G = (V, E)$ be a connected graph with minimum degree $\delta \geq 2$ and girth g. Let $W \subset E$ be a minimum nontrivial edge-cut, let H_i be a component of $G - W$, and let $W_i \subset V(H_i)$ be the set of vertices of H_i which are incident with some edge in W, $i = 0, 1$. Then there exists some vertex $x_i \in V(H_i)$ such that*

(a) [1, 13] *$d(x_i, W_i) \geq \lfloor (g-1)/2 \rfloor$, if $|W_i| \leq \delta - 1$.*

(b) [2] *$d(x_i, W_i) \geq \lceil (g-3)/2 \rceil$, if $|W| \leq \xi - 1$, where $\xi = \min\{deg(u) + deg(v) - 2 : uv \in E\}$ is the minimum edge-degree of G.*

For every minimum edge-cut W of G such that H_0, H_1 are the two components of $G - W$, we will write henceforth $W = [W_0, W_1]$ with $W_0 \subset V(H_0)$ and $W_1 \subset V(H_1)$ containing all endvertices of the edges in W. Note that $|W_i| \leq |W|$, $i = 0, 1$. From now on, let

$$\mu_i = \max\{d(x, W_i) : x \in V(H_i)\}, \quad i = 0, 1.$$

When W is nontrivial and $|W| \leq \xi - 1$, it follows from Proposition 3 that $\mu_i \geq \lceil (g-3)/2 \rceil$. Likewise, μ_0 and μ_1 satisfy some other basic properties shown in next lemma.

Lemma 4 *Let $G = (V, E)$ be a connected graph with minimum degree $\delta \geq 3$ and odd girth $g \geq 5$. Let $W = [W_0, W_1] \subset E$ be a minimum nontrivial edge-cut with cardinality $|W| \leq \delta$. Let $G - W = H_0 \cup H_1$, where $W_i \subset V(H_i)$. If $\mu_i = (g-3)/2$ the following statements hold:*

(i) $|W_i| = |W| = \delta$, *and every* $a \in W_i$ *is incident to a unique edge of* W.

(ii) *Every vertex* $z \in V(H_i)$ *such that* $d(z, W_i) = \mu_i$ *has* $deg(z) = \delta$.

(iii) *For every* $a \in W_i$ *there exists a vertex* $x \in V(H_i)$ *such that* $d(x, W_i) = d(x, a) = \mu_i$ *and* $N_{(g-3)/2}(x) \cap W_i = \{a\}$. *Further,* $N(x)$ *can be labeled as* $\{u_1, u_2, \ldots, u_\delta\}$, *and* W_i *can be labeled as* $\{a_1, a_2, \ldots, a_\delta\}$, *where* $a_1 = a$, *so that* $N_{(g-5)/2}(u_1) \cap W_i = \{a_1\}$ *and* $N_{(g-3)/2}(u_k) \cap W_i = \{a_k\}$ *for every* $k > 1$. *Consequently* $|[N_{(g-3)/2}(x) \cap W_i, W_{i+1}]| = 1$ *and* $|[N_{(g-3)/2}(u_k) \cap W_i, W_{i+1}]| = 1$ *(with subscripts taken mod 2). See Figure 2.*

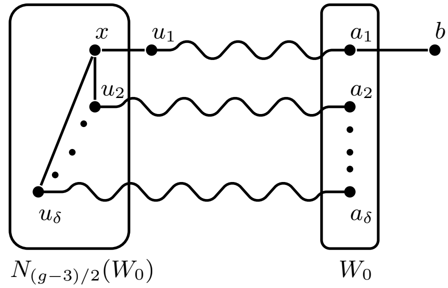

Figure 1: Lemma 4.

Proof: (i) Since $\mu_i = (g-3)/2$, $d(x, W_i) \leq \mu_i = (g-3)/2 < (g-1)/2$ for all $x \in V(H_i)$. Hence from Proposition 3 (a), it follows that $|W_i| \geq \delta$, yielding $|W_i| = \delta$ because $|W_i| \leq |W| \leq \delta$. Observe that $\delta = |W_i| = |W|$ means that $|N(a) \cap W_{i+1}| = 1$ for each vertex $a \in W_i$ (taking the subscripts mod 2).

(ii) First observe that $\mu_i = (g-3)/2 \geq 1$ since $g \geq 5$. Let us define the following partition of $N(v)$ for all $v \in V(H_i)$

$$S^-(v) = \begin{cases} \{z \in N(v) : d(z, W_i) = d(v, W_i) - 1\} & \text{if } v \notin W_i; \\ W_{i+1} \cap N(v) & \text{if } v \in W_i. \end{cases}$$

$$S^+(v) = \{z \in N(v) : d(z, W_i) = d(v, W_i) + 1\}$$
$$S^=(v) = \{z \in N(v) : d(z, W_i) = d(v, W_i)\}.$$

Let z be a vertex of H_i such that $d(z, W_i) = \mu_i = (g-3)/2$. Then we have

$$\begin{array}{rcl} N(z) & = & S^{=}(z) \cup S^{-}(z); \\ |N_{(g-3)/2}(S^{=}(z)) \cap W_i| & \geq & |S^{=}(z)|; \\ |N_{(g-5)/2}(S^{-}(z)) \cap W_i| & \geq & |S^{-}(z)|; \\ N_{(g-3)/2}(S^{=}(z)) \cap N_{(g-5)/2}(S^{-}(z))) & = & \emptyset, \end{array} \quad (1)$$

because otherwise cycles of length less than the girth g appear. Since

$$\begin{array}{rcl} \delta \leq deg(z) & = & |S^{=}(z)| + |S^{-}(z)| \\ & \leq & |N_{(g-3)/2}(S^{=}(z)) \cap W_i| + |N_{(g-5)/2}(S^{-}(z)) \cap W_i| \\ & \leq & |W_i| = \delta \end{array}$$

it follows that $\delta = deg(z)$. Therefore item (ii) holds.

(iii) First let us prove that there exists an edge zz' such that $d(z, W_i) = d(z', W_i) = (g-3)/2$. Otherwise, $S^{=}(z) = \emptyset$ for all z with $d(z, W_i) = (g-3)/2$. This implies that for all $u \in N(z)$, $u \in S^{-}(z)$ and $S^{=}(S^{+}(u)) = \emptyset$. Further, $|N_{(g-5)/2}(u) \cap W_i| = 1$ for all $u \in N(z)$, because $\delta = |W_i| = \sum_{u \in N(z)} |N_{(g-5)/2}(u) \cap W_i| \geq \delta$. Hence $|S^{-}(u)| = 1$, and so $|S^{+}(u)| + |S^{=}(u)| = deg(u) - 1 \geq \delta - 1 \geq 2$. Suppose that $|S^{=}(u)| \geq 1$ for some $u \in N(z)$. Then as $N_{(g-3)/2}(z) \cap W_i$ and $N_{(g-5)/2}(S^{=}(u)) \cap W_i$ are two vertex disjoint sets we have $|W| \geq |N_{(g-3)/2}(z) \cap W_i| + |N_{(g-5)/2}(S^{=}(u)) \cap W_i| \geq \delta + 1$ which is a contradiction because $|W| = \delta$. Then we must assume that for all $u \in N(z)$, $|S^{+}(u)| = deg(u) - 1 \geq \delta - 1 \geq 2$. Let $t \in S^{+}(u) - z$, according to our first assumption $S^{=}(t) = \emptyset$ meaning that $N(t) = S^{-}(t)$. Since t has the same behavior as z we have $W_i = N_{(g-3)/2}(S^{-}(z)) = N_{(g-3)/2}(S^{-}(t))$, and as $2 < \delta \leq deg(z) = deg(t)$, there exist cycles through $\{z, u, t, w\}$ for some $w \in W_i$ of length less than g which is a contradiction.

Hence we may assume that there exists an edge zz' such that $d(z, W_i) = d(z', W_i) = (g-3)/2$. Since $N_{(g-5)/2}(S^{-}(z)) \cap W_i$, $N_{(g-5)/2}(S^{-}(z')) \cap W_i$ and $N_{(g-3)/2}(S^{=}(z') - z) \cap W_i$ are three pairwise disjoint sets because $g \geq 5$, and taking into account (1) we have

$$\begin{array}{rcl} \delta = |W| & \geq & |N_{(g-5)/2}(S^{-}(z)) \cap W_i| + |N_{(g-5)/2}(S^{-}(z')) \cap W_i| \\ & & + |N_{(g-3)/2}(S^{=}(z') - z) \cap W_i| \\ & \geq & |S^{-}(z)| + |S^{-}(z')| + |S^{=}(z') - z| \\ & = & deg(z) - 1 + |S^{-}(z)| \geq \delta. \end{array}$$

Therefore, all inequalities become equalities, i.e., $|S^-(z)| = 1 = |N_{(g-5)/2}(S^-(z)) \cap W_i|$. So $S^-(z) = \{z_1\}$ and $N(z) - z_1 = S^=(z)$ yielding a partition of W_i:

$$W_i = \left(N_{(g-5)/2}(z_1) \cap W_i\right) \cup \left(\cup_{z' \in N(z)-z_1} N_{(g-3)/2}(z') \cap W_i\right),$$

because for all $z' \in N(z) - z_1$ the sets $N_{(g-3)/2}(z') \cap W_i$ and the set $N_{(g-5)/2}(z_1) \cap W_i$ are mutually disjoint. Thus, $|N_{(g-3)/2}(z') \cap W_i| = 1$ for all $z' \in N(z) - z_1$. Therefore, for every vertex $a \in W_i$ there exists a vertex $x \in (N(z) - z_1) \cup \{z\}$ such that $d(x, W_i) = d(x, a) = (g-3)/2$ and $N_{(g-3)/2}(x) \cap W_i = \{a\}$. Furthermore, since every vertex $z' \in N(z) - z_1$ has the same behavior as z, $N(x)$ can be labeled as $\{u_1, u_2, \ldots, u_\delta\}$, and W_i can be labeled as $\{a_1, a_2, \ldots, a_\delta\}$, where $a_1 = a$, so that $N_{(g-5)/2}(u_1) \cap W_i = \{a_1\}$ and $N_{(g-3)/2}(u_k) \cap W_i = \{a_k\}$ for every $k > 1$. Finally, using (i) we obtain $|[N_{(g-3)/2}(x) \cap W_i, W_{i+1}]| = 1$ and $|[N_{(g-3)/2}(u_k) \cap W_i, W_{i+1}]| = 1$, which finishes the proof. □

A semiregular cage is known to be maximally edge-connected [3]. Now, we are ready to prove that semiregular cages with odd girth are edge-superconnected. As will be seen, *Hall's Theorem* is a key point of this study. Recall that if S is a set of vertices in a graph G, the set of all neighbors of the vertices in S is denoted by $N(S)$.

Theorem 5 ([12] **Hall's Theorem**) *A bipartite graph with bipartition (X_1, X_2) has a matching which covers every vertex in X_1 if and only if*

$$|N(S)| \geq |S| \text{ for all } S \subset X_1.$$

Using Hall's Theorem Jiang [14] proved the following result.

Lemma 6 [14] *Let G be a bipartite graph with bipartition (X_1, X_2) where $|X_1| = |X_2| = r$. If G contains at least $r^2 - r + 1$ edges, then G contains a perfect matching.*

The following lemma is an stronger version of Lemma 6, which is also proved using Hall's Theorem.

Lemma 7 *Let $\mathcal{B}$ be a bipartite graph with bipartition (X_1, X_2) where $|X_1| = |X_2| = r$. If $\delta(\mathcal{B}) \geq 1$ and $|E(\mathcal{B})| \geq r^2 - r$, then $\mathcal{B}$ contains a perfect matching.*

Proof: Let $\mathcal{B} = (X_1, X_2)$ be a bipartite graph with $|X_1| = |X_2| = r$, $\delta(\mathcal{B}) \geq 1$ and $|E(\mathcal{B})| \geq r^2 - r$. We shall apply Hall's Theorem to prove the lemma; we shall show that for a subset $S \subset X_1$, $|N(S)| \geq |S|$. Notice that if $|S| = 1$, then $|N(S)| \geq 1 = |S|$ because $\delta(\mathcal{B}) \geq 1$; and if $S = X_1$, $N(S) = X_2$ because $\delta(\mathcal{B}) \geq 1$ implies that each vertex $u \in X_2$ must have a neighbor in S, hence $|S| = |N(S)|$.

Therefore we continue the proof reasoning by contradiction and so assuming that $1 \leq |N(S)| < |S| = t \leq r - 1$. Then the number of edges in $\mathcal{B}$ is at most

$$|E(\mathcal{B})| = |[S, N(S)]| + |[X_1 \setminus S, X_2]| \leq t(t-1) + (r-t)r,$$

and by hypothesis $|E(\mathcal{B})| \geq r^2 - r$. Thus $r^2 - r \leq t(t-1) + (r-t)r$, yielding $0 \leq (t-r)(t-1)$, which is an absurdity because $1 < t < r$. Therefore $|N(S)| \geq |S|$ for all $S \subset X_1$, and by Hall's Theorem the lemma follows. □

Theorem 8 *Let G be a $(\{d, d+1\}; g)$-cage with odd girth $g \geq 5$, and $d \geq 3$. Then G is edge-superconnected.*

Proof: Let us assume that G is a non edge-superconnected $(\{d, d+1\}; g)$-cage, and we will arrive at a contradiction. To this end, let us take a minimum nontrivial edge-cut $W = [W_0, W_1] \subset E(G)$ such that $|W| \leq \delta$. Let $G - W = H_0 \cup H_1$, and let $W_i \subset V(H_i)$ be the set of vertices of H_i which are incident with some edge in W, $i = 0, 1$. From Proposition 3 it follows that $\mu_i = \max\{d(x, W_i) : x \in V(H_i)\} \geq (g-3)/2$, $i = 0, 1$. Let $x_i \in V(H_i) \cap N_{\mu_i}(W_i)$. As G is a $(\{d, d+1\}; g)$-cage, the diameter is at most $diam(G) \leq g$ by Theorem 2, so we get the following chain of inequalities:

$$g \geq diam(G) \geq d(x_0, x_1) \geq d(x_0, W_0) + 1 + d(x_1, W_1) = \mu_0 + 1 + \mu_1 \geq g - 2.$$

If we assume henceforth $\mu_0 \leq \mu_1$ (without loss of generality), then either $(g-3)/2 = \mu_0 \leq \mu_1 \leq (g+1)/2$, or $\mu_0 = \mu_1 = (g-1)/2$. We proceed to study each one of these cases.

In what follows, let X_0, X_1 be two subsets of $V(G)$ such that $|X_0| = |X_1|$. Let $\mathcal{B}_\Gamma$ denote the bipartite graph with bipartition (X_0, X_1) and $E(\mathcal{B}_\Gamma) = \{u_i v_j : u_i \in X_0, v_j \in X_1, d_\Gamma(u_i, v_j) \geq g - 1\}$, where Γ is a certain subgraph of G.

Case (a): $\mu_0 = (g-3)/2$.

From Lemma 4 (i), $|W_0| = d = |W|$ so that each vertex of W_0 is incident to a unique edge of W, yielding that every vertex $a \in W_0$ has $deg_{H_0}(a) \in \{d-1, d\}$. Also by Lemma 4 (ii), every vertex $x \in N_{(g-3)/2} \cap V(H_0)$ has $deg(x) = d$. And by Lemma 4 (iii), for every $a \in W_0$ there exists a vertex $x_0 \in N_{(g-3)/2} \cap V(H_0)$ such that $N(x_0) = \{u_1, u_2, \ldots, u_d\}$ and $W_0 = \{a_1, a_2, \ldots, a_d\}$, where $a_1 = a$, in such a way that $d(u_1, a_1) = d(u_1, W_0) = (g-5)/2$, $d(u_j, W_0) = d(u_j, a_j) = (g-3)/2$, and by (ii), $deg(u_j) = d$ for every $j \geq 2$. This implies that $d_{G-x_0}(u_1, a_j) \geq (g-1)/2$ for all $j \geq 2$, because the shortest (u_1, a_j)-path in $G - x_0$, the shortest (u_j, a_j)-path in G, and the path $u_j x_0 u_1$ in G of length two, form a closed walk containing a cycle. Reasoning analogously, $d_{G-x_0}(u_j, a_1) \geq (g+1)/2$ for all $j \geq 2$ and $d_{G-x_0}(u_j, a_i) \geq (g-1)/2$ for $j \neq i$, $j, i \in \{2, \ldots, d\}$. Furthermore, $[N_{(g-3)/2}(x_0) \cap W_0, W_1] = \{a_1 b_1\}$ for some $b_1 \in W_1$.

Subcase (a.1): $\mu_1 = (g+1)/2$.

Let $x_1 \in V(H_1)$ be any vertex such that $d(W_1, x_1) = (g+1)/2$. Let $X_0 = \{u_2, \ldots, u_d\} \cup \{x_0\}$ and $X_1 = \{v_1, v_2, \ldots, v_d\} \subseteq N(x_1)$. As $d(u_i, W_0) = (g-3)/2$ for $i \geq 2$ and $d_{G-x_1}(W_1, N(x_1)) \geq (g-1)/2$, then $d_{G-x_1}(X_0, X_1) \geq g-1$, so $|E(\mathcal{B}_\Gamma)| = d^2$, where $\Gamma = G - x_1$. Clearly $\mathcal{B}_\Gamma$ is a complete bipartite graph, so there is a perfect matching M which covers every vertex in X_0 and if $deg(x_1) = d$, also covers $N(x_1)$. Hence, in this case the graph $G^* = (G - \{x_1\} - \{x_0 u_d\}) \cup M$ has girth at least g and the vertices $u_2, \ldots, u_{d-1}$ have degree $d+1$ in G^* as they had degree d in G; for the same reason x_0 and u_d have degree d in G^*. The remaining vertices have the same degree they had in G. As G^* is a $(\{d, d+1\}; g^*)$-graph with girth $g^* \geq g$ and $|V(G^*)| < |V(G)|$, we get a contradiction to the monotonocity Theorem 1. If $deg(x_1) = d+1$, since $d_{G^*}(u_d, v_{d+1}) \geq g-1$ where $v_{d+1} \in N(x_1) \setminus X_1$, we can add the new edge $u_d v_{d+1}$ to G^* without decreasing the girth. Then $G^* \cup \{u_d v_{d+1}\}$ gives us again a contradiction.

Subcase (a.2): $\mu_1 = (g-3)/2$.

By Lemma 4, given $b_1 \in W_1$ there exists $x_1 \in V(H_1) \cap N_{(g-3)/2}(W_1)$ of $deg(x_1) = d$ such that $N(x_1) = \{v_1, v_2, \ldots, v_d\}$, $W_1 = \{b_1, b_2, \ldots, b_d\}$ and each vertex of W_1 is incident to a unique edge of W, hence $W = \{a_1 b_1, a_2 b_2, \ldots, a_d b_d\}$. Also, $d(b_1, v_1) = d(W_1, v_1) = (g-5)/2$, and $d(W_1, v_j) = d(b_j, v_j) = (g-3)/2$ for every $j \geq 2$ and besides $deg(v_j) = d$. Then $d(x_0, x_1) = d(x_0, a_1) + 1 + d(b_1, x_1) = g-2$, and if $g = 5$ it is easy to see that the shortest (x_0, x_1)-path of length three is unique, clearly $x_0 a_1 b_1 x_1$.

Now let $\Gamma = G - \{x_0, x_1\}$. We have

$$\begin{aligned} d_\Gamma(u_1, N(x_1) - v_1) &= \min\{d_\Gamma(u_1, a_1) + 1 + d_\Gamma(b_1, N(x_1) - v_1);\\ &\qquad d_\Gamma(u_1, a_j) + 1 + d_\Gamma(b_j, N(x_1) - v_1), j \geq 2\}\\ &\geq \min\{\frac{g-5}{2} + 1 + \frac{g+1}{2}; \frac{g-1}{2} + 1 + \frac{g-3}{2}\} = g - 1, \end{aligned}$$

since $d_\Gamma(b_1, v_j) \geq (g+1)/2$ for all $j \geq 2$, because the shortest (b_1, v_j)-path in Γ, the shortest (b_1, v_1)-path in Γ, and the path $v_j x_1 v_1$ in G of length two, form a closed walk containing a cycle. Reasoning in the same way, it follows for all $j \geq 2$ that

$$\begin{aligned} &d_\Gamma(u_j, N(x_1) - v_j) =\\ &= \min\{d_\Gamma(u_j, a_j) + 1 + d_\Gamma(b_j, N(x_1) - v_j); d_\Gamma(u_j, a_h) + 1\\ &\quad + d_\Gamma(b_h, N(x_1) - v_j), h \neq j\}\\ &\geq \min \left\{ \begin{array}{l} \left\{\frac{g-3}{2} + 1 + \frac{g-1}{2}; \frac{g-1}{2} + 1 + \frac{g-3}{2}\right\} \text{ if } h \geq 2, h \neq j \\ \left\{\frac{g-3}{2} + 1 + \frac{g-1}{2}; \frac{g+1}{2} + 1 + \frac{g-5}{2}\right\} \text{ if } h = 1 \end{array} \right\}\\ &= g - 1. \end{aligned}$$

Analogously, $d_\Gamma(N(x_0) - u_1, v_1) \geq g - 1$ and $d_\Gamma(N(x_0) - u_j, v_j) \geq g - 1$ for all $j \geq 2$. Let $X_0 = N(x_0)$ and $X_1 = N(x_1)$. The bipartite graph $\mathcal{B}_\Gamma = (X_0, X_1)$ has $|E(\mathcal{B}_\Gamma)| = d^2 - d$ and $deg_{\mathcal{B}_\Gamma}(w) \geq 1$ for all $w \in X_0 \cup X_1$. From Lemma 7, there is a perfect matching M between $X_0 = N(x_0)$ and $X_1 = N(x_1)$. Hence $G^* = (G - \{x_0, x_1\}) \cup M$ is a $(\{d, d+1\}; g^*)$-graph (because every vertex in G^* has the same degree it had in G and the removed vertices x_0, x_1 had degree d, as well as the vertices u_j, v_k for every $j, k \geq 2$) with $g^* \geq g$ and $|V(G^*)| \leq |V(G)|$, which contradicts the monotonocity Theorem 1, and we are done.

Subcase (a.3): $\mu_1 = (g-1)/2$. In this case we distinguish two other possible subcases.

Subcase (a.3.1): There exists $x_1 \in V(H_1) \cap N_{(g-1)/2}(W_1)$ such that $d(b, v) \leq (g-1)/2$ for all $b \in W_1$ and for all $v \in N(x_1)$.

Then, every $b \in W_1$ has $deg_{H_1}(b) = deg(x_1) \in \{d, d+1\}$ because $d(b, v) \leq (g-1)/2$ and $|N_{(g-3)/2}(v) \cap N(b)| \leq 1$ for all $v \in N(x_1)$ (otherwise cycles of length less than g appear). Hence $deg(x_1) = d$ and $deg(b) = d+1$ for all $b \in W_1$. Thus $N(x_1) = \{v_1, \ldots, v_d\}$ and $W = [W_0, W_1]$ is a matching, i.e., $W = \{a_1b_1, \ldots, a_db_d\}$. Therefore the subgraph H_1 gives a contradiction unless H_1 is d-regular. In this case let us consider the graph $\hat{G} = (G - x_1 - W) \cup \{a_1v_1, \ldots, a_dv_d\}$ which clearly has girth at least g. Moreover $deg_{\hat{G}}(b_i) = deg(b_i) - 1 = d$ and every vertex different from b_i has the same degree it had in G. Thus we may suppose that $\hat{G}$ is d-regular because otherwise $\hat{G}$ would be a $(\{d, d+1\}; g^*)$-graph with girth $g^* \geq g$ and smaller than G, a contradiction. Moreover, we may assume that $d_{H_1}(b_1, v_1) = (g-3)/2$ and $d_{H_1}(b_1, N(x_1) - v_1) = (g-1)/2$. Thus we have

$$\begin{aligned} d_{\hat{G}}(b_1, u_2) &\geq \min\{d_{H_1}(b_1, v_2) + |\{v_2a_2\}| \\ &\quad + d_{H_0}(a_2, u_2); d_{H_1}(b_1, v_1) + |\{v_1a_1\}| + d_{H_0}(a_1, u_2)\} \\ &\geq \min\{\frac{g-1}{2} + 1 + \frac{g-3}{2}; \frac{g-3}{2} + 1 + \frac{g+1}{2}\} \\ &= g-1, \end{aligned}$$

which implies that we can add to $\hat{G}$ the edge u_2b_1 to obtain a graph without decreasing the girth g. As this new graph is smaller than G and has degrees $\{d, d+1\}$ we get a contradiction to the monotonicity Theorem 1, and we are done.

Subcase (a.3.2): For all $z \in V(H_1) \cap N_{(g-1)/2}(W_1)$ there exists $v \in N(x_1)$ and $b \in W_1$ such that $d(b, v) \geq (g+1)/2$.

Let $x_1 \in V(H_1) \cap N_{(g-1)/2}(W_1)$, $v_1 \in N(x_1)$ and $b^* \in W_1$ be such that $d(b^*, v_1) \geq (g+1)/2$. By Lemma 4, there exists a unique edge $a^*b^* \in W$ to which the vertex $a^* \in W_0$ is incident, and there exists a vertex $x^* \in V(H_0)$ of $deg(x^*) = d$ such that $d(x^*, W_0) = d(x^*, a^*) = (g-3)/2$ and $N_{(g-3)/2}(x^*) \cap W_0 = \{a^*\}$. Further, $N(x^*)$ can be labeled as $\{z_1, z_2, \ldots, z_d\}$, and W_0 can be labeled as $\{a_1, a_2, \ldots, a_d\}$, where $a_1 = a^*$, so that $N_{(g-5)/2}(z_1) \cap W_i = \{a_1\}$, $N_{(g-3)/2}(z_k) \cap W_i = \{a_k\}$ and $deg(z_k) = d$ for every $k > 1$. Furthermore, $[N_{(g-3)/2}(x^*) \cap W_0, W_1] = \{a_1b^*\}$

Let $\Gamma = G - \{x^*, x_1\}$. We obtain

$$\begin{aligned} &d_\Gamma(z_1, v_1) \\ &= \min\{d_\Gamma(z_1, a_1) + 1 + d_\Gamma(b^*, v_1);\ d_\Gamma(z_1, a_j) + 1 + d_\Gamma(b', v_1), j \geq 2, a_jb' \in W\} \\ &\geq \min\{\frac{g-5}{2} + 1 + \frac{g+1}{2};\ \frac{g-1}{2} + 1 + \frac{g-3}{2}\} = g-1. \end{aligned}$$

Moreover, $d_{H_0}(z_k, W_0) = (g-3)/2$ for all $z_k \in N(x^*) - z_1$ and for $k > 1$ there exists a unique vertex say $b_k \in W_1$ for which $a_k b_k \in W$. As for each $b \in W_1$, $|N_{(g-3)/2}(b) \cap N(x_1)| \leq 1$ (otherwise cycles of length less than g appear) we may denote by v_k the vertex in $N(x_1) - v_1$ such that $d(b_k, v_k) = (g-3)/2$, if any. Thus we obtain

$$d_\Gamma(z_k, N(x_1) \setminus \{v_1, v_k\}) = d(z_k, a_k) + 1 + d(b_k, N(x_1) \setminus \{v_1, v_k\})$$
$$\geq \tfrac{g-3}{2} + 1 + \tfrac{g-1}{2} = g - 1.$$

Let us consider $X_0 = N(x^*) - z_1$ and $X_1 \subseteq N(x_1) - v_1$, with $|X_1| = d - 1$. It is clear that $|deg_{\mathcal{B}_\Gamma}(z_k)| \geq d - 2 \geq 1$ for all $z_k \in N(x^*) - u_1$ yielding $|E(\mathcal{B}_\Gamma)| \geq (d-2)(d-1) = (d-1)^2 - (d-1)$.

First, suppose that $|deg_{\mathcal{B}_\Gamma}(v)| \geq 1$ for all $v \in N(x_1) - v_1$. From Lemma 7, there is a matching M which covers every vertex in $N(x^*) - z_1$ and every vertex in $N(x_1) - v_1$ if $deg(x_1) = d$. In this case $G^* = (G - \{x^*, x_1\}) \cup M \cup \{z_1 v_1\}$ is a graph with girth $g^* \geq g$ and smaller than G whose vertices have the same degree they had in G; thus G^* is a $(\{d, d+1\}; g^*)$-graph and we are done. Thus suppose that $deg(x_1) = d + 1$ and that after adding the matching $M \cup \{z_1 v_1\}$ to $G - \{x^*, x_1\}$ the vertex $v_{d+1} \in (N(x_1) - v_1) \setminus X_1$ remains of degree $d - 1$. By Lemma 4 every z_k, $k > 1$, has degree d in G, and we have proved that $d(z_k, N(x_1) \setminus \{v_1, v_k\}) \geq g - 1$. Then we add one extra edge $z_k v_{d+1}$ to G^* obtaining a new $(\{d, d+1\}; g^*)$-graph with $g^* \geq g$ and smaller than G, a contradiction to the monotonicity Theorem 1, so we are done.

Therefore we must suppose that there exists $v_2 \in N(x_1) - v_1$ such that $|deg_{\mathcal{B}_\Gamma}(v_2)| = 0$. This implies that $d(v_2, b) = (g-3)/2$ for all $b \in W_1 - b^*$, hence $d(v, W_1 - b^*) = (g-1)/2$ for all $v \in N(x_1) - v_2$. First suppose that $d(v_2, b^*) \geq (g+1)/2$; then $d_\Gamma(z_1, v_2) \geq g - 1$, $d_\Gamma(z_k, N(x_1) - v_2) = g - 1$ for all $k \geq 2$, thus we consider the set $X_1 \subseteq N(x_1) - v_2$ with $|X_1| = d - 1$. It is clear that $|deg_{\mathcal{B}_\Gamma}(w)| \geq d - 1$ for all $w \in X_0 \cup X_1$. Using Lemma 7 and reasoning as before we get a contradiction. Therefore we must suppose that $d(v_2, b^*) \leq (g-1)/2$. Since $N(x_1) - v_2 \subseteq N_{(g-1)/2}(W_1) \cap V(H_1)$ we have by hypothesis that for all $v \in N(x_1) - v_2$ there exists $\hat{v}_1 \in N(v)$ and $\hat{b}^* \in W_1$ such that $d(\hat{b}^*, \hat{v}_1) \geq (g+1)/2$. As the behavior of any $v \in N(x_1) - v_2$ is the same as vertex x_1, reasoning as before we get a contradiction unless for all $v \in N(x_1) - v_2$ there exists $\hat{v}_2 \in N(v) - \hat{v}_1$ such that $|deg_{\mathcal{B}_{\hat{\Gamma}}}(\hat{v}_2)| = 0$ satisfying $d(\hat{v}_2, b) = (g-3)/2$ for all $b \in W_1 - \hat{b}^*$ and $d(\hat{v}_2, \hat{b}^*) \leq (g-1)/2$. Therefore we conclude that every vertex $b \in W_1$ has

$deg_{H_1}(b) = deg(x_1) \in \{d, d+1\}$. Now considering the same graph as in Subcase (a.3.1) we get a contradiction.

Case (b): $\mu_0 = \mu_1 = (g-1)/2$.

Let $x_0 \in V(H_0)$ and $x_1 \in V(H_1)$ satisfy $d(x_i, W_i) = (g-1)/2$, $i = 0, 1$.

First of all note that there must exist a vertex in $N(x_0)$ of degree d, otherwise $G - x_0$ would be either a $\{d, d+1\}$-graph or a d-regular graph. In the former case we get a contradiction because $G - x_0$ is smaller than G and has girth at least g. And in the latter case we consider the graph $(G - x_0) \cup \{u_i x_1\}$ with $u_i \in N(x_0)$, which gives again a contradiction. Similarly, note that there must exist a vertex in $N(x_1)$ of degree d.

Suppose that $deg(x_0) = deg(x_1) = r$ with $r \in \{d, d+1\}$. Let $X_0 = N(x_0), X_1 = N(x_1)$ and $\Gamma = G - \{x_0, x_1\}$. Define $A = \{u_i v_j : u_i \in X_0, v_j \in X_1, d_\Gamma(u_i, v_j) \leq g-2\}$ and consider $\mathcal{B}_\Gamma = K_{|X_0|,|X_1|} - A$. Note that every (u_i, v_j)-path in G goes through an edge of W. Therefore every edge in W gives rise to at most one element in A, otherwise G would contain a cycle of length at most $2(g-3)/2 + 2 = g - 1$. Hence $|A| \leq |W| \leq d$ and $|E(\mathcal{B}_\Gamma)| = |K_{r,r}| - |A| \geq r^2 - d$.

If $r = d+1$ then $|E(\mathcal{B}_\Gamma)| \geq (d+1)^2 - d = d^2 + d + 1$ and by Lemma 6, the graph $\mathcal{B}_\Gamma$ contains a perfect matching M. Therefore the graph $G' = G - \{x_0, x_1\} \cup M$ has fewer vertices than G and girth at least g producing a contradiction unless G' is regular of degree d. In this case we consider the graph $G'' = G' \cup \{uv\}$ where $u \in N(x_0)$ is such that $d(u, W_0) = (g-1)/2$ (such a vertex must exist because $deg(x_0) = d+1$ and $|W_1| \leq d$) and $v \in N(x_1)$ such that $uv \notin M$. As G'' is a $(\{d, d+1\}; g)$-graph with fewer vertices than G and girth g a contradiction is again obtained.

Suppose $r = d$. If $deg_{\mathcal{B}_\Gamma}(z) \geq 1$ for all $z \in \mathcal{B}_\Gamma$, then by Lemma 7 there exists a perfect matching M between X_0 and X_1; reasoning as before we obtain again a contradiction. Hence, we may assume that $deg_{\mathcal{B}_\Gamma}(u_1) = 0$ for some $u_1 \in X_0$. This implies that $d_\Gamma(u_1, v_j) = g-2$ for all $v_j \in N(x_1)$, or equivalently $d_\Gamma(v_j, W_1) = (g-3)/2$ for all $v_j \in N(x_1)$. From this, and because $g \geq 5$, we get $|W_1| \geq |N(x_1)| = d$, yielding $|W_1| = d$ (since $d = |W| \geq |W_1|$), and also $N_{(g-3)/2}(v_j) \cap W_1 = \{b_j\}$ for all $v_j \in N(x_1)$. That is, $|N(b_j) \cap W_0| = 1$ for every $b_j \in W_1$. Also we have $N_{(g-1)/2}(u_1) \cap W_1 = W_1$, hence $N_{(g-3)/2}(u_1) \cap W_0 = W_0$ and thus $d(u_i, W_0) = (g-1)/2$ for $i \geq 2$.

Let $u_k \in N(x_0)$, $k \geq 2$, define $\Gamma_k = G - \{u_k, x_1\}$ and consider the sets

$$X_k = \begin{cases} N(u_k) & \text{if } deg(u_k) = d; \\ N(u_k) - x_0 & \text{if } deg(u_k) = d+1; \end{cases}$$

$$X_1 = N(x_1);$$

$$A_k = \{z_i v_j : z_i \in X_k, v_j \in X_1, d_{\Gamma_k}(z_i, v_j) \leq g-2\}.$$

Let $\mathcal{B}_{\Gamma_k} = K_{|X_k|,|X_1|} - A_k$.

If $deg_{\mathcal{B}_{\Gamma_k}}(z) \geq 1$ for all $z \in X_k$, we get a perfect matching M between X_k and $N(x_1)$ by Lemma 7; if $deg(u_k) = d$ the graph $\Gamma_k \cup M$ yields a contradiction; if $deg(u_k) = d+1$ the graph $\Gamma_k \cup M \cup \{x_0 v_j\}$, where v_j is a vertex of $N(x_1)$ with degree d, yields again a contradiction. Therefore we can suppose that for every $u_k \in N(x_0) - u_1$ there exists $\hat{z}_k \in N(u_k)$ such that $d_{\Gamma_k}(\hat{z}_k, v_j) = g-2$ for all $v_j \in N(x_1)$. Hence, $N_{(g-3)/2}(\hat{z}_k) \cap W_0 = W_0$, that is $d_{\Gamma_k}(\hat{z}_k, a_j) = (g-3)/2$ for each $a_j \in W_0$. Therefore $deg_{H_0}(a_j) = d$, $deg(a_j) = d+1$ and $[W_0, W_1]$ is a matching (recall that $|N(b_j) \cap W_0| = 1$ for every $b_j \in W_1$). We can now use the same graph $\hat{G} = (G - \{x_0\} - W) \cup \{b_1 u_1, \ldots, b_d u_d\}$ as used in Case (a.3.2), arriving again at a contradiction.

The only remaining case occurs when x_0 and x_1 have different degrees. Let us suppose $deg(x_0) = d$ and $deg(x_1) = d+1$. As $deg(x_1) = d+1 > |W_1|$, there exists, say $v_{d+1} \in N(x_1)$, such that $d(v_{d+1}, W_1) = (g-1)/2$. We proceed as before, with the sets $X_0 = N(x_0)$ and $X_1 = N(x_1) - v_{d+1}$, finding a graph G' with fewer vertices and the same girth and degrees as G, except for the vertex v_{d+1}. Recall that there must exist a vertex $y \in N(x_0)$ such that $deg(y) = d$. Then we construct the graph $G^* = G' \cup \{y v_{d+1}\}$, which is a new $\{d, d+1\}$-graph with girth g, arriving at a contradiction. This ends the proof of the theorem. □

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