Subdivisions in a bipartite graph

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Abstract

Given a bipartite graph G with m and n vertices, respectively, in its vertices classes, and given two integers s, t such that $2 \leq s \leq t$, $0 \leq m-s \leq n-t$, and $m+n \leq 2s+t-1$, we prove that if G has at least $mn - (2(m-s)+n-t)$ edges then it contains a subdivision of the complete bipartite $K_{(s,t)}$ with s vertices in the m-class and t vertices in the n-class. Furthermore, we characterize the corresponding extremal bipartite graphs with $mn - (2(m-s)+n-t+1)$ edges for this topological Turan type problem.

1 Introduction

Throughout this paper, only undirected simple graphs without loops or multiple edges are considered. Unless otherwise stated, we follow [5] for terminology and definitions.

Two well-known extensions of the Turán problem [19] are the Turán topological problem and the Zarankiewicz problem. The former one consists of estimating the extremal function $ex(n, TK_p)$ which denotes the maximum number of edges of a graph on n vertices free of a topological minor TK_p of a complete graph on p vertices (see Bollobás' excellent monograph [4] devoted to this subject and the contributions on this topic [1, 13, 11, 16, 15, 18]). The second was stated by Zarankiewicz [20] who studied the maximum size of a bipartite graph on (m, n) vertices, denoted by $z(m, n; s, t)$ that contains no bipartite complete $K_{(s,t)}$ subgraph with s vertices in the m-class and t vertices in the n-class. For a survey of this problem we also refer the reader to Section VI.2 of [4]. Most of the contributions are bounds for the function $z(m, n; s, t)$ when s, t are fixed and m, n are much larger than s, t (see, for example, [6, 7, 8]). Other contributions provide exact values of the extremal function [2, 9, 10].

Recent results on some problems involving the contention of a complete bipartite graph or a subdivision of a complete bipartite graph can be found in the literature [3, 12, 14, 17]. Böhme et al. [3] studied the size of a k-connected graph free of either an induced path of a given length or a subdivision of a complete bipartite graph. Kühn and Osthus [12] proved that for any graph H and for every integer s there exists a function $f = f(H, s)$ such that every graph of size at least f contains either a $K_{s,s}$ as a subgraph or an induced subdivision of H. Meyer [17] also relates the size of a graph with the property of containing a minor of $K_{s,t}$. Other problems involving the contention of maximum matching in graphs are considered in [14].

Combining the topological version of the Turán problem for complete graphs with the Zarankiewicz problem, we introduce the extremal function $tz(m, n; s, t)$ as a natural extension. The function $tz(m, n; s, t)$ is defined as *the maximum size of a (m, n)-bipartite graph free of a topological minor $TK_{(s,t)}$ of a complete bipartite $K_{(s,t)}$ with s vertices in the m-class and t vertices in the n-class.* The objective of this paper is to obtain exact values for this extremal function $tz(m, n; s, t)$ and to characterize the corresponding extremal bipartite graphs for infinitely many related values of m, n, s, t. Namely, we determine the exact value of $tz(m, n; s, t)$ and we characterize the family $TZ(m, n; s, t)$ of extremal graphs for any values of m, n, s, t satisfying $2 \le m - s \le n - t$ and $m + n \le 2s + t - 1$.

A subdivision of a graph H is a graph TH obtained from H by replacing the edges of H with internally disjoint paths. The *branch vertices* of TH are all those vertices that correspond to vertices of H. The complete bipartite graph $K_{(s,t)}$ is said to be *a topological minor* of a bipartite graph G if

$TK_{(s,t)} \subseteq G$.

Given two positive integers, m, n, a bipartite graph G with vertex classes X and Y of cardinalities $|X| = m$ and $|Y| = n$, is denoted by $G = (X, Y)$. The sets of vertices and edges of G are denoted by $V(G) = X \cup Y$ and $E(G)$, respectively, whereas $v(G)$ and $e(G)$ stand for the corresponding cardinalities.

For a bipartite graph $H = (X, Y)$, the degree of a vertex v in the graph H is denoted by $d_H(v)$ whereas $\Delta_X(H)$ (resp. $\Delta_Y(H)$) stand for the maximum degree among vertices in the first class (resp. second class). Thus, $\Delta(H) = \max\{\Delta_X(H), \Delta_Y(H)\}$ is the maximum degree of H. Let us consider two subsets of vertices $\{x_1, x_2, \ldots, x_p\} \subseteq X$ and $\{y_1, y_2, \ldots, y_p\} \subseteq Y$. Let us denote by $H_{0,0} = H$, $H_{1,0} = H - \{x_1\}$, $H_{1,1} = H_{1,0} - \{y_1\}$, and for all $i = 2, \ldots, p$, let us denoted by $H_{i,i-1} = H_{i-1,i-1} - \{x_i\}$ and $H_{i,i} = H_{i,i-1} - \{y_i\}$. Next we introduce the notion of *decreasing sequence of vertices* in a bipartite graph $H = (X, Y)$.

Definition 1 Given an integer $p \geq 1$ and a bipartite graph $H = (X, Y)$, a subset of vertices of H, $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$, with $\{x_1, \ldots, x_p\} \subseteq X$ and $\{y_1, \ldots, y_p\} \subseteq Y$, is called a decreasing sequence of H if the following assertions hold:

(i) $d_{H_{i-1,i-1}}(x_i) = \Delta_X(H_{i-1,i-1})$, for $i = 1, \ldots, p$.

(ii) $d_{H_{i,i-1}}(y_i) = \Delta_Y(H_{i,i-1})$, for $i = 1, \ldots, p$.

(iii) For each $i = 1, \ldots, p$, either $x_i y_i \notin E(H)$ or every vertex $y \in V(H_{i,i-1}) \cap Y$ with degree $d_{H_{i,i-1}}(y) = \Delta_Y(H_{i,i-1})$ is adjacent to vertex x_i in H.

Note that

$$d_{H_{0,0}}(x_1) \geq d_{H_{1,1}}(x_2) \geq \ldots \geq d_{H_{p-1,p-1}}(x_p) \geq \Delta_X(H_{p,p})$$

and

$$d_{H_{1,0}}(y_1) \geq d_{H_{2,1}}(y_2) \geq \ldots \geq d_{H_{p,p-1}}(y_p) \geq \Delta_Y(H_{p,p}),$$

and furthermore,

$$e(H) = \sum_{i=1}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}).$$

2 Exact values

Let G be a bipartite graph $G = (X, Y)$ on m and n vertices in X and Y respectively. We will henceforth use H to denote the bipartite complement of G, i.e., the bipartite graph $H = (X, Y) = K_{(m,n)} - E(G)$.

The problem of finding a $TK_{(s,t)}$ in a bipartite graph G can be formulated in terms of its bipartite complement H. Indeed, if $G = (X, Y)$ contains a $TK_{(s,t)}$ with set of branch vertices $S \cup T$, $S \subset X$, $T \subset Y$, then the edges of the graph $H[S \cup T]$ are missing in G and thus they must be replaced in G with internally disjoint paths passing through vertices of $X \setminus S$ and vertices of $Y \setminus T$. Since each of these paths must have odd length at least 3, it follows that $e\,(H[S \cup T]) \leq \min\{|X \setminus S|, |Y \setminus T|\}$. Hence, the following necessary but not sufficient condition on the induced subgraph $H[S \cup T]$ in order to determine whether $K_{(s,t)}$ is a topological minor of G is immediate.

Remark 2 Let $G = (X, Y)$ be with $|X| = m$ and $|Y| = n$ and let H be the bipartite complement of G. If G contains a $TK_{(s,t)}$, then there exist $S \subseteq X$ and $T \subseteq Y$ with $|S| = s$, $|T| = t$, such that the number of edges of the subgraph induced by $S \cup T$ in the bipartite complement of G satisfies

$$e\,(H[S \cup T]) \leq \min\{m - s, n - t\}.$$

By using Remark 2, the following proposition provides a lower bound on the maximum size of a (m, n)-bipartite graph free of a topological minor $TK_{(s,t)}$ of $K_{(s,t)}$.

Proposition 3 *Let m, n, s, t be integers such that $2 \leq s \leq t$, $0 \leq m - s \leq n - t$, and $m + n \leq 2s + t - 1$. Then the bipartite graph $G = K_{(m,n)} - M$, where M is any matching of cardinality $2(m - s) + n - t + 1$, does not contain $TK_{(s,t)}$ and therefore,*

$$tz(m, n; s, t) \geq mn - (2(m - s) + n - t + 1)\,.$$

Proof: First, let us see that $K_{(m,n)}$ has a matching of cardinality $2(m - s) + n - t + 1$. This is clear because from $2 \leq s \leq t$ and $0 \leq m - s \leq n - t$, it follows that $m \leq n$, and from the hypothesis $m + n \leq 2s + t - 1$ it follows that $2(m - s) + n - t + 1 = (m + n) + m - 2s - t + 1 \leq m \leq n$. Therefore, we may consider the bipartite graph $G = (X, Y) = K_{(m,n)} - M$ where M is a matching of cardinality $2(m-s)+n-t+1$ in $K_{(m,n)}$. Next let us see that $K_{(s,t)}$ is not a topological minor of G. For that, from Remark 2 it is enough to prove that $e\left(H[S\cup T]\right) > m-s$ for any subsets $S\subseteq X$ and $T\subseteq Y$ of cardinalities s and t, respectively, with $s\le t$. Observe that the number of isolated vertices in the class Y of H is exactly $n-(2(m-s)+n-t+1)$. It follows that the number of edges of $H[X\cup T]$ is

$$e(H[X\cup T]) \ge t-(n-(2(m-s)+n-t+1)) = 2m-2s+1.$$

But since $e(H[(X\setminus S)\cup T]) \le m-s$, then we have

$$\begin{aligned} e(H[S\cup T]) &= e(H[X\cup T]) - e(H[(X\setminus S)\cup T]) \\ &\ge 2m-2s+1-(m-s) \\ &= m-s+1 > m-s. \end{aligned}$$

Thus the result holds. $\square$

Lemma 4 *Let $p\ge 1$ be an integer and let $G=(X,Y)$ be a bipartite graph, with $|X|\ge p$ and $|Y|\ge p$, and denote by $H=(X,Y)$ the bipartite complement of G. Let $\{x_1,y_1,x_2,y_2,\dots,x_p,y_p\}$ be any decreasing sequence of H and denote by $r=e(H_{p,p})$. If $r\ge 1$ and $e(H)\le 3p$, then:*

(i) $r\le p$.

(ii) $\Delta(H_{p,p})=1$.

(iii) $\{x_{p-(r-1)}y_{p-(r-1)},\dots,x_py_p\}\cap E(H)=\emptyset$.

(iv) $\{ay_{p-(r-1)},\dots,ay_p\}\cap E(H)=\emptyset$, *for each* $a\in X\setminus\{x_1,\dots,x_p\}$ *of degree* $d_{H_{p,p}}(a)=1$.

(v) *If* $r\ge 2$, *then* $\{x_{p-(r-2)}b,\dots,x_pb\}\cap E(H)=\emptyset$, *for each* $b\in Y\setminus\{y_1,\dots,y_p\}$ *of degree* $d_{H_{p,p}}(b)=1$.

Proof: Since $e(H_{p,p})=r\ge 1$ we deduce $\Delta_X(H_{p,p})\ge 1$, $\Delta_Y(H_{p,p})\ge 1$, following that $d_{H_{i-1,i-1}}(x_i)\ge 1$ and $d_{H_{i,i-1}}(y_i)\ge 1$ for $i=1,\dots,p$, and therefore

$$e(H_{p,p}) = e(H) - \sum_{i=1}^{p}\left(d_{H_{i-1,i-1}}(x_i)+d_{H_{i,i-1}}(y_i)\right) \le 3p-2p = p,$$

thus item (i) is proved.

If $\Delta_X(H_{p,p}) \geq 2$, then $d_{H_{i-1,i-1}}(x_i) \geq 2$ for each $i = 1, \dots, p$, hence,

$$e(H) = \sum_{i=1}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}) \geq 3p + r > 3p,$$

which is a contradiction. Analogously, we arrive at a contradiction if $\Delta_Y(H_{p,p}) \geq 2$. Thus, $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies $\Delta(H_{p,p}) = 1$, hence item (ii) is shown.

(iii) Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \dots, e_r = a_rb_r$, $a_i \in X \setminus \{x_1, \dots, x_p\}$ and $b_i \in Y \setminus \{y_1, \dots, y_p\}$, for $i = 1, \dots, r$. By item (i) we know that $r \leq p$. We reason by way of contradiction supposing that there exists $j \in \{0, \dots, r-1\}$ such that $x_{p-j}y_{p-j} \in E(H)$. First we claim that $d_{H_{p-j,p-j-1}}(y_{p-j}) = 1$. Otherwise, if $d_{H_{p-j,p-j-1}}(y_{p-j}) \geq 2$ then $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \dots, p-j$ and therefore, by (ii) we have

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) \\ &+ \sum_{i=p-j+1}^{p} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r \\ &= 3p + (r-j) \\ &> 3p, \end{aligned}$$

the last inequality due to the fact that $j \leq r-1$. Since this is a contradiction with the hypothesis, then $\Delta_Y(H_{p-j,p-j-1}) = d_{H_{p-j,p-j-1}}(y_{p-j}) = 1$, yielding to $d_{H_{i,i-1}}(y_i) = 1$, for $i = p-j, \dots, p$ and $d_{H_{p,p}}(b_i) = 1$, for $i = 1, \dots, r$. As $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ is a decreasing sequence of H, it follows that x_{p-j} is adjacent in H to each one of the vertices of the set $\{y_{p-j}, \dots, y_p, b_1, \dots, b_r\}$ because of point (iii) of Definition 1. That is, $d_{H_{p-j-1,p-j-1}}(x_{p-j}) \geq j+1+r$, which means that $d_{H_{i-1,i-1}}(x_i) \geq j+1+r \geq$

2, for $i = 1, \dots, p-j$ and therefore,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-j+1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r = 3p + (r-j) > 3p, \end{aligned}$$

again a contradiction. Thus $x_{p-j}y_{p-j} \notin E(H)$ for all $j \in \{0, \dots, r-1\}$, hence item (iii) is valid.
(iv) Note that $r \geq 1$ implies $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \dots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \dots, r-1\}$ such that $ay_{p-j} \in E(H)$ for a vertex $a \in X \setminus \{x_1, \dots, x_p\}$ of degree $d_{H_{p,p}}(a) = 1$. Then $d_{H_{p-j-1,p-j-1}}(a) \geq 2$ and hence, $d_{H_{i-1,i-1}}(x_i) \geq 2$, for $i = 1, \dots, p-j$. Thus,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-(j-1)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r = 3p + (r-j) > 3p, \end{aligned}$$

because $j \leq r-1$, against the hypothesis.
(v) Since $r \geq 2$ then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \dots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \dots, r-2\}$ such that $x_{p-j}b \in E(H)$ for a vertex $b \in Y \setminus \{y_1, \dots, y_p\}$ of degree $d_{H_{p,p}}(b) = 1$. Then $d_{H_{p-j-1,p-j-2}}(b) \geq 2$ and hence, $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \dots, p-j-1$. Thus,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-j}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3(p-j-1) + 2(j+1) + r = 3p + (r-j-1) > 3p, \end{aligned}$$

because $j \leq r-2$, again a contradiction. $\square$

Lemma 5 *Let $p \geq 2$ be an integer. Let $G = (X, Y)$ be a bipartite graph with $|X| \geq p$ and $|Y| \geq p$, and denote by $H = (X, Y)$ the bipartite complement of G. Suppose that there exists a decreasing sequence of vertices $U = \{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ of H such that $E(H_{p,p}) = \{ab\}$ with $a \in X$ and $b \in Y$. If $e(H) \leq 3p$ then there exists an (a, b)-path in G with its internal vertices belonging to U.*

Proof: Since $E(H_{p,p}) = \{ab\}$, then $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies that $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. If G contains the path a, y_p, x_p, b, then we are done. So assume that some of the edges ay_p, x_py_p, x_pb is an edge of H. We know by Lemma 4 (iii) that $x_py_p \notin E(H)$. If $ay_p \in E(H)$, then $d_{H_{p-1,p-1}}(a) \geq 2$, because $\{ay_p, ab\} \subset E(H_{p-1,p-1})$. Then $d_{H_{i-1,i-1}}(x_i) \geq 2$ and we get

$$e(H) = \sum_{i=1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \geq 3p + 1,$$

which is a contradiction. Therefore we can suppose that $x_pb \in E(H)$ and $ay_p \notin E(H)$. Then $\{x_pb, ab\} \subset E(H_{p-1,p-2})$, following that $d_{H_{p-1,p-2}}(b) \geq 2$, which implies that $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \ldots, p-1$. Since $d_{H_{p,p-1}}(y_p) \geq 1$ and $d_{H_{i-1,i-1}}(x_i) \geq 1$ for $i = 1, \ldots, p$, it follows that

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \big(d_{H_{p-1,p-1}}(x_p) + d_{H_{p,p-1}}(y_p)\big) + e(H_{p,p}) \\ &\geq 3(p-1) + 2 + 1 = 3p. \end{aligned}$$

This means that all the above inequalities become equalities, that is,

$$\begin{cases} d_{H_{i,i-1}}(y_i) = 2, \text{ for } i = 1, \ldots, p-1, \text{ and } d_{H_{p,p-1}}(y_p) = 1; \\ d_{H_{i-1,i-1}}(x_i) = 1, \text{ for } i = 1, \ldots, p. \end{cases} \tag{1}$$

Therefore we obtain that:

- $x_py_{p-1} \notin E(H)$, because otherwise, $\{x_pb, x_py_{p-1}\} \subset E(H_{p-2,p-2})$ and thus, $d_{H_{p-2,p-2}}(x_{p-1}) = \Delta_X(H_{p-1,p-1}) \geq 2$, contradicting (1).
- $x_{p-1}b \notin E(H)$, for if not, $\{x_{p-1}b, x_pb, ab\} \subset E(H_{p-2,p-3})$ and hence, $d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq 3$, against (1).

- $x_{p-1}y_{p-1} \notin E(H)$, because otherwise, $d_{H_{p-2,p-3}}(y_{p-1}) \geq 3$ and therefore, $d_{H_{p-2,p-3}}(y_{p-2}) \geq 3$, contradicting (1).

Thus, it follows that $\{ay_p, x_py_p, x_py_{p-1}, x_{p-1}y_{p-1}, x_{p-1}b\} \cap E(H) = \emptyset$. Consequently, there exists in G the path $a, y_p, x_p, y_{p-1}, x_{p-1}, b$, hence the result holds. $\square$

Lemma 6 *Let m, n, p be integers such that $p \geq 2$, $m > p$ and $n > p$. Let $G = (X, Y)$ be a bipartite graph with $|X| = m$ and $|Y| = n$, and denote by $H = (X, Y)$ the bipartite complement of G. If $e(H) \leq 3p$, then $K_{(m-p,n-p)}$ is a topological minor of G.*

Proof: Let $U = \{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ be a decreasing sequence of H. The graph $H_{p,p}$ is a bipartite graph with vertex classes $X^* = X \setminus \{x_1, \ldots, x_p\}$ and $Y^* = Y \setminus \{y_1, \ldots, y_p\}$, so $|X^*| = m - p$ and $|Y^*| = n - p$. If $e(H_{p,p}) = 0$ then the bipartite complement of $H_{p,p}$ is $K_{(m-p,n-p)}$ and the result follows. We may henceforth assume that $e(H_{p,p}) > 0$, or in other words $\Delta_X(H_{p,p}) \geq 1$ and $\Delta_Y(H_{p,p}) \geq 1$, thus $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Then by Lemma 4 we have $e(H_{p,p}) = r \leq p$ and $\Delta(H_{p,p}) = 1$. Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X^*$ and $b_i \in Y^*$, for $i = 1, \ldots, r$. In order to prove that G contains $TK_{(m-p,n-p)}$ with set of branch vertices $X^* \cup Y^*$, we will show the existence of vertex disjoint (a_i, b_i)-paths in G, $i = 1, \ldots, r$, with internal vertices from U. As $e(H) \leq 3p$, if $r = 1$ then the bipartite complement of $H_{p,p}$ is $K_{m-p,n-p} - e_1$. Thus, by Lemma 5, the bipartite graph G contains $TK_{m-p,n-p}$ and we are done. Hence assume that $2 \leq r \leq p$, then by Lemma 4 (iii), (iv), (v), for each $i = 1, \ldots, r$ and $j = 0, \ldots, r-2$, there exists in G the path $a_i, y_{p-j}, x_{p-j}, b_i$. Thus, we only must show that there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Otherwise, since $x_{p-(r-1)}y_{p-(r-1)} \in E(G)$ and $a_iy_{p-(r-1)} \in E(G)$ for all $i = 1, \ldots, r$, because of Lemma 4, we deduce that $x_{p-(r-1)}b_i \in E(H)$ for all $i = 1, \ldots, r$, that is, $d_{H_{p-r,p-r}}(x_{p-(r-1)}) \geq r$ and therefore, $d_{H_{i-1,i-1}}(x_i) \geq r$

for $i = 1, \ldots, p-(r-1)$. Then since $2 \le r \le p$ it follows that

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-(r-1)} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-(r-2)}^{p} \big(d_{H_{i-1,i-1}}(i_k) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\ge (r+1)(p-(r-1)) + 2(r-1) + r \\ &= 3p + 1 + (r-2)(p-r+1) \\ &> 3p \end{aligned}$$

which is a contradiction. Hence there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Without loss of generality we may assume that $i = r$. Then there exist in G the vertex-disjoint paths $a_j, y_{p-(j-1)}, x_{p-(j-1)}, b_j$ for $j = 1, \ldots, r$. Thus, G contains $TK_{(m-p,n-p)}$ and this finishes the proof. $\square$

The following lemma gives a sufficient condition on the size of a bipartite graph in order to contain a complete bipartite graph as a topological minor.

Lemma 7 *Let m, n, s, t be integers such that $2 \le m-s \le n-t$. Let $G = (X, Y)$ be a bipartite graph with $|X| = m$, $|Y| = n$. If the bipartite complement H of G has size $e(H) \le 2(m-s) + n - t$, then $K_{(s,t)}$ is a topological minor of G.*

Proof: Set $p = m - s$ and $q = n - t$, then $2 \le p \le q$ and $e(H) \le 2p + q$. First, suppose that $p = q$. Thus the bipartite graph H has size at most $3p$, and by Lemma 6, we obtain that $K_{(m-p,n-p)} = K_{(s,t)}$ is a topological minor of G. Hence, assume that $p < q$. Without loss of generality, we may assume that the vertices of the partite set Y are ordered in such a way that $d_H(y_1) \ge d_H(y_2) \ge \cdots \ge d_H(y_n)$. Set $Y' = \{y_1, \ldots, y_{q-p}\} \subseteq Y$ and let us consider the bipartite graph $H' = (X, Y \setminus Y')$. Observe that $|X| = m$ and $|Y \setminus Y'| = n-(q-p) = t+p$. If $e(H') = 0$ then the bipartite complement G' of H' is the complete bipartite graph $K_{(m,t+p)}$. Since G' is a subgraph of G and $K_{(s,t)} \subseteq K_{(m,t+p)}$, then G contains a $K_{(s,t)}$ and we are done. So, we may assume that $e(H') > 0$, which implies that $d_H(y_i) \ge 1$ for $i = 1, \ldots, q-p$. Hence, $e(H') = e(H) - \sum_{i=1}^{q-p} d_H(y_i) \le 2p + q - (q-p) \le 3p$, and therefore, from Lemma 6, it follows that $K_{(m-p,t+p-p)} = K_{(s,t)}$ is a topological minor of G. $\square$

Combining Proposition 3 and Lemma 7 the following theorem is immediate.

Theorem 8 *Let m, n, s, t be integers such that $2 \le s \le t$, $2 \le m-s \le n-t$, and $m + n \le 2s + t - 1$. Then*

$$tz(m, n; s, t) = mn - (2(m - s) + n - t + 1) .$$

3 Family of extremal graphs

When an extremal problem is studied, it is not only important to know the exact value of the extremal function, but also characterize the family of extremal graphs. In this section we characterize the extremal family $TZ(m, n; s, t)$ for integers m, n, s, t such that $2 \le s \le t$, $2 \le m - s \le n - t$, and $m + n \le 2s + t - 1$.

Lemma 9 *Let $p \ge 2$ be an integer and let $G = (X, Y)$ be a bipartite graph with $|X| \ge p$ and $|Y| \ge p$, and denote by $H = (X, Y)$ the bipartite complement of G. Let $\{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ be any decreasing sequence of H and denote by $r = e(H_{p,p})$. If $e(H) \le 3p + 1$ and $\Delta_X(H) \ge 2$ then*

(i) *$r \le p$.*

(ii) *$\Delta(H_{p,p}) \le 1$.*

(iii) *If $r = 1$ then $\{x_{p-(r-1)}y_{p-(r-1)}, \ldots, x_p y_p\} \cap E(H) = \emptyset$.*

(iv) *If $r \ge 2$ then $\{ay_{p-(r-2)}, \ldots, ay_p\} \cap E(H) = \emptyset$, for each $a \in X \setminus \{x_1, \ldots, x_p\}$ of degree $d_{H_{p,p}}(a) = 1$, if any.*

(v) *If $r \ge 2$ then $\{x_{p-(r-2)}b, \ldots, x_p b\} \cap E(H) = \emptyset$, for each $b \in Y \setminus \{y_1, \ldots, y_p\}$ of degree $d_{H_{p,p}}(b) = 1$, if any.*

Proof: If $e(H_{p,p}) = r = 0$, then both items (i) and (ii) hold. Hence we may assume that $0 < r = e(H_{p,p}) \le 3p + 1$, which implies $\Delta_X(H_{p,p}) \ge 1$, $\Delta_Y(H_{p,p}) \ge 1$, following that $d_{H_{i-1,i-1}}(x_i) \ge 1$ for $i = 2, \ldots, p$ and

$d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Moreover, $d_{H_{0,0}}(x_1) \geq 2$, because $\Delta_X(H) \geq 2$. Therefore

$$\begin{aligned} e(H_{p,p}) &= e(H) - \big(d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)\big) \\ &- \sum_{i=2}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &\leq 3p + 1 - 3 - 2(p-1) = p, \end{aligned}$$

thus item (i) is proved.

If $\Delta_X(H_{p,p}) \geq 2$, then $e(H_{p,p}) \geq 2$ and $d_{H_{i-1,i-1}}(x_i) \geq 2$ for each $i = 1, \ldots, p$, hence,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3p + e(H_{p,p}) \geq 3p + 2 > 3p + 1, \end{aligned}$$

which is a contradiction. Analogously, we arrive at a contradiction if $\Delta_Y(H_{p,p}) \geq 2$. Thus, $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies $\Delta(H_{p,p}) = 1$, hence item (ii) is shown.
(iii) From item (i) it follows that $r \leq p$. Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X \setminus \{x_1, \ldots, x_p\}$ and $b_i \in Y \setminus \{y_1, \ldots, y_p\}$, for $i = 1, \ldots, r$. Since $e(H_{p,p}) = r \geq 1$, then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-1\}$ such that $x_{p-j}y_{p-j} \in E(H)$. Then $d_{H_{i,i-1}}(y_i) \geq 2$ for $i = 1, \ldots, p-j-1$, because $d_{H_{p-j,p-j-1}}(y_{p-j}) \geq 1$ and $x_{p-j}y_{p-j} \in E(H)$. We have two cases:

Case 1. Assume that $d_{H_{p-j,p-j-1}}(y_{p-j}) \geq 2$, then $d_{H_{i,i-1}}(y_i) \geq 2$ for $i = 1, \ldots, p-j$. Since $d_{H_{0,0}}(x_1) = \Delta_X(H) \geq 2$ and $j \leq r-1$ it follows that

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-(j-1)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 4 + 3(p-j-1) + 2j + r \\ &= 3p + 1 + (r-j) \\ &> 3p + 1, \end{aligned}$$

which is a contradiction.

Case 2. Assume that $d_{H_{p-j,p-j-1}}(y_{p-j}) = 1$, then $d_{H_{i,i-1}}(y_i) = 1$, for $i = p-j,\dots,p$. Moreover, $d_{H_{p,p}}(b_i) = 1$, for $i = 1,\dots,r$, because $\Delta(H_{p,p}) = 1$. As $\{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ is a decreasing sequence of H and $x_{p-j}y_{p-j} \in E(H)$, it follows that x_{p-j} is adjacent in H to each one of the vertices of the set $\{y_{p-j},\dots,y_p, b_1,\dots,b_r\}$ because of point (iii) of Definition 1. That is, $d_{H_{p-j-1,p-j-1}}(x_{p-j}) \ge j+1+r$, which means that $d_{H_{i-1,i-1}}(x_i) \ge j+1+r$ for $i = 1,\dots,p-j$. If $j = 0$ then $d_{H_{i-1,i-1}}(x_i) \ge 1+r$ for $i = 1,\dots,p$, and therefore

$$\begin{aligned}
e(H) &= \sum_{i=1}^{p-1}\big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big)\\
&+ (d_{H_{p-1,p-1}}(x_p) + d_{H_{p,p-1}}(y_p)) + e(H_{p,p})\\
&\ge (3+r)(p-1) + (r+2) + r\\
&= 3p+1+r(p+1)-2 > 3p+1,
\end{aligned}$$

because $r \ge 1$ and $p \ge 2$, which is a contradiction. If $j = r-1$ then $d_{H_{i-1,i-1}}(x_i) \ge j+1+r = 2r$ for $i = 1,\dots,p-(r-1)$, and therefore

$$\begin{aligned}
e(H) &= \sum_{i=1}^{p-r}\big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big)\\
&+ (d_{H_{p-r,p-r}}(x_{p-(r-1)}) + d_{H_{p-(r-1),p-r}}(y_{p-(r-1)}))\\
&+ \sum_{i=p-(r-2)}^{p}\big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p})\\
&\ge (2r+2)(p-r) + (2r+1) + 2(r-1) + r\\
&= 3p+1+(2rp-2r^2-p+3r-2)\\
&\ge 3p+1+(p-1)\\
&> 3p+1,
\end{aligned}$$

because $p \geq 2$, which also contradicts the hypothesis. Finally, if $1 \leq j \leq r-2$ then $d_{H_{i-1,i-1}}(x_i) \geq j+1+r \geq 3$ for $i = 1, \ldots, p-j$, and therefore

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ (d_{H_{p-j-1,p-j-1}}(x_{p-j}) + d_{H_{p-j,p-j-1}}(y_{p-j})) \\ &+ \sum_{i=p-(j-1)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 5(p-j-1) + 4 + 2j + r \\ &= 3p + 1 + (2p - 3j - 2 + r) \\ &\geq 3p + 1 + (3r - 3j - 2) > 3p + 1, \end{aligned}$$

because $p \geq r$ and $j \leq r-2$, again a contradiction.

Thus $x_{p-j}y_{p-j} \notin E(H)$ for all $j \in \{0, \ldots, r-1\}$, hence item (iii) is valid.

(iv) Assume $e(H_{p,p}) = r \geq 2$. Then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-2\}$ such that $ay_{p-j} \in E(H)$ for a vertex $a \in X \setminus \{x_1, \ldots, x_p\}$ of degree $d_{H_{p,p}}(a) = 1$. Then $d_{H_{p-j-1,p-j-1}}(a) \geq 2$ and hence, $d_{H_{i-1,i-1}}(x_i) \geq 2$, for $i = 1, \ldots, p-j$. Thus,

$$\begin{aligned} e(H) &= \sum_{i=1}^{p-j} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-(j-1)}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 3(p-j) + 2j + r = 3p + (r-j) > 3p+1, \end{aligned}$$

because $j \leq r-2$, against the hypothesis.

(v) Assume $e(H_{p,p}) = r \geq 2$. Then $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Moreover, $d_{H_{0,0}}(x_1) \geq 2$, due to the fact that $\Delta_X(H) \geq 2$. We reason by way of contradiction supposing that there exists $j \in \{0, \ldots, r-2\}$ such that $x_{p-j}b \in E(H)$ for a vertex $b \in Y \setminus \{y_1, \ldots, y_p\}$ of degree $d_{H_{p,p}}(b) = 1$. Then $d_{H_{p-j-1,p-j-2}}(b) \geq 2$ and hence, $d_{H_{i,i-1}}(y_i) \geq 2$, for

$i = 1, \dots, p-j-1$. Thus,

$$\begin{aligned} e(H) &= (d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)) + \sum_{i=2}^{p-j-1} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) \\ &+ \sum_{i=p-j}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \\ &\geq 4 + 3(p-j-2) + 2(j+1) + r = 3p + (r-j) > 3p+1, \end{aligned}$$

because $j \leq r-2$, again a contradiction. $\square$

Lemma 10 *Let $p \geq 4$ be an integer. Let $G = (X, Y)$ be a bipartite graph with $|X| \geq p$ and $|Y| \geq p$, and denote by $H = (X, Y)$ the bipartite complement of G. Suppose that $\Delta_X(H) \geq 2$ and there exists a decreasing sequence of vertices $U = \{x_1, y_1, x_2, y_2, \dots, x_p, y_p\}$ of H such that $E(H_{p,p}) = \{ab\}$ with $a \in X$ and $b \in Y$. If $e(H) \leq 3p+1$ then there exists an (a, b)-path in G with its internal vertices belonging to U.*

Proof: Assume that $e(H) \leq 3p+1$. Note that $d_{H_{0,0}}(x_1) = \Delta_X(H) \geq 2$. Since $E(H_{p,p}) = \{ab\}$, then $\Delta_X(H_{p,p}) = \Delta_Y(H_{p,p}) = 1$, which implies that $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \dots, p$. If G contains the path a, y_p, x_p, b, then we are done. So assume that some of the edges ay_p, x_py_p, x_pb is an edge of H. We know by Lemma 9 that $x_py_p \notin E(H)$. So, let us distinguish two cases.

Case 1. Suppose that $ay_p \in E(H)$. Then $d_{H_{p-1,p-1}}(a) \geq 2$, because $ab \in E(H)$. Then $d_{H_{i-1,i-1}}(x_i) \geq 2$ and we get

$$e(H) = \sum_{i=1}^{p} \big(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\big) + e(H_{p,p}) \geq 3p+1 \geq e(H).$$

Thus, all the inequalities become equalities, that is,

$$d_{H_{i-1,i-1}}(x_i) = 2 \text{ and } d_{H_{i,i-1}}(y_i) = 1, \text{ for } i = 1, \dots, p. \tag{2}$$

Hence, we obtain that:

- $x_{p-1}y_{p-1} \notin E(H)$. Otherwise, since
$$\Delta_Y(H_{p-1,p-2}) = d_{H_{p-1,p-2}}(y_{p-1}) = 1$$
and both y_p and b have also degree 1 in $H_{p-1,p-2}$, applying point (iii) of Definition 1, it follows that $\{x_{p-1}y_{p-1}, x_{p-1}y_p, x_{p-1}b\} \subset E(H)$ and therefore, $d_{H_{p-2,p-2}}(x_{p-1}) \geq 3$, which contradicts (2).

- $ay_{p-1} \notin E(H)$, because otherwise,

$$d_{H_{p-2,p-2}}(x_{p-1}) = \Delta_X(H_{p-2,p-2}) \geq d_{H_{p-2,p-2}}(a) \geq 3,$$

  contradicting (2).

- $x_{p-1}b \notin E(H)$, for if not,

$$d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq d_{H_{p-2,p-3}}(b) \geq 2,$$

  against (2).

As a consequence, we get that the path a, y_{p-1}, x_{p-1}, b of G connects the vertices a and b.

Case 2. Suppose that $x_p b \in E(H)$ and $ay_p \notin E(H)$. Thus, $d_{H_{p-1,p-2}}(y_p) \geq 2$, which implies that $d_{H_{i,i-1}}(y_i) \geq 2$, for $i = 1, \dots, p-1$. Since $d_{H_{p,p-1}}(y_p) \geq 1$, $d_{H_{0,0}}(x_1) \geq 2$ and $d_{H_{i-1,i-1}}(x_i) \geq 1$ for $i = 2, \dots, p$, it follows that

$$\begin{aligned} e(H) &= (d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)) + \sum_{i=2}^{p-1} \left(d_{H_{i-1,i-1}}(x_i) + d_{H_{i,i-1}}(y_i)\right) \\ &+ (d_{H_{p-1,p-1}}(x_p) + d_{H_{p,p-1}}(y_p)) + e(H_{p,p}) \\ &\geq 4 + 3(p-2) + 2 + 1 = 3p + 1 = e(H), \end{aligned}$$

which means that all the above inequalities become equalities, that is,

$$\begin{cases} d_{H_{0,0}}(x_1) = 2 \text{ and } d_{H_{i-1,i-1}}(x_i) = 1 \text{ for } i = 2, \dots, p; \\ d_{H_{i,i-1}}(y_i) = 2 \text{ for } i = 1, \dots, p-1, \text{ and } d_{H_{p,p-1}}(y_p) = 1. \end{cases} \tag{3}$$

Therefore, we have:

- $x_{p-1}b \notin E(H)$, because on the contrary,

$$d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq d_{H_{p-2,p-3}}(b) \geq 3$$

  against (3).

- $x_p y_{p-1} \notin E(H)$, for if not,

$$d_{H_{p-3,p-3}}(x_{p-2}) = \Delta_X(H_{p-3,p-3}) \geq d_{p-3,p-3}(x_p) \geq 2$$

  and this contradicts (3), since $p \geq 4$.

- $x_{p-1}y_{p-1} \notin E(H)$, because otherwise, taking into account that $d_{H_{p-1,p-2}}(y_{p-1}) = 2$, we have
$$d_{H_{p-2,p-3}}(y_{p-2}) = \Delta_Y(H_{p-2,p-3}) \geq d_{H_{p-2,p-3}}(y_{p-1}) \geq 3,$$
contradicting (3).

Thus, in this case, it follows that $\{ay_p, x_py_p, x_py_{p-1}, x_{p-1}y_{p-1}, x_{p-1}b\} \cap E(H) = \emptyset$. Consequently, there exists in G the path $a, y_p, x_p, y_{p-1}, x_{p-1}, b$, and the result also holds in this case. □

Lemma 11 *Let m, n, p be integers such that $p \geq 4$, $m > p$ and $n > p$. Let $G = (X, Y)$ be a bipartite graph with $|X| = m$ and $|Y| = n$, and denote by $H = (X, Y)$ the bipartite complement of G. If $\Delta(H) \geq 2$ and $e(H) \leq 3p+1$, then $K_{(m-p,n-p)}$ is a topological minor of G.*

Proof: Without loss of generality we may assume that $\Delta(H) = \Delta_X(H)$ (otherwise it is enough to interchange the classes X with Y). Let $U = \{x_1, y_1, x_2, y_2, \ldots, x_p, y_p\}$ be a decreasing sequence of H. The graph $H_{p,p}$ is a bipartite graph with vertex classes $X^* = X \setminus \{x_1, \ldots, x_p\}$ and $Y^* = Y \setminus \{y_1, \ldots, y_p\}$, so $|X^*| = m - p$ and $|Y^*| = n - p$. If $e(H_{p,p}) = 0$ then the bipartite complement of $H_{p,p}$ is $K_{(m-p,n-p)}$ and the result follows. So, we may henceforth assume that $e(H_{p,p}) \geq 1$ or in other words, $\Delta_X(H_{p,p}) \geq 1$ and $\Delta_Y(H_{p,p}) \geq 1$, thus $d_{H_{i-1,i-1}}(x_i) \geq 1$ and $d_{H_{i,i-1}}(y_i) \geq 1$ for $i = 1, \ldots, p$. Then by Lemma 9 we have $e(H_{p,p}) = r \leq p$ and $\Delta(H_{p,p}) = 1$. Let us denote the edges of $H_{p,p}$ by $e_1 = a_1b_1, \ldots, e_r = a_rb_r$, $a_i \in X^*$ and $b_i \in Y^*$, for $i = 1, \ldots, r$. In order to prove that G contains a $TK_{(m-p,n-p)}$ with set of branch vertices $X^* \cup Y^*$, we will show the existence of vertex disjoint (a_i, b_i)-paths in G, $i = 1, \ldots, r$, with internal vertices in U. As $e(H) \leq 3p + 1$, we are done if $r = 1$ by applying Lemma 10, hence assume that $2 \leq r \leq p$.

First, suppose that $2 \leq r \leq p - 1$. Then, by Lemma 9 (iii), (iv), (v), for each $i = 1, \ldots, r$ and $j = 0, \ldots, r - 2$, there exists in G the path $a_i, y_{p-j}, x_{p-j}, b_i$. Thus, we only must show that there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. We reason by way of contradiction supposing that for all $i = 1, \ldots, r$ the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ does not exist in G. From Lemma 9 it follows that $x_{p-(r-1)}, y_{p-(r-1)} \in E(G)$, thus $a_iy_{p-(r-1)} \in E(H)$ or $x_{p-(r-1)}b_i \in E(H)$ for each $i = 1, \ldots, r$. We will distinguish three possible cases:

Case 1. Assume that $x_{p-(r-1)}b_i \in E(H)$ for all $i = 1,\dots,r$, then $d_{H_{p-r,p-r}}(x_{p-(r-1)}) \geq r$ and thus, $d_{H_{j-1,j-1}}(x_j) \geq r$ for $j = 1,\dots,p-(r-1)$. Moreover, $d_{H_{p-r,p-(r+1)}}(y_{p-r}) = \Delta_Y(H_{p-r,p-(r+1)}) \geq d_{H_{p-r,p-(r+1)}}(b_i) \geq 2$, which means that $d_{H_{j,j-1}}(y_j) \geq 2$ for $j = 1,\dots,p-r$. Thus,

$$\begin{aligned}
e(H) &= \sum_{j=1}^{p-r}\big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big)\\
&+ \Big(d_{H_{p-r,p-r}}(x_{p-(r-1)}) + d_{H_{p-(r-1),p-r}}(y_{p-(r-1)})\Big)\\
&+ \sum_{j=p-(r-2)}^{p}\big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p})\\
&\geq (r+2)(p-r) + (r+1) + 2(r-1) + r\\
&= 3p + 1 + (r-2)(p-r) + p - 2\\
&> 3p+1,
\end{aligned}$$

since $2 \leq r < p$ and $p > 2$, which is a contradiction.

Case 2. Assume that $a_i y_{p-(r-1)} \in E(H)$ for all $i = 1,\dots,r$, then, reasoning as in Case 1, we have $d_{H_{j,j-1}}(y_j) \geq r$ for $j = 1,\dots,p-(r-1)$, and $d_{H_{j-1,j-1}}(x_j) \geq 2$ for $j = 1,\dots,p-(r-1)$. Thus,

$$\begin{aligned}
e(H) &= \sum_{j=1}^{p-(r-1)}\big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big)\\
&+ \sum_{j=p-(r-2)}^{p}\big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p})\\
&\geq (r+2)(p-(r-1)) + 2(r-1) + r\\
&= 3p + 1 + (r-2)(p-r) + p - 1\\
&> 3p+1,
\end{aligned}$$

since $2 \leq r < p$ and $p > 1$, which is a contradiction.

Case 3. Assume that there exist $i_0, j_0 \in \{1,\dots,r\}$ such that $x_{p-(r-1)}b_{i_0} \notin E(H)$ and $a_{j_0}y_{p-(r-1)} \notin E(H)$. Clearly $i_0 \neq j_0$, because $x_{p-(r-1)}y_{p-(r-1)} \notin E(H)$ (by Lemma 9) and by hypothesis, the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ does not exist in G for all $i = 1,\dots,r$. Since

$x_{p-(r-1)}y_{p-(r-1)} \notin E(H)$, it follows that $x_{p-(r-1)}b_{j_0} \in E(H)$, for if not, we find in G the path $a_{j_0}, y_{p-(r-1)}, x_{p-(r-1)}, b_{j_0}$ against our assumption. Analogously, $a_{i_0}x_{p-(r-1)} \in E(H)$. Observe that $\{a_{i_0}x_{p-(r-1)}, a_{i_0}b_{i_0}\} \subset E(H_{p-r,p-r})$ and therefore,

$$d_{H_{p-r,p-r}}(x_{p-(r-1)}) = \Delta_X(H_{p-r,p-r}) \geq d_{H_{p-r,p-r}}(a_{i_0}) \geq 2$$

, which implies that $d_{H_{i-1,i-1}}(x_i) \geq 2$ for $i = 1, \ldots, p-(r-1)$. Moreover, observe also that $\{y_{p-(r-1)}b_{j_0}, a_{j_0}b_{j_0}\} \subset E(H_{p-r,p-(r+1)})$ and therefore, $d_{H_{p-r,p-(r+1)}}(y_{p-r}) = \Delta_X(H_{p-r,p-(r+1)}) \geq d_{H_{p-r,p-(r+1)}}(b_{j_0}) \geq 2$, which means that $d_{H_{i,i-1}}(y_i) \geq 2$ for $i = 1, \ldots, p-r$. Hence,

$$\begin{aligned}
e(H) &= \sum_{j=1}^{p-r} \left(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\right) \\
&+ \Big(d_{H_{p-r,p-r}}(x_{p-(r-1)}) + d_{H_{p-(r-1),p-r}}(y_{p-(r-1)})\Big) \\
&+ \sum_{j=p-(r-2)}^{p} \left(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\right) + e(H_{p,p}) \\
&\geq 4(p-r) + 3 + 2(r-1) + r \\
&= 4p + 1 - r \\
&= 3p + 1 + (p-r) \\
&> 3p + 1,
\end{aligned}$$

since $r \leq p-1$. Then, if $2 \leq r \leq p-1$, in all the possible cases, we arrive at a contradiction with the assumption that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ does not exist in G for all $i = 1, \ldots, r$. Thus, if $2 \leq r \leq p-1$ there exists $i \in \{1, \ldots, r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Without loss of generality we may assume that $i = r$. Then there exist in G the vertex-disjoint paths $a_j, y_{p-(j-1)}, x_{p-(j-1)}, b_j$ for $j = 1, \ldots, r$.

Second, assume that $r = p$. Then, from Lemma 9 it follows that

$$\begin{cases}
\{x_1y_1, \ldots, x_py_p\} \cap E(H) = \emptyset; \\
\{a_iy_2, \ldots, a_iy_p\} \cap E(H) = \emptyset \text{ for } i = 1, \ldots, p; \\
\{x_2b_i, \ldots, x_pb_i\} \cap E(H) = \emptyset \text{ for } i = 1, \ldots, p.
\end{cases} \tag{4}$$

This means that for each $i = 1, \ldots, p$ and $j = 0, \ldots, p-2$, there exists in G the path $a_i, y_{p-j}, x_{p-j}, b_i$. Thus, we only must show that there exists $i \in$

$\{1,\ldots,p\}$ such that the path a_i, y_1, x_1, b_i is contained in G. We reason by way of contradiction supposing that for all $i = 1,\ldots,p$ the path a_i, y_1, x_1, b_i does not exist in G. Since $x_1y_1 \in E(G)$ we deduce that for each $i = 1,\ldots,p$, $a_iy_1 \in E(H)$ or $x_1b_i \in E(H)$. If $\{a_iy_1, a_{i^*}y_1\} \subset E(H)$ for two indices $i, i^* \in \{1,\ldots,p\}$, with $i \neq i^*$, then $d_{H_{1,0}}(y_1) \geq 2$. Since $d_{H_{0,0}}(x_1) = \Delta_X(H) \geq 2$ we have

$$\begin{aligned}
e(H) &= \big(d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)\big)\\
&+ \sum_{j=2}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p})\\
&\geq 4 + 2(p-1) + p\\
&= 3p + 2\\
&> 3p + 1,
\end{aligned}$$

a contradiction. Thus, in the set $\{a_1,\ldots,a_p\}$ there is at most one vertex adjacent to y_1 in H, which means that x_1 must be adjacent in H to at least $p-1$ vertices of the set $\{b_1,\ldots,b_p\}$, due to the fact that for each $i = 1,\ldots,p$, $a_iy_1 \in E(H)$ or $x_1b_i \in E(H)$. Then $d_{H_{0,0}}(x_1) \geq p-1$ and therefore,

$$\begin{aligned}
e(H) &= \big(d_{H_{0,0}}(x_1) + d_{H_{1,0}}(y_1)\big)\\
&+ \sum_{j=2}^{p} \big(d_{H_{j-1,j-1}}(x_j) + d_{H_{j,j-1}}(y_j)\big) + e(H_{p,p})\\
&\geq p + 2(p-1) + p\\
&= 3p + 1 + (p-3)\\
&> 3p + 1,
\end{aligned}$$

since $p \geq 4$, again a contradiction with the hypothesis. Hence, there exists $i \in \{1,\ldots,r\}$ such that the path $a_i, y_{p-(r-1)}, x_{p-(r-1)}, b_i$ is contained in G. Without loss of generality we may assume that $i = r$. Then there exist in G the vertex-disjoint paths $a_j, y_{p-(j-1)}, x_{p-(j-1)}, b_j$ for $j = 1,\ldots,r$, and the result holds. □

Theorem 12 *Let m, n, s, t be integers such that $2 \leq s \leq t$, $4 \leq m - s \leq n - t$, and $m + n \leq 2s + t - 1$. Then $G = (X, Y) \in TZ(m, n; s, t)$ iff $G = K_{(m,n)} - M$ where M is any matching of cardinality $2(m-s)+n-t+1$.*

Proof: By Proposition 3 and Theorem 8, if $G = K_{(m,n)} - M$ where M is any matching of cardinality $2(m - s) + n - t + 1$, then $G \in TZ(m, n; s, t)$. Thus, we only must show that there are no more extremal bipartite graphs. For that, it is enough to prove that the bipartite complement $H = (X, Y)$ of every extremal bipartite graph $G = (X, Y) \in TZ(m, n; s, t)$ has maximum degree $\Delta(H) = 1$.

Let $G = (X, Y) \in TZ(m, n; s, t)$ satisfy the hypothesis of the theorem and let us denote by $H = (X, Y)$ the bipartite complement of G. Set $p = m - s$ and $q = n - t$, then $4 \leq p \leq q$ and $e(H) = 2p + q + 1$. If $p = q$ then $\Delta(H) = 1$, follows from Lemma 11. Thus, assume that $p < q$. Without loss of generality, we may assume that the vertices of the partite set Y are ordered in such a way that $d_H(y_1) \geq d_H(y_2) \geq \cdots \geq d_H(y_n)$. Set $Y' = \{y_1, \ldots, y_{q-p}\} \subseteq Y$ and let us consider the bipartite graph $H' = (X, Y \setminus Y')$. Observe that $|X| = m$ and $|Y \setminus Y'| = n - (q - p) = t + p$. If $e(H') = 0$ then the bipartite complement G' of H' is the complete bipartite graph $K_{(m,t+p)}$. Since G' is a subgraph of G and $K_{(s,t)} \subseteq K_{(m,t+p)}$, then G contains a $K_{(s,t)}$, against the assumption. So, we may assume that $e(H') > 0$, which means that $d_H(y_i) \geq 1$ for $i = 1, \ldots, q - p$. Hence,

$$e(H') = e(H) - \sum_{i=1}^{q-p} d_H(y_i) \leq 2p + q + 1 - (q - p) \leq 3p + 1. \qquad (5)$$

Then the following facts can be concluded:

- $E(H') = 3p + 1$. Otherwise if $E(H') < 3p + 1$ then, from Lemma 6, it follows that G' contains $TK_{(m-p,n-(q-p)+p)} = TK_{(m-p,n-q)} = TK_{(s,t)}$, but this contradicts the fact that $G \in TZ(m, n; s, t)$.
- $d_H(y_i) = 1$, for $i = 1, \ldots, q - p$, thus $\Delta_Y(H) = 1$, because $\Delta_Y(H) = d_H(y_1)$. This is directly derived because all the inequalities (5) become equalities since $E(H') = 3p + 1$.

Next let us see that $\Delta_X(H) = 1$. Otherwise, there is a vertex $x \in X$ having two distinct neighbors $y, y^* \in N_H(x)$. Since $\Delta_Y(H) = 1$, then $N_H(y) = N_H(y^*) = \{x\}$, and besides, there are exactly $e(H) = 2p+q+1 > q-p+2$

vertices of degree 1 in the class Y. Let us consider the bipartite graph $G^* = (X^*, Y^*)$ whose bipartite complement $H^* = (X^*, Y^*)$ is obtained from H by removing any $q-p$ vertices of $Y \setminus \{y, y^*\}$ of degree 1. The graph H^* satisfies that $|X^*| = |X| = m > p$, $|Y^*| = |Y| - (q-p) = t + p > p$, $e(H^*) = e(H) - (q-p) = 3p+1$. Further, observe that $d_{H^*}(x) \geq 2$, because $\{y, y^*\} \subset Y^*$ and $\{xy, xy^*\} \subset E(H^*)$, which means that $\Delta(H^*) \geq 2$. Then, by applying Lemma 11, the bipartite complement G^* of H^* contains a $TK_{(m-p,t+p-p)} = TK_{(s,t)}$. Since G^* is a subgraph of G, we deduce that G contains $TK_{(s,t)}$, and this contradicts the fact that $G \in TZ(m, n; s, t)$. Hence, $\Delta(H) = \min\{\Delta_X(H), \Delta_Y(H)\} = 1$ and this proves the result. $\square$

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References

[1] C. Balbuena, M. Cera, A. Dinez and P. García-Vázquez. New exact values of the maximum size of graphs free of topological complete subgraphs. *Discrete Math.*, 307(9-10):1038–1046, 2007.

[2] C. Balbuena, P. García-Vázquez, X. Marcote and J.C. Valenzuela. New results on the Zarankiewicz problem. *Discrete Math.*, 307(17-18):2322–2327, 2007.

[3] T. Böhme, B. Mohar, R. Škrekovski and M. Stiebitz. Subdivisions of large complete bipartite graphs and long induced paths in k-connected graphs. *J. Graph Theory*, 45:270–274, 2004.

[4] B. Bollobás. *Extremal Graph Theory.* Academic Press, New York, 1978.

[5] R. Diestel. *Graph Theory.* Springer-Verlag Berlin Heildeberg, 2005.

[6] Z. Furëdi. New asymptotics for Bipartite Turán Numbers. *J. Combin. Theory Ser. A*, 75:141–144, 1996.

[7] A.P. Godbole and H.C. Graziano. Contributions to the problem of Zarankiewicz. *J. Statist. Plann. Inference*, 95:197–208, 2001.

[8] A.P. Godbole, B. Lamorte and E.J. Sandquist. Threshold functions for the bipartite Turán property. *Electron. J. Combin.*, 4:R18, 1997.

[9] W. Goddard, M.A. Henning and O.R. Oellermann. Bipartite Ramsey numbers and Zarankiewicz numbers. *Discrete Math.*, 219:85–95, 2000.

[10] J. Griggs and H. Chih-Chang. On the half-half case of the Zarankiewicz problem. *Discrete Math.*, 249:95–104, 2002.

[11] D. Kühn and D. Osthus. Improved bounds for topological cliques in graphs of large girth. *SIAM J. Discrete Math.*, 20(1):62–78, 2006.

[12] D. Kühn and D. Osthus. Induced subdivisions in $K_{s,s}$-free graphs of large average degree. *Combinatorica*, 24(2):287–304, 2004.

[13] D. Kühn and D. Osthus. Topological minors in graphs of large girth. *J. Combin. Theory Ser. B*, 86(2):364–380, 2002.

[14] Y. Liu and G.Y. Yan. Graphs isomorphic to their maximum matching graphs. *Acta Mathematica Sinica, English Series,* 29(9):15071516, 2009.

[15] W. Mader. Graphs with $3n-6$ edges not containing a subdivision of K_5. *Combinatorica*, 25(4):425–438, 2005.

[16] W. Mader. $3n-5$ edges do force a subdivision of K_5. *Combinatorica*, 18(4):569–595, 1998.

[17] J.S. Myers. The extremal function for unbalanced bipartite minors. *Discrete Math.*, 271:209–222, 2003.

[18] M. Simonovits. *Extremal graph theory, Selected topics in graph theory, 2.* Academic Press, London, 1983.

[19] P. Turán. On an extremal problem in graph theory. *Mat. Fiz. Lapok*, 48:436–452, 1941.

[20] K. Zarankiewicz. Problem P 101. *Colloq. Math.*, 2:301, 1951.