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Abstract

Let $(\mathcal{P}, \mathcal{L}, I)$ be a partial linear space and $X \subseteq \mathcal{P} \cup \mathcal{L}$. Let us denote by $(X)_I = \bigcup_{x \in X}\{y : yIx\}$ and by $[X] = (X)_I \cup X$. With this terminology a *partial linear space* $(\mathcal{P}, \mathcal{L}, I)$ *is said to admit a* $(1, \leq k)$*-identifying code* if the sets $[X]$ are mutually different for all $X \subseteq \mathcal{P} \cup \mathcal{L}$ with $|X| \leq k$. In this paper we give a characterization of k-regular partial linear spaces admitting a $(1, \leq k)$-identifying code. Equivalently, we give a characterization of k-regular bipartite graphs of girth at least six admitting a $(1, \leq k)$-identifying code. That is, k-regular bipartite graphs of girth at least six admitting a set C of vertices such that the sets $N[x] \cap C$ are nonempty and pairwise distinct for all vertex $x \in X$ where X is a subset of vertices of $|X| \leq k$. Moreover, we present a family of k-regular partial linear spaces on $2(k-1)^2+k$ points and $2(k-1)^2 + k$ lines whose incidence graphs do not admit a $(1, \leq k)$-identifying code. Finally, we show that the smallest $(k; 6)$-graphs known up to now for $k-1$ not a prime power admit a $(1, \leq k)$-identifying code.

1 Introduction

We only consider undirected simple graphs without loops or multiple edges. Unless otherwise stated, we follow the book by Godsil and Royle [18] for terminology and definitions.

Let G be a graph with vertex set $V = V(G)$ and edge set $E = E(G)$. The *distance* between two vertices u, v in G, $d_G(u, v)$ or simply $d(u, v)$, is the length of a shortest path joining u and v. The *degree* of a vertex $v \in V$, denoted by $d_G(v)$ or $d(v)$, is the number of edges incident with v. The *minimum degree* of G is denoted by $\delta(G)$, and a graph is said to be *k-regular* if all its vertices have the same degree k. The *neighborhood* $N(v)$ of a vertex v is the set of all vertices that are adjacent to v. The closed neighborhood of v is defined by $N[v] = N(v) \cup \{v\}$. For a vertex subset $X \subseteq V$, the neighborhood of X is defined as $N(X) = \cup_{x \in X} N(x)$, and $N[X] = N(X) \cup X$. The *girth* of a graph G is the length of a shortest cycle and a $(k; g)$-graph is a k-regular graph with girth g. A *$(k; g)$-cage* is a smallest $(k; g)$-graph.

Let C be a nonempty subset of V. For $X \subseteq V$ the set of vertices $I(C) = I(C; X)$ is defined as follows

$$I(C) = \bigcup_{x \in X} N[x] \cap C.$$

If all the sets $I(C)$ are different for all subset $X \subseteq V$ where $|X| \leq k$, then C is said to be a *$(1, \leq k)$-identifying code* in G. In 1998, Karpovsky, Chakrabarty and Levitin [22] introduced $(1, \leq k)$-identifying codes in graphs. Identifying codes appear motivated by the problem of determining faulty processors in a multiprocessor system. We say that a graph G *admits a $(1, \leq k)$-identifying code* if there exists such a code $C \subseteq V$ in G. Not all graphs admit $(1, \leq k)$-identifying codes, for instance Laihonen [23] pointed out that a graph formed by a set of independent edges cannot admit a $(1, \leq 1)$-identifying code, because clearly for all $uv \in E$, $N[u] = \{u, v\} = N[v]$. It is not difficult to see that if G admits $(1, \leq k)$-identifying codes, then $C = V$ is also a $(1, \leq k)$-identifying code. Hence a graph admits $(1, \leq k)$-identifying codes if and only if the sets $N[X]$ are mutually different for all $X \subseteq V$ with $|X| \leq k$. Results on identifying codes in specific families on graphs as well as results on the smallest cardinality of an identifying code can be seen in [6, 9, 13, 14].

Laihonen and Ranto [24] proved that if G is a connected graph with at least three vertices admitting a $(1, \leq k)$-identifying code, then the minimum degree is $\delta(G) \geq k$. Gravier and Moncel [17] showed the existence of a graph with minimum degree exactly k admitting a $(1, \leq k)$-identifying code. Recently, Laihonen [23] proved the following result.

Theorem 1 [23] *Let $k \geq 2$ be an integer.*

(i) *If a k-regular graph has girth $g \geq 7$, then it admits a $(1, \leq k)$-identifying code.*

(ii) *If a k-regular graph has girth $g \geq 5$, then it admits a $(1, \leq k-1)$-identifying code.*

According to item (ii) of Theorem 1, all $(k;6)$-graphs admit a $(1, \leq k-1)$-identifying code. The main aim of this paper is to approach the problem of characterizing bipartite $(k;g)$-graphs for $g \geq 6$ admitting $(1, \leq k)$-identifying codes. To do that we consider a bipartite graph as the incidence graph of a partial linear space $(\mathcal{P}, \mathcal{L}, I)$ [18]. A point $p \in \mathcal{P}$ and a line $L \in \mathcal{L}$ are said to be incident if $(p, L) \in I \subseteq \mathcal{P} \times \mathcal{L}$ and for short this is denoted by pIL or LIp. A *partial linear space* is an incidence structure in which any two points of $\mathcal{P}$ are incident with at most one line of $\mathcal{L}$. This implies that any two lines are incident with at most one point. The *incidence graph* $\mathcal{B}$ of a partial linear space $(\mathcal{P}, \mathcal{L}, I)$ is the graph with vertex set $V(\mathcal{B}) = \mathcal{P} \cup \mathcal{L}$ and edge set $E(\mathcal{B}) = I$, i.e., two vertices are adjacent if and only they are incident. It is easy to check that $\mathcal{B}$ is a bipartite graph of girth at least 6. A partial linear space $(\mathcal{P}, \mathcal{L}, I)$ is said to be *k-regular* if every line is incident with k points and every point is incident with k lines. Obviously the incidence graph of a k-regular partial linear space is a k-regular bipartite graph.

First, we define a partial linear space admitting a $(1, \leq k)$-identifying code. In our main theorem we give a characterization of k-regular partial linear spaces admitting a $(1, \leq k)$-identifying code. As a consequence of this result, we show that minimal $(k;6)$-cages, which are the incidence graphs of projective planes of order $k-1$, do not admit a $(1, \leq k)$-identifying code. Moreover, we present a family of k-regular partial linear space on $2(k-1)^2 + k$ points and $2(k-1)^2 + k$ lines whose incidence graphs do not admit a $(1, \leq k)$-identifying code. Finally, we show that the smallest

$(k;6)$-graphs known up to now and constructed in [1, 2, 3, 4, 5, 7, 16] for $k-1$ not a prime power admit a $(1,\leq k)$-identifying code.

The paper is organized as follows. In the next section we present our main theorem and we give a construction of a family of k-regular partial linear spaces without $(1,\leq k)$-identifying codes. In the final section we apply the theorem to show certain families of small $(k;6)$-graphs that have $(1,\leq k)$-identifying codes.

2 Main theorem

Let $(\mathcal{P},\mathcal{L},I)$ be a partial linear space and $X\subseteq\mathcal{P}\cup\mathcal{L}$. Following Dembowski [10], let us denote by $(X)_I=\bigcup_{x\in X}\{y : yIx\}$ and by $[X]=(X)_I\cup X$. With this terminology we give the following definition.

Definition 2 A partial linear space $(\mathcal{P},\mathcal{L},I)$ is said to admit a $(1,\leq k)$-identifying code if and only if the sets $[X]$ are mutually different for all $X\subseteq\mathcal{P}\cup\mathcal{L}$ with $|X|\leq k$.

As an immediate consequence of Theorem 1 we can write the following corollary.

Corollary 3 *Let $k\geq 2$ be an integer. A k-regular partial linear space $(\mathcal{P},\mathcal{L},I)$ admits a $(1,\leq k-1)$-identifying code.*

Next, we present a characterization of k-regular partial linear spaces admitting a $(1,\leq k)$-identifying code as well as some consequences.

Theorem 4 *Let $k\geq 2$ be an integer. A k-regular partial linear space $(\mathcal{P},\mathcal{L},I)$ admits a $(1,\leq k)$-identifying code if and only if the following two conditions hold:*

(i) *For every two collinear points $u,p\in\mathcal{P}$ there exists a point $z\in\mathcal{P}$ which is collinear with just one of u,p. Equivalently, for every $u,p\in\mathcal{P}$ such that $|(u)_I\cap(p)_I|=1$, there exists $z\in\mathcal{P}$ such that $|(u)_I\cap(z)_I|+|(p)_I\cap(z)_I|=1$.*

(ii) *For every two concurrent lines $L,M\in\mathcal{L}$ there exists a line $\Lambda\in\mathcal{L}$ which is concurrent with just one of L,M. Equivalently, for every $L,M\in\mathcal{L}$ such that $|(L)_I\cap(M)_I|=1$, there exists $\Lambda\in\mathcal{L}$ such that $|(L)_I\cap(\Lambda)_I|+|(M)_I\cap(\Lambda)_I|=1$.*

Proof: Suppose that $(\mathcal{P},\mathcal{L},I)$ admits a $(1,\leq k)$-identifying code and that there exist two concurrent lines $M,L\in\mathcal{L}$ such that

$$\text{for every line } \Lambda\in\mathcal{L}, |(M)_I\cap(\Lambda)_I|=1 \text{ iff } |(L)_I\cap(\Lambda)_I|=1. \qquad (1)$$

Let $(M)_I\cap(L)_I=\{p\}$ and consider the sets $X=\{M\}\cup((L)_I-p)\subset\mathcal{P}\cup\mathcal{L}$ and $Y=\{L\}\cup((M)_I-p)\subset\mathcal{P}\cup\mathcal{L}$. Observe that $X\neq Y$ and $|X|=|Y|=k$ because $(\mathcal{P},\mathcal{L},I)$ is k-regular. Then

$$\begin{aligned}[X] &= [M]\cup((L)_I-p)\cup\textstyle\bigcup_{h\in(L)_I-p}\{\Lambda\in\mathcal{L}:\Lambda Ih\},\\ [Y] &= [L]\cup((M)_I-p)\cup\textstyle\bigcup_{h\in(M)_I-p}\{\Lambda\in\mathcal{L}:\Lambda Ih\}.\end{aligned}$$

Clearly $[X]\cap\mathcal{P}=(M)_I\cup((L)_I-p)=[Y]\cap\mathcal{P}$; and $[X]\cap\mathcal{L}=\{M,L\}\cup\bigcup_{h\in(L)_I-p}\{\Lambda\in\mathcal{L}:\Lambda Ih\}$ and $[Y]\cap\mathcal{L}=\{M,L\}\cup\bigcup_{h\in(M)_I-p}\{\Lambda\in\mathcal{L}:\Lambda Ih\}$. Assumption (1) yields to $[X]\cap\mathcal{L}=[Y]\cap\mathcal{L}$ meaning that $[X]=[Y]$, which is a contradiction with the hypothesis that $(\mathcal{P},\mathcal{L},I)$ admits a $(1,\leq k)$-identifying code. We may reason analogously to prove that there are no two collinear points $p,q\in\mathcal{P}$ such that for every point $r\in\mathcal{P}$, $|(p)_I\cap(r)_I|=1$ iff $|(q)_I\cap(r)_I|=1$.

Conversely, suppose that $(\mathcal{P},\mathcal{L},I)$ does not admit a $(1,\leq k)$-identifying code and let us assume that for every two elements $u,v\in\mathcal{P}\cup\mathcal{L}$ such that $|(u)_I\cap(v)_I|=1$, there exists $z\in\mathcal{P}\cup\mathcal{L}$, for which

$$|(u)_I\cap(z)_I|+|(v)_I\cap(z)_I|=1.$$

By Corollary 5, $(\mathcal{P},\mathcal{L},I)$ admits $(1,\leq k-1)$-identifying codes and hence $[X]\neq[Y]$ holds for all $X,Y\subseteq\mathcal{P}\cup\mathcal{L}$ such that $|X|,|Y|\leq k-1$. According to our assumption, there must exist two different sets $X,Y\subseteq\mathcal{P}\cup\mathcal{L}$ such that $\max\{|X|,|Y|\}=k$ and $[X]=[Y]$. Without loss of generality, we may assume that $X,Y\subseteq\mathcal{P}\cup\mathcal{L}$, $X\neq Y$, $|X|=k$, $|Y|\leq k$ and $[X]=[Y]$.

First, let us see that $|Y|=k$. Let $x\in X\setminus Y$, then $(x)_I\subset[X]=[Y]$. Since $x\notin Y$ it follows that $([w]-x)\cap Y\neq\emptyset$ for all $w\in(x)_I$. Moreover as two points are incident with at most one line and two lines are incident with at most one point, we have $([w]-x)\cap([w']-x)=\emptyset$ for all $w,w'\in(x)_I$, $w\neq w'$. Therefore $|Y|\geq|(x)_I|=k$, giving $|Y|=k$.

Now let us see that each X and Y must contain both points and lines. Otherwise suppose that $X\subseteq\mathcal{P}$, then $[X]\cap\mathcal{P}=X$. In this case if $Y\subseteq\mathcal{P}$ then $[Y]\cap\mathcal{P}=Y$ yielding that $X=Y$ because $[X]=[Y]$, which is a contradiction. Therefore there exists $L \in Y \cap \mathcal{L}$, hence $(L)_I \subseteq [Y] \cap \mathcal{P} = [X] \cap \mathcal{P} = X$, which implies $(L)_I = X$ because $|(L)_I| = k$, as $(\mathcal{P}, \mathcal{L}, I)$ is k-regular, and $|X| = k$. As two lines have at most one common point and $k \geq 2$ we have $Y \cap \mathcal{L} = \{L\}$. Further, $Y \cap \mathcal{P} \subseteq [Y] \cap \mathcal{P} = [X] \cap \mathcal{P} = X$, hence we may assume that $Y = \{x_1, \ldots, x_{k-1}, L\}$ and $X = \{x_1, \ldots, x_k\} = (L)_I$. As $k \geq 2$ we can take $L' \neq L$ such that $(L')_I \cap (L)_I = \{x_k\}$, i.e., $L' \notin Y$ and $L' \notin (x_i)_I$ for $i = 1, \ldots, k-1$, yielding that $L' \in [X] \setminus [Y]$, a contradiction because $[X] = [Y]$. Thus $X \not\subseteq \mathcal{P}$. Analogously, $Y \not\subseteq \mathcal{P}$, and changing points for lines we may check that $X \not\subseteq \mathcal{L}$, and $Y \not\subseteq \mathcal{L}$.

Henceforth, let us assume that

$$X \cap \mathcal{P} = \{x_1, \ldots, x_s\}, X \cap \mathcal{L} = \{L_{s+1}, \ldots, L_k\},$$

$$Y \cap \mathcal{P} = \{y_1, \ldots, y_r\}, Y \cap \mathcal{L} = \{M_{r+1}, \ldots, M_k\}$$

and let us prove the following claim.

Claim 1 (i) $(x_i)_I \cap \{L_{s+1}, \ldots, L_k\} = \emptyset$ *for all* $i = 1, \ldots, s$.

(ii) $(y_i)_I \cap \{M_{r+1}, \ldots, M_k\} = \emptyset$ *for all* $i = 1, \ldots, r$.

Proof: First, suppose that $y_j \notin \{x_1, \ldots, x_s\}$ for some $j \in \{1, \ldots, r\}$. As $y_j \in Y$ we have

$$(y_j)_I \subseteq [Y] \cap \mathcal{L} = [X] \cap \mathcal{L} = \{L_{s+1}, \ldots, L_k\} \cup (x_1)_I \cup \cdots \cup (x_s)_I.$$

As $|(y_j)_I| = k$ and $|(y_j)_I \cap (x_i)_I| \leq 1$, then $\{L_{s+1}, \ldots, L_k\} \subset (y_j)_I$, $|(y_j)_I \cap (x_i)_I| = 1$ for all $i = 1, \ldots, s$, and $(y_j)_I \cap (x_i)_I \notin \{L_{s+1}, \ldots, L_k\}$. Hence $(x_i)_I \cap \{L_{s+1}, \ldots, L_k\} = \emptyset$, so item (i) of the claim is true in this case. Second, suppose $\{y_1, \ldots, y_r\} \subseteq \{x_1, \ldots, x_s\}$, then there exists a line $M_j \notin \{L_{s+1}, \ldots, L_k\}$ because $X \neq Y$. We have $(M_j)_I \subseteq [X] \cap \mathcal{P} = [Y] \cap \mathcal{P}$. Therefore changing points for lines and reasoning as before it follows that $\{x_1, \ldots, x_s\} \subset (M_j)_I$, $|(M_j)_I \cap (L_i)_I| = 1$ for all $i = s+1, \ldots, k$, and $(M_j)_I \cap (L_i)_I \notin \{x_1, \ldots, x_s\}$, hence $(x_i)_I \cap \{L_{s+1}, \ldots, L_k\} = \emptyset$, so item (i) of the claim holds. The proof of (ii) is analogous. ■

Now, suppose that $Y \cap \mathcal{L} = \{M_{r+1}, \ldots, M_k\} \subseteq \{L_{s+1}, \ldots, L_k\}$. Without loss of generality assume that $M_j = L_j$, $j = r+1, \ldots, k$. Hence $[X] \cap \mathcal{P} = \{x_1, \ldots, x_s\} \cup (L_{s+1})_I \cup \cdots \cup (L_k)_I = [Y] \cap \mathcal{P} = \{y_1, \ldots, y_r\} \cup (L_{r+1})_I \cup \cdots \cup (L_k)_I$. Claim 1, yields that $\{x_1, \ldots, x_s\} \subset \{y_1, \ldots, y_r\}$ and $\{L_{s+1}, \ldots, L_r\} \cap$

$Y = \emptyset$, otherwise $X = Y$ which is a contradiction. Therefore, $|(L_{s+1})_I \cap \{y_1, \ldots, y_r\}| \leq r - s$, and as $|(L_{s+1})_I \cap (L_j)_I| \leq 1$ for all $j = r+1, \ldots, k$, we have $|(L_{s+1})_I| \leq r-s+k-r = k-s < k$ which is a contradiction. Therefore $\{M_{r+1}, \ldots, M_k\} \not\subseteq \{L_{s+1}, \ldots, L_k\}$ and in analogous way it is proved that $\{L_{s+1}, \ldots, L_k\} \not\subseteq \{M_{r+1}, \ldots, M_k\}$.

Next, suppose that $s \geq 2$ and take $M \in \{M_{r+1}, \ldots, M_k\} \setminus \{L_{s+1}, \ldots, L_k\}$. We have $(M)_I \subset [Y] \cap \mathcal{P} = [X] \cap \mathcal{P} = \{x_1, \ldots, x_s\} \cup (L_{s+1})_I \cup \cdots \cup (L_k)_I$. As $|(M)_I| = k$, $\{x_1, \ldots, x_s\} \subset (M)_I$ and $|(M)_I \cap (L_i)_I| = 1$ for all $i = s+1, \ldots, k$; thus M must be unique because $s \geq 2$. Therefore $Y \cap \mathcal{L} = \{M_{r+1}, \ldots, M_k\} \subseteq \{L_{s+1}, \ldots, L_k\} \cup \{M\}$. Without loss of generality assume that $Y \cap \mathcal{L} = \{M, L_{r+2}, \ldots, L_k\}$. Again, $(y_j)_I \subseteq [X] \cap \mathcal{L} = \{L_{s+1}, \ldots, L_k\} \cup (x_1)_I \cup \cdots \cup (x_s)_I$. By Claim 1, $(y_j)_I \cap \{L_{r+2}, \ldots, L_k\} = \emptyset$ and as $|(y_j)_I \cap \bigcup_{i=1}^{s} (x_i)_I| \leq s$, then $k = |(y_j)_I| \leq (r+1-s)+s = r+1$, so $r \geq k-1$. Hence $Y = \{y_1, \ldots, y_{k-1}\} \cup \{M\}$. Now, take $L \in X \cap \mathcal{L}$, $L \neq M$. As $(L)_I \subseteq [Y] \cap \mathcal{P}$, reasoning as before we obtain that $(L)_I = \{y_1, \ldots, y_{k-1}\} \cup ((L)_I \cap (M)_I)$ yielding that L must be unique, so $X = \{x_1, \ldots, x_{k-1}\} \cup \{L\}$. As $[X] \cap \mathcal{P} = [Y] \cap \mathcal{P} = \{x_1, \ldots, x_{k-1}\} \cup (L)_I = \{y_1, \ldots, y_{k-1}\} \cup (M)_I$, it follows that $(M)_I = \{x_1, \ldots, x_{k-1}\} \cup ((L)_I \cap (M)_I)$. Hence L and M are two concurrent lines such that every line Λ is concurrent with L if and only if Λ is concurrent with M because $[X] \cap \mathcal{L} = [Y] \cap \mathcal{L}$. In other words, L and M satisfy (1), which is a contradiction with the hypothesis (ii).

It remains to study the case $s = 1$ so that $X = \{x_1, L_2, \ldots, L_k\}$. If $r \geq 2$ reasoning as for the case $s \geq 2$ we get that $s \geq k-1$ meaning that $k = 2$ which is a contradiction with the fact that $2 \leq r < k$. Thus we get that $r = 1$ and so $Y = \{y_1, M_2, \ldots, M_k\}$. By Claim 1, $(x_1)_I = \{M_2, \ldots, M_k\} \cup ((x_1)_I \cap (y_1)_I)$ and $(y_1)_I = \{L_2, \ldots, L_k\} \cup ((x_1)_I \cap (y_1)_I)$. Hence x_1 and y_1 are two collinear points such that every point z is collinear with x_1 if and only if z is collinear with y_1, contradicting the hypothesis (i). □

As an immediate consequence of Theorem 4 we get the following theorem which is a characterization of k-regular bipartite graphs of girth at least 6 admitting a $(1, \leq k)$-identifying code.

Theorem 5 *A k-regular bipartite graph $\mathcal{B}$ of girth at least 6 admits a $(1, \leq k)$-identifying code if and only if for every two vertices $u, v \in V(\mathcal{B})$ such that $|N(u) \cap N(v)| = 1$, there exists $z \in V(\mathcal{B})$ in such a way that $|N(u) \cap N(z)| + |N(v) \cap N(z)| = 1$.*

3 Families of small $(k, 6)$-graphs without $(1, \leq k)$-identifying codes

A projective plane of order $k-1$ is a k-regular partial linear space such that any two distinct points are collinear and any two distinct lines are concurrent. A minimal $(k;6)$-cage is a bipartite graph which can be obtained as the incidence graph of a projective plane of order $k-1$. Using the properties of projective planes it is not difficult to check that a projective plane of order $k-1$ does not admit a $(1, \leq k)$-identifying code as a consequence of Theorem 4. And in the same way it is shown that a minimal $(k;6)$-cage has no $(1, \leq k)$-identifying code as a consequence of Theorem 5.

Corollary 6 (i) *A projective plane of order* $k-1$ *does not admit a* $(1, \leq k)$*-identifying code.*

(ii) *A minimal* $(k;6)$*-cage does not admit a* $(1, \leq k)$*-identifying code.*

Projective planes are not the unique partial linear spaces which do not admit a $(1, \leq k)$-identifying code. For instance, Figure 1 depicts on the right side a partial linear space of 11 points and 11 lines which does not admit $(1, \leq 3)$-identifying codes. On the left side we can see the corresponding $(3;6)$-bipartite graph on 22 vertices. It is easy to find two different lines L and M satisfying condition (1) of the proof of Theorem 4. So this graph does not admit $(1, \leq 3)$-identifying codes. In the next theorem we construct a family of k-regular partial linear spaces without $(1, \leq k)$-identifying codes. The partial plane of Figure 1 belongs to this family.

Theorem 7 *Let* $(\mathcal{P}, \mathcal{L}, I)$ *be a projective plane of order* $k-1 \geq 2$ *and consider a point* $p_0 \in \mathcal{P}$ *and a line* $L_0 \in (p_0)_I \cap \mathcal{L}$*. Let* $\mathcal{L}_0 = \mathcal{L} \setminus (p_0)_I$ *and* $\mathcal{P}_0 = \mathcal{P} \setminus (L_0)_I$ *and take* $\mathcal{L}_0'$*,* $\mathcal{P}_0'$ *disjoint copies of* $\mathcal{L}_0$ *and* $\mathcal{P}_0$*, respectively. Observe that* $|\mathcal{L}_0| = |\mathcal{P}_0| = (k-1)^2$*, thus we can consider a bijection* $f : \mathcal{P}_0' \to \mathcal{L}_0'$*. Let us define a new incidence structure* $\left(\mathcal{P} \cup \mathcal{P}_0', \mathcal{L} \cup \mathcal{L}_0', I_f'\right)$ *as follows.*

1. *For all* $(z', M) \in (\mathcal{P} \cup \mathcal{P}_0') \times (\mathcal{L} \setminus \mathcal{L}_0)$*,* $z' I_f' M$ *iff* $z' \in \mathcal{P}$ *and* $z' I M$*.*
2. *For all* $(z', M) \in (\mathcal{P} \cup \mathcal{P}_0') \times \mathcal{L}_0$*,* $z' I_f' M$ *iff*

$$\begin{cases} z' \in \mathcal{P} \setminus \mathcal{P}_0 & \text{and } z' I M; \\ z' \in \mathcal{P}_0' & \text{and } z I M, \text{ where } z \in \mathcal{P}_0 \text{ is the copy of } z'. \end{cases}$$

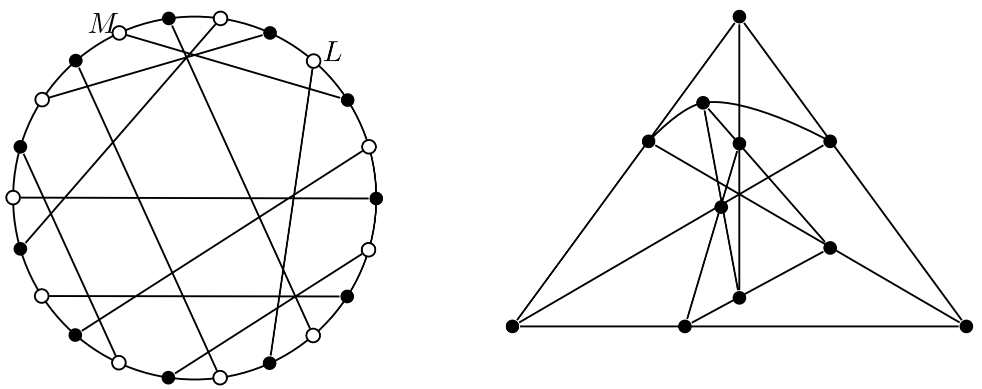

Figure 1: A (3,6)-bipartite graph on 22 vertices without $(1, \leq 3)$ codes and its corresponding partial linear space.

3 *For all* $(z', M') \in (\mathcal{P} \cup \mathcal{P}'_0) \times \mathcal{L}'_0$, $z' I'_f M'$ *iff*

$$\begin{cases} z' \in \mathcal{P}_0 & \text{and } z'IM \text{ where } M \in \mathcal{L}_0 \text{ is the copy of } M'; \\ z' \in \mathcal{P}'_0 & \text{and } f(z') = M'. \end{cases}$$

Then $\left(\mathcal{P} \cup \mathcal{P}'_0, \mathcal{L} \cup \mathcal{L}'_0, I'_f\right)$ *is a* k*-regular partial linear space on* $2(k-1)^2+k$ *points and* $2(k-1)^2+k$ *lines without* $(1, \leq k)$*-identifying codes.*

Proof: First let us see that $\left(\mathcal{P} \cup \mathcal{P}'_0, \mathcal{L} \cup \mathcal{L}'_0, I'_f\right)$ is a partial linear space. To do that let us show that two distinct lines $A', B' \in \mathcal{L} \cup \mathcal{L}'_0$ have at most one point in common. Let z' be a point such that $z' I'_f A'$ and $z' I'_f B'$. Due to the rules given in 1 and 2 and from the fact that $(\mathcal{P}, \mathcal{L}, I)$ is a projective plane it follows that z' is unique if both A' and B' are in $\mathcal{L}$. If both lines A' and B' are in $\mathcal{L}'_0$, then $z' \in \mathcal{P}_0$ because the rule 3, so z' is unique. And finally if $A' \in \mathcal{L}_0$ and $B' \in \mathcal{L}'_0$ the unique possible point in common is $z' \in \mathcal{P}'_0$ such that $f(z') = B'$ and $A'Iz$ (in the projective plane) where z is the copy of z'. By duality it can be shown that there exists at most one line through two distinct points. (In Figure 2 it is depicted the incidence graph corresponding to $\left(\mathcal{P} \cup \mathcal{P}'_0, \mathcal{L} \cup \mathcal{L}'_0, I'_f\right)$, where $(\mathcal{P}, \mathcal{L}, I)$ is the projective plane of order 2. This graph is also depicted in Figure 1.)

Next let us see that $\left(\mathcal{P} \cup \mathcal{P}'_0, \mathcal{L} \cup \mathcal{L}'_0, I'_f\right)$ is k-regular. It is clear that $(p_0)_{I'_f} = (p_0)_I$, i.e., every line in the set $\{M \in \mathcal{L} \cup \mathcal{L}'_0 : M I'_f p_0\}$ is incident with the same k points as in the projective plane $(\mathcal{P}, \mathcal{L}, I)$. Moreover, a line $M \in \mathcal{L}_0$ is incident with one point from $\mathcal{P} \setminus \mathcal{P}_0$ and $k-1$ points from $\mathcal{P}'_0$ because the rule 2. And a line $M \in \mathcal{L}'_0$ is incident with $k-1$ points from $\mathcal{P}_0$ and one point from $\mathcal{P}'_0$ due to the rule 3.

Finally observe that $\left(\mathcal{P} \cup \mathcal{P}'_0, \mathcal{L} \cup \mathcal{L}'_0, I'_f\right)$ has no $(1, \leq k)$-identifying codes because any two lines from the set $\{M \in \mathcal{L} \cup \mathcal{L}'_0 : M I'_f p_0,\ M \neq L_0\}$ satisfy the property (1) given in the proof of Theorem 4. □

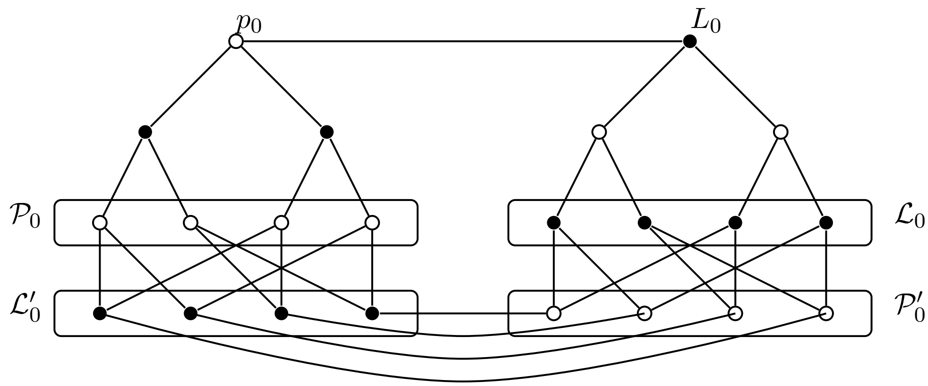

Figure 2: The incidence graph of $\left(\mathcal{P} \cup \mathcal{P}'_0, \mathcal{L} \cup \mathcal{L}'_0, I'_f\right)$, where $(\mathcal{P}, \mathcal{L}, I)$ is the projective plane of order 2.

4 Families of small $(k, 6)$-graphs with $(1, \leq k)$-identifying codes

Minimal $(k;6)$-cages are known to exist when $k-1$ is a prime power. The order of any $(k;6)$-cage is denoted by $n(k;6)$. A new way for constructing projective planes via its incidence matrices is given in [5]. By removing some rows and columns from these matrices some new bipartite $(k;6)$-graphs with $2(qk-1)$ vertices are obtained for all $k \leq q$ where q is a prime power [5]. The same result is also obtained in [3], but finding these graphs as subgraphs of the incidence graph of a known projective plane. For $k = q$ the same result is obtained in [1], also using incidence matrices. Moreover in [5] the incidence matrix of a $(q-1;6)$-regular balanced bipartite graph on $2(q(q-1)-2)$ vertices was obtained. When q is a square and is the smallest prime power greater than or equal to $k-1$, $(k;6)$-regular graphs with order $2(kq-(q-k)(\sqrt{q}+1)-\sqrt{q})$ have been constructed in [16]. Recently, these results have been improved finding new bipartite $(k;6)$-graphs with $2(qk-2)$ vertices for all $k \leq q$ where q is a prime power [2]. These graphs have the smallest number of vertices known so far among the regular graphs with girth 6 yielding that $n(k;6) \leq 2(qk-2)$ is the best upper bound known up to now. More details about constructions of cages can be found in the survey by Wong [25] or in the survey by Holton and Sheehan [21] or in the more recent dynamic cage survey by Exoo and Jajcay [12]. In this later survey some of the above mentioned constructions are described in a geometric way.

The main aim of this section is to prove that the mentioned new small bipartite $(k;6)$-graphs for all $k \leq q$ where q is a prime power constructed in [1, 2, 3, 4, 5, 7, 16] admit a $(1, \leq k)$-identifying code. With this aim we shall verify that the corresponding partial k-regular linear space admits $(1, \leq k)$-identifying code by means of Theorem 4. We recall some geometric notions introduced in [2, 16]. *A generalized d-gon of order $k-1$* is a partial linear space whose incidence graph is a k-regular bipartite graph with girth $2d$ and diameter d. Finite generalized d-gons exist only for $d \in \{3, 4, 6\}$ (see [8, 18]). When $d = 3$, a 3-gon of order $k-1$ is a projective plane of order $k-1$ (see [8, 18]). A *t-good* structure in a generalized d-gon (see [16]) is a pair $(\mathcal{P}^*, \mathcal{L}^*)$ consisting of a set of points $\mathcal{P}^*$ and a set of lines $\mathcal{L}^*$ satisfying the following conditions:

1. Any point not belonging to $\mathcal{P}^*$ is incident with t lines contained in $\mathcal{L}^*$.

2. Any line not belonging to $\mathcal{L}^*$ is incident with t points contained in $\mathcal{P}^*$.

Clearly, by removing the points and lines of a t-good structure from a $(q+1)$-regular generalized d-gon, we obtain a $(q+1-t)$-regular partial linear space. Its incidence graph is a balanced bipartite $(q+1-t)$-regular graph of girth at least $2d$.

Let $(\mathcal{P}, \mathcal{L}, I)$ be a partial linear space, we say that an incidence pIL is deleted if the point p is not removed from $\mathcal{P}$, but the line L of $\mathcal{L}$ is replaced with the new line $L-p$. The point p is said to be *separated* from the line L. In [2], $(t+1)$-*good* structures were generalized by defining $(t+1)$-*coregular* structures using this removal incidence. An ordered triple

$(\mathcal{P}_0, \mathcal{L}_0, \mathcal{I}_0)$, whose elements are a set of points $\mathcal{P}_0$, a set of lines $\mathcal{L}_0$ and a set of incidences $\mathcal{I}_0$, is said to be a $(t+1)$-*coregular* structure in a generalized d-gon (see [2]) if the removal from a $(q+1)$-regular d-gon of the points in $\mathcal{P}_0$, the lines in $\mathcal{L}_0$ and the incidences in I_0 leads to a new $(q-t)$-regular partial linear space. Obviously, its incidence graph is a bipartite $(q-t)$-regular graph with girth at least $2d$. More precisely, in [2] the following $(t+1)$-coregular structures in projective planes of order q for $t \leq q-2$ were found.

Theorem 8 [2] *Let $(\mathcal{P}, \mathcal{L}, I)$ be a projective plane of order q and $L^* \in \mathcal{L}$ such that $(L^*)_I = \{p, x_1, \ldots, x_q\}$. Let $(p)_I = \{L^*, L_p^1, \ldots, L_p^q\}$ be the set of lines passing through p. The following structures $(\mathcal{P}_0, \mathcal{L}_0, \mathcal{I}_0)$ are $(t+1)$-coregular for $0 \leq t \leq q-2$:*

$$t = 0 \ : \mathcal{P}_0 = \{x_1\} \cup (L_p^1)_I; \quad \mathcal{L}_0 = \{L_p^1\} \cup (x_1)_I; \quad \mathcal{I}_0 = \emptyset.$$

$$t \geq 1 \ : \mathcal{P}_0 = \{x_1, x_2, \ldots, x_{t+1}\} \cup (L_p^1)_I \cup (L_p^2)_I \cup \cdots \cup (L_p^t)_I \cup (M)_I \quad \text{where } M \in (x_{t+2})_I - L^*;$$

$$\begin{aligned}
\mathcal{L}_0 &= \{L_p^1, L_p^2, \ldots, L_p^t, M\} \cup (x_1)_I \cup \cdots \cup (x_t)_I \\
&\cup \begin{cases} (x_2)_I & \text{if } t = 1 \\ (x_{t+1})_I - \{A_1, \ldots, A_{t-1}\} & \text{if } t \geq 2, \text{ where } A_i \in (x_{t+1})_I - L^* \\ & \text{is the line connecting } x_{t+1} \text{ and} \\ & M \cap L_p^i, i = 1, \ldots, t-1; \end{cases} \\
\mathcal{I}_0 &= \{x_j I L : L \in (x_j)_I \text{ such that } M \cap L_p^i \in (L)_I \text{ for some } i \in \{1, \ldots, t\}, \\
&\quad j = t+3, \ldots, q\} \\
&\quad \cup \{a_{ij} I L_p^j : a_{ij} = A_i \cap L_p^j, j = t+1, \ldots, q, i = 1, \ldots, t-1, \ t \geq 2\}.
\end{aligned}$$

It is not difficult to check that the partial linear spaces whose incidence graphs are the bipartite graphs constructed in [1, 2, 3, 4, 5, 7, 16] are obtained by removing $(t+1)$-good or $(t+1)$-coregular structures from projective planes. For all the constructions contained in these papers it is not difficult to verify the following remark:

Remark 9 If Π' is a partial linear space obtained by removing a t-good or a t-coregular structure from a projective plane Π and p is a removed or separated point, then p is incident to either $q-t+1$ or to $q-t+2$ lines in Π'. Moreover, in a special construction using Baer Subplanes and t-good structures in projective planes of order square prime powers (see [16]), the removed points are incident with exactly $q - \sqrt{q} - t + 1$ lines in Π'.

It is worth noting that in all the constructions of k-regular partial linear spaces contained in [1, 2, 3, 4, 5, 7, 16], the smallest prime power q with $k \leq q$ and an integer $t \geq 1$ such that $k = q + 1 - t$ are considered. Then, using the following result concerning with the existence of prime numbers in short intervals, we prove Theorem 11.

Theorem 10 [11]

(i) *If* $k \geq 3275$ *then the interval* $[k, k(1 + \frac{1}{2ln^2(k)})]$ *contains a prime number.*

(ii) *If* $6 \leq k \leq 3276$ *then the interval* $[k, \frac{7k}{6}]$ *contains a prime power.*

The Bertran's postulate states (see [19]) that for every $k > 2$ there exists a prime q verifying the inequality $k < q < 2k$. In this work we will take advantage from Theorem 10, because we only need to check the less restrictive inequality $q < 2k - 2$.

Theorem 11 *Let* $q > 2$ *be a prime power and* $t < q+1$ *an integer. Suppose that* $2t < q$ *or if* q *is a square prime power that* $t \in (q', q)$ *where* q' *is also a prime power such that there is no prime power in the interval* (q', q)*. If* Π' *is a* $(q + 1 - t)$*-regular partial plane constructed by removing a* t*-good or a* t*-coregular structure from a projective plane* Π *of order* q*, then* Π' *admits a* $(1, \leq k)$*-identifying code.*

Proof: Assume that Π' does not admit a $(1, \leq k)$-identifying code and let L and M be two concurrent lines in Π' that satisfy the condition (1) in the proof of Theorem 4 with $\{p\} = (L)_I \cap (M)_I$. Let p_1 be a removed or separated point from $L - p$. Suppose that there are exactly a lines incident to p_1 in Π' (without considering L). If some of these lines had a common point with M in Π', then Π' would admit a $(1, \leq k)$–identifying code by Theorem 4 which is a contradiction with our assumption. Then any of these lines have in common with M points that are not in Π' or that have been separated from M. As M is incident to exactly t points in the projective plane which are not incident to M in Π' (they are removed or separated points), then a must be equal to t.Therefore, by Remark 9, we have the following three cases:

- If p_1 is incident to $q-t+1$ lines in Π', then $a = q-t$ (the number of lines in Π' except L). Hence $q-t=t$, i.e. $q=2t$. This is a contradiction with the hypothesis $2t < q$.

- If p_1 is incident to $q-t+1$ lines in Π', then $a = q-t+1 = t$, which is again a contradiction .

- If q is a square prime power, then p_1 is incident to $q-\sqrt{q}-t+1$ lines in Π' and $2t = q-\sqrt{q}$. Then $q = 2^{2\alpha}$ and $t = 2^{2\alpha-1} - 2^{\alpha-1}$, which is a contradiction to the hypothesis $t \in (\sqrt{q}, q)$, because $\sqrt{q} = 2^{\alpha}$ is also a prime power.

Reasoning as above and taking into account the dual of Remark 9 it is straightforward to prove that there are not two concurrent points p and q in Π' such that for any point r in Π' we have $|(p)_I \cap (r)_I| = 1$ iff $|(q)_I \cap (r)_I| = 1$.

Then, we can conclude that Π' admits a $(1, \leq k)$-identifying code. □

As an immediate consequence of Theorem 11, we can write the following corollary.

Corollary 12 (i) *The k-regular parcial linear spaces whose incidence graphs are the $(k;6)$-graphs constructed in [1, 2, 3, 4, 5, 7, 16] admit a $(1, \leq k)$-identifying code.*

(ii) *The $(k;6)$-graphs constructed in [1, 2, 3, 4, 5, 7, 16] admit a $(1, \leq k)$-identifying code.*

In Figure 3, a 3-regular linear space of 8 points and 8 lines is depicted. It is obtained by removing from a projective plane of order 3 a 1-coregular structure, see [2]. On the right side it is shown its corresponding bipartite graph on 16 vertices.

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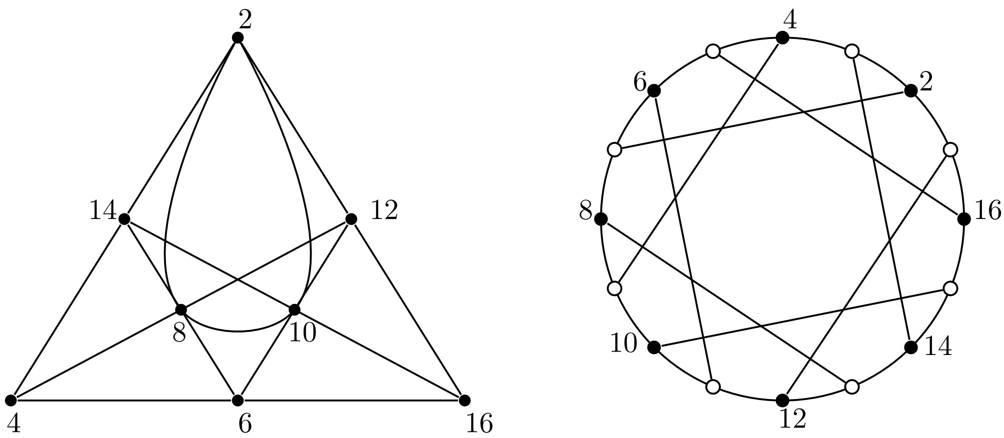

Figure 3: A 3-regular partial linear space of 8 points and 8 lines admitting $(1, \leq 3)$-identifying code and its corresponding (3,6)-bipartite graph on 16 vertices.

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