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Abstract

For a strongly connected digraph D the restricted arc-connectivity $\lambda'(D)$ is defined as the minimum cardinality of an arc-cut over all arc-cuts S satisfying that $D - S$ has a non trivial strong component D_1 such that $D - V(D_1)$ contains an arc. Let S be a subset of vertices of D. We denote by $\omega^+(S)$ the set of arcs uv with $u \in S$ and $v \notin S$, and by $\omega^-(S)$ the set of arcs uv with $u \notin S$ and $v \in S$. A digraph $D = (V, A)$ is said to be λ'-optimal if $\lambda'(D) = \xi'(D)$, where $\xi'(D)$ is the minimum arc-degree of D defined as $\xi(D) = \min\{\xi'(xy) : xy \in A\}$, and $\xi'(xy) = \min\{|\omega^+(\{x,y\})|, |\omega^-(\{x,y\})|, |\omega^+(x) \cup \omega^-(y)|, |\omega^-(x) \cup \omega^+(y)|\}$. In this paper a sufficient condition for a s-geodetic strongly connected digraph D to be λ'-optimal is given in terms of its diameter. Further we see that the h-iterated line digraph $L^h(D)$ of a s-geodetic digraph is λ'-optimal for certain iteration h.

1 Introduction

We consider finite digraphs without loops and multiple edges. Let $D = (V, A)$ be a strongly connected digraph, with vertex set $V = V(D)$ and arc set $A = A(D)$. For any pair F, F' of proper vertex subsets of a digraph D, we define $[F, F'] = \{xy \in A : x \in F, y \in F'\}$. If $F' = \overline{F} = V \setminus F$, we write

$\omega^+(F)$ or $\omega^-(\overline{F})$ instead of $[F,\overline{F}]$. When $F=\{x\}$ we abbreviate $\omega^+(\{x\})$ and $\omega^-(\{x\})$ to $\omega^+(x)$ and $\omega^-(x)$, respectively. Clearly, $d^+(x)=|\omega^+(x)|$ and $d^-(x)=|\omega^-(x)|$.

A subset $S\subseteq A$ of arcs is an arc-cut if $D-S$ is not strongly connected. Each minimum arc-cut has the form $\omega^+(F)$, where F is a proper subset of V. Thus, the *arc-connectivity* of a digraph D can be defined as

$$\lambda(D)=\min\{|\omega^+(F)| : F\subset V,\ F\neq\emptyset,\ F\neq V\}.$$

It is well-known that for any digraph D, $\lambda(D)\leq\delta(D)$ [10]. Hence, D is said to be *maximally arc-connected* if $\lambda(D)=\delta(D)$. Following Hamidoune [12, 13], a subset F of vertices of a strongly connected digraph D with arc-connectivity λ is a *positive α-fragment* if $|\omega^+(F)|=\lambda$ and, similarly, F is a *negative α-fragment* if $|\omega^-(F)|=\lambda$. Note that F is a positive α-fragment if an only if $\overline{F}=V(D)\setminus F$ is a negative α-fragment.

When the underlying topology of an interconnection network is modeled by a connected graph or a strongly connected digraph D, where $V(D)$ is the set of processors and $A(D)$ is the set of communication links, the edge-connectivity or arc-connectivity of D are important measurements for fault tolerance of the network. However, one might be interested in more refined indices of reliability. Even two graphs or digraphs with the same edge/arc-connectivity λ may be considered to have different reliabilities, since the number or type of minimum arc-cuts is different.

The study of fault-tolerance of networks modeled by an undirected graph has been intense in recent years. By restricting the forbidden fault set to be the sets of neighboring edges of any spanning subgraph with no more than k-vertices in the faulty networks, Fàbrega and Fiol [9, 8] introduced the k-extra-edge-connectivity of interconnection networks (where k is a positive integer) as follows. Given a graph G and a non-negative integer k, the k-extra-edge-connectivity $\lambda_k(G)$ of G is the minimum cardinality of a set of edges of G, if any, whose deletion disconnects G and every remaining component contains at least k vertices. More information and results on the k-extra-edge-connectivity can be found [3, 6]. The *restricted edge-connectivity* $\lambda'(G)$, introduced by Esfahanian and Hakimi [7] for a graph G, corresponds to the 2-extra-edge-connectivity and it is the minimum cardinality over all *restricted edge-cuts* S, i.e., those such that there are no isolated vertices in $G-S$. A restricted edge-cut S is called a *λ'-cut* if $|S|=\lambda'(G)$. Obviously for any λ'-cut S, the graph $G-S$ consists of exactly two components. A connected graph G is called *λ'-connected* if $\lambda'(G)$ exists. Esfahanian and Hakimi [7] showed that each connected graph G of order $n(G) \geq 4$ except a star, is λ'-connected and satisfies $\lambda'(G) \leq \xi(G)$, where $\xi(G)$ denotes the minimum *edge-degree* of G defined as $\xi(G) = \min\{d(u) + d(v) - 2 : uv \in E(G)\}$. More information and recent results on restricted edge-connectivity of graphs can be found in the survey by Hellwig and Volkmann, [15]. All these concepts of the extraconnectivity and restricted connectivity were inspired by the definition of conditional connectivity introduced by Harary [14] who asked for the minimum cardinality of a set of edges of G, if any, whose deletion disconnects G such that every remaining component satisfies some prescribed property.

Volkmann [17] extended the notion of *restricted edge-connectivity* to digraphs. Given a strongly connected digraph D, an arc set S of D is a *restricted arc-cut* of D if $D-S$ has a non-trivial strong component D_1 such that $D-V(D_1)$ contains an arc. The restricted arc-connectivity $\lambda'(D)$ is defined as the minimum cardinality over all restricted arc-cuts S. A strongly connected digraph D is called λ'-connected if $\lambda'(D)$ exists. A restricted arc-cut S is called a *λ'-cut* if $|S| = \lambda'(D)$. In the same paper, Volkmann proved that each strong digraph D of order $n \geq 4$ and girth $g = 2$ or $g = 3$ except some families of digraphs is λ'-connected and satisfies $\lambda(D) \leq \lambda'(D) \leq \xi(D)$, where $\xi(D)$ is defined as follows. If $C_g = u_1u_2 \ldots u_gu_1$ is a shortest cycle of D, then $\xi(C_g) = \min\{|\omega^+(C_g)|, |\omega^-(C_g)|\}$, and $\xi(D) = \min\{\xi(C_g) : C_g \text{ is a shortest cycle of } D\}$.

More recently, Wang and Lin [18] have focused in studying the λ'-optimal digraphs by considering the notion of *arc-degree*. For any arc $xy \in A(D)$, the arc-degree of xy is defined as

$$\xi'(xy) = \min\{|\omega^+(\{x,y\})|, |\omega^-(\{x,y\})|, |\omega^+(x) \cup \omega^-(y)|, |\omega^-(x) \cup \omega^+(y)|\}.$$

The minimum arc-degree of D is $\xi'(D) = \min\{\xi'(xy) : xy \in A(D)\}$. Similar to the definition of λ'-optimal graphs, in [18] a λ'-connected digraph D is called λ'-optimal if $\lambda'(D) = \xi'(D)$. In the aforementioned paper [18], Wang and Lin proved the following useful theorem.

Theorem A [18] *Let D be a strongly connected digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then D is λ'-connected and $\lambda'(D) \leq \xi'(D)$.*

Starting from this result, Wang and Lin introduced the notion of *λ'-optimality* to denote the digraphs D for which $\lambda'(D) = \xi'(D)$. Then, they provided an example of a digraph having a λ'-cut which can not be written as $\omega^+(F)$ for any proper subset $F \subset V(D)$. And further in the same paper they proved that if D has no minimum restricted arc-cut of the form $\omega^+(F)$ where F is a proper subset of $V(D)$, then D is λ'-optimal.

In this paper we prove that every λ'-cut S of a λ'-connected digraph D with cardinality $|S| < \xi'(D)$ is necessarily of the form $S = \omega^+(F)$. Furthermore, both induced subdigraphs $D[F]$ and $D[\overline{F}]$ of D are shown to have an arc. These structural results allows us to give a sufficient condition for a s-geodetic digraph to have $\lambda'(D) = \xi'(D)$, i.e. to be λ'-optimal. A digraph D with diameter $diam(D)$ is said to be s-geodetic if for any two (not necessarily different) vertices $x, y \in V$, there exists at most one $x \to y$ path of length at most s. Obviously, if $d(x, y) \leq s$ then there exists exactly one such path. Note that $1 \leq s \leq g - 1 \leq diam(D)$, where $g \geq 2$ is the girth of D. Our interest is in the maximum integer s for which D is s-geodetic. If $s = diam(D)$, the digraph D is called strongly geodetic [16]. In this reference it was proved that all strongly geodetic digraphs are either complete digraphs or cycles.

Sufficient conditions for a s-geodetic digraph with minimum degree δ to be maximally arc connected have been given in terms of its diameter $diam(D)$ and the parameter s. In this regard, the following result is contained in [4]:

$$\lambda = \delta \text{ if } diam(D) \leq 2s.$$

The k-extra-connectivity was studied for s-geodetic digraphs in [2]. In this work we prove that $\lambda'(D) = \xi'(D)$ if $diam(D) \leq 2s - 1$, and we also show that D is λ'-optimal if $diam(D) = 2s$ when D satisfies an additional hypothesis. Furthermore, we see that the h-iterated line digraph $L^h(D)$ of a s-geodetic digraph is λ'-optimal for certain iteration h.

2 Results

Following the book by Harary (see [11], pg. 199), each vertex of a digraph is in exactly one strong component and an arc lies in one strong component depending on whether or not it is in some cycle. It follows from the maximality of strong components that the strong components of a digraph D can be labeled $D_1, \ldots, D_k$ such that there is no arc from D_j to D_i unless $j < i$. Such an ordering is called an *acyclic ordering* of the strong components of D. In order to obtain our main result we require the following lemma.

Lemma 1 *Let $D = (V, A)$ be a λ'-connected digraph and S a λ'-cut of D such that $|S| < \xi'(D)$. Then the set V can be partitioned into two subsets, F, $\overline{F}$ such that $S = \omega^+(F) = \omega^-(\overline{F})$ and both induced subdigraphs $D[F]$ and $D[\overline{F}]$ of D contain an arc.*

Proof: Let S be a λ'-cut of D and let $D_1, \ldots, D_k$, $k \geq 2$, be an acyclic ordering of the strong components of $D - S$. Since S is a restricted arc-cut some strong component D_j of $D - S$ must be non trivial, i.e. $|V(D_j)| \geq 2$ and $D - V(D_j)$ contains an arc. Suppose that D_j is the unique non-trivial strong component of $D - S$. As $D - V(D_j)$ contains an arc yz, then $k \geq 3$. If $j = 1$ then by considering $F = \cup_{i=2}^{k} V(D_i)$ and $\overline{F} = V(D_1)$, it follows that $\omega^+(F)$ is a restricted arc-cut of D. Since $\omega^+(F) \subseteq S$ and S is a λ'-cut, then $\omega^+(F) = S$ and clearly both induced subdigraphs $D[F]$ and $D[\overline{F}]$ of D contain an arc. The prof is analogous if $j = k$, hence, assume that $2 \leq j \leq k - 1$. If $\{y, z\} \subseteq \cup_{i=1}^{j-1} V(D_i)$ then it is enough to consider $F = \cup_{i=j}^{k} V(D_i)$ and $\overline{F} = \cup_{i=1}^{j-1} V(D_i)$ and clearly $S = \omega^+(F)$ and both induced subdigraphs $D[F]$ and $D[\overline{F}]$ of D contain an arc. The prof is also analogous if $\{y, z\} \subseteq \cup_{i=j+1}^{k} V(D_i)$. Thus, we may assume that $y \in \cup_{i=1}^{j-1} V(D_i)$ and $z \in \cup_{i=j+1}^{k} V(D_i)$, yielding that $\omega^+(z) \cup \omega^-(y) \subseteq S$ or is there the previous situation for another arc. Clearly $\omega^+(z) \cup \omega^-(y)$ is a restricted arc-cut of D because D_j is a strong component of $D - (\omega^+(z) \cup \omega^-(y))$ and the arc yz belongs to $D - D_j$. Then $\omega^+(z) \cup \omega^-(y) = S$ and hence $\xi'(D) \leq |\omega^+(z) \cup \omega^-(y)| = |S|$ which is a contradiction with the hypothesis. Therefore $D - S$ has at least two distinct non-trivial strong components D_t and D_j, meaning that $D[\cup_{i=1}^{j-1} V(D_i)]$ contains an arc or $D[\cup_{i=j+1}^{k} V(D_i)]$ contains an arc. In the former case let $F = \cup_{i=j}^{k} V(D_i)$ and $\overline{F} = \cup_{i=1}^{j-1} V(D_i)$. Since there is no arc from F to $\overline{F}$ in $D - S$, then $\omega^+(F) = S$ and we are done because clearly both $D[F]$ and $D[\overline{F}]$ contain an arc. Similarly if $D[\cup_{i=j+1}^{k} V(D_i)]$ contains an arc, then $F = \cup_{i=j+1}^{k} V(D_i)$ and $\overline{F} = \cup_{i=1}^{j} V(D_i)$ satisfy the lemma. □

We will henceforth denote the set of arcs $\omega^+(F)$ by $[X, \overline{X}]$, where $X \subseteq F$ and $\overline{X} \subseteq \overline{F}$ are, respectively, the sets of out and in vertices of the arcs of $\omega^+(F)$.

The following remark is immediate from the definition of s-geodetic digraphs.

If uv is an arc of a s-geodetic digraph D with $s \geq 2$, then $N_i^+(u) \cap N_i^+(v) = \emptyset$ and $N_{i+1}^+(u) \cap N_i^+(v) = \emptyset$ for all $i = 1, \ldots, s-1$.

Some properties on the s-geodetic digraphs not being λ'-optimal are provided in the following results.

Lemma 2 *Let D be a λ'-connected s-geodetic digraph and $\omega^+(F) = [X, \overline{X}]$ a λ'-cut. If $\lambda'(D) < \xi'(D)$ then there exists some vertex $u \in F$ such that $d(u, X) \geq s-1$ and there exists some vertex $\overline{u} \in \overline{F}$ such that $d(\overline{X}, \overline{u}) \geq s-1$.*

Proof: When $s = 1$ the assertion is obvious, hence assume $s \geq 2$. Let us denote by $\mu = \max\{d(u, X) : u \in F\}$. We reason by way of contradiction by supposing $\mu \leq s-2$. First assume that $\mu = 0$. This implies that every vertex of F is an initial of some arc of $[X, \overline{X}]$, that is $F = X$. By Lemma 1, we can consider an arc uv in $D[F]$ and since $N^+(u) \cap N^+(v) = \emptyset$ because $s \geq 2$, then

$$\begin{aligned} \lambda'(D) = \left|[X, \overline{X}]\right| &\geq |[\{u, v\}, \overline{X}]| + \left|[(N^+(u) - v) \cap X, \overline{X}]\right| \\ &\quad + \left|[(N^+(v) - u) \cap X, \overline{X}]\right| \\ &\geq |N^+(u) - v| + |N^+(v) - u| \\ &= |\omega^+(\{u, v\})| \geq \xi'(uv) \geq \xi'(D), \end{aligned}$$

which is a contradiction. Hence, assume that $1 \leq \mu \leq s-2$, which means that $s \geq 3$.

Case 1: There exists an arc uv in $D[F]$ such that $d(u, X) = d(v, X) = \mu$.

Let us denote by $A_u = (N^+(u) - v) \cap N_\mu^-(X)$, $A_v = (N^+(v) - u) \cap N_\mu^-(X)$, $B_u = N^+(u) \cap N_{\mu-1}^-(X)$ and $B_v = N^+(v) \cap N_{\mu-1}^-(X)$ and observe that $N^+(u) - v = A_u \cup B_u$ and $N^+(v) - u = A_v \cup B_v$. It is clear by Remark 1 that $N^+(u) \cap N^+(v) = \emptyset$ because $s \geq 3$, and therefore, the sets A_u, A_v, B_u, B_v are pairwise disjoint. Let us see that the sets $N_\mu^+(A_u) \cap X$, $N_\mu^+(A_v) \cap X$, $N_\mu^+(u) \cap X$ and $N_\mu^+(v) \cap X$ are pairwise disjoint. Note that every vertex x belonging to any of the previous sets is at distance at most $\mu + 2 \leq s$ from u. Hence, the existence of some vertex x belonging to two of these sets implies the existence of two paths $u \to x$ of length at most s, which contradicts the hypothesis that D is s-geodetic. The same argument justifies that $|N_\mu^+(A_u) \cap X| \geq |A_u|$, $|N_\mu^+(A_v) \cap X| \geq |A_v|$, $|N_\mu^+(u) \cap X| \geq |N^+(u) \cap N_{\mu-1}^-(X)| = |B_u|$ and $|N_\mu^+(v) \cap X| \geq |N^+(v) \cap N_{\mu-1}^-(X)| = |B_v|$, since D is s-geodetic. Hence,

$$\begin{aligned}\lambda'(D) = \big|[X,\overline{X}]\big| \geq |X| &\geq |N^+_\mu(A_u)\cap X| + |N^+_\mu(A_v)\cap X| \\ &\quad + |N^+_\mu(u)\cap X| + |N^+_\mu(v)\cap X| \\ &\geq |A_u| + |A_v| + |B_u| + |B_v| \\ &= |N^+(u)-v| + |N^+(v)-u| \\ &= |\omega^+(\{u,v\})| \geq \xi'(uv) \geq \xi'(D),\end{aligned}$$

against the fact that $\lambda'(D) < \xi'(D)$.

Case 2: There is no arc uv in $D[F]$ such that $d(u,X) = d(v,X) = \mu$.

Let $u \in N^-_\mu(X)$ and take any $v \in N^+(u) \cap N^-_{\mu-1}(X)$. Let us denote by $A = (N^+(v)-u)\cap N^-_\mu(X)$, $B = N^+(v)\cap N^-_{\mu-1}(X)$ and $C = (N^+(A)-v)\cap N^-_{\mu-1}(X)$. As $s \geq 3$ the girth of D is at least 4, and thus it is clear by Remark 1 that the sets $N^+(u)$, B, C are pairwise disjoint. Since $s \geq 3$ and the induced subdigraph $D[N^-_\mu(X)\cap F]$ contains no arc, then $|C| \geq |A|$. Let us see that the sets $N^+_{\mu-1}(N^+(u))\cap X$, $N^+_{\mu-1}(B)\cap X$ and $N^+_{\mu-1}(C)\cap X$ are pairwise disjoint. Note that every vertex x belonging to any of these sets is at distance at most $\mu+2 \leq s$ from u. Hence, the existence of some vertex x belonging to two of these sets implies the existence of two paths of length at most s from u to x, which contradicts the hypothesis that D is s-geodetic. The same argument justifies that $|[N^+_{\mu-1}(N^+(u))\cap X,\overline{X}]| \geq |N^+(u)-v| + (|N^+(v)-u| - |B| - |A|)$, $|[N^+_{\mu-1}(B)\cap X,\overline{X}]| \geq |N^+_{\mu-1}(B)\cap X| \geq |B|$ and $|[N^+_{\mu-1}(C)\cap X,\overline{X}]| \geq |N^+_{\mu-1}(C)\cap X| \geq |C| \geq |A|$, since D is s-geodetic. Hence,

$$\begin{aligned}\lambda'(D) = \big|[X,\overline{X}]\big| &\geq |[N^+_{\mu-1}(N^+(u))\cap X,\overline{X}]| + |[N^+_{\mu-1}(B)\cap X,\overline{X}]| \\ &\quad + |[N^+_{\mu-1}(C)\cap X,\overline{X}]| \\ &\geq |N^+(u)-v| + (|N^+(v)-u| - |B| - |A|) + |B| + |A| \\ &= |N^+(u)-v| + |N^+(v)-u| \\ &= |\omega^+(\{u,v\})| \geq \xi'(uv) \geq \xi'(D),\end{aligned}$$

against the fact that $\lambda'(D) < \xi'(D)$.

Thus, $\mu \geq s-1$. The prof of $\overline{\mu} \geq s-1$, being $\overline{\mu} = \max\{d(\overline{X}, \overline{u}) : \overline{u} \in \overline{F}\}$ is analogous. So the result holds. $\square$

Lemma 3 *Let $D=(V,A)$ be a λ'-connected s-geodetic digraph and $\omega^+(F) = [X, \overline{X}]$ a λ'-cut such that $\max\{d(u,X) : u \in F\} = s-1$ and $\max\{d(\overline{X}, \overline{u}) : \overline{u} \in F\} = s-1$. If $\lambda'(D) < \xi'(D)$ then the following assertions hold:*

(i) *The induced subdigraphs $D[N^-_{s-1}(X) \cap F]$ and $D[N^+_{s-1}(\overline{X}) \cap \overline{F}]$ contain some arc.*

(ii) *There exist $u_0 \in N^-_{s-1}(X) \cap F$, $\overline{u}_0 \in N^+_{s-1}(\overline{X}) \cap \overline{F}$ such that $|[N^+_{s-1}(u_0) \cap X, \overline{X}]| = 1$ and $|[X, N^-_{s-1}(\overline{u}_0) \cap \overline{X}]| = 1$.*

Proof: (i) Clearly, the result holds if $s=1$, because of Lemma 1. Therefore, assume that $s \geq 2$ and reason by way of contradiction supposing that $D[N^-_{s-1}(X) \cap F]$ has no arc. Then every vertex $u \in N^-_{s-1}(X) \cap F$ satisfies that $N^+(u) \cap N^-_{s-1}(X) = \emptyset$ which means that $d(u,X) = s-1$, and $|N^+_{s-1}(u) \cap X| \geq d^+(u)$ because D is s-geodetic. Let us consider a vertex $u \in N^-_{s-1}(X) \cap F$ such that $d^+(u) \leq d^+(u')$ for all $u' \in N^-_{s-1}(X) \cap F$. Take any vertex $v \in N^+(u)$ and let us consider the subsets of F, $A = (N^+(v) - u) \cap N^-_{s-1}(X)$, $B = N^+(v) \cap N^-_{s-2}(X)$. It is clear by Remark 1 and due to $s \geq 2$ that the sets $N^+(u) - v$, A, B are pairwise disjoint. Moreover, since $2 \leq s \leq g-1$ where g is the girth of D, then there is no symmetric arc in D, yielding that $v \notin N^+(a)$ for all $a \in A$. As $d^+(a) \geq d^+(u)$ and $D[N^-_{s-1}(X) \cap F]$ has no arc, then $|(N^+(a) \setminus N^+(u)) \cap N^-_{s-2}(X)| \geq 1$ for all $a \in A$, hence the set $C = (N^+(A) \setminus N^+(u)) \cap N^-_{s-2}(X)$ satisfies that $|C| \geq |A|$. Let us see that the sets $N^+_{s-2}(N^+(u) - v) \cap X$, $N^+_{s-2}(B) \cap X$, $N^+_{s-2}(C) \cap X$ and $N^+_{s-2}(v) \cap X$ are pairwise disjoint. Note that every vertex x belonging to any of these sets is at distance at most $s-2+2 = s$ from v. Hence, if some vertex x belongs to two of these sets, then two distinct directed paths of length at most s from v to x exist, which contradicts the hypothesis that D is s-geodetic. The same argument justifies that $|[N^+_{s-2}(N^+(u) - v) \cap X, \overline{X}]| \geq |N^+(u) - v|$, $|[N^+_{s-2}(B) \cap X, \overline{X}]| \geq |N^+_{s-2}(B) \cap X| \geq |B|$, $|[N^+_{s-2}(C) \cap X, \overline{X}]| \geq |N^+_{s-2}(C) \cap X| \geq |C| \geq |A|$ and $|[N^+_{s-2}(v) \cap X, \overline{X}]| \geq |N^+_{s-2}(v) \cap X| \geq |N^+(v) - u| - |A| - |B|$, since D is

s-geodetic. Hence,

$$\begin{aligned}\lambda'(D) = \big|[X,\overline{X}]\big| &\geq |[N^+_{s-2}(N^+(u)-v)\cap X,\overline{X}]| + |[N^+_{s-2}(B)\cap X,\overline{X}]|\\ &\quad +|[N^+_{s-2}(C)\cap X,\overline{X}]| + |[N^+_{s-2}(v)\cap X,\overline{X}]|\\ &\geq |N^+(u)-v| + |B| + |A| + |N^+(v)-u| - |A| - |B|\\ &= |N^+(u)-v| + |N^+(v)-u|\\ &= |\omega^+(\{u,v\})| \geq \xi'(uv) \geq \xi'(D),\end{aligned}$$

against the fact that $\lambda'(D) < \xi'(D)$. Thus, $D[N^-_{s-1}(X)\cap F]$ must contain some arc. Analogously it is proved that $D[N^+_{s-1}(\overline{X})\cap\overline{F}]$ contains some arc.

(ii) First assume that $s = 1$, which means that $F = X$. Let uv be an arc of $D[F]$ and suppose that $|[\{z\},\overline{X}]| \geq 2$ for all $z \in X$. Then

$$\begin{aligned}\lambda'(D) &= |[X,\overline{X}]|\\ &\geq |[\{u\},\overline{X}]| + |[\{v\},\overline{X}]| + |[((N^+(u)\cup N^+(v))\setminus\{u,v\})\cap X,\overline{X}]|\\ &\geq |[\{u\},\overline{X}]| + |[\{v\},\overline{X}]| + 2|\left((N^+(u)\cup N^+(v))\setminus\{u,v\}\right)\cap X|\\ &\geq |[\{u\},\overline{X}]| + |[\{v\},\overline{X}]| + |(N^+(u)-v)\cap X|\\ &\qquad + |(N^+(v)-u)\cap X|\\ &\geq \xi'(uv) \geq \xi'(D),\end{aligned}$$

which is a contradiction. Hence assume that $s \geq 2$. Let $u \in F\cap(N^-_{s-1}(X)\cup N^-_{s-2}(X))$ be such that $|N^+(u)\cap N^-_{s-1}(X)| \geq 1$ is maximum in $F\cap(N^-_{s-1}(X)\cup N^-_{s-2}(X))$. Two cases need to be distinguished:

Case 1. Assume that $u \in N^-_{s-1}(X)\cap F$. Take any $v \in N^+(u)\cap N^-_{s-2}(X)$ and denote by $U = N^+(u)\cap N^-_{s-1}(X)$ and $W = (N^+(v)-u)\cap N^-_{s-1}(X)$. Since $2 \leq s \leq g-1$ then D has no symmetric arc, hence $W = N^+(v)\cap N^-_{s-1}(X)$. Observe that $|U| \geq |W|$ because the way that vertex u has been selected. Further notice that $|U| + |W| \geq 1$ for if not, $\lambda'(D) = |[X,\overline{X}]| \geq |X| \geq |N^+_{s-1}(u)\cap X| + |N^+_{s-1}(v)\cap X| \geq |N^+(u)-v| + |N^+(v)-u| \geq \xi'(uv) \geq \xi'(D)$ and this is a contradiction.

Suppose that $|[N^+_{s-1}(z)\cap X,\overline{X}]| \geq 2$ for all $z \in U\cup W$. Since D is s-geodetic, then the sets $N^+_{s-1}(u)\cap X$, $N^+_{s-2}\left(N^+(v)\cap N^-_{s-2}(X)\right)\cap X$ and $N^+_{s-1}(U)\cap X$ are pairwise disjoint. Furthermore, the inequalities

$$\begin{aligned}
|[N^+_{s-1}(u)\cap X,\overline{X}]| &\geq |N^+_{s-1}(u)\cap X|\\
&= |N^+_{s-2}(N^+(u)-v)\cap X| + |N^+_{s-2}(v)\cap X|\\
&\geq |N^+(u)-v| - |U| + |N^+(v)-u|\\
&\quad -|N^+(v)\cap N^-_{s-2}(X)| - |W|,
\end{aligned}$$

$$\begin{aligned}
|[N^+_{s-2}\left(N^+(v)\cap N^-_{s-2}(X)\right)\cap X,\overline{X}]| &\geq |N^+_{s-2}\left(N^+(v)\cap N^-_{s-2}(X)\right)\cap X|\\
&\geq |N^+(v)\cap N^-_{s-2}(X)|
\end{aligned}$$

and

$$|[N^+_{s-1}(U)\cap X,\overline{X}]| \geq 2|U| \geq |U| + |W|$$

hold. Hence,

$$\begin{aligned}
\lambda'(D) &\geq |X|\\
&\geq |[N^+_{s-1}(u)\cap X,\overline{X}]| + |[N^+_{s-2}\left(N^+(v)\cap N^-_{s-2}(X)\right)\cap X,\overline{X}]|\\
&\quad +|[N^+_{s-1}(U)\cap X,\overline{X}]|\\
&\geq |N^+(u)-v| - |U| + |N^+(v)-u| - |N^+(v)\cap N^-_{s-2}(X)| - |W|\\
&\quad +|N^+(v)\cap N^-_{s-2}(X)| + |U| + |W|\\
&= |N^+(u)-v| + |N^+(v)-u|\\
&= |\omega^+(\{u,v\})| \geq \xi'(uv) \geq \xi'(D),
\end{aligned}$$

a contradiction. Then there must exists a vertex $u_0\in U\cup W$ such that $|[N^+_{s-1}(u_0)\cap X,\overline{X}]| = 1$.

Case 2. Assume that $u\in N^-_{s-2}(X)\cap F$. Note that $|N^+(u)\cap N^-_{s-1}(X)| \geq |N^+(v)\cap N^-_{s-1}(X)| + 1$ for all $v\in N^-_{s-1}(X)\cap F$ may be assumed because if for some $v\in N^-_{s-1}(X)\cap F$, $|N^+(u)\cap N^-_{s-1}(X)| = |N^+(v)\cap N^-_{s-1}(X)|$ the result follows from Case 1. Take any $v\in N^+(u)\cap N^-_{s-1}(X)$ and denote by $U=(N^+(u)-v)\cap N^-_{s-1}(X)$ and $W=N^+(v)\cap N^-_{s-1}(X)$ and observe that $u\notin W$ because $u\in N^-_{s-2}(X)\cap F$. Observe that $|U| = |(N^+(u)-v)\cap N^-_{s-1}(X)| = |N^+(u)\cap N^-_{s-1}(X)| - 1 \geq |W|$ because the way that vertex u has been selected. Further notice that $|U|+|W|\geq 1$ because otherwise

$|N^+(u) \cap N^-_{s-1}(X)| = 1$ and $|N^+(v) \cap N^-_{s-1}(X)| = 0$ for all $v \in N^-_{s-1}(X) \cap F$ yielding that the subdigraph $D[N^-_{s-1}(X) \cap F]$ has no arc, which contradicts item (i).

As in the above case suppose that $|[N^+_{s-1}(z) \cap X, \overline{X}]| \geq 2$ for all $z \in U \cup W$. The sets $N^+_{s-2}(u) \cap X$, $N^+_{s-2}\left(N^+(u) \cap N^-_{s-2}(X)\right) \cap X$, $N^+_{s-1}(U) \cap X$ and $N^+_{s-1}(v) \cap X$ are pairwise disjoint, since D is s-geodetic. Furthermore, the inequalities

$$|[N^+_{s-2}(u) \cap X, \overline{X}]| \geq |N^+_{s-2}(u) \cap X| \geq |N^+(u) - v| - |U| - |N^+(u) \cap N^-_{s-2}(X)|,$$

$$\begin{aligned} |[N^+_{s-2}\left(N^+(u) \cap N^-_{s-2}(X)\right) \cap X, \overline{X}]| &\geq |N^+_{s-2}\left(N^+(u) \cap N^-_{s-2}(X)\right) \cap X| \\ &\geq |N^+(u) \cap N^-_{s-2}(X)|, \end{aligned}$$

$$|[N^+_{s-1}(U) \cap X, \overline{X}]| \geq 2|U| \geq |U| + |W|$$

and

$$|[N^+_{s-1}(v) \cap X, \overline{X}]| \geq |N^+_{s-1}(v) \cap X| \geq |N^+(v) - u| - |W|$$

hold. Hence,

$$\begin{aligned} \lambda'(D) \geq |X| &\geq |[N^+_{s-2}(u) \cap X, \overline{X}]| \\ &\quad + |[N^+_{s-2}\left(N^+(u) \cap N^-_{s-2}(X)\right) \cap X, \overline{X}]| \\ &\quad + |[N^+_{s-1}(U) \cap X, \overline{X}]| + |[N^+_{s-1}(v) \cap X, \overline{X}]| \\ &\geq |N^+(u) - v| - |U| - |N^+(u) \cap N^-_{s-2}(X)| \\ &\quad + |N^+(u) \cap N^-_{s-2}(X)| + |U| + |W| + |N^+(v) - u| - |W| \\ &= |N^+(u) - v| + |N^+(v) - u| \\ &= |\omega^+(\{u, v\})| \geq \xi'(uv) \geq \xi'(D), \end{aligned}$$

again a contradiction. Then there must exists a vertex $u_0 \in U \cup W$ such that $|[N^+_{s-1}(u_0) \cap X, \overline{X}]| = 1$.

The prof of the existence of a vertex $\overline{u}_0 \in \overline{F}$ such that $|[X, N^-_{s-1}(\overline{u}) \cap \overline{X}]| = 1$ is analogous. □

As a consequence of all the above previous result, a sufficient condition for a s-geodetic digraph to be λ'-optimal is given in the following theorem.

Theorem 4 *Let D be a strongly connected s-geodetic digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$. Then D is λ'-optimal if the diameter is $diam(D) \leq 2s-1$.*

Proof: From Theorem A it follows that D is λ'-connected and $\lambda'(D) \leq \xi'(D)$. Suppose that D is non λ'-optimal and let S be a λ'-cut. Then from Lemma 1 it follows that $S = \omega^+(F) = [X, \overline{X}]$. Moreover, from Lemma 2 there is a vertex $u_0 \in F$ such that $d(u_0, X) \geq s-1$ and there is a vertex $\overline{u}_0 \in \overline{F}$ such that $d(\overline{X}, \overline{u}_0) \geq s-1$. Hence $diam(D) \geq d(u_0, X) + 1 + d(\overline{X}, \overline{u}_0) \geq 2s-1$, which is a contradiction unless $diam(D) = 2s-1$. In this case, all the former inequalities become equalities, that is, $\max\{d(u, X) : u \in F\} = \max\{d(\overline{X}, \overline{u}) : \overline{u} \in F\} = s-1$. Then by Lemma 3 we may assume that $|[N^+_{s-1}(u_0) \cap X, \overline{X}]| = 1$ and $|[X, N^-_{s-1}(\overline{u}_0) \cap \overline{X}]| = 1$. Let us denote by $[N^+_{s-1}(u_0) \cap X, \overline{X}] = [x_0, \overline{x}_0]$, for some $x_0 \in X$, $\overline{x}_0 \in \overline{X}$; and let us denote by $[X, N^-_{s-1}(\overline{u}_0) \cap \overline{X}] = [y_0, \overline{y}_0]$, for some $y_0 \in X$, $\overline{y}_0 \in \overline{X}$.

From $d(u_0, \overline{u}_0) = 2s-1$, it follows that $x_0 = y_0$ and $\overline{x}_0 = \overline{y}_0$. Notice also that $|N^+(u_0) \cap N^-_{s-1}(X)| \geq d^+(u_0) - 1$ because $|N^+(u_0) \cap N^-_{s-2}(X)| \leq |N^+_{s-1}(u_0) \cap X| = 1$; analogously $|N^-(\overline{u}_0) \cap N^+_{s-1}(X)| \geq d^-(\overline{u}_0) - 1$. First, suppose $\delta^+(D) \geq 3$, then there exists a vertex $v \in N^+(u_0) \cap N^-_{s-1}(X)$. Observe that $d(v, \overline{u}_0) = 2s-1$, yielding that $|[N^+_{s-1}(v) \cap X, \overline{x}_0]| \geq 1$, that is, $x_0 \in N^+_{s-1}(v) \cap X$. Therefore the shortest $u_0 \to x_0$ path together with the arc u_0v and the shortest $v \to x_0$ path are two distinct $u_0 \to x_0$ directed paths of length at most s, which is a contradiction. A similar contradiction is obtained supposing $\delta^-(D) \geq 3$. Hence, D is λ'-optimal. $\square$

Our next goal is to study sufficient conditions for λ'-optimality in s-geodetic digraphs of diameter $diam(D) = 2s$.

Theorem 5 *Let D be a strongly connected s-geodetic digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$ and diameter $diam(D) = 2s$. Then D is λ'-optimal if $|N^+_s(u) \cap N^-_s(v)| \geq 3$ for all pair u, v of vertices at distance $d(u, v) = 2s$.*

Proof: From Theorem A it follows that D is λ'-connected and $\lambda'(D) \leq \xi'(D)$. Let S be a λ'-cut of D and suppose that D is non λ'-optimal, that is, $|S| < \xi'(D)$. A contradiction will be obtained by proving the existence of two vertices $u, v \in V(D)$ such that $d(u, v) = 2s$ and $|N^+_s(u) \cap N^-_s(v)| < 3$. From Lemma 1 it follows that $S = \omega^+(F) = [X, \overline{X}]$. Let us denote by $\mu = \max\{d(u, X) : u \in F\}$ and $\overline{\mu} = \max\{d(\overline{X}, \overline{u}) : \overline{u} \in \overline{F}\}$. From Lemma 2 it follows that $\mu \geq s-1$ and $\overline{\mu} \geq s-1$. If $\mu + \overline{\mu} \geq 2s$ then it is enough to take two vertices u (at distance μ to X) and $\overline{u}$ (at distance $\overline{\mu}$ from $\overline{X}$), yielding that $2s = diam(D) \geq d(u,\overline{u}) \geq d(u,X)+1+d(\overline{X},\overline{u}) = \mu+\overline{\mu}+1 \geq 2s+1$, which is a contradiction, hence, $\mu = s-1$ or $\overline{\mu} = s-1$.

First assume that $\mu = s-1$ and $\overline{\mu} = s$. By Lemma 3, there exists a vertex $u_0 \in N^-_{s-1}(X) \cap F$ such that $[N^+_{s-1}(u_0) \cap X, \overline{X}] = \{x\overline{x}\}$. Given any vertex $\overline{u} \in \overline{F}$ at distance $\overline{\mu} = s$ from $\overline{X}$, we have $2s = diam(D) \geq d(u_0,\overline{u}) \geq d(u_0,X)+1+d(\overline{X},\overline{u}) = s-1+1+s = 2s$, following that $d(u_0,\overline{u}) = 2s$. Notice that $N^+_s(u_0) \cap \overline{F} = \{\overline{x}\}$ whereas $F \cap N^-_s(\overline{u}) = \emptyset$, since $d(\overline{X},\overline{u}) = s$. Hence $N^+_s(u) \cap N^-_s(\overline{u}) = \{\overline{x}\}$ which contradicts the hypothesis that $|N^+_s(u) \cap N^-_s(v)| \geq 3$ for all pair u, v of vertices at distance $d(u,v) = 2s$.

Second assume that $\mu = s-1$ and $\overline{\mu} = s-1$. By Lemma 3, there exists a vertex $u_0 \in N^-_{s-1}(X) \cap F$ such that $[N^+_{s-1}(u_0) \cap X, \overline{X}] = \{x\overline{x}\}$, and there is a vertex $\overline{u}_0 \in N^+_{s-1}(\overline{X}) \cap \overline{F}$ such that $[X, N^-_{s-1}(\overline{u}_0) \cap \overline{X}] = \{y\overline{y}\}$. Notice that $N^+_s(u_0) \cap \overline{F} = \{\overline{x}\}$ and $N^-_s(\overline{u}_0) \cap F = \{y\}$. Then

$$N^+_s(u_0) \cap N^-_s(\overline{u}_0) \subseteq \left(N^+_s(u_0) \cap \overline{F}\right) \cup \left(N^-_s(\overline{u}_0) \cap F\right) = \{\overline{x}, y\}$$

and therefore, $|N^+_s(u_0) \cap N^-_s(\overline{u}_0)| < 3$, again a contradiction.

Hence, D is λ'-optimal and the result holds. □

We recall here that in the line digraph $L(D)$ of a digraph D, each vertex represents an arc of D. Thus, $V(L(D)) = \{uv : (u,v) \in A(D)\}$; and a vertex uv is adjacent to a vertex xz if and only if $v = x$, that is, when the arc (u,v) is adjacent to the arc (x,z) in D. For any $h \geq 1$ the *h-iterated line digraph*, $L^h(D)$, is defined recursively by $L^h(D) = L(L^{h-1}(D))$. From the definition it follows that the minimum degrees $\delta(L(D)) = \delta(D) = \delta$. Moreover, the diameter of any strongly connected digraph other than a directed cycle [1] satisfies

$$diam(L^h(D)) = diam(D) + h. \qquad (1)$$

Moreover, if D is s-geodetic then $L^h(D)$ is s'-geodetic with $s' = \min\{s+h, g-1\}$, where g denotes the girth of D [5].

Theorem 6 *Let D be a strongly connected s-geodetic digraph with $\delta^+(D) \geq 3$ or $\delta^-(D) \geq 3$ and girth $g \geq s+1$. Then $L^{g-1-s}(D)$ is λ'-optimal if $diam(D) \leq g+s-2$.*

Proof: Observe that the iterated line digraph $L^{g-1-s}(D)$ is s'-geodetic with $s' = s+g-1-s = g-1$. From (1) and the hypothesis $diam(D) \leq$

$g+s-2$ it follows that

$$diam(L^{g-1-s}(D)) = diam(D) + g - 1 - s \leq 2(g-1) - 1 = 2s' - 1.$$

Hence the result follows directly from Theorem 4. □

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