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On the Order Type L-valued Relations on L-powersets

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Abstract

The research in the field of the so called Fuzzy Mathematics can be conditionally devided into two mainstreams: the first one emphasizes on the study of different fuzzy structures (topological, algebraic, analytical, etc.) on an ordinary set X, while L-valued sets X (that are sets equipped with some L-valued equalities $E : X \times X \to L$, or, more generally, with L-valued relations $R : X \times X \to L$) are the starting point for the second one. (L being a lattice usually with an additionally algebraic structure). The aim of this work is to discuss the problem how an L-valued relation given on a set X can be extended to the L-valued relation $\mathcal{R}$ on the L-powerset L^X. This problem, is important, among other for the theory of L-fuzzy topological spaces in the sense of [15], [16].
Keywords: L-relations, L-valued equalities, L-valued sets.

Introduction

In our previous works [17], [18], we have introduced the concept of an L-valued L-topological space, which can be considered as a synthesis of the concept of an L-topological space in the sense of Chang-Goguen [2], [6] and the concept of a many-valued set in the sense of Höhle [8], see also [9]. Our next aim is to introduce the concept of an L-valued L-*fuzzy* topological space, which would be an analogous synthesis of the concept of an L-fuzzy topological space in the sense of [15], [16], see also [10], that is a pair $(X, \mathcal{T})$ where X is a set and $\mathcal{T} : L^X \to L$ is an L-fuzzy topology on X, and the concept of a many-valued set, that is a pair (X, E) where X is a set and $E : X \times X \to L$ is an L-valued equality on it and to develop the corresponding theory. However, for realizing this plan we have an additional problem. Namely, since L-fuzzy topology on a set X is a mapping $\mathcal{T} : L^X \to L$ (and not a family $\tau \subseteq L^X$ as in case of Chang-Goguen L-topology), and since X is equiped with an L-valued equality $E : X \times X \to L$, it is natural to request some kind of extensionality for a mapping $\mathcal{T} : L^X \to L$. Therefore the problem appears how to "lift" the L-valued equality $E : X \times X \to L$ from X to an L-valued equality on the L-powerset L^X, that is to get an L-valued equality $\mathcal{E} : L^X \times L^X \to L$.

However, since an L-valued equality $E : X \times X \to L$ is a special type of an L-valued relation $R : X \times X \to L$, we decided first to study the problem of extension of an L-valued preoder type relations

$$R : X \times X \to L$$

to analogous L-valued preoder type structures

$$\mathcal{R} : L^X \times L^X \to L.$$

Further, having an L-valued equality $E : X \times X \to L$ we can extend it to an L-valued relation $\mathcal{R}$ on L^X and, then by "symmetrizing" it we get an L-valued equality $\mathcal{E}$ on L^X.

1 Prerequisities

Let $(L, \leq, \wedge, \vee)$ be a complete lattice, i.e. $(L, \leq)$ is a partially ordered set such that for every subset $A \subset L$ the join $\bigvee A$ and the meet $\bigwedge A$ are defined. In particular, $\bigvee L =: 1$ and $\bigwedge L =: 0$ are respectively the universal upper and the universal lower bounds in L. We assume that $1 \neq 0$, i.e. L has at least two elements.

Further, let $* : L \times L \to L$ be a binary operation on L such that

1. $\alpha * \beta = \beta * \alpha$ for all $\alpha, \beta \in L$;
2. $\alpha * (\beta * \gamma) = (\alpha * \beta) * \gamma$ for all $\alpha, \beta, \gamma \in L$;
3. $\alpha * 1 = \alpha$ and $\alpha * 0 = 0$ for all $\alpha \in L$;
4. $\alpha * \left(\bigvee_{j \in J} \beta_j \right) = \bigvee_{j \in J} (\alpha * \beta_j) \quad \forall \alpha \in L$ and $\forall \{\beta_j : j \in J\} \subset L$.

In what follows the 5-tuple $(L, \leq, \wedge, \vee, *)$ satisfying the above conditions will be referred to as a *commutative cl-monoid* (cf. e.g. [8]).

It is well known that a further binary operation $\mapsto : L \times L \to L$ (residuation) is defined on a commutative cl-monoid L which is connected with $*$ by Galois correspondence, that is

$$\alpha * \beta \leq \gamma \Longleftrightarrow \alpha \leq \beta \mapsto \gamma \text{ for all } \alpha, \beta, \gamma \in L.$$

Explicitely residuation $\mapsto$ is given by

$$\alpha \mapsto \beta = \bigvee \{\lambda \in L \mid \alpha * \lambda \leq \beta\}.$$

It is known that the following properties hold in a commutative *cl*-monoid $(L, \leq, \wedge, \vee)$ (cf e.g. [8]).

Proposition 1.1 *Let α, β, γ, α_i, β_i be arbitrary elements from a commutative cl-monoid L. Then:*

1. $\left(\bigvee_{i\in\mathcal{I}} \alpha_i\right) \mapsto \beta = \bigwedge_{i\in\mathcal{I}} (\alpha_i \mapsto \beta)$;
2. $\alpha \mapsto \left(\bigwedge_{i\in\mathcal{I}} \beta_i\right) = \bigwedge_{i\in\mathcal{I}} (\alpha \mapsto \beta_i)$;
3. *if* $\alpha \le \beta$ *then* $\alpha \mapsto \beta = 1$;
4. $\alpha * \beta \le \alpha \wedge \beta$;
5. $(\alpha \mapsto \beta) * (\beta \mapsto \gamma) \le \alpha \mapsto \gamma$;
6. $(\alpha * \beta) \mapsto (\gamma \mapsto \delta) \ge (\alpha \mapsto \gamma) * (\beta \mapsto \delta)$;
7. $(\alpha \mapsto \beta) \wedge (\beta \mapsto \alpha) = 1 \Rightarrow \alpha = \beta$;
8. $(\alpha * \beta) \mapsto \gamma = \alpha \mapsto (\beta \mapsto \gamma)$.

In what follows $L = (L, \le, \wedge, \vee, *)$ *always denotes a commutative cl-monoid.*

2 L-valued preodered sets, category PROSET(L) and some related categories

Definition 2.1 *An L-valued relation (or a fuzzy relation) on a set X is a map* $R : X \times X \to L$.

An L-valued relation R is called

1. *reflexive if* $R(x, x) = 1$ *for all* $x \in X$*;*
2. *transitive, if* $R(x, y) * R(y, z) \le R(x, z)$ *for all* $x, y, z \in X$;
3. *symmetric, if* $R(x, y) = R(y, x)$ *for all* $x, y \in X$;
4. *separated, if* $R(x, y) = R(y, x) = 1$ *implies that* $x = y$ *for all* $x, y \in X$.

Different authors have used different terminology to describe fuzzy relations with special properties. We shall use the following names:

A transitive L-valued relation is called an L-valued quasipreoder. A reflexive transitive L-valued relation is called an L-valued preoder. A separated L-valued preoder is called an L-valued partial order. A symmetric L-valued preorder is called an L-valued equality. The corresponding pair (X, R) *will be refereed to as an L-valued quasipreodered set, L-valued preodered set, an L-valued partially ordered set, and an L-valued set resp.*

If R is an L-valued preoder on a set, then given $x, y \in X$ the value $R(x, y)$ is interpreted as the degree to which x is greater than or equal to y. In case R is an L-valued equality on X, the intuitive meaning of the value $R(x, y)$ is the degree to which x and y are equal.

Remark 2.2 L-valued relations, usually in case when $L = [0, 1]$ and when $*$ is a left-semicontinuous t-norm (see e.g. [12]) were considered by many authors and they used different terminology. In particular, a fuzzy relation $R : X \times X \to [0, 1]$ satisfying (1), (2) and (3) is called *a fuzzy equality* in [8], [9] *a fuzzy equivalence* in [11], [13], or an *indistinguishability operator* [19]. In [3], [4], [5] a fuzzy relation $R : X \times X \to L$ is called *a fuzzy equality* if it satisfies all conditions (1) – (4).

Examples 2.3

1. Let $X = L$. Then by setting $R(x, y) = x \mapsto y$ we define a canonical L-valued partial oder on X and by setting $E(x, y) = R(x, y) \wedge R(y, x)$ we define a canonical L-valued separated equality on X (cf. e.g. [19]).
2. Let (X, ρ) be a pseudo-quasimetric space such that $\rho(x, y) \leq 1$ for all $x, y \in X$. Then by setting $R(x, y) = 1 - \rho(x, y)$ we define an L-valued preoder on X where L is the unit interval [0,1] endowed with the Łukasiewicz conjunction $*$. Moreover, if ρ is a pseudometric, then R is an L-valued equality, and in case ρ is a metric, the L-valued equality R is separated (cf e.g. [8]).
3. Let $\mathcal{A} \subseteq L^X$ be a family of L-subsets of X. Then, by setting
$$R(\mathcal{A})(x, y) = \bigwedge_{A \in \mathcal{A}} (A(x) \mapsto A(y))$$
we obtain an L-valued preoder on X.

Definition 2.4 *Given L-valued (quasi)preodered sets* (X, R_X) *and* (Y, R_Y) *a mapping* $f : X \to Y$ *is called extensional if*

$$R_X(x_1, x_2) \leq R_Y(f(x_1), f(x_2)) \textit{ for all } x_1, x_2 \in X.$$

L-valued quasi-preodered sets and extensional mappings between them form a category which will be denoted **QPROSET**(L). Its full subcategories consisting of L-valued preodered sets and L-valued sets will be denoted resp. by **PROSET**(L) and **SET**(L). To denote the subcategories of these categories determined by separated L-valued relation we use notations **SQPROSET**(L), **SPROSET**(L) and **SSET**(L) resp. However for the category of separated L-valued partial ordered sets **SPROSET**(L) which are separated by definition and which play a special role in our work an alternative notation **PAOSET**(L) will be also used. In the sequel our main interest here will be in categories **PROSET**(L) and **PAOSET**(L). Categories **SET**(L) and **SSET**(L) will be discussed in Section 6.

Proposition 2.5 *Let* X *be a set and* $\mathfrak{R}(X, L)$ *be the family of* L*-valued preoders on* X*. Then* $\mathfrak{R}(X, L)$ *is a complete lattice. Its bottom* $\inf \mathfrak{R}$ *is the discrete (or crisp) (L-valued) preoder*

$$R_{dis}(x, y) = \begin{cases} 1 \textit{ if } x = y; \\ 0 \textit{ if } x \neq y. \end{cases}$$

The top $\sup\mathfrak{R}$ *of the lattice* $\mathfrak{R}(X,L)$ *is the indiscrete (L-valued) preoder*

$$R_{ind}(x,y) = 1 \text{ for all } x,y \in X.$$

3 L-valued preoder on the L-powerset of an L-valued preodered set

Let (X,R) be an L-valued preodered set. Our first aim is to lift the L-valued preoder R from X to the L-valued quasipreoder $\mathcal{R}$ on the L-powerset L^X of X. We do it as follows.

Given $A, C \in L^X$ we set

$$\mathcal{R}(A,C) = \bigwedge_{x,z\in X} ((R(x,z) * A(x)) \mapsto C(z)).$$

Thus we obtain an L-valued relation

$$\mathcal{R} : L^X \times L^X \to L.$$

From the Proposition 1.1(7) it follows that equivalently $\mathcal{R}(A,C)$ can be defined by

$$\mathcal{R}(A,C) = \bigwedge_{x,z\in X} (R(x,z) \mapsto \big(A(x) \mapsto C(z)\big)).$$

Remark 3.1 The "defuzzified" meaning of the formulae

$$(R(x,z) * A(x)) \mapsto C(z) \text{ and } R(x,z) \mapsto (A(x) \mapsto C(z))$$

can be explained as follows:
If x is grater than or equal to z and x belongs to A then z should belong to C. In particular, in this case, taking $x = z$ we get $A(x) \le C(x)$ for every $x \in X$. By verifying this condition for all $x, z \in X$ we conclude whether A is greater than or equal to C – this is the "defuzzified" meaning of the value $\mathcal{R}(A,C)$.

In case $A, C \subseteq X$, that is A, C are crisp subsets of X

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } x \in A \text{ and } R(x,z) > 0 \text{ implies } z \in C \\ 0 \text{ otherwise }. \end{cases}$$

In particular, in case R is a crisp preoder $\le$ on X, then

$$\mathcal{R}(A,C) = 1 \text{ iff } x \in A \text{ and } z \le x \text{ implies that } z \in C$$

and $\mathcal{R}(A,C) = 0$ otherwise.

Proposition 3.2 *If* $R : X \times X \to L$ *is an L-valued reflexive relation on* X*, then*

$$\mathcal{R}(A,C) \ge \mathcal{R}(A,B) * \mathcal{R}(B,C) \text{ for all } A,B,C \in L^X,$$

and hence $\mathcal{R} : L^X \times L^X \to L$ *is an L-valued quasipreorder on* L^X.

Proof
To prove the statement we define an auxiliary relation

$$\mathcal{Q} : L^X \times L^X \to L$$

as follows: given $A, C \in L^X$ let

$$\mathcal{Q}(A,C) = \bigwedge_{x,y,z \in X} ((R(x,y) * R(y,z)) \mapsto (A(x) \mapsto C(z)))\,.$$

Obviously $\mathcal{Q}(A,C) \leq \mathcal{R}(A,C)$: just take $y = z$ and apply reflexivity of R according to which $R(z,z) = 1$.

On the other hand

$$\mathcal{Q}(A,C) \geq \mathcal{R}(A,B) * \mathcal{R}(B,C) \text{ for any } B \in L^X.$$

Indeed, fix any $x, y, z \in X$. Then

$$\begin{aligned} &(R(x,y) * R(y,z)) \mapsto (A(x) \mapsto C(z)) \geq \\ &\geq (R(x,y) * R(y,z)) \mapsto ((A(x) \mapsto B(y)) * (B(y) \mapsto C(z))) \geq \\ &\geq (R(x,y) \mapsto (A(x) \mapsto B(y))) * (R(y,z) \mapsto (B(y) \mapsto C(z)))\,. \end{aligned}$$

Now, taking infimum on the both sides of the obtained inequalities by $x, y, z \in X$ and taking into account that $\mathcal{Q}(A,C) \leq \mathcal{R}(A,C)$, we get the required inequality

$$\mathcal{R}(A,C) \geq \mathcal{R}(A,B) * \mathcal{R}(B,C) \quad \forall\ A, B, C \in L^X.$$

□

Corollary 3.3 *If $R : X \times X \to L$ is an L-valued preoder on X, thus R it is reflexive and transitive, then $\mathcal{R} : L^X \times L^X \to L$ is an L-valued quasipreorder on L^X.*

Remark 3.4 As a referee has noticed, in case R is an L-valued preoder, then $\mathcal{R} = \mathcal{Q}$. Indeed, the equality $\mathcal{Q} \leq \mathcal{R}$ is proved above. Conversely, by transitivity of R we have $R(x,y) * R(y,z) \leq R(x,z)$, and hence

$$(R(x,y) * R(y,z)) \mapsto (A(x) \mapsto C(z)) \geq R(x,z) \mapsto (A(x) \mapsto C(z)).$$

By taking infimum on $x, y, z \in X$ we get the inequality $\mathcal{Q} \geq \mathcal{R}$. Hence $\mathcal{R} = \mathcal{Q}$.

Remark 3.5 In analogy with $\mathcal{Q} : L^X \times L^X \to L$, we can define a relation $\mathcal{R}_n : L^X \times L^X \to L$ by setting

$$\mathcal{R}_n(A,C) = \bigwedge_{y_0, \dots y_n} ((R(y_0,y_1) * \ldots * R(y_{n-1},y_n)) \mapsto (A(x) \mapsto C(z)))\,,$$

where $y_0 = x, \ldots, y_n = z$. In these notations $\mathcal{R} = \mathcal{R}_1$ and $\mathcal{Q} = \mathcal{R}_2$.
Analogously, as above, one can show that for every $n \geq 2$ and for every $k, 1 < k < n$ the inequality

$$\mathcal{R}_k(A,C) \geq \mathcal{R}_n(A,C) \geq \mathcal{R}_k(A,B) * \mathcal{R}_{n-k}(B,C)$$

holds for all $A, B, C \in L^X$ and hence, in particular $\mathcal{R}_n = \mathcal{R}$ for all n in case R is an L-valued preoder.

Remark 3.6 Let us call an L-set A R-extensional, if

$$R(x,z) * A(x) \leq A(z) \text{ for all } x,z \in X.$$

(A similar property, in case R is an L-valued equality was considered by U. Höhle see e.g. [8] and other authors.)
The intuitive "defuzzified" meaning of this condition is the requirement that z should belong to A whenever x belongs to A and z is less than or equal to x.

Let R ve an L-valued quasipreoder on X and let L_R^X be the set of all R-extensional L-sets. In case $A,B,C \in L_R^X$ we have additionally that

$$\mathcal{R}(A,C) = \mathcal{Q}(A,C) \quad \forall\ A,C \in L^X.$$

Indeed, in the obtained inequality

$$\mathcal{R}(A,C) \geq \mathcal{Q}(A,C) \geq \mathcal{R}(A,B) * \mathcal{R}(B,C)$$

just take $B = A$.

In the proposition 3.2., we have proved that the relation $\mathcal{R}$ on L^X is an L-valued quasipreoder. Unfortunately, the reflexivity cannot be ensured by this relation if all L-sets were considered (even if R itself was reflexive). Nevertheless, the reflexivity can be proved if we restrict the domain of $\mathcal{R}$ to the set L_R^X of all R-extensional L-sets.

Theorem 3.7 *If*

$$R : X \times X \to L$$

is an L-valued preoder on X, then

$$\mathcal{R} : L_R^X \times L_R^X \to L$$

is a separated L-valued preoder on L_R^X.
Moreover $\mathcal{R} = \mathcal{Q}$ when restricted to L_R^X.

Proof From proposition 3.2 it follows that $\mathcal{R} : L_R^X \times L_R^X \to L$ is transitive. Further, by definitions and known properies, we conclude that under these assumptions for evry $A \in L_R^X$

$$\mathcal{R}(A,A) = \bigwedge_{x,z \in X} ((R(x,z) * A(x)) \mapsto A(z)) \geq \bigwedge_{x \in X} (A(x) \mapsto A(x)) = 1,$$

and hence $\mathcal{R}$ is reflexive.
Finally, to prove that $\mathcal{R} : L_R^X \times L_R^X \to L$ is separated let $A, C \in L^X$ and assume that $\mathcal{R}(A,C) = 1$. Then

$$\mathcal{R}(A,C) = \bigwedge_{x,z \in X} ((R(x,z) * A(x)) \mapsto C(z)) = 1.$$

This means that

$$\forall x, z \in X \quad (R(x,z) * A(x)) \mapsto C(z) = 1,$$

and in particular

$$\forall x \in X \quad (R(x,x) * A(x)) \mapsto C(x) = 1,$$

however this means that $A(x) \leq C(x)$ for all $x \in X$, that is $A \leq C$. In a similar way from the assumption $\mathcal{R}(C,A) = 1$ we conclude that $C \leq A$. Thus if $\mathcal{R}(A,C) = \mathcal{R}(C,A) = 1$, then $A = C$.
Now from the inequality

$$\mathcal{R}(A,C) \geq \mathcal{Q}(A,C) \geq R(A,B) * \mathcal{R}(B,C)$$

we get

$$\mathcal{R}(A,C) = \mathcal{Q}(A,C):$$

just take $B = A$.
□

From Propositions 3.7 and 3.2 we get

Theorem 3.8 *If*

$$R : X \times X \to L$$

is an L-valued preoder on X then

$$\mathcal{R} : L^X \times L^X \to L$$

is an L-valued quasipreoder on the powerset L^X and an L-valued partial oder on the extensional powerset L^X_R.

Examples 3.9 In all these examples

$$\mathcal{R} : L^X \times L^X \to L$$

is an L-valued quasipreoder on L^X induced by an L-valued preoder

$$R : X \times X \to L$$

unless specified. By α_X we denote the constant function $\alpha_X : X \to L$ with value $\alpha \in L$.

1. Let $A \in L^X_R$. Then
$\mathcal{R}(A, 0_X) = \left(\bigvee_{x \in X} A(x) \right) \to 0.$

2. $\mathcal{R}(A, 1_X) = 1$ for any $A \in L^X$.

3. $\mathcal{R}(1_X, A) = 1 \to \bigwedge_{x \in X} A(x).$

4. Given $a \in X$ let 1_a stand for the characteristic function of the set $\{a\}$. Then

$$\mathcal{R}(A, 1_a) = \left(\bigvee_{x \neq a} A(x) \right) \to 0.$$

In particular, if $a \neq b, a, b \in X$, then $\mathcal{R}(1_a, 1_b) = 0$.

5. For every $a \in X$ we define an L-set

$$s_a : X \to L \quad \text{ by } s_a(x) = R(a, x).$$

This is the so called singleton generated by a. Since

$$s_a(x) * R(x, z) = R(a, x) * R(x, z) \leq R(a, z) = s_a(z),$$

singletons are extensional. Moreover, it is easy to notice that s_a is the smallest one of all extensional L-sets, which are greater than or equal to the L-set 1_a. Let $a, b \in X$. Then

$$\mathcal{R}(s_a, s_b) = \bigwedge_{x,z \in X} ((R(a, x) * R(x, z)) \mapsto R(b, z)) =$$

$$\bigwedge_{z \in X} (R(a, z) \mapsto R(b, z)) \leq$$

$$\leq R(a, a) \mapsto R(b, a) = R(b, a).$$

On the other hand, since

$$R(a, b) * R(b, z) \leq R(a, z)$$

from the Galois connection we conclude that for all $a, b \in X$ and every $z \in X$ it holds

$$R(b, z) \mapsto R(a, z) \geq R(a, b),$$

and, since this holds for any $z \in X$, by taking infimum on x we obtain:

$$\mathcal{R}(s_a, s_b) \geq R(b, a),$$

and hence

$$\mathcal{R}(s_a, s_b) = R(b, a).$$

This equality can be interpreted as follows. Let $\mathcal{R}^c$ stand for the order on L^X obtained by reversing of $\mathcal{R}$. That is

$$\mathcal{R}^c(A, C) = \mathcal{R}(C, A).$$

Now the obtained equality means that by assigning to each $a \in X$ its singleton $s_a \in L_E^X$ we may identify (X, R) with the L-valued partially odered subset $(S, \mathcal{R}_S^c)$ of the L-valued partially ordered set $(L_R^X, \mathcal{R})$ where $S = \{s_a : a \in X\}$ and $\mathcal{R}_S^c$ is the restriction of $\mathcal{R}^c$ to S.

4 Powerset functor $\Phi : \mathbf{PROSET}(L) \to \mathbf{PAOSET}(L)^{op}$

In this section we show that the construction assigning to an L-valued preodered set (X, R) its extensional powerset $(L^X_R, \mathcal{R})$ can be considered as a contravariant functor Φ from the category **PROSET**(L) into the category **PAOSET**(L) that is as a functor

$$\Phi : \mathbf{PROSET}(L) \to \mathbf{PAOSET}(L)^{op}.$$

We shall discuss some properties of this functor. We start with the following

Proposition 4.1 *Let (X, R_X), (Y, R_Y) be L-valued preodered sets and*

$$f : X \to Y$$

be an extensional mapping. Then for every $C, D \in L^Y$ it holds

$$\mathcal{R}_X(f^{-1}(C), f^{-1}(D)) \geq \mathcal{R}_Y(C, D).$$

Recall that the preimage of an L-set $C : Y \to L$ under a function $f : X \to Y$ is defined by the equality $f^{-1}(C)(x) = (f \circ C)(x)$.

Proof follows from the next series of inequalities:

$$\mathcal{R}_X(f^{\leftarrow}(C), f^{\leftarrow}(D)) =$$

$$= \bigwedge_{x,x' \in X} \big(R_X(x, x') \mapsto \big(f^{-1}(C)(x) \mapsto f^{-1}(D)(x')\big)\big) =$$

$$= \bigwedge_{x,x' \in X} (R_X(x, x') \mapsto (C(f(x)) \mapsto D(f(x')))) \geq$$

$$\geq \bigwedge_{x,x' \in X} (R_Y(f(x), f(x')) \mapsto (C(f(x)) \mapsto D(f(x'))) \geq$$

$$\geq \bigwedge_{y,y' \in Y} (R_Y(y, y') \mapsto (C(y) \mapsto D(y'))) = \mathcal{R}_Y(C, D).$$

From Proposition 4.1 and Theorem 3.8 we get

Theorem 4.2 *By assigning to each L-valued preodered set*

$$(X, R) \in Ob(\mathbf{PROSET}(L))$$

its extensional powerset $(L^X_E, \mathcal{R})$ and to each extensional mapping

$$f : (X, R_X) \to (Y, R_Y)$$

the mapping

$$f^{\leftarrow} : (L^Y_R, \mathcal{R}_X) \to (L^X_R, \mathcal{R}_Y)$$

we define a functor

$$\Phi : PROSET(L) \to PAOSET(L)^{op}.$$

(Here $f^{\leftarrow}(C) = f^{-1}(C)$ for $C \in L^Y$, cf. e.g. [14].)

Theorem 4.3 *Functor*

$$\Phi : \mathbf{PROSET}(L) \to \mathbf{PAOSET}(L)^{op}$$

is one-to-one on objects. The restriction Φ' of the functor Φ to $\mathbf{PAOSET}(L)$, that is the functor

$$\Phi' : \mathbf{PAOSET}(L) \to \mathbf{PAOSET}(L)^{op}$$

is an embedding.

Proof Let R_1 and R_2 be L-valued relations on a set X and $R_1 \neq R_2$. Then there exist $a, b \in X$ such that $R_1(a,b) \neq R_2(a,b)$. However, as it was shown above, $\mathcal{R}_1(s_a, s_b) = R_1(b,a)$ and $\mathcal{R}_2(s_a, s_b) = R_2(b,a)$ (where s_a, s_b are singletons corresponding to the points a, b).
Hence $\mathcal{R}_1 \neq \mathcal{R}_2$.
□

Remark 4.4 In a similar way as functor Φ one can consider a functor

$$\tilde{\Phi} : \mathbf{PROSET}(L) \to \mathbf{QPROSET}(L)^{op}$$

assigning to each (X, R) the L-valued quasipreoder set $(L^X, \mathcal{R})$. The image $\tilde{\Phi}(\mathbf{PROSET}(L))$ is a subcategory of the category $\mathbf{QPROSET}(L)^{op}$. We shall not go into details of this construction here.

Remark 4.5 *Functors Φ and $\tilde{\Phi}$ are order reversing.*

Indeed, assume that R_1 and R_2 are two L-valued preoders on X and $R_1 \leq R_2$. Then for any $A, C \in L^X$

$$\mathcal{R}_1(A,C) = \bigwedge_{x,z \in X} ((R_1(x,z) * A(x)) \mapsto C(z)) \geq$$

$$\geq \bigwedge_{x,z \in X} ((R_2(x,z) * A(x)) \mapsto C(z)) = \mathcal{R}_2(A,C)$$

and hence $\mathcal{R}_1 \geq \mathcal{R}_2$.

It would be interesting to study the properties of these functors. In particular, we have the following hypothesis:
Hypothesis 1. Let Z be a set, (X_i, R_i) be a family of sets endowed with some order type relation, and

$$f_i : Z \to X_i, i \in \mathcal{I}$$

be a family of mappings. Further, let R_0 be an order-type relation on Z, initial for this family of mappings. Then the corresponding L-valued relation on the powerset L^Z (or L^Z_R) $\mathcal{R}_0$ is the *final* order type relation for the family of mappings

$$f_i^{\leftarrow} : (L^X, \mathcal{R}_i) \to L^Z.$$

Hypothesis 2 Let Z be a set, (X_i, R_i) be a family of sets endowed with some order type relation, and

$$f_i : X_i \to Z, i \in \mathcal{I}$$

be a family of mappings. Further, let R^0 be an order-type relation on Z, final for this family of mappings. Then the corresponding L-valued relation on the powerset L^Z (or L^Z_R) $\mathcal{R}^0$ is the *initial* order type relation on L^Z (or L^Z_R) for the family of mappings

$$f_i^{\leftarrow} : L^Z \to (L^X, \mathcal{R}_i).$$

A related problem, how do these functors behave on products and coproducts?

5 Lattices $QPR(L^X)$ and $PR(L^X)$

Given a set X we denote by $PR(L^X)$ the family of all L-valued preoders $\mathcal{R}$ on L^X obtained from L-valued preoders R on X. In other words $\mathcal{S} \in PR(L^X)$ if and only if $(L^X_E, \mathcal{S}) \in Ob(\Phi(\mathbf{PROSET}(L))$. In a similar way $\mathcal{S} \in QPR(L^X)$ if and only if $(L^X, \mathcal{S}) \in Ob(\Phi(\mathbf{QPROSET}(L))$.

From the previous results it follows, that $QPR(L^X)$ and $PR(L^X_R)$ are bounded lattices where the greatest element $\mathcal{R}_\top$ is induced by the discrete (L-valued) preoder R_{dis} on X and the smallest element $\mathcal{R}_\perp$ is induced by indiscrete L-valued preoder R_{ind} on X. Explicitely, for the largest element $\mathcal{R}_\top$: given $A, C \in L^X_R$

$$\mathcal{R}_\top(A, C) = \bigwedge_{x \in X} (A(x) \mapsto C(x)) \,.$$

Indeed,

$$\mathcal{R}_\top(A, C) = \bigwedge_{x,z \in X} (R_{dis}(x, z) \mapsto (A(x) \mapsto C(z)))$$

and

$$R_{dis}(x, z) \mapsto (A(x) \mapsto C(z)) = 1 \text{ if } x \neq z$$

while

$$R_{dis}(x, x) \mapsto (A(x) \mapsto C(z)) = A(x) \mapsto C(z).$$

For the smallest element $\mathcal{R}_\perp$: given $A, C \in L^X_R$

$$\mathcal{R}_\perp(A, C) = \bigvee_{x \in X} A(x) \mapsto \bigwedge_{z \in X} C(z).$$

Indeed

$$\mathcal{R}(A, C) = \bigwedge_{x,z \in X} (R_{ind}(x, z) \mapsto (A(x) \mapsto C(z))) =$$

$$= \bigwedge_{x,z \in X} (1 \mapsto (A(x) \mapsto C(z))) = \bigwedge_{x,z \in X} (A(x) \mapsto C(z)) =$$

$$= \bigvee_{x \in X} A(x) \mapsto \bigwedge_{z \in X} C(z).$$

Note that in case A is R_{ind}-extensional, then

$$\mathcal{R}_{\perp}(A,A) = \bigwedge_{x\in X} (A(x) \mapsto A(x)) = 1,$$

and hence $\mathcal{R}_{\perp}$ is an L-valued preoder, but generally $\mathcal{R}_{\perp}$ is only a quasi-preoder.

Examples 5.1

1. Let $L = [0,1]$ and $* = \wedge$ in $(L, \leq, \wedge, \vee, *)$, that is

$$(L, \leq, \wedge, \vee)$$

is viewed as a Heyting algebra. Recall that the corresponding residium is defined by

$$\alpha \mapsto \beta = \begin{cases} 1 \text{ if } \alpha \leq \beta \text{ and} \\ 0 \text{ otherwise} \end{cases}$$

for $\alpha, \beta \in L$.

(a) Let $R = R_{ind}$ be the indiscrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A,C) = \bigvee_{x\in X} A(x) \mapsto \bigwedge_{z\in X} C(z).$$

Hence

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } \sup\limits_{x\in X} A(x) \leq \inf\limits_{x]\in X} C(x) \text{ and} \\ \inf\limits_{x\in X} C(x) \text{ otherwise .} \end{cases}$$

In particular, for $A, C \subseteq X$

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } A = \emptyset \text{ or } C = X \text{ and} \\ 0 \text{ otherwise .} \end{cases}$$

(Note that X and $\emptyset$ are the only extensional sets in this case.)

(b) Let $R = R_{dis}$ be the discrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A,C) = \bigwedge_{x\in X} (A(x) \mapsto C(x)).$$

Hence

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } A(x) \leq C(x) \quad \forall x \in X \text{ and} \\ \inf\limits_{x}\{C(x) \mid x \in X, A(x) \geq C(x)\} \text{ otherwise.} \end{cases}$$

In particular, for $A, C \subseteq X$

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } A \subseteq C \text{ and} \\ 0 \text{ if } A \not\subseteq C. \end{cases}$$

2. Let $L = [0,1]$ and $*$ be the Łukasiewicz conjunction that is

$$\alpha * \beta = \max\{\alpha + \beta - 1, 0\} \text{ for } \alpha, \beta \in [0,1]$$

and hence $(L, \leq, \wedge, \vee, *)$ is an MV-algebra. Recall that the corresponding residium is defined by

$$\alpha \mapsto \beta = \min\{1 - \alpha + \beta,\ 1\}.$$

(a) Let $R = R_{ind}$ be the indiscrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A,C) = \bigwedge_{x,z \in X} \min\{1 - A(x) + C(z),\ 1\}.$$

Hence

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } \sup\limits_{x \in X} A(x) \leq \inf\limits_{x \in X} C(x) \text{ and} \\ 1 - \sup\limits_{x \in X} A(x) + \inf\limits_{x \in X} C(x) \ \text{ otherwise.} \end{cases}$$

(b) Let $R = R_{dis}$ be the discrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A,C) = \bigwedge_{x,z \in X} ((R(x,z) * A(x)) \mapsto C(z)) =$$

$$= \bigwedge_{x \in X} (A(x) \mapsto\ C(x))$$

Hence

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } A(x) \leq C(x)\ \forall x \in X \text{ and} \\ \inf\limits_{x \in X} \{1 - A(x) + C(x)\} \text{ otherwise.} \end{cases}$$

3. Let $L = [0,1]$ and $*$ be the product on $[0,1]$ that is $\alpha * \beta = \alpha \cdot \beta$ for $\alpha, \beta \in [0,1]$. Recall that the corresponding residium in this case is defined by

$$\alpha \mapsto \beta = \begin{cases} 1 \text{ if } \alpha \leq \beta \text{ and} \\ \frac{\beta}{\alpha} \text{ otherwise .} \end{cases}$$

(a) Let $R = R_{ind}$ be the indiscrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A,C) = \bigvee_{x \in X} A(x) \mapsto \bigwedge_{x \in X} C(x).$$

Hence

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } \sup\limits_{x \in X} A(x) \leq \inf\limits_{x \in X} C(x) \text{ and} \\ \dfrac{\bigwedge_{x \in X} C(x)}{\bigvee_{x \in X} A(x)} \text{ otherwise.} \end{cases}$$

(b) Let $R = R_{dis}$ be the discrete L-valued preoder on X and $A, C \in L^X$. Then

$$\mathcal{R}(A,C) = \bigwedge_{x\in X} (A(x) \mapsto C(x))$$

Hence

$$\mathcal{R}(A,C) = \begin{cases} 1 \text{ if } A(x) \le C(x)\ \forall x \in X \text{ and} \\ \frac{\bigwedge_{x\in X: A(x)\ge C(x)} C(x)}{\bigwedge_{x\in X: A(x)\ge C(x)} A(x)} \text{ otherwise.} \end{cases}$$

6 L-valued equality on the L-powerset of an L-valued set

Let X be a set and $E : X \times X \to L$ be an L-valued equality on X, that is a symmetric preoder. Refering to Section 3 by setting

$$\mathcal{R}(A,C) = \bigwedge_{x,z\in X} (E(x,z) \mapsto (A(x) \mapsto C(z)))$$

we obtain a separated L-valued preoder on L^X_E (where L^X_E is the family of all extensional L-subsets of X) and an L-valued quasipreoder on L^X. In the next theorem we symmetrize this relation in order to get an L-valued equality on L^X_E.

Theorem 6.1 *For $A, C \in L^X$ let*

$$\mathcal{E}(A,C) = \mathcal{R}(A,C) \wedge \mathcal{R}(C,A).$$

Then $\mathcal{E} : L^X_E \times L^X_E \mapsto L$ is an L-valued equality on L^X_E.

Proof The reflexivity of $\mathcal{E}$ follows from the reflexivity of $\mathcal{R}$.
The symmetry of $\mathcal{E}$ is obvious from the definition.
The transitivity follows from the next series of (in)equalities (see Proposition 1.1):

$$\mathcal{E}(A,B) * \mathcal{E}(B,C) =$$
$$= (\mathcal{R}(A,B) \wedge \mathcal{R}(B,A)) * (\mathcal{R}(B,C) \wedge \mathcal{R}(C,B)) \le$$
$$\le (\mathcal{R}(A,B) * \mathcal{R}(B,C)) \wedge (\mathcal{R}(C,B) * \mathcal{R}(B,A)) \le$$
$$\le \mathcal{R}(A,C) \wedge \mathcal{R}(C,A) = \mathcal{E}(A,C).$$

Hence the pair $(L^X_E, \mathcal{E})$ is a separated L-valued set.
□

Thus, assigning to an L-valued set (X,E) the pair $(L^X_E, \mathcal{E})$ we obtain a functor

$$\Psi : \mathbf{SET}(L) \to \mathbf{SSET}(L)^{op},$$

where $\mathbf{SSET}(L)$ is the category of all separated L-valued L-sets.

One can get results about L-valued equalities on the L-powerset and the funcor Ψ analogous to the results about L-valued preoders on the L-powersets and the functor Φ discussed in sections 3, 4 and 5.

Remark 6.2 There are alternative ways how one can extend an L-valued equality $E : X \times X \to L$ to the L-powerset L^X_E. In particular, let

$$\mathcal{E}' : L^X_E \times L^X_E \to L$$

be defined by setting $\mathcal{E}'(A, C) = \mathcal{R}(A, C) * \mathcal{R}(C, A)$. One can easily notice that $\mathcal{E}'$ is an L-valued equality on L^X_E and that $\mathcal{E}' \leq \mathcal{E}$. However, the equality generally does not hold.

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