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On Two Conditional Entropies without Probability

D.Vivona - M.Divari *
Sapienza - Universitá di Roma
Dip. di Metodi e Modelli Matematici per le Scienze Applicate
16, Via A.Scarpa - 00161 Roma (Italy)
vivona@dmmm.uniroma1.it

Abstract

We generalize the conditional entropy without probability given by Benvenuti in [1] and we recognize that this form is the most general compatible with the given properties.

Then we compare our form of conditional entropy given in [4] with Benvenuti's one.

Key words: Entropy, Conditional entropy, Functional equations.

1 Introduction

In a probabilistic setting, Khinchin and Yaglom proposed a form of conditional entropy, [3, 5].

Later, Benvenuti defined the conditional entropy without probability, [1].

In this paper, by using the variables of Benvenuti's form, we give a generalization of conditional entropy.

Then, we point out the link between Benvenuti's expression and the form found by us in a recent paper, [4].

2 Preliminaries

In the crisp setting, following Forte, [2], we consider the following model.

1) *Setting.* X is an abstract space, $\mathcal{A}$ a σ-algebra of crisp sets $A \subset X$, π_A is a partition of A :

$$\pi_A = \{A_1, ..., A_i, ..., A_n / A_i \cap A_h = \emptyset, i \neq h, A_i \neq \emptyset, A_i \in \mathcal{A},\ \cup_{\rm i=1}^{\rm n} \rm A_i = A\} \quad , \qquad (1)$$

A is the support of π_A, $\mathcal{E}$ is the class of all partitions of subsets A of X. This class is not empty because it contains, at least, the partition consisting of the only set A, which will be indicated with $\{A\}$.

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2) *Order*. The partition π_A is less fine than π'_A ($\pi_A \preceq \pi'_A$) if every element of π'_A is included in an element of π_A.

3) *Algebraical independence*. Given two partitions π_A as in (1) and

$$\pi_B = \{B_1, ..., B_j, ..., B_m / B_j \cap B_k = \emptyset, j \neq k, B_j \neq \emptyset, B_j \in \mathcal{A}, \cup_{j=1}^{m} B_j = B\} , \quad (2)$$

they are algebraically independent if $A_i \cap B_j \neq \emptyset, \ \forall \ i = 1, ...n, \ j = 1, ...m$.

4) *Entropy measure*. The entropy H without probability is a map $H : \mathcal{E} \to I\!R_0^+$ with the following properties: $\forall \ \pi_A, \pi'_A, \pi_B \in \mathcal{E}$:
(i) $\pi_A \preceq \pi'_A \Rightarrow H(\pi_A) \leq H(\pi'_A)$.
Furthermore: $H(\{X\}) = 0$ and $H(\emptyset) = +\infty$.
(ii) $H(\pi_A \cap \pi_B) = H(\pi_A) + H(\pi_B)$,
if π_A and π_B are algebraically independent.

3 Conditional entropy

Benvenuti in [1] defined the conditional entropy without probability of a partition $\pi_A \in \mathcal{E}$ in axiomatic way as

$$D(\pi_A) = H(\pi_A) - H(\{A\}) = H(\pi_A) - J(A), \quad (3)$$

where $J(A)$ is the information of the support A of π_A :
the conditional entropy $D(\pi_A)$ of the partition π_A is defined as the gap between the unconditional entropy $H(\pi_A)$ and the entropy of the support A.
The conditional entropy (3) enjoys the following properties: $\forall \ \pi_A, \pi'_A, \pi_k$ ($k = 1, ..., n$) $\in \mathcal{E}$:
(I) $D(\{A\}) = 0$,
(II) $\pi_A \preceq \pi'_A \Rightarrow D(\pi_A) \leq D(\pi'_A)$,
(III) $D\Big(\cap_{k=1}^{n} \pi_k \Big) = \Sigma_{k=1}^{n} \ D(\pi_k)$,
if the partitions π_k are algebraically independent.
In the setting of Benvenuti's axioms, we give a generalization of (3), putting

$$D'(\pi_A) = \Psi \Big(H(\pi_A), J(A) \Big) , \quad (4)$$

where $\Psi : I\!R_0^+ \times I\!R_0^+ \to I\!R_0^+$ must satisfy the following properties for all $\pi_A, \pi'_A, \pi_B \in \mathcal{E}$:
(I') $\Psi \Big(H(\{A\}), J(A) \Big) = 0$,
(II') $\pi_A \preceq \pi'_A \Rightarrow \Psi \Big(H(\pi_A), J(A) \Big) \leq \Psi \Big(H(\pi'_A), J(A) \Big)$,
(III') $\Psi \Big(H(\pi_A \cap \pi_B), J(A \cap B) \Big) = \Psi \Big(H(\pi_A), J(A) \Big) + \Psi \Big(H(\pi_B), J(B) \Big)$,
if π_A and π_B are algebraically independent.

Putting $x = H(\pi_A), x' = H(\pi_A^{'}), y = J(A), z = H(\pi_B), t = J(B)$ with $x, x', y, z, t \in [0, +\infty)$ and $x > y, x' > y, z > t$, from (I')-(III') we obtain the following system of functional equations:

$$\begin{cases} (a) \quad \Psi(y,y) = 0 \\ (b) \quad x \leq x' \Rightarrow \Psi(x,y) \leq \Psi(x',y) \\ (c) \quad \Psi(x+z, y+t) = \Psi(x,y) + \Psi(z,t) \quad . \end{cases}$$

4 Solution of the problem

First of all, we recognize that the system is satisfied by the function:

$$\Psi(x,y) = x - y \quad , \tag{5}$$

so we find again the Benvenuti formulation (3).

Now, we look for other solutions, restricting ourselves to functions of the kind

$$\Psi(x,y) = h^{-1}\Big(h(x) - h(y)\Big) \tag{6}$$

where the function h is strictly increasing with $h(0) = 0$.

It is immediate verify that every function Ψ of the kind (6) is solution of the equations (a) and (b).

The equation (c) becomes

$$h^{-1}\Big(h(x+z) - h(y+t)\Big) = h^{-1}\Big(h(x) - h(y)\Big) + h^{-1}\Big(h(z) - h(t)\Big) . \tag{7}$$

Putting

$$y = 0, \quad z = t, \tag{8}$$

the equation (7) is

$$h^{-1}\Big(h(x+t) - h(t)\Big) = h^{-1}(h(x)) + h^{-1}(0) = x$$

and therefore

$$h(x+t) = h(x) + h(t) : \tag{9}$$

this is the well-known Cauchy equation whose solution is $h(u) = c\ u,\ c \in \mathbb{R}_0^+$.

From (6), we deduce immediately

$$\Psi(x,y) = x - y \ ,$$

and our generalization coincides with (3).

Therefore, when we use as variables the entropy $H(\pi_A)$ and the information $J(A)$ and we restrict ourselves to the case described in 6, we have a unique conditional entropy

$$D'(\pi_A) = \Psi\Big(H(\pi_A), J(A)\Big) = H(\pi_A) - J(A) = D(\pi_A)$$

which coincides with the conditional entropy given by Benvenuti.

5 Conclusion

In [4], in the crisp case, the authors have characterized an entropy $H_{\pi'}(\pi)$ for a partition π conditioned by a partition π' as function of $H(\pi \cap \pi)$ and $H(\pi')$:

$$H_{\pi'}(\pi) = \Phi(H(\pi \cap \pi'), H(\pi')).$$

We have proved that

$$H_{\pi'}(\pi) = H(\pi \cap \pi') - H(\pi').$$

That means that if the conditioning partition π' is the support set A the conditional entropy is exactly Benvenuti's conditional entropy (3).

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