Mathware & Soft Computing 12 (2005) 107-120

Majority Multiplicative Ordered Weighting Geometric Operators and Their Use in the Aggregation of Multiplicative Preference Relations

J.I. Peláez, J.M. Doña and A. Mesas
Department of Languages and Computer Sciences
University of Malaga, Malaga 29071, Spain
jignacio@lcc.uma.es

Abstract

In this paper, we introduced the majority multiplicative ordered weighted geometric (MM-OWG) operator and its properties. This is a general type of the aggregate dependent weights which we have applied in geometric environment. The MM-OWG operator is based on the OWG operators and on the majority operators. We provide the MM-OWG operators to aggregate in a multiplicative environment, i.e. when it's necessary to aggregate information given on a ratio scale. Therefore, it allows us to incorporate the concept of majority in problems where the information is provided using a ratio scale. Its properties are studied and an application for multicriteria decision making problems with multiplicative preference relations is presented.

1 Introduction

Decision making is a usual task in human activities where a set of experts work in a decision process to obtain a final value which is representative of the group. The first step of this decision process is constituted by the individual evaluations of the experts; each decision maker rates each alternative on the basis of an adopted evaluation scheme [4, 6, 13, 19]. It is possible to use different scales to represent the information of each experts, this scales are [19] the nominal scale, which consists essentially of assigning labels to objects; the ordinal scale which gives a rank order of objects and is invariant under monotone increasing transformations; the interval scale, unique up to positive linear transformation of the form $y = ax + b$; $a > 0$; the difference scale, invariant under a transformation of the form $y = x + b$; and finally the ratio scale, invariant under positive linear transformations of the form $y = ax$; $a > 0$. We assume that at the end of this step each alternative has associated a performance judgment on the linguistic scale (or numeric scale).

The second step consists in determining for each alternative a consensual value which synthesizes the individual evaluation. This value must be representative of a collective estimation and is obtained by the aggregation of the opinions of the experts [3, 12, 14, 19]. Finally, the process concludes with the selection of the best alternative/s as the most representative value of solution of the problem.

One of the main problems in decision making is how to define a fusion method which considers the majority opinions from the individual opinions. To obtain a value of synthesis of the alternatives which is representative of the opinions of the experts exist diverse approaches in which are realized an aggregation guided by the concept of majority, where majority is defined as a collective evaluation in which the opinions of the most of the experts involved in the decision problem are considered. In these approaches the result is not necessarily of unanimity, but it must be obtained a solution with agreement among a fuzzy majority of the decision makers [7, 10, 11, 15, 23].

To obtain a majority value in the decision making exits different operators [14, 20, 23], i.e Yager in [20] introduced a new aggregation technique based on the ordered weighted averaging (OWA) operators. An OWA operator of dimension n is a mapping $F : R^n \rightarrow R$ that has an associated n vector $W = [w_1, w_2, \ldots, w_n]^T$ such that $w_i \in [0, 1]$ and $\sum\limits_{i=1}^{n} w_i = 1$.

Furthermore $F(a_1, a_2, \ldots, a_n) = \sum\limits_{j=1}^{n} w_j \cdot b_j$where b_j is the jth largest of the a_i.

A fundamental aspect of this operator is the re-ordering step, in particular an aggregate a_i is not associated with a particular weight w_i, but rather a weight is associated with a particular ordered position of aggregate.

It is noted that different OWA operators are distinguished by their weighting function. In [20] Yager pointed out three important special cases of OWA aggregations:

1.F^*. In this case $W = W^* = [1, 0, ..., 0]^T$ and $F^*(a_1,a_2, ..., a_n) = \text{Max}(a_i)$

2. F_*. In this case $W = W_* = [0, 0, ..., 1]^T$ and $F^*(a_1,a_2, ..., a_n) = \text{Max}(a_i)$

3. F_{Ave}. In this case $W = W_{ave} = \left[\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right]^T$ and $F_{ave}(a_1, a_2, \ldots, a_n) = \frac{1}{n} \cdot \sum\limits_{i=1}^{n} a_i$

The OWA operators have been applied in different areas using the different scales [22]. In this contribution we are interested in problems where the ratio scale is used to model the information, i.e. Analytic Hierarchy Process (AHP) [19].

As shown in [1, 2], the proper aggregation operator of ratio-scale measurements is the geometric mean and is not the arithmetic mean. However, this operator does not allow incorporating the concept of fuzzy majority in the decision processes. We could use the OWA operator, but this is not possible because it presents a similar behaviour to the arithmetic mean. For this reason in [5] a new ordered weighted geometric operator is defined for synthesizing ratio-scale judgements. This operator combines the definition of the OWA weighting vector with the geometric mean. For this reason this operator presents the same advantages and problems to characterize the majority concept in the aggregation which appears in OWA operators [16, 17, 23].

In this contribution, we introduced the majority multiplicative ordered weighted geometric (MM-OWG) operator for synthesizing values in multiplicative environment. This operator combines the definition of the majority operators MA-OWA with OWG operator. It allows incorporating the concept of majority in the aggregation process. We study its properties and present an application for multicriteria decision making processes with multiplicative preference relations. In order to do this, the paper is set out as follows. In section 2, the concept of aggregate dependent weights and the OWG and the Majority operators are introduced; in section 3 the MM-OWG operator and its properties are presented; in section 4 an example of its use in multicriteria decision making is shown; and finally, in section 5, the conclusions are pointed out.

2 Preliminaries: Aggregate Dependent Weights, OWG, and MA-OWA Operators

We start this section by providing the definitions that are needed to justify the introduction of the MM-OWG operator.

2.1 Aggregate dependent weights

Usually the OWA operators are defined as a function where the weights depend on aggregates.

In the definition for the OWA operator is indicated that

$$F\left(a_1, a_2, \ldots, a_n\right) = \sum_{i=1}^{n} w_i \cdot b_i$$

Where the b_i is the ith largest of the a_i. The weights were requires to satisfy two conditions:

(1) $w_i \in [0,\ 1]$

(2) $\sum\limits_{i=1}^{n} w_i = 1$.

Habitually it is assumed that the weights were fixed given constant values. But it's possible generalize the concept of OWA aggregation by allowing the weights to be a function of the aggregates or more precisely the ordered aggregates, the b_i. It is still required that the weights satisfy the conditions (1) and (2) [21]. Thus we allow that

$$w_i = f_i\left(b_1, b_2, \ldots, b_n\right)$$

and then

$$F\left(a_1, a_2, \ldots, a_n\right) = \sum_{i=1}^{n} f_i\left(b_1, b_2, \ldots, b_n\right) \cdot b_i$$

In this case where the weights depend on aggregates, many, but not all, of the properties of the OWA operator still hold:

(1) All aggregate dependent operators still lie between F^* and F_*.

(2) The operator is still idempotent $F(a,\ a,\ \ldots,\ a) = a$.

(3) The operator is still commutative, the indexing of the a_i's is not important.

One property that is not necessarily retained is that monotonicity. Assume $A = (a_1, ..., a_n)$ and $A' = (a'_1, ..., a'_n)$ are two collections of aggregates such that $a_i >= a'_i$ for all i. If the weights are constant then $F(A) >= F(A')$.

We subsequently see that this is not the case when the weights are aggregate dependent.

In this way, an OWA aggregation is defined as neat if the aggregated value is independent of the ordering. Let $A = (a_1, ..., a_n)$ be our inputs, let $B = (\mathrm{b}_1, ..., \mathrm{b}_n)$ be the inputs ordered and $C = (\mathrm{c}_1, ..., c_n) = \mathrm{Perm}(a_1, ..., a_n)$ be any permutation of the input. Formally the neat OWA operator is defined if

$$F(a_1, a_2, \ldots, a_n) = \sum_{i=1}^{n} w_i \cdot b_i$$

Is the same for the any assignment $C = B$.

One simple example of neat OWA operator is when $w_i = \frac{1}{n}$. In this case

$$F(a_1, a_2, \ldots, a_n) = \frac{1}{n} \cdot \sum_{i=1}^{n} a_i$$

If the weights w_i are fixed, this is the only possible neat OWA operator. However, when we allow aggregate dependent weights we can get more neat operators.

One important characteristic of the neat OWA aggregators is that they don't need to be ordered. This implies that the formulation for the neat OWA aggregators can be written using the arguments a_i with introducing the ordered inputs b_i.

There are several family the neat operator [21]. A family of aggregate dependent weights that we shall study in this work are the called MA-OWA operators introduced by Peláez and Doña [14].

2.2 The OWG Operator

The decision problem when the experts express their preferences using multiplicative preference relations has been solved by Saaty using the decision analytic hierarchical process (AHP), which obtains the set of solution alternatives by means of the eigenvector method [19]. However, this decision process is not guided by the concept of fuzzy majority. Chiclana et al. [4] obtained the transformation function between multiplicative and fuzzy preference relations, which is given in the following result.

Proposition 1. Suppose that we have a set of alternatives $X = \{x_1, \ldots, x_n\}$ and associated with it, a multiplicative reciprocal preference relation $\mathrm{A} = (a_{ij})$, with $a_{ij} \in [1/9, 9]$ and $a_{ij} \cdot a_{ji} = 1$, $\forall i, j$. Then the corresponding fuzzy reciprocal preference relation $\mathrm{P} = (p_{ij})$, associated with A, with $p_{ij} \in [0, 1]$ and $p_{ij} + p_{ji} = 1$, $\forall i, j$ is given as follows: $p_{ij} = f(a_{ij}) = 1/2\ (1 + log_9 a_{ij})$.

The foregoing transformation function is bijective and, therefore, allows us to transpose concepts that have been defined for fuzzy preference relations to multiplicative preference relations. Based on this, Chiclana et al. [4, 8] considered

GDM problems where the information about the alternatives is represented using multiplicative preference relations and designed the OWG operator [5, 9].

Definition. An OWG operator of dimension n is a function $F^G : R^n \rightarrow R$, to which a set of weights or weighting vectors is associated, W = (w_1, ... , w_n), such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$, and it is defined to aggregate a list of values {a_1, ..., a_n} according to the following expression:

$$F^G\left(a_1, a_2, \ldots, a_n\right) = \prod_{i=1}^{n} \left(a_{\sigma(i)}\right)^{w_i}$$

where σ: {1, ..., n} → {1, ..., n} is a permutation such that $a_{\sigma(i)} \geq a_{\sigma(i+1)}$, $\forall i =$ 1, ..., n, i.e., $a_{\sigma(i)}$ is the ith highest value in the set {a_1, ..., a_n}.

The OWG operators are continuous, compensative, commutative, and idempotent and are comprised between the maximum and the minimum [5, 9, 24]. Because the OWG operator is based on the OWA operator, it is clear that the weighting vector W can be obtained by the same method used in the case of the OWA operator, i.e., the vector may be obtained using a fuzzy quantifier Q. For this reason, the OWG operator present the same problems to represent the semantic of majority in group decision making problems exposed in [16, 17, 23].

Example 1. We suppose that we have a set of eight experts who express their opinions by means of the Saaty scale about an alternative a.

A = [9, 9, 9, 3, 3, 1, 1/2, 1/2]

Then using the quantifier *most* with the pair (0.4, 0.9) and weighting vector, W = [0 0 0 0.2 0.25 0.25 0.25 0.05] the OWG produces the aggregation value 1.75. This value does not characterize the value of the majority, which is intuitively a value closer to 4 (more than 62% of the opinion value are higher than the produced result), the OWG operator do not produce a representative solution of the quantity expressed by the quantifier, due to the weights are interpreted as the increase in satisfaction in having i+1 criteria fully satisfied with respect to having fully satisfied i criteria.

2.3 Majority Additive OWA Operators.

Majority operators MA-OWA [14, 15, 16] arise because it is necessary to obtain representative values of the majority of the elements to aggregate in some aggregation processes without omission of minority values. The most common aggregation operators [5, 21] over-emphasize the opinion of the minority as the expense of those of the majority creating an aggregation that can be considered imprecise for group decision making problems.

The *MA-OWA* is based in majority groups and it is defined in [14] as:

$$F_{MA}\left(a_1, a_2, \ldots, a_n\right) = \sum_{i=1}^{n} w_i \cdot b_i = \sum_{i=1}^{n} f_i\left(b_1, b_2, \ldots, b_n\right) \cdot b_i$$

where

$w_i \in [0,1]$ with $\sum_{i=1}^{n} w_i = 1$

b_i is the ith element of $a_1, \ldots, a_n$ that is ordered in ascender order by cardinalities.

The weights of MA-OWA operator are calculated:

Let δ_i the importance for the element i with $\delta_i > 0$, then

$$w_i = f_i(b_1, \ldots, b_n) = \frac{\gamma_i^{\delta_{\min}}}{\theta_{\delta_{\max}} \cdot \theta_{\delta_{\max-1}} \cdot \ldots \cdot \theta_{\delta_{\min+1}} \cdot \theta_{\delta_{\min}}} +$$

$$+ \frac{\gamma_i^{\delta_{\min+1}}}{\theta_{\delta_{\max}} \cdot \theta_{\delta_{\max-1}} \cdot \ldots \cdot \theta_{\delta_{\min+1}}} + \ldots + \frac{\gamma_i^{\delta_{\max}}}{\theta_{\delta_{\max}}}$$

where

$$\gamma_i^k = \begin{cases} 1 & if\ \delta_i \geq k \\ 0 & otherwise \end{cases}$$

and

$$\theta_i = \begin{cases} (number\ of\ item\ with\ \ \delta \geq i) + 1 & if\ i \neq \delta_{\min} \\ number\ of\ item\ with\ \ \delta \geq i & otherwise \end{cases}$$

The majority operators aggregate in function of δ_i that generally represents the importance of the element i using its cardinality. In the majority processes are considered the formation of discussion or majority groups depending on similarities or distances among the experts' opinions. All values with a minimum of separation are considered inside the same group. The calculation method for the value δ_i is independent from the definition of the majority operators. Usually the importance value δ_i is calculated using the distance function:

$$dist(a_i, a_j) = \begin{cases} 1 & if\ \ |a_i - a_j| \leq x \\ 0 & otherwise \end{cases}$$

The cardinality of a_i is the sum of all values $dist(a_i, a_j)$ for $j = 1 \ldots n$ being n the number of elements to aggregate.

$$\delta_i = \sum_{j=1}^{n} dist(a_i, a_j)$$

The value x model the final size of each group. Socially this grade is measured by the flexibility of the decision maker for grouping and reinforcing his/her opinions.

3 The MM-OWG Operator and Its Properties

In this section, we present the MM-OWG operator to aggregate ratio-scale judgements. It is based on the OWG operator [5] and on the Majority operator [14], and therefore, incorporates the advantage of OWG to deal with ratio-scale judgements and the advantage of the MA-OWA operator to represent the concept of majority. It is defined as follows.

Definition. A MM-OWG operator of dimension n is a function, $F^G : R^n \to R$, that has associated a set of weights or exponential weighting vector $W = (w_1, ..., w_n)$, such that $w_i \in [0,\ 1]$ and $\sum_{i=1}^{n} w_i = 1$, and it is defined to aggregate a list of values $\{a_1,\ ...,\ a_n\}$ according to the following expression:

$$F^{MM}\left(a_1, a_2, \ldots, a_n\right) = \prod_{i=1}^{n} (b_i)^{w_i} = \prod_{i=1}^{n} (b_i)^{f_i(b_1, b_2, \ldots, b_n)}$$

The weights of MM-OWG operator and the importance function are calculated using the same functions defined for the MA-OWA operator.

Example 2. If we again take the same experts and opinions about the alternative a as in example 1, the MM-OWG operator considers four values [9, 3, 1, 1/2] with cardinalities [3, 2, 1, 2] respectively (we consider a distance function with $x = 0$). Then the weight vector (in ascending order by cardinalities) is $W = [0.032, 0.156, 0.156, 0.656]$

$$F^{MM}(0.5, 1, 3, 9) = 0.5^{0.032} \cdot 1^{0.156} \cdot 3^{0.156} \cdot 9^{0.656} = 4.5$$

In this case the obtained value is representative of the majority, we can see that it is nearest of the intuitive majority value (4) than the produced result in the example 1 using OWG operator with *most* (1.75).

The main properties of the MM-OWG operator are the following:

Property 1. The MM-OWG is a “orand” operator

$$F_*(a_1, a_2, ..., a_n) \le F^{MM}(a_1, a_2, ..., a_n) \le F^*(a_1, a_2, ..., a_n)$$

Thus the upper and lower star MM-OWG operator are boundaries. From the above it became clear that for any F^{MM}

$$\text{Min}(a_i) \le F^{MM}(a_1, a_2,\ ...,\ a_n) \le \text{Max}(a_i)$$

Property 2 (Commutative). The MM-OWG operator can be seen to commutative. Let $< a_1, a_2,\ ...,\ a_n >$ be a bag aggregates and let $< d_1, d_2,\ ...,\ d_n >$ be a permutation of the a_i. Then

$$F^{MM}(a_1, a_2,\ ...,\ a_n) = F^{MM}(d_1, d_2,\ ...,\ d_n)$$

Property 3 (Monotonicity). A third characteristic associated with the operators is monotonicity. Assume a_i and c_i are collection of aggregates, i=1,...,n such that for each i, $c_i \le a_i$. Then

$$F^{MM}\left(c_1, c_2, \ldots, c_n\right) \le F^{MM}\left(a_1, a_2, \ldots, a_n\right)$$

where importance vector is same to the both collection.

Property 4 (Idempotency). Another characteristic associated with these operators is idempotency. If $a_i = a$ for all i =1,. . . , n, then for any OWG operator

$$F^{MM}\left(a, a, \ldots, a\right) = a$$

Property 5. The operator is reduced to the geometric mean operator when all elements in the aggregation have the same importance, then $w_i = 1/n \quad \forall i$.

$$F^{MM}(a_1, a_2, \ldots, a_n) = \prod_{i=1}^{n} (a_i)^{w_i} = \prod_{i=1}^{n} (a_i)^{\frac{1}{n}} = GM(a_1, a_2, \ldots, a_n)$$

Property 6 (Simplification). A new characteristic associated with the operators is simplification. Let a_i is collection of aggregates, i= 1,..., n, and $\delta_i \quad > 1$ for all i = 1, ..., n. Then

$$F^{MM}(a_1,a_2,\ ...,\ a_n) = F^{MM}(c_1,c_2,\ ...,\ c_n)$$

Where the new δ_i associated to c_i is calculated as $\delta_i = \delta_i - \text{Min}\{\delta_i\} - 1$ for all $i =$ 1,. . . , n.

Property 7 (constant product). If all values to aggregate are increased by a constant k, the MM-OWG is increased by this constant k.

$$\prod_{i=1}^{n} (k \cdot a_i)^{w_i} = \prod_{i=1}^{n} (k^{w_i} \cdot a_i^{w_i}) = k \cdot F^{MM}$$

with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$

The next properties are very interesting to use the MM-OWG operator in multicriteria decision making problems with multiplicative preference relations, i.e. Analytic Hierarchical Process (AHP), where it's need to aggregate matrix with multiplicative reciprocal preference relation which satisfies the consistency property.

Property 8 (Reciprocity). The reciprocity of the MM-OWG is equal to the MM-OWG of the reciprocity of the values to aggregate.

$$\frac{1}{F^{MM}} = \frac{1}{\prod\limits_{i=1}^{n} (a_i)^{w_i}} = \prod_{i=1}^{n} (\frac{1}{a_i})^{w_i}$$

And then

$$M^1 = \begin{bmatrix} 1 & \cdots & a_{1n}^1 \\ \vdots & \ddots & \vdots \\ a_{n1}^1 & \cdots & 1 \end{bmatrix}, M^2 = \begin{bmatrix} 1 & \cdots & a_{1n}^2 \\ \vdots & \ddots & \vdots \\ a_{n1}^2 & \cdots & 1 \end{bmatrix} \cdots$$

$$\cdots M^m = \begin{bmatrix} 1 & \cdots & a_{1n}^k \\ \vdots & \ddots & \vdots \\ a_{n1}^k & \cdots & 1 \end{bmatrix}$$

where $a_{ii}^r = 1$, $a_{ij}^r = \frac{1}{a_{ji}^r}$ and the function δ_i must be calculate using the distance between matrixes. Then if $a_{ij}^{F^{MM}} = F^{MM}(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^m) = \prod_{k=1}^{m} (a_{ij}^k)^{w_k}$ then

$$a_{ij}^{F^{MM}} = \prod_{r=1}^{m} (a_{ij}^r)^{w_r} = \prod_{r=1}^{m} (\frac{1}{a_{ji}^r})^{w_r} = \frac{1}{\prod_{r=1}^{m} (a_{ji}^r)^{w_r}} = \frac{1}{a_{ji}^{F^{MM}}}$$

Property 9 (Consistency). The MM-OWG of a set of consistent values is also consistent

$$M^1 = \begin{bmatrix} 1 & \cdots & a_{1n}^1 \\ \vdots & \ddots & \vdots \\ a_{n1}^1 & \cdots & 1 \end{bmatrix}, M^2 = \begin{bmatrix} 1 & \cdots & a_{1n}^2 \\ \vdots & \ddots & \vdots \\ a_{n1}^2 & \cdots & 1 \end{bmatrix} \cdots$$

$$\cdots M^m = \begin{bmatrix} 1 & \cdots & a_{1n}^k \\ \vdots & \ddots & \vdots \\ a_{n1}^k & \cdots & 1 \end{bmatrix}$$

where $a_{ii}^r = 1$, $a_{ij}^r \cdot a_{jk}^r = a_{ik}^r$, $\forall i, j, k, r$ and the function δ_i must be calculate using the distance between matrixes. Then

if $a_{ij}^{F^{MM}} = F^{MM}(a_{ij}^1, a_{ij}^2, \ldots, a_{ij}^m) = \prod_{k=1}^{m} (a_{ij}^{\sigma(k)})^{w_k}$ then

$$a_{ij}^{F^{MM}} = \prod_{r=1}^{m} (a_{ij}^{\sigma(r)})^{w_r}; \quad a_{jk}^{F^{MM}} = \prod_{r=1}^{m} (a_{jk}^{\sigma(r)})^{w_r}; \quad a_{ik}^{F^{MM}} = \prod_{r=1}^{m} (a_{ik}^{\sigma(r)})^{w_r}$$

$$a_{ij}^{F^{MM}} \cdot a_{jk}^{F^{MM}} = \prod_{r=1}^{m} (a_{ij}^{\sigma(r)})^{w_r} \cdot \prod_{r=1}^{m} (a_{jk}^{\sigma(r)})^{w_r} =$$

$$= \prod_{r=1}^{m} (a_{ij}^{\sigma(r)} \cdot a_{jk}^{\sigma(r)})^{w_r} = \prod_{r=1}^{m} (a_{ik}^{\sigma(r)})^{w_r} = a_{ik}^{F^{MM}}$$

Example 3: Suppose a set of three experts provide the following multiplicative preference relations on a set of three alternatives (M_1, M_2, M_3) which satisfy the consistency and reciprocal properties

$$M^1 = \begin{bmatrix} 1 & 3 & 6 \\ 1/3 & 1 & 2 \\ 1/6 & 1/2 & 1 \end{bmatrix}, M^2 = \begin{bmatrix} 1 & 3 & 9 \\ 1/3 & 1 & 3 \\ 1/9 & 1/3 & 1 \end{bmatrix}, M^3 = \begin{bmatrix} 1 & 3 & 6 \\ 1/3 & 1 & 2 \\ 1/6 & 1/2 & 1 \end{bmatrix}$$

where we use a distance function for matrixes with factor of grouping equal to 0. For the distance function we must remark which the distance between 1/5 and 1/3 is the same that between 5 and 3 (distance 2). Then we consider 2 matrixes with cardinality 2 (M^1 and M^3) and 1 (M^2).

Then the aggregation matrix is

$$M^{F^{MM}} = \begin{bmatrix} 1 & 3 & 6.640 \\ 0.333 & 1 & 2.213 \\ 0.150 & 0.451 & 1 \end{bmatrix}$$

the values $a_{12}^{F^{MM}}$,$a_{13}^{F^{MM}}$,$a_{21}^{F^{MM}}$ are calculated as follow

$$a_{12}^{F^{MM}} = F^{MM}(3,\ 3) = 3^{1/4} \cdot 3^{3/4} = 3;$$

$$a_{13}^{F^{MM}} = F^{MM}(9,\ 6) = 9^{1/4} \cdot 6^{3/4} = 6.640;$$

$$a_{21}^{F^{MM}} = F^{MM}(3,\ 2) = 3^{1/4} \cdot 2^{3/4} = 2.213$$

¿From properties 8 and 9 we can infer that if we aggregate a set of multiplicative preference relations which are reciprocal and consistent, then the collective multiplicative preference relation obtained by using the MM-OWG is also reciprocal and maintain the consistency.

Remark. The MM-OWG operator models the concept of majority using a semantic where all the elements in the aggregation process are considers. The MM-OWG operator aggregates creating majority groups using a distance function, however we can use a fuzzy majority with this operator adding fuzzy quantifiers in the aggregation [17].

4 Solving a Multicriteria Decision Making Problem Using the MM-OWG Operator

Let $X = \{x_1, x_2, ..., x_n, (n >= 2)\}$ be a finite set of alternatives. The alternatives must be classified from best to worst (ordinal ranking), using the information known according to a finite set of general criteria or experts $E = \{e_1, e_2, ..., e_m, (m >= 2)\}$. We assume that the experts' preferences over the set of alternatives, X, are represented by means of the multiplicative preference relations on X, i.e.,

$$A^k \subset X \times X, A^k = \left[a_{ij}^k\right]$$

where a_{ij}^k indicates a ratio of preference intensity for alternative x_i to that of x_j, i.e., it is interpreted as x_i is a_{ij}^k times as good as x_j. Each a_{ij}^k is assessed using the ratio scale proposed by Saaty, that is, precisely the 1 to 9 scale [19]: $a_{ij}^k = 1$ indicates indifference between x_i and x_j, $a_{ij}^k = 9$ indicates that x_i is absolutely preferred to x_j. That is:

1 equally important.
3 weakly more important.
5 strongly more important.
7 demonstrably or very strongly more important.
9 absolutely more important.
2, 4, 6, and 8 compromise between slightly differing judgments.

In order to guarantee that A^k is "self-consistent", only some pairwise comparison statements are collected to construct it. The rest of the values are what satisfy the following conditions [19]:

1. Multiplicative Reciprocity Property: $a_{ij}^r = \frac{1}{a_{ji}^r}$
2. Saaty's Consistency Property: $a_{ij}^r \cdot a_{jk}^r = a_{ik}^r, \ \forall i, j, k, r$

Then, we consider multiplicative preference relations assessed in Saaty's discrete scale, which has only the following set of values: {1/9, 1/8, ..., 1/2, 1, 2, ..., 8, 9}.

The multicriteria decision making problem when the experts express their preferences using multiplicative preference relations have been solved by Saaty using the decision AHP, which obtains the set of solution alternatives by means of the eigenvector method [19]. However, this decision process is not guided by the concept of majority. Here, we present an alternative decision process to the AHP proposed by Saaty in order to show the application of the MM-OWG operator. Following the choice scheme proposed in [18], i.e.,

$$\text{Aggregation} + \text{Exploitation}$$

1. Aggregation phase.

This phase defines a collective multiplicative preference relation, M^C, which indicates the global preference according to the majority of experts' opinions. M^C is obtained from $\{A^1, \ldots, A^m\}$ by means of the application of an aggregation operator.

2. Exploitation phase.

Using the aggregation operator for multiplicative preference relations, this phase transforms the aggregated or global information about the alternatives into a global ranking of them, supplying the set of solution alternatives.

Finally the application of choice degree of alternatives over this global ranking allows us to obtain the following solution set of alternatives whose elements are called maximum dominance ones.

$$\text{Sol} = \max \{x^c_{1,\ldots,}x^c_n\}$$

where $x^c_i = \text{MM-OWG}\ [a^c_{i1}, \ldots, a^c_{in}]$

4.1 Example

In this example we use as aggregation operator the MM-OWG.

Assume that we have a set of six experts, E = $\{e_1, e_2, e_3, e_4, e_5, e_6\}$, and a set of three alternatives, X = $\{x_1, x_2, x_3\}$. Suppose that experts supply their opinions by means of the following multiplicative preference relations:

$$A^1 = \begin{bmatrix} 1 & 2 & 7 \\ 1/2 & 1 & 5 \\ 1/7 & 1/5 & 1 \end{bmatrix}, A^2 = \begin{bmatrix} 1 & 2 & 7 \\ 1/2 & 1 & 5 \\ 1/7 & 1/5 & 1 \end{bmatrix}, A^3 = \begin{bmatrix} 1 & 2 & 7 \\ 1/2 & 1 & 5 \\ 1/7 & 1/5 & 1 \end{bmatrix},$$

$$A^4 = \begin{bmatrix} 1 & 9 & 3 \\ 1/9 & 1 & 1/3 \\ 1/3 & 3 & 1 \end{bmatrix}, A^5 = \begin{bmatrix} 1 & 3 & 1 \\ 1/3 & 1 & 3 \\ 1 & 1/3 & 1 \end{bmatrix}, A^6 = \begin{bmatrix} 1 & 3 & 1 \\ 1/3 & 1 & 3 \\ 1 & 1/3 & 1 \end{bmatrix}$$

In the decision process with MM-OWG we use a distance function for matrixes with factor of grouping equal to 0, then we work with three matrixes ($AG^1 = A^1$, A^2,A^3;$AG^2 = A^5$, A^6;and $AG^3 = A^4$) with cardinalities 3, 2 and 1 respectively.

1. *Aggregation phase.*

The collectives multiplicative preference relation obtained in this phase are the following:

$$M^C = \begin{bmatrix} 1 & 2.37 & 4.33 \\ 1/2.37 & 1 & 3.84 \\ 1/4.33 & 1/3.84 & 1 \end{bmatrix}$$

where the values a^C_{12},a^C_{13},a^C_{21} are calculated as follow

$$a^C_{12} = F^{MM}(2,\ 3,\ \ 9) = 2^{13/18} \cdot 3^{4/18} \cdot 9^{1/18} = 2.37$$

$$a^C_{13} = F^{MM}(7,\ 1,\ 3) = 7^{13/18} \cdot 1^{13/18} \cdot 3^{1/18} = 4.33$$

$$a^C_{23} = F^{MM}(5,\ 3,\ 1/3) = 5^{13/18} \cdot 3^{13/18} \cdot 1/3^{1/18} = 3.84$$

2. *Exploitation phase.*

$x_1 = F^{MM}$ (1, 2.37, 4.33) = 2.17
$x_2 = F^{MM}$ (1/2.37, 1, 3.84) = 1.17
$x_3 = F^{MM}$ (1/4.33, 1/3.84, 1) = 0.391
$Sol = max\ \{x_1 = 2.17;\ x_2 = 1.17;\ x_3 = 0.391\} = x_1$

5 Conclusion

In this contribution, we have presented a new aggregation operator for multiplicative environment, called MM-OWG operator. This operator has been designed to deal with ratio judgements and to model the majority concept in the aggregation processes. We have studied its properties; also we have shown how this operator is adequate for the synthesis of ratio judgements in multicriteria decision making problems with multiplicative preference relations, i.e. Analytic Hierarchical Process (AHP), where it's need to aggregate multiplicative reciprocal preference relation which satisfies the consistency property. Finally we have illustrated its use in a multicriteria decision making problem with multiplicative preference relations.

In the future, we will research the use of the MM-OWG operator for designing variations of AHP process where we need to use a majority semantic.

References

[1] Azcel, J., Alsina, C. (1983) Procedures for synthesizing ratio judgements. Journal of Mathematical Psychology. 27, 93-102.

[2] Azcel, J., Alsina, C. (1987) Synthesizing judgements: A functional equations approach, Mathematical Modelling. 9, 311-320.

[3] Chiclana F., Herrera-Viedma E., Herrera F., Alonso S.,2004. Induced Ordered Weighted Geometric Operators and Their Use in the Aggregation of Multiplicative Preferences Relations. International Journal of Intelligent Systems, 19, 233-255.

[4] Chiclana F, Herrera F, Herrera-Viedma E. Integrating multiplicative preference relations in a multipurpose decision making model based on fuzzy preference relations. Fuzzy Sets and Systems. 2001;112:277–291.

[5] Chiclana F, Herrera F, Herrera-Viedma E. The ordered weighted geometric operator: Properties and application. In: Proc of 8th International Conference on Information Processing and Management of Uncertainty in Knowledge-based Systems, Madrid, 2000. pp 985–991.

[6] Filev D. Yager R. 1996. On the Issue of Obtaining OWA Operator Weights. Fuzzy Sets and Systems. 94. 157-169.

[7] Fodor J. Marichal J-L. Roubens M. 1995. Characterization of the Ordered Weighted Averaging Operators. IEEE Transactions on Fuzzy Systems. Vol 3. n. 2.

[8] Herrera F, Herrera-Viedma E, Chiclana F. Multiperson decision making based on multiplicative preference relations. European Journal and Operational Research 2001;129:372–385.

[9] Herrera F, Herrera-Viedma E, Chiclana F. A study of the origin and uses of the ordered weighted geometric operator in multicriteria decision making. International Journal of Intelligent Systems 2003;18. 689–707.

[10] Herrera F. Herrera-Viedma E and Verdegay J.I. 1996. Direct Approach Processes in Group Decision Making Using Linguistic OWA Operators. Fuzzy Sets and Systems. Vol 79. 175-190.

[11] Marimin. Motohide Umano. Itsuo Hatono. Hiroyuki Tamura. 1998. Linguistic Labels for Expressing Fuzzy Preference Relations in Fuzzy Group Decision Making. IEEE Transactions on Systems, Man, and Cybernetics. Vol 28. n. 2.

[12] Nurmi H. and Kacprzyk J. 1991. On Fuzzy Tournaments and Their Solution Concepts in Group Decision Making. European Journal of Operational Research.

[13] O'Hagan. 1988. Aggregating template rule antecedents in real-time expert systems with fuzzy set logic. Proc. 22^{nd} Annual IEEE Asilomar Conf. On Signals, Systems and Computers, Pacific Grove, CA.

[14] Pelaez J.I., Doña J.M., Majority Additive-Ordered Weighting Averaging: A New Neat Ordered Weighting Averaging Operators Based on the Majority Process, International Journal of Intelligent Systems, 18, 4 (2003) 469-481.

[15] Pelaez J.I., Doña J.M., LAMA: A Linguistic Aggregation of Majority Additive Operator, International Journal of Intelligent Systems, (2003).

[16] Pelaez J.I., Doña J.M., La Red D., Analysis of the Majority Process in Group Decision Making Process, 9th International Conference on Fuzzy Theory and Technology, North Carolina. USA. (2003).

[17] Pelaez J.I., Doña J.M., La Red D., Mesas A. Majority Opinion in Group Decision Making using the QMA-OWA Operator. Proceeding of ESTYLF 2004. 449-454. ISBN: 84-609-2160-3.

[18] Roubens, M. (1997) Fuzzy sets and decision analysis. Fuzzy Sets and Systems. 90, 199-206.

[19] Saaty T, L. 1980. The Analytic Hierarchy Process. Macgraw Hill.

[20] Yager R. 1988. On ordered weighted averaging aggregation operators in multi-criteria decision making. IEEE Trans. On Systems, Man and Cybernetics 18. 183-190.

[21] Yager R. 1993. Families of OWA operators. Fuzzy Sets and Systems. 59. 125-148.

[22] Yager R. Kacprzyk J. 1997. The ordered weighted averaging operators. Theory and Applications. Kluwer Academic Publishers, Boston.

[23] Yayer R. Pasi G. 2002. Modelling Majority Opinion in Multi-Agent Decision Making. International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems. ISBN. 2-9516453-5-X.

[24] Xu ZS, Da QL. The ordered weighted geometric averaging operators. International Journal of Intelligent Systems 2002;17. 709 –716.