QÜESTIIÓ, vol. 26, 1-2, p. 283-287, 2002

GEOMETRICAL UNDERSTANDING OF THE CAUCHY DISTRIBUTION

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Advanced calculus is necessary to prove rigorously the main properties of the Cauchy distribution. It is well known that the Cauchy distribution can be generated by a tangent transformation of the uniform distribution. By interpreting this transformation on a circle, it is possible to present elementary and intuitive proofs of some important and useful properties of the distribution.

Keywords: Cauchy distribution, distribution on a circle, central limit theorem

AMS Classification (MSC 2000): 60-01; 60E05

–Received December 2001.
–Accepted April 2002.

The Cauchy distribution is a good example of a continuous stable distribution for which mean, variance and higher order moments do not exist. Despite the opinion that this distribution is a source of counterexamples, having little connection with statistical practice, it provides a useful illustration of a distribution for which the law of large numbers and the central limit theorem do not hold. Jolliffe (1995) illustrated this theorem with Poisson and binomial distributions, and he pointed out the difficulties in handling the Cauchy (see also Lienhard, 1996). In fact, one would need the help of the characteristic function or to compute suitable double integrals. It is shown in this article that these properties can be proved in an informal and intuitive way, which may be useful in an intermediate course.

If U is a uniform random variable on the interval $I = (-\pi/2, \pi/2)$, it is well known that $Y = \tan(U)$ follows the standard Cauchy distribution, i.e., with probability density

$$f(y) = \frac{1}{\pi}\frac{1}{1+y^2} \qquad -\infty < y < \infty.$$

Also $Z = \tan(nU)$, where n is any positive integer, has this distribution.

The tangent transformation can be described using the geometric analogy of a rotating diameter of a circle (Figure 1). Suppose, using usual rectangular co-ordinates, that the circle has center O and radius 1, and suppose a diameter POP' of the circle is a needle which rotates uniformly round the circle. Suppose P is the endpoint with positive value for x, and let OP make angle U with the x-axis. Then U lies in the interval $I = (-\pi/2, \pi/2)$. Let $Y = \tan(U)$; Y has the Cauchy distribution. Note also that $Y = \tan(U + \pi)$, so we only need to look at the half-right part of the final position of the needle. In the following, we will take all angles in the interval I modulo π and use the property that if V is any arbitrary random value, then $U + V$ (mod π) is also uniform on the interval I and is independent of U.

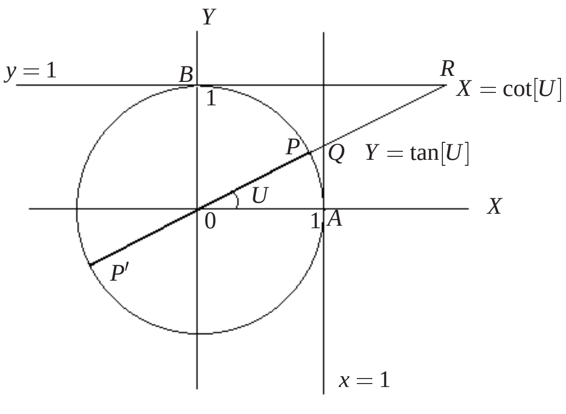

Figure 1.

In Figure 1, Y is the vertical coordinate of the point Q, where Q is the intersection of the line through the half-needle OP and the line $x = 1$. An interesting physical illustration is obtained by taking the origin as a radioactive source of α-particles impacting on a fixed line (Rao, 1973, p. 169).

The use of the circle and the rotating needle can give intuitive demonstrations of several features of the Cauchy distribution. Some examples are:

1) *Y has heavy tails, i.e., Y takes extreme values with high probability.*

   This feature empirically distinguishes Y from the standard normal and many other distributions. It is easily seen by observing that an angle U close to $\pi/2$ (or $-\pi/2$), which will give a large tangent value Y, has the same probability density $(= 1/\pi)$ as an angle close to 0.

2) *$X = Y^{-1}$ is also distributed as a standard Cauchy distribution.*

   By symmetry, using the rotating needle analogy, the Cauchy variable could as well be generated by using the angle between the needle and the y-axis and projecting on the line $y = 1$ (giving the value BR in Figure 1). But this is the same as taking $X = \cot(U) = 1/Y$.

3) *The mean does not exist*.

   Actually, it is easy to give an analytic proof of this result, as taking the expectation of Y gives an indeterminate integral. Using the analogy of the circle, we should consider the position of the needle after several successive rotations. It is clear that the mean position might be anywhere around the circle, and a formal «mean» is not clearly defined.

4) *The distribution of the mean of any number of independent observations has the same distribution as Cauchy. Consequently, the law of the large numbers describing the convergent behavior of the mean does not hold for this distribution.*

   The analogy in 3) can be used again. Rotating the needle several times, we obtain axes uniformly distributed around the circle, and the intuitive average axis is also uniformly distributed. Taking U as its angle with the x-axis and constructing $Y = \tan(U)$ gives the standard Cauchy distribution. A rigorous definition of the mean direction and a proof of its uniform distribution needs, of course, more advanced arguments.

   The above analogy is clearly not a proof of 4), because the tangent of the angle of the mean axis is not the mean of the tangents. Instead, the Cauchy distribution for the mean of the Cauchy sample can be illustrated using the following simulation. Choose an integer m, generate independent uniform angles $u_1, \ldots, u_m$ and compute

$$\bar{u} = \text{atan}\left(\frac{1}{m}\sum_{i=1}^{m}\tan(u_i)\right)$$

Repeat this operation n times and plot a histogram of the obtained sample $\bar{u}_1, \ldots, \bar{u}_n$. The uniform distribution of $\bar{u}$ will be quite apparent (Figure 2), showing that $\tan(\bar{u})$ is Cauchy. Note that to recognize the Cauchy distribution by a histogram of a Cauchy sample is less evident, due to the distortion produced by the very large values.

This geometrical approach, and the associated trigonometry, can give the proof of other properties, for example:

5) $Z = (Y^{-1} - Y)/2$ *has the standard Cauchy distribution*.

   This is a consequence of the trigonometric identity $[\cot(u) - \tan(u)]/2 = \cot(2u)$.

6) *If W is any random variable then $T = (Y + W)/(1 - YW)$ has the standard Cauchy distribution*.

   Using the formula $\tan(u+v) = [\tan(u) + \tan(v)]/[1 - \tan(u)\tan(v)]$, write $W = \tan(V)$, where V is any angle in I. Then $U + V$ (mod π) is uniformly distributed and $T = \tan(U + V)$.

7) *If U_1, U_2 are independently uniform on I, and if $Y_1 = \tan(U_1)$, $Y_2 = \tan(U_2)$, and $Y_3 = -\tan(U_1 + U_2)$, then Y_1, Y_2, Y_3 are pairwise independent with standard Cauchy distributions but are jointly dependent*.

   From the trigonometric relation in 6) above, we have $Y_3 = -(Y_1 + Y_2)/(1 - Y_1Y_2)$. Thus the relation $Y_1Y_2Y_3 = Y_1 + Y_3$ exists and the Y-values are not independent.

   Further formulas were proved in a similar way by Jones (1999) for the normal and Cauchy distribution. See also Cuadras (2000). For example:

8) *Using that* $\tan(U)$ *is distributed as* $\tan(nU)$ for $n = 2, 3, 4$, *then if Z is standard Cauchy so is*

$$2Z/(1 - Z^2), \quad Z/(3 - Z^2)/(1 - 3Z^2) \quad \text{and} \quad 4Z/(1 - Z^2)/(1 - 6Z^2 + Z^4).$$

9) *Using* $\tan(2U + c)$ *or equivalently combining* $2Z/(1 - Z^2)$ *and* $(Z + B)/(1 - BZ)$, *where B is independent of Z (see above), then*

$$\frac{2Z + B(1 - Z^2)}{1 - B^2 - 2BZ}$$

   is also standard Cauchy.

Finally, while it is relatively easy to prove, using only geometry, that the sum of independent N(0,1) is also normal (see Mantel, 1972), it is an open question to give a geometric but conscientious proof, elementary enough for teaching purposes, (i.e., without using double integrals or the characteristic function), that the mean of the tangents of uniform angles is the tangent of an angle also uniformly distributed in I, i.e, following the Cauchy distribution. See Cohen (2000).

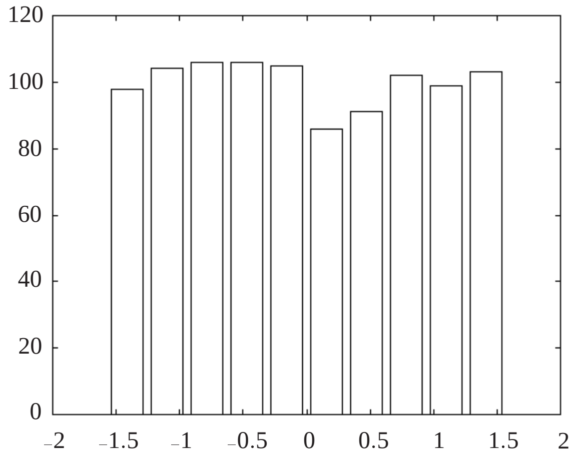

Figure 2. Histogram of a sample of $\bar{u}$ revealing a uniform distribution, indicating that $\tan(\bar{u})$ has the standard Cauchy distribution.

APPENDIX

MATLAB code to generate samples giving the histogram of Figure 2.

```
m = 100; n = 1000; rand('seed',2002);
for i = 1 : n, u = rand(m,1) * pi - pi/2; c = tan(u); mc(i) = atan(mean(c)); end
hist(mc)
```

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