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Extremes of periodic moving averages of random variables with regularly varying tail probabilities

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Abstract

We define a family of local mixing conditions that enable the computation of the extremal index of periodic sequences from the joint distributions of k consecutive variables of the sequence. By applying results, under local and global mixing conditions, to the $(2m-1)$–dependent periodic sequence $X_n^{(m)} = \sum_{j=-m}^{m-1} c_j Z_{n-j}$, $n \geq 1$, we compute the extremal index of the periodic moving average sequence $X_n = \sum_{j=-\infty}^{\infty} c_j Z_{n-j}$, $n \geq 1$, of random variables with regularly varying tail probabilities.
This paper generalizes the theory for extremes of stationary moving averages with regularly varying tail probabilities.

MSC: 60G70

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1 Introduction

The moving average process of the form

$$X_n = \sum_{j=-\infty}^{\infty} c_j Z_{n-j}, \quad n \geq 1, \tag{1.1}$$

with iid real-valued innovations or noise variables $(Z_j)_{j\in\mathbb{Z}}$, includes the popular $ARMA(p,q)$ and $AR(p)$ processes considered in classical time series analysis. Studies of the extreme value behaviour of such processes have been carried out, among others, by Cline (1983), Davis and Resnick (1985, 1988) and Chernick *et al.* (1991).

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In this paper we are concerned with moving average processes of the form (1.1) but with $\mathbf{Z} = \{Z_j\}_{j\in\mathbb{Z}}$ a $T-$periodic sequence of independent real-valued variables, such that $\overline{F}_i(x) = P(|Z_i| > x)$, $i = 1, \ldots, T$, are regularly varying with exponent $-\alpha$, i.e.,

$$\overline{F}_i(x) = x^{-\alpha} L_i(x), \quad x > 0, \quad i = 1, \ldots, T, \tag{1.2}$$

for some $\alpha > 0$ and $L_i : \mathbb{R}_+ \to \mathbb{R}_+$ slowly varying functions. We also assume the tail balance conditions

$$\lim_{x\to\infty} \frac{P(Z_i > x)}{\overline{F}_i(x)} = p_i, \quad \lim_{x\to\infty} \frac{P(Z_i < -x)}{\overline{F}_i(x)} = q_i, \quad i = 1, \ldots, T, \tag{1.3}$$

for some p_i and $q_i \in [0, 1]$ such that $p_i + q_i = 1$, $i = 1, \ldots, T$, and tail equivalence in the following way

$$\lim_{x\to\infty} \frac{P(Z_i > x)}{P(Z_j > x)} = \gamma_{i,j}^{(+)} > 0, \quad \lim_{x\to\infty} \frac{P(Z_i < -x)}{P(Z_j < -x)} = \gamma_{i,j}^{(-)} > 0, \quad i, j = 1, \ldots, T. \tag{1.4}$$

The sequence of real constants $\mathbf{c} = \{c_j\}_{j\in\mathbb{Z}}$ will be taken to satisfy

$$\sum_{j=-\infty}^{\infty} |c_j|^{\delta} < \infty, \tag{1.5}$$

for some $\delta < \min\{\alpha, 1\}$, in order to guarantee the a.s. convergence of (1.1). Notice that conditions (1.3) and (1.4) imply the existence of $\gamma_{i,j} = \lim_{x\to\infty} \frac{P(|Z_i|>x)}{P(|Z_j|>x)}$, $i, j = 1, \ldots, T$.

Extreme value theory known for periodic sequences can then be applied to this moving average sequence $\mathbf{X} = \{X_n\}_{n\geq1}$, since it is also a $T-$periodic sequence. Alpuim (1988) showed that under Leadbetter's global mixing condition D, the only possible limit laws for the normalized maxima of a $T-$periodic sequence are the three extreme value distributions. Under local mixing conditions $D_T^{(k)}$, $k = 1, 2$, Ferreira (1994) studied the extremal behaviour of periodic sequences, and under the weaker local mixing conditions $D_T^{(k)}$, $k \geq 3$, Ferreira and Martins (2003) obtained the expression for the extremal index of a $T-$periodic sequence from the joint distribution of k consecutive variables of the sequence.

We say that for a fixed integer $k \geq 1$ and a sequence of real constants $\mathbf{u} = \{u_n\}_{n\geq1}$ the condition $D_T^{(k)}(u_n)$ holds for a $T-$periodic sequence $\mathbf{X}$ satisfying Leadbetter's condition $D(u_n)$ (see Leadbetter *et al.* (1983)) with mixing coefficients $\beta_{n,l}$, when there exists a sequence of integers $\mathbf{k} = \{k_n\}_{n\geq1}$ such that $\lim_{n\to\infty} k_n = \infty$, $\lim\limits_{n\to\infty} k_n \frac{l_n}{n} = 0$, $\lim_{n\to\infty} k_n \beta_{n,l_n} = 0$, and

$$\lim_{n\to\infty} n \frac{1}{T} \sum_{i=1}^{T} \sum_{j=i+k}^{\left[\frac{n}{k_n T}\right]T} P(X_i > u_n \geq M_{i+1,i+k-1}, X_j > u_n) = 0, \tag{1.6}$$

where $M_{i,j} = \max\{X_i, X_{i+1}, \ldots, X_j\}$ and $M_{i,j} = -\infty$ for $i > j$.

Under this local dependence condition the extremal index of **X**,

$$\theta_{\mathbf{X}} = \frac{-\log(\lim_{n\to\infty} P(\max_{1\le i\le n} X_i \le u_n))}{\tau},$$

where

$$\tau = \lim_{n\to\infty} n\frac{1}{T}\sum_{i=1}^{T} P(X_i > u_n), \tag{1.7}$$

can be computed from

$$\theta_{\mathbf{X}} = \lim_{n\to\infty} \frac{n\frac{1}{T}\sum_{i=1}^{T} P(X_i > u_n \ge M_{i+1,i+k-1})}{\tau}. \tag{1.8}$$

A sequence **u** satisfying (1.7) is usually denoted by $\mathbf{u}^{(\tau)} = \{u_n^{(\tau)}\}$ for **X**, and its elements are called normalized levels for **X**.

Observe that, when $k \ge 2$, condition (1.6) is implied by

$$\lim_{n\to\infty} S^{(k)}_{\left[\frac{n}{k_n T}\right]} = \lim_{n\to\infty} n\frac{1}{T}\sum_{i=1}^{T}\sum_{j=i+k}^{\left[\frac{n}{k_n T}\right]T} P(X_i > u_n, X_{j-1} \le u_n < X_j) = 0,$$

which limits the distance between exceedances of level u_n, that is, in each interval there can only be more than one exceedance of u_n if separated by less than $k-1$ non-exceedances of u_n. Consequently, the local dependence condition $D_T^{(k)}$, $k \ge 1$, become weaker as the value of k increases.

Our aim in this paper is to use the previous results, that generalize the ones obtained by Chernick *et al.* (1991) for stationary sequences, to obtain the expression for the extremal index of the $T-$periodic moving average sequence of random variables with regularly varying tail probabilities **X** defined by (1.1) and satisfying certain balance and tail equivalence conditions. To attain this we start by characterizing in Section 2 the behaviour of each tail $P(X_i) > x$, $i = 1, \ldots, T$ as $x \to \infty$, and by obtaining sufficient conditions that allow the application of our results to a finite moving average sequence $\mathbf{X}^{(m)}$ that "approximates" **X** as $m \to \infty$. In Section 3 we present our main result which gives the expression of the extremal index of the $T-$periodic moving average sequence **X**.

The proofs of all theorems presented are given in the Appendix.

2 First results

The first result we present is a simple modification of a theorem found in Resnick (1987) for the stationary case, but crucial for the characterization of the behaviour of each tail $P(X_i > x)$, $i = 1, \ldots, T$, as $x \to \infty$, which we present ahead.

Theorem 2.1 *Let* $\mathbf{Z} = \{Z_n\}_{n\in\mathbb{Z}}$ *be a* $T-$*periodic sequence of independent random variables satisfying (1.2), (1.3) and (1.4) and* $\mathbf{c} = \{c_j\}_{j\in\mathbb{Z}}$ *a sequence of real constants satisfying (1.5). Then for* $i = 1, \ldots, T$ *when* $x \to \infty$

$$\lim_{x\to\infty} \frac{P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right)}{P(|Z_i| > x)} = p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)}\ p_{i-s}^{-1} \sum_{j=-\infty}^{\infty} |c_{jT+s}|^{\alpha}. \tag{2.9}$$

The behaviour of each tail $P(X_i > x)$, $i = 1, \ldots, T$, as $x \to \infty$ is vital for the extremal behaviour of the periodic moving average process $\mathbf{X}$. As Embrechts *et al.* (1997), we prove how every r.v. Z_j, $j \in \mathbb{Z}$, has a contribution to each tail $P(X_i > x)$, $i = 1, \ldots, T$.

Theorem 2.2 *Let* $\mathbf{Z} = \{Z_n\}_{n\in\mathbb{Z}}$ *be a* $T-$*periodic sequence of independent variables satisfying (1.2), (1.3) and (1.4) and* $\mathbf{c} = \{c_j\}_{j\in\mathbb{Z}}$ *a sequence of real constants satisfying (1.5). Then for* $i = 1, \ldots, T$

$$P(X_i > x) \sim x^{-\alpha} L_i(x) \left\{ p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \sum_{j=-\infty}^{\infty} \left[c_{jT+s}^{+}\right]^{\alpha} + q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} \sum_{j=-\infty}^{\infty} \left[c_{jT+s}^{-}\right]^{\alpha} \right\},$$

where $c_j^+ = max\{c_j, 0\}$ *and* $c_j^- = max\{-c_j, 0\}$.

As we can see, the contribution of the random variables $\mathbf{Z} = \{Z_n\}_{n\in\mathbb{Z}}$ to each tail depends on the size and sign of the respective weight c_j associated to them.

The computation of the extremal index using expression (1.8) requires the validation of a long range and a local mixing condition, which is often a difficult task when considering some sequences, namely moving average sequences. To overcome this difficulty it's useful to consider in these cases an "approximating" sequence $\mathbf{X}^{(m)} = \{X_n^{(m)}\}_{n\geq 1}$ for a fixed integer m, then apply a Slutsky argument and let $m \to \infty$. We can then use the extremal index of this sequence $\mathbf{X}^{(m)}$ to estimate that of $\mathbf{X}$.

Sufficient conditions, to take into consideration such a sequence $\mathbf{X}^{(m)}$, in the periodic case, can be found in the next result, analogous to the one found in Chernick *et al.* (1991) for the stationary case.

Theorem 2.3 *Suppose* $\mathbf{X}$ *and* $\mathbf{X}^{(m)}$, $m \geq 1$ *are* $T-$*periodic sequences defined on the same probability space such that for some sequences of constants* $\mathbf{u} = \{u_n\}_{n\geq 1}$ *and* $i = 1, \ldots, T$,

$$\lim_{\epsilon\to 0} \limsup_{n\to\infty} nP\left((1-\epsilon)u_n < X_i \leq (1+\epsilon)u_n\right) = 0, \tag{2.10}$$

$$\lim_{m\to\infty} \limsup_{n\to\infty} nP\left(|X_i - X_i^{(m)}| > \epsilon u_n\right) = 0, \quad \epsilon > 0. \tag{2.11}$$

Then

(i) $\lim_{m\to\infty} \limsup_{n\to\infty} \left|P(M(I_n) \leq u_n) - P(M^{(m)}(I_n) \leq u_n)\right| = 0$, *where the supremum is taken over all index sets* $I_n \subset \{1, \ldots, n\}$.

(ii) If condition $D(\mathbf{u})$ holds for $\mathbf{X}^{(m)}$ for each m, then it holds for $\mathbf{X}$ as well.

(iii) $\lim_{m\to\infty}\lim_{n\to\infty} n|P(X_i > u_n \geq M_{i+1,i+k-1}) - P(X_i^{(m)} > u_n \geq M_{i+1,i+k-1}^{(m)})| = 0,\ k \geq 2,$

with $M(I) = \max_{j\in I} X_j$ *and* $M^{(m)}(I) = \max_{j\in I} X_j^{(m)}$, *for* $I \subset \{1, \dots, n\}$.

Remark 1 If (2.10) and (2.11) hold with $\mathbf{u}^{(\tau)}$ and $\mathbf{X}^{(m)}$, $m \geq 1$ has extremal index $\theta_{\mathbf{X}^{(m)}}$, then by Theorem 1.3(i) with $I_n = \{1, \dots, n\}$, $\mathbf{X}$ has extremal index $\theta_{\mathbf{X}}$ if and only if $\theta_{\mathbf{X}^{(m)}} \to \theta_{\mathbf{X}}$ as $m \to \infty$.

3 Main result

We are now in conditions to state our main theorem which computes the extremal index $\theta_{\mathbf{X}}$ of a periodic sequence $\mathbf{X}$ of moving averages of random variables with regularly varying tail probabilities. For this, we need to consider a sequence of constants $\mathbf{u} = \{u_n\}_{n\in\mathbb{N}}$ satisfying

$$\lim_{n\to\infty} nP(|Z_i| > u_n) = \tau_i \Bigg/ \left\{ p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \sum_{j=-\infty}^{\infty} \left[c_{jT+s}^{+}\right]^{\alpha} + q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} \sum_{j=-\infty}^{\infty} \left[c_{jT+s}^{-}\right]^{\alpha} \right\}, \quad (3.1)$$

for given $\tau_i > 0$, $i = 1, \dots, T$. Such a sequence exists by the assumption of regular variation of each $\overline{F}_i$, $i = 1, \dots, T$, and implies, by Theorem 2.2, that $nP(X_i > u_n) \to \tau_i$, $x \to \infty$, $i = 1, \dots, T$, therefore $\mathbf{u} = \mathbf{u}^{(\tau)}$ for $\mathbf{X}$ with $\tau = \frac{1}{T}\sum_{i=1}^{T} \tau_i$.

Theorem 3.1 *Let* $\mathbf{X} = \{X_n\}_{n\geq 1}$ *be a* $T-$*periodic moving average sequence as defined in (1.1). Then* $\mathbf{X}$ *has extremal index*

$$\theta_{\mathbf{X}} = \frac{\sum_{i=1}^{T} \gamma_{i,1} \left\{ p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} c_s^{+}(\alpha) + q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} c_s^{-}(\alpha) \right\}}{\sum_{i=1}^{T} \gamma_{i,1} \left\{ p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \sum_{j=-\infty}^{\infty} \left[c_{jT+s}^{(+)}\right]^{\alpha} + q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} \sum_{j=-\infty}^{\infty} \left[c_{jT+s}^{-}\right]^{\alpha} \right\}},$$

where

$$c_s^{+}(\alpha) = \sum_{j=-\infty}^{\infty} \left(\left[c_{jT+s}^{+}\right]^{\alpha} - \max_{r > jT+s} \{c_r^{+}\}^{\alpha} \right)_{+}$$

and

$$c_s^{-}(\alpha) = \sum_{j=-\infty}^{\infty} \left(\left[c_{jT+s}^{-}\right]^{\alpha} - \max_{r > jT+s} \{c_r^{-}\}^{\alpha} \right)_{+}.$$

This result shows how the balance and tail equivalence parameters influence the value of the mean number of clustered exceedances in these processes.

4 Appendix

Proof (Theorem 2.1). We begin by showing a weaker result, namely for each $i = 1, \ldots, T$,

$$\lim_{x\to\infty} \frac{P(|c_1 Z_{i-1}| + |c_2 Z_{i-2}| + |c_3 Z_{i-3}| > x)}{P(|Z_i| > x)} = \gamma^{(+)}_{i-1,i}\, p^{-1}_{i-1}\, p_i |c_1|^\alpha + \gamma^{(+)}_{i-2,i}\, p^{-1}_{i-2}\, p_i |c_2|^\alpha + \gamma^{(+)}_{i-3,i}\, p^{-1}_{i-3}\, p_i |c_3|^\alpha. \tag{4.2}$$

We restrict ourselves to three summands with non-zero c_1, c_2, c_3 to show the method, the general case can be proved analogously by induction. For $\delta \in (0, 1/3)$ and $i = 1, \ldots, T$,

$$\begin{aligned}
&\{|c_1 Z_{i-1}| + |c_2 Z_{i-2}| + |c_3 Z_{i-3}| > x\} \\
&\quad \subset \{|c_1 Z_{i-1}| > (1-\delta)^2 x\} \;\cup\; \{|c_2 Z_{i-2}| > (1-\delta)^2 x\} \;\cup\; \{|c_3 Z_{i-3}| > (1-\delta)^2 x\} \\
&\quad \cup \{|c_1 Z_{i-1}| > \delta(1-\delta)x, |c_2 Z_{i-2}| > \delta(1-\delta)x\} \\
&\quad \cup \{|c_1 Z_{i-1}| > \delta(1-\delta)x, |c_3 Z_{i-3}| > \delta(1-\delta)x\} \\
&\quad \cup \{|c_2 Z_{i-2}| > \delta(1-\delta)x, |c_3 Z_{i-3}| > \delta(1-\delta)x\} \\
&\quad \cup \{|c_1 Z_{i-1}| > \delta^2 x, |c_2 Z_{i-2}| > \delta^2 x, |c_3 Z_{i-3}| > \delta^2 x\}.
\end{aligned}$$

Hence, by conditions (1.2), (1.3) and (1.4)

$$\begin{aligned}
\limsup_{x\to\infty} \frac{P(|c_1 Z_{i-1}| + |c_2 Z_{i-2}| + |c_3 Z_{i-3}| > x)}{P(|Z_i| > x)} &\le |c_1|^\alpha (1-\delta)^{-2\alpha} p^{-1}_{i-1}\, \gamma^{(+)}_{i-1,i}\, p_i \\
&+ |c_2|^\alpha (1-\delta)^{-2\alpha} p^{-1}_{i-2}\, \gamma^{(+)}_{i-2,i}\, p_i + |c_3|^\alpha (1-\delta)^{-2\alpha} p^{-1}_{i-3}\, \gamma^{(+)}_{i-3,i}\, p_i.
\end{aligned} \tag{4.3}$$

Moreover,

$$\begin{aligned}
&\{|c_1 Z_{i-1}| + |c_2 Z_{i-2}| + |c_3 Z_{i-3}| > x\} \\
&\quad \supset \{|c_1 Z_{i-1}| > (1+\delta)^2 x, |c_2 Z_{i-2}| + |c_3 Z_{i-3}| \le \delta x\} \\
&\quad \cup \{|c_1 Z_{i-1}| \le \delta x, |c_2 Z_{i-2}| > (1+\delta)^2 x, |c_3 Z_{i-3}| \le \delta(1+\delta)x\} \\
&\quad \cup \{|c_1 Z_{i-1}| \le \delta x, |c_2 Z_{i-2}| \le \delta(1+\delta)x, |c_3 Z_{i-3}| > (1+\delta)^2 x\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\liminf_{x\to\infty} \frac{P(|c_1 Z_{i-1}| + |c_2 Z_{i-2}| + |c_3 Z_{i-3}| > x)}{P(|Z_i| > x)} &\ge |c_1|^\alpha (1+\delta)^{-2\alpha} p^{-1}_{i-1}\, \gamma^{(+)}_{i-1,i}\, p_i \\
&+ |c_2|^\alpha (1+\delta)^{-2\alpha} p^{-1}_{i-2}\, \gamma^{(+)}_{i-2,i}\, p_i + |c_3|^\alpha (1+\delta)^{-2\alpha} p^{-1}_{i-3}\, \gamma^{(+)}_{i-3,i}\, p_i.
\end{aligned} \tag{4.4}$$

Letting $\delta \to 0$ in (4.3) and (4.4) we obtain (4.2). Notice that in the case $T = 2$ we have $\gamma^{(+)}_{i-1,i} = \gamma^{(+)}_{i-3,i}$, $p_{i-1} = p_{i-3}$, for $i = 1, 2$ and $p_1 = p_3$.

We must leap now from (4.2) to (2.9). For $x > 0$ and $I_1 = \{0, \dots, T-1\}$, write

$$\begin{aligned}
P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right) &= P\left(\sum_{s=0}^{T-1}\sum_{j=-\infty}^{\infty} |c_{jT+s} Z_{i-jT-s}| > x\right) \\
&= P\left(\sum_{s=0}^{T-1}\sum_{j=-\infty}^{\infty} |c_{jT+s} Z_{i-jT-s}| > x, \max_{s\in I_1}\max_{j\in\mathbb{R}} |c_{jT+s} Z_{i-jT-s}| > x\right) \\
&+ P\left(\sum_{s=0}^{T-1}\sum_{j=-\infty}^{\infty} |c_{jT+s} Z_{i-jT-s}| > x, \max_{s\in I_1}\max_{j\in\mathbb{R}} |c_{jT+s} Z_{i-jT-s}| \le x\right) \\
&\le P\left(\bigcup_{s=0}^{T-1}\bigcup_{j=-\infty}^{\infty} \{|c_{jT+s} Z_{i-jT-s}| > x\}\right) \\
&+ P\left(\sum_{s=0}^{T-1}\sum_{j=-\infty}^{\infty} |c_{jT+s} Z_{i-jT-s}| \mathbb{1}_{\{|c_{jT+s} Z_{i-jT-s}| \le x\}} > x\right).
\end{aligned}$$

Applying Markov's inequality to the second term on the right hand side, we obtain for $i = 1, \dots, T$,

$$\begin{aligned}
&P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right) \Big/ P(|Z_i| > x) \\
&\quad \le \sum_{s=0}^{T-1}\sum_{j=-\infty}^{\infty} P(|Z_{i-s}| > |c_{jT+s}|^{-1} x) \Big/ P(|Z_i| > x) \\
&\quad + \frac{1}{x}\sum_{s=0}^{T-1}\sum_{j=-\infty}^{\infty} |c_{jT+s}| E(|Z_{i-s}| \mathbb{1}_{\{|Z_{i-s}| \le |c_{jT+s}|^{-1} x\}}) \Big/ P(|Z_i| > x) \\
&\quad = I(x) + J(x).
\end{aligned} \tag{4.5}$$

For $I(x)$, since for $i = 1, \dots, T$ and $s \in I_1$, $P(|Z_{i-s}| > x) \in RV_{-\alpha}$, we have that for all $s \in I_1$ and $j \in \mathbb{R}$ such that $|c_{jT+s}| < 1$ (i.e., all but a finite number) there exists x_0 such that $x > x_0$ implies

$$\frac{P(|Z_{i-s}| > |c_{jT+s}|^{-1} x)}{P(|Z_i| > x)} \le (1+\rho)|c_{jT+s}|^{\rho} p_{i-s}^{-1} \gamma_{i-s,i}^{(+)} p_i,$$

for each $i = 1, \dots, T$. This bound is summable because of (1.5) and hence by dominated convergence

$$\lim_{x\to\infty} I(x) = p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} p_{i-s}^{-1} \sum_{j=-\infty}^{\infty} |c_{jT+s}|^{\alpha}.$$

For $J(x)$ lets start by considering $0 < \alpha < 1$. From an integration by parts

$$\frac{E(|Z_{i-s}|\mathbb{1}_{\{|Z_{i-s}|\le x\}})}{xP(|Z_{i-s}| > x)} = \frac{\int_0^x P(|Z_{i-s}| > u)du}{xP(|Z_{i-s}| > x)} - 1,$$

and since $P(|Z_{i-s}| > x) \in RV_{-\alpha}$ for all $i = 1, \ldots, T$, $s \in I_1$, by applying Karamata's Theorem this converges as $x \to \infty$ to $\alpha(1-\alpha)^{-1}$. Thus $E(|Z_{i-s}|\mathbb{1}_{\{|Z_{i-s}|\le x\}}) \in RV_{1-\alpha}$ and hence we have, for all but a finite number of s and j's, that for x sufficiently large and some constant $K' > 0$,

$$\begin{aligned}|c_{jT+s}|\frac{E(|Z_{i-s}|\mathbb{1}_{\{|Z_{i-s}|\le |c_{jT+s}|^{-1}x\}})}{xP(|Z_i| > x)} &\le K'|c_{jT+s}|(|c_{jT+s}|^{-1})^{1-\alpha+\alpha-\rho}\ p_{i-s}^{-1}\,\gamma_{i-s,i}^{(+)}\ p_i \\ &= K'|c_{jT+s}|^{\rho}\ p_{i-s}^{-1}\,\gamma_{i-s,i}^{(+)}\ p_i,\end{aligned}$$

which is summable in s and j. So we conclude

$$\limsup_{x\to\infty} J(x) \le K'\ p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)}\ p_{i-s}^{-1} \sum_{j=-\infty}^{\infty} |c_{jT+s}|^{\alpha},$$

and hence with $0 < \alpha < 1$ for some $K' > 0$

$$\limsup_{x\to\infty} \frac{P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right)}{P(|Z_i| > x)} \le (K'+1)p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)}\ p_{i-s}^{-1} \sum_{j=-\infty}^{\infty} |c_{jT+s}|^{\alpha}. \tag{4.6}$$

If $\alpha \ge 1$, we get a similar inequality by reduction to the case $0 < \alpha < 1$ as follows: Pick $\beta \in (\alpha, \alpha\delta^{-1})$ and consider $c = \sum_j |c_j|$ and $p_j = |c_j|/c$. By Jensen's inequality we get

$$\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}|\right)^{\beta} = c^{\beta}\left(\sum_{j=-\infty}^{\infty} p_j|Z_{i-j}|\right)^{\beta} \le c^{\beta-1}\sum_{j=-\infty}^{\infty} |c_j|\ |Z_{i-j}|^{\beta}.$$

Then, by (4.5) we can write for $i = 1, \ldots, T$ and $\beta \in (\alpha, \alpha\delta^{-1})$

$$\begin{aligned}&P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right)\Big/P(|Z_i| > x) \\ &\quad\le P\left(\sum_{s=0}^{T-1}\sum_{j=-\infty}^{\infty} |c_{jT+s}|\ |Z_{i-jT-s}|^{\beta} > c^{1-\beta}\, x^{\beta}\right)\Big/P(|Z_i|^{\beta} > x^{\beta}) \\ &\quad\le \sum_{s=0}^{T-1}\sum_{j=-\infty}^{\infty} P(|Z_{i-s}|^{\beta} > |c_{jT+s}|^{-1}\, c^{1-\beta}\, x^{\beta})\Big/P(|Z_i|^{\beta} > x^{\beta}) \\ &\quad+\frac{1}{c^{1-\beta}\, x^{\beta}}\sum_{s=0}^{T-1}\sum_{j=-\infty}^{\infty} |c_{jT+s}|E(|Z_{i-s}|^{\beta}\ \mathbb{1}_{\{|Z_{i-s}|^{\beta}\le |c_{jT+s}|^{-1}\, c^{1-\beta}\, x^{\beta}\}})\Big/P(|Z_i|^{\beta} > x^{\beta}).\end{aligned}$$

Now, as before, since $P(|Z_{i-s}|^{\beta} > x) \in RV_{-\alpha\beta^{-1}}$ with $\delta < \alpha\beta^{-1} < 1$, for all $i = 1, \ldots, T$, $s \in I_1$,

$$\limsup_{x\to\infty} \frac{P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right)}{P(|Z_i| > x)} \leq (1+K'')p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \, p_{i-s}^{-1} \sum_{j=-\infty}^{\infty} |c_{jT+s}|^{\alpha\beta^{-1}} \, c^{\alpha(1-\beta^{-1})} < \infty, \tag{4.7}$$

for some constant $K'' > 0$, which is similar to (4.6).

We are now in conditions to prove (2.9): For any integer $m = KT$ with $K \geq 1$ we have the obvious extension of (4.2)

$$\frac{P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right)}{P(|Z_i| > x)} \geq \frac{P\left(\sum_{j=-m}^{m-1} |c_j Z_{i-j}| > x\right)}{P(|Z_i| > x)}$$

$$\underset{x\to\infty}{\longrightarrow} \quad p_i \sum_{j=-m}^{m-1} \gamma_{i-j,i}^{(+)} \, p_{i-j}^{-1} \, |c_j|^{\alpha} = p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \, p_{i-s}^{-1} \sum_{j=-K}^{K-1} |c_{jT+s}|^{\alpha},$$

and since K is arbitrary

$$\liminf_{x\to\infty} \frac{P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right)}{P(|Z_i| > x)} \geq p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \, p_{i-s}^{-1} \sum_{j=-\infty}^{\infty} |c_{jT+s}|^{\alpha}.$$

On the other hand, for any $\epsilon > 0$, $I_2 = \{-m, \ldots, m-1\}$ and $I_2^* = \mathbb{N}_0 \backslash I_2$

$$\frac{P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right)}{P(|Z_i| > x)} \leq \frac{P\left(\sum_{j\in I_2} |c_j Z_{i-j}| > (1-\epsilon)x\right)}{P(|Z_i| > x)} + \frac{P\left(\sum_{j\in I_2^*} |c_j Z_{i-j}| > \epsilon x\right)}{P(|Z_i| > x)}$$

and so from (4.2) and (4.6) for some $K' > 0$

$$\limsup_{x\to\infty} \frac{P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right)}{P(|Z_i| > x)} \leq (1-\epsilon)^{-\alpha} \, p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \, p_{i-s}^{-1} \sum_{j=-K}^{K-1} |c_{jT+s}|^{\alpha}$$

$$+(K'+1)\epsilon^{-\alpha} \, p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \, p_{i-s}^{-1} \sum_{j\notin\{-K,\ldots,K-1\}} |c_{jT+s}|^{\alpha},$$

for the case $0 < \alpha < 1$, with a similar bound provided by (4.7) when $\alpha \geq 1$. Let $K \to \infty$ and then send $\epsilon \to 0$ to obtain for $i = 1, \ldots, T$

$$\limsup_{x\to\infty} \frac{P\left(\sum_{j=-\infty}^{\infty} |c_j Z_{i-j}| > x\right)}{P(|Z_i| > x)} \leq p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \, p_{i-s}^{-1} \sum_{j=-\infty}^{\infty} |c_{jT+s}|^{\alpha},$$

which combined with the lim inf statement proves (2.9). □

Proof (Theorem 2.2). For $m = KT$ with $K \geq 1$ arbitrary, lets consider the $T-$periodic sequence $\mathbf{X}^{(m)} = \{X_n^{(m)}\}_{n\geq 1}$ of finite moving averages of the form

$$X_n^{(m)} = \sum_{j=-m}^{m-1} c_j \, Z_{n-j}, \quad X_n^{*(m)} = X_n - X_n^{(m)}. \tag{4.8}$$

For $X_i^{(m)}, i = 1, \ldots, T, m$ defined in this way we have for $\epsilon \in (0, 1)$,

$$P(X_i^{(m)} > (1+\epsilon)x) - P\left(\sum_{j \notin \{-m,\ldots,m-1\}} |c_j Z_{i-j}| \geq \epsilon x\right) \tag{4.9}$$

$$\leq P(X_i^{(m)} > (1+\epsilon)x) - P(X_i^{*(m)} \leq -\epsilon x)$$

$$\leq P(X_i^{(m)} > (1+\epsilon)x, X_i^{*(m)} > -\epsilon x)$$

$$\leq P(X_i > x)$$

$$\leq P(X_i^{(m)} > (1-\epsilon)x) + P(X_i^{*(m)} > \epsilon x)$$

$$\leq P(X_i^{(m)} > (1-\epsilon)x) + P\left(\sum_{j \notin \{-m,\ldots,m-1\}} |c_j Z_{i-j}| > \epsilon x\right). \tag{4.10}$$

Theorem 1.1 implies that

$$\lim_{m\to\infty} \lim_{x\to\infty} P\left(\sum_{j \notin \{-m,\ldots,m-1\}} |c_j Z_{i-j}| \geq \epsilon x\right) \Big/ P(|Z_i| > x)$$

$$= \lim_{K\to\infty} p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} p_{i-s}^{-1} \sum_{j \notin \{-K,\ldots,K-1\}} |c_{jT+s}|^\alpha = 0.$$

The latter relation, (4.9) and (4.10) show that it's suffice to prove, for every $m = KT$, $K \geq 1$ that

$$P(X_i^{(m)} > x) \sim x^{-\alpha} L_i(x) \left\{ p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \sum_{j=-K}^{K-1} \left[c_{jT+s}^{+}\right]^\alpha + q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} \sum_{j=-K}^{K-1} \left[c_{jT+s}^{-}\right]^\alpha \right\}.$$

As in the proof of Theorem 1.1, by applying (1.2), (1.3) and (1.4), we have for $\delta \in (0, 1/3)$ and $i = 1, \ldots, T$,

$$\limsup_{x\to\infty} \frac{P\left(c_1 Z_{i-1} + c_2 Z_{i-2} + c_3 Z_{i-3} > x\right)}{P(|Z_i| > x)}$$

$$\leq (1-\delta)^{-2\alpha} p_i \sum_{j=1}^{3} \gamma_{i-j,i}^{(+)} \left[c_j^+\right]^\alpha + (1-\delta)^{-2\alpha} q_i \sum_{j=1}^{3} \gamma_{i-j,i}^{(-)} \left[c_j^-\right]^\alpha, \tag{4.11}$$

and

$$\liminf_{x\to\infty} \frac{P\left(c_1 Z_{i-1} + c_2 Z_{i-2} + c_3 Z_{i-3} > x\right)}{P(|Z_i| > x)}$$

$$\geq (1+\delta)^{-2\alpha} p_i \sum_{j=1}^{3} \gamma_{i-j,i}^{(+)} \left[c_j^+\right]^\alpha + (1-\delta)^{-2\alpha} q_i \sum_{j=1}^{3} \gamma_{i-j,i}^{(-)} \left[c_j^-\right]^\alpha. \tag{4.12}$$

Letting $\delta \to 0$ in (4.11) and (4.12) concludes the proof. □

Proof (Theorem 2.3). *(i)* For $\epsilon > 0$

$$
\begin{aligned}
&|P(M(I_n) \leq u_n) - P(M^{(m)}(I_n) \leq u_n)| \\
&\quad \leq P\left((1-\epsilon)u_n < M(I_n) \leq (1+\epsilon)u_n\right) + P(|M(I_n) - M^{(m)}(I_n)| > \epsilon u_n) \\
&\quad \leq n\sum_{i=1}^{T} P\left((1-\epsilon)u_n < X_i \leq (1+\epsilon)u_n\right) + n\sum_{i=1}^{T} P(|X_i - X_i^{(m)}| > \epsilon u_n).
\end{aligned}
$$

Following *(i)* from (2.10) and (2.11).

(ii) Let $\lambda \in (0,1)$, $A \subset \{1,\ldots,k\}$ and $B \subset \{k+[n\lambda],\ldots,n\}$, $k \leq n - [n\lambda]$. Taking the suprema over all A and B we have by the triangle inequality

$$
\begin{aligned}
&\limsup_{n\to\infty} \{|P(M(A\cup B) \leq u_n) - P(M^{(m)}(A\cup B) \leq u_n)|\} \\
&\quad \leq \lim_{m\to\infty} \limsup_{n\to\infty} \{|P(M(A\cup B) \leq u_n) - P(M^{(m)}(A\cup B) \leq u_n)|\} \\
&\quad + \lim_{m\to\infty} \limsup_{n\to\infty} \{|P(M^{(m)}(A) \leq u_n)P(M^{(m)}(B) \leq u_n) - P(M(A) \leq u_n)P(M(B) \leq u_n)|\} \\
&\quad + \lim_{m\to\infty} \limsup_{n\to\infty} \{|P(M^{(m)}(A\cup B) \leq u_n) - P(M^{(m)}(A) \leq u_n)P(M^{(m)}(B) \leq u_n)|\} = o(1),
\end{aligned}
$$

by *(i)* and the fact that $D(\mathbf{u})$ holds $\mathbf{X}^{(m)}$, for all m. Thus, by Lemma 3.2.1 of Leadbetter *et al.* (1983), $D(\mathbf{u})$ holds for $\mathbf{X}$, with $\beta_{n,[n\lambda]} = 0$.

(iii) Since

$$
\begin{aligned}
&n|P(X_i > u_n \geq M_{i+1,i+k-1}) - P(X_i^{(m)} > u_n \geq M^{(m)}_{i+1,i+k-1})| \\
&\quad \leq n|P(M_{i+1,i+k-1} \leq u_n) - P(M^{(m)}_{i+1,i+k-1} \leq u_n)| + n|P(M_{i,i+k-1} \leq u_n) - P(M^{(m)}_{i,i+k-1} \leq u_n)|,
\end{aligned}
$$

(iii) follows immediately from *(i)*. □

Proof (Theorem 3.1). Lets consider again the $T-$periodic sequence $\mathbf{X}^{(m)}, m \geq 1$ of finite moving averages defined in (4.8). Since $\mathbf{X}^{(m)}$ is $(2m-1)$-dependent it verifies $D(\mathbf{u})$ with mixing coefficient $\beta_{n,l_n} = 0$, for $l_n \geq 2m$.

From (3.1) and Theorem 1.2, it follows that

$$nP(X_i > u_n) \underset{n\to\infty}{\longrightarrow} \tau_i, \quad i = 1,\ldots,T.$$

Hence, by considering $c_j = 0$ for $j \notin I_2 = \{-m,\ldots,m-1\}$, we also have

$$nP(X_i^{(m)} > u_n) \underset{n\to\infty}{\longrightarrow} \tau_i, \quad i = 1,\ldots,T.$$

In this way, $D_T^{(2m)}(\mathbf{u})$ also holds for $\mathbf{X}^{(m)}$ since for $\mathbf{k} = \{k_n\}_{n\in\mathbb{N}}$ as in the definition of $D_T^{(2m)}$ with $l_n \equiv 2m$ we have for $r'_n = \left[\frac{n}{k_n T}\right]$

$$S_{r_n'}^{(2m)} = n\frac{1}{T}\sum_{i=1}^{T}\sum_{j=i+2m}^{r_n' T} P(X_i^{(m)} > u_n, X_{j-1}^{(m)} \leq u_n < X_j^{(m)})$$
$$\leq r_n' n\frac{1}{T}\sum_{i=1}^{T} P(X_i^{(m)} > u_n)P(X_{i+2m}^{(m)} > u_n) = o(1).$$

We can then use (1.8) to compute the extremal index of $\mathbf{X}^{(m)}$.
For $i = 1, \ldots, T$,

$$\begin{aligned} P(X_i^{(m)} > u_n \geq M_{i+1,i+2m-1}^{(m)} &= P(M_{i+1,i+2m-1}^{(m)} \leq u_n, \max_{j\in I_2}\{c_j Z_{i-j}\} > u_n) \\ &- P(M_{i,i+2m-1}^{(m)} \leq u_n, \max_{j\in I_2}\{c_j Z_{i-j}\} > u_n) \\ &+ P(X_i^{(m)} > u_n \geq M_{i+1,i+2m-1}^{(m)}, \max_{j\in I_2}\{c_j Z_{i-j}\} \leq u_n). \quad (4.13) \end{aligned}$$

Let us first note that for any $\epsilon > 0$,

$$P(X_i^{(m)} > u_n \geq M_{i+1,i+2m-1}^{(m)}, \max_{j\in I_2}\{c_j Z_{i-j}\} \leq u_n)$$
$$\leq P(X_i^{(m)} > u_n, \max_{j\in I_2}\{c_j Z_{i-j}\} \leq (1-\epsilon)u_n) + P((1-\epsilon)u_n < \max_{j\in I_2}\{c_j Z_{i-j}\} \leq u_n),$$

and

$$\begin{aligned} &P(X_i^{(m)} > u_n, \max_{j\in I_2}\{c_j Z_{i-j}\} \leq (1-\epsilon)u_n) \\ &= P\left(\bigcup_{s=-m}^{m-1}\{X_i^{(m)} > u_n, c_s Z_{i-s} = \max_{j\in I_2} c_j Z_{i-j} \leq (1-\epsilon)u_n\}\right) \\ &\leq \sum_{s=-m}^{m-1} P(X_i^{(m)} > u_n, c_s Z_{i-s} \leq (1-\epsilon)u_n) \\ &\leq \sum_{s=-m}^{m-1} P\left(\sum_{\substack{k=-m \\ k\neq s}}^{m-1} \min\{c_k Z_{i-k}, c_s Z_{i-s}\} > \epsilon u_n\right) \\ &\leq \sum_{s=-m}^{m-1} P\left(\sum_{\substack{k=-m \\ k\neq s}}^{m-1} c_k Z_{i-k} > \epsilon u_n, \sum_{\substack{k=-m \\ k\neq s}}^{m-1} c_s Z_{i-s} > \epsilon u_n\right) \\ &\leq \sum_{s=-m}^{m-1}\sum_{\substack{k=-m \\ k\neq s}}^{m-1} P\left(c_k Z_{i-k} > \epsilon u_n(2m-1)^{-1}\right) P\left(c_s Z_{i-s} > \epsilon u_n(2m-1)^{-1}\right) = O(n^{-2}). \end{aligned} \quad (4.14)$$

On the other hand, for $\rho > 0$

$$\lim_{n\to\infty} nP\left(\max_{j\in I_2} c_j Z_{i-j} > \rho u_n\right) = \lim_{n\to\infty} n\left\{\sum_{j\in S_+} P(Z_{i-j} > \rho c_j^{-1} u_n) + \sum_{j\in S_-} P(Z_{i-j} < \rho c_j^{-1} u_n)\right\}$$
$$= \frac{\tau_i\, p_i\, \rho^{-\alpha} \sum_{j\in S_+} c_j^{\alpha}\, \gamma_{i-j,i}^{(+)} + \tau_i\, q_i\, \rho^{-\alpha} \sum_{j\in S_-} (-c_j)^{\alpha}\, \gamma_{i-j,i}^{(-)}}{p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \sum_{j=-\infty}^{\infty} \left[c_{jT+s}^{+}\right]^{\alpha} + q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} \sum_{j=\infty}^{\infty} \left[c_{jT+s}^{-}\right]^{\alpha}}$$

where $S_+ = \{j : c_j \geq 0, j \in I_2\}$ and $S_- = \{j : c_j < 0, j \in I_2\}$.
Hence,

$$\lim_{\epsilon\to 0}\lim_{n\to\infty} nP\left((1-\epsilon)u_n < \max_{j\in I_2} c_j Z_{i-j} \leq u_n\right)$$
$$= \lim_{\epsilon\to 0}\lim_{n\to\infty} nP\left(\max_{j\in I_2} c_j Z_{i-j} > (1-\epsilon)u_n\right) - \lim_{n\to\infty} nP\left(\max_{j\in I_2} c_j Z_{i-j} > u_n\right) = 0, \quad (4.15)$$

and so from (4.14) and (4.15) it follows that for $i = 1, \ldots, T$,

$$\lim_{n\to\infty} nP\left(X_i^{(m)} > u_n \geq M_{i+1,i+2m-1},\ \max_{j\in I_2} c_j Z_{i-j} \leq u_n\right) = 0. \quad (4.16)$$

By a similar analysis we deduce that

$$\lim_{n\to\infty} nP\left(M_{i,i+2m-1} \leq u_n,\ \max_{j\in I_2} c_j Z_{i-j} > u_n\right) = 0. \quad (4.17)$$

Combining (4.13), (4.16) and (4.17) we obtain

$$\lim_{n\to\infty} nP\left(X_i^{(m)} > u_n \geq M_{i+1,i+2m-1}^{(m)}\right) \quad (4.18)$$
$$= \lim_{n\to\infty} nP\left(M_{i+1,i+2m-1}^{(m)} \leq u_n,\ \max_{j\in I_2} c_j Z_{i-j} > u_n\right)$$
$$= \lim_{n\to\infty} n\left\{\sum_{j\in S_+} P\left(c_j Z_{i-j} > u_n \geq M_{i+1,i+2m-1}^{(m)}\right) + \sum_{j\in S_-} P\left(c_j Z_{i-j} > u_n \geq M_{i+1,i+2m-1}^{(m)}\right)\right\}.$$

If $j \in S_+$ and $c_j Z_{i-j} > u_n$ for each $i = 1, \ldots, T$, then the condition $M_{i+1,i+2m-1} \leq u_n$ is essentially $\max(0, \max_{i+1\leq s\leq i+2m-1}\ c_{s-i+j})Z_{i-j} = c_j^{+}(2m)Z_{i-j}$. In the same way, if $j \in S_-$ and $c_j Z_{ij} > u_n$ for each $i = 1, \ldots, T$, then the condition $M_{i+1,i+2m-1} \leq u_n$ can be replaced by $\max(0, \max_{i+1\leq s\leq i+2m-1} (-c_{s-i+j})(-Z_{i-j})) = -c_j^{-}(2m)Z_{i-j}$. Using these arguments it is straightforward that (4.18) equals

$$= \lim_{n\to\infty} n \left\{ \sum_{j\in S_+} P(c_j Z_{i-j} > u_n \geq c_j^+(2m) Z_{i-j}) + \sum_{j\in S_-} P(c_j Z_{i-j} > u_n \geq (-c_j^-(2m)) Z_{i-j}) \right\}$$

$$= \frac{\tau_i \left\{ p_i \sum_{j=-m}^{m-1} \gamma_{i-j,i}^{(+)} ([c_j^+]^\alpha - [c_j^+(2m)]^\alpha)_+ + q_i \sum_{j=-m}^{m-1} \gamma_{i-j,i}^{(-)} ([c_j^-]^\alpha - [c_j^-(2m)]^\alpha)_+ \right\}}{p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \sum_{j=-\infty}^{\infty} [c_{jT+s}^+]^\alpha + q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} \sum_{j=\infty}^{\infty} [c_{jT+s}^-]^\alpha}$$

where in the definition of $c_j^\pm(2m)$ we use the convention $c_j = 0$ for $j \notin I_2 = \{-m, \dots, m-1\}$, with in particular, $c_{m-1}^\pm(2m) = 0$.

From this, it follows immediately the subsequent expression for the extremal index of $\mathbf{X}^{(m)}$

$$\theta_{\mathbf{X}}^{(m)} = \frac{\sum_{i=1}^{T} \frac{\tau_i \left\{ p_i \sum_{j=-m}^{m-1} \gamma_{i-j,i}^{(+)} ([c_j^+]^\alpha - [c_j^+(2m)]^\alpha)_+ + q_i \sum_{j=-m}^{m-1} \gamma_{i-j,i}^{(-)} ([c_j^-]^\alpha - [c_j^-(2m)]^\alpha)_+ \right\}}{p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \sum_{j=-\infty}^{\infty} [c_{jT+s}^+]^\alpha + q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} \sum_{j=-\infty}^{\infty} [c_{jT+s}^-]^\alpha}}{\sum_{i=1}^{T} \tau_i} \tag{4.19}$$

Now considering

$$A_i = p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \sum_{j=-\infty}^{\infty} [c_{jT+s}^+]^\alpha + q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} \sum_{j=-\infty}^{\infty} [c_{jT+s}^-]^\alpha, \quad i = 1, \dots, T,$$

by (3.1) we can establish the following relation

$$\frac{\tau_i}{A_i} = \frac{\tau_j}{A_j} \gamma_{i,j}, \quad i, j = 1, \dots, T.$$

Using this relation in (4.19) we obtain the next simplified expression for the extremal index of $\mathbf{X}^{(m)}$

$$\theta_{\mathbf{X}}^{(m)} = \frac{\sum_{i=1}^{T} \left\{ \gamma_{i,1}\, p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} c_s^{(K)}(\alpha)^+ + \gamma_{i,1}\, q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} c_s^{(K)}(\alpha)^- \right\}}{\sum_{i=1}^{T} \left\{ \gamma_{i,1}\, p_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(+)} \sum_{j=-\infty}^{\infty} [c_{jT+s}^{(+)}]^\alpha + \gamma_{i,1}\, q_i \sum_{s=0}^{T-1} \gamma_{i-s,i}^{(-)} \sum_{j=-\infty}^{\infty} [c_{jT+s}^-]^\alpha \right\}}$$

where $c_s^{(K)}(\alpha)^+ = \sum_{j=-K}^{K-1} ([c_{jT+s}^+]^\alpha - [c_{jT+s}^+(2m)]^\alpha)_+$ and $c_s^{(K)}(\alpha)^- = \sum_{j=-K}^{K-1} ([c_{jT+s}^-]^\alpha - [c_{jT+s}^-(2m)]^\alpha)_+$, $s = 0, \dots, T-1$.

It follows by an easy check that $\theta_{\mathbf{X}^{(m)}} \to \theta_{\mathbf{X}}$ as $m = KT \to \infty$, hence by Theorem 1.3 and the remark immediately following it we obtain the result upon showing (2.10) and (2.11) which is straightforward. $\square$

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