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Abstract

In this paper we study the graphs such that the deletion of any edge does not increase the diameter. We give some upper bounds for the order of such a graph with given maximum degree and diameter. On the other hand construction of graphs provide lower bounds. As usual, for this kind of problems, there is often a gap between these two bounds.

1 Introduction

A A graph is said *edge-non-vulnerable* if its diameter is unchanged after deletion of any one of its edges.

Such graphs do exist. For example the graph on 4 vertices on the left of figure 1 has diameter 2, and the removal of an edge gives a graph isomorphic to one of the other graphs in the picture, both have diameter 2.

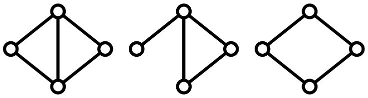


Figure 1: A (toy) edge-non-vulnerable graph

An obvious upper bound for these edge-non-vulnerable graphs with given maximum degree Δ and diameter D is the classical Moore bound, namely $1 + \Delta \sum_{k=0}^{D-1} (\Delta - 1)^k$. But this can be easily improved, since the condition imposes that between any pair of distinct vertices at least two paths of length $\leq D$ exist, and this implies the upper bound

$$n \leq n(\Delta, D) = 1 + \frac{1}{2}\Delta \sum_{k=0}^{D-1} (\Delta - 1)^k$$

Clearly, no graph of diameter 1 (in other words no complete graph) is edge-non-vulnerable, since the removal of the edge betwen x and y either disconnects the graph (if its order is 2) or increases its diameter to 2 (if the order is larger than 2.

2 Diameter 2, upper bound

For diameter 2, we have $n(\Delta, D) = 1 + \Delta^2/2$. This bound obviously cannot be attained if Δ is odd! So what about $(\Delta^2 + 1)/2$? This number is odd, therefore it is not compatible with a regular graph of degree Δ. Moreover, if some vertex has degree $< \Delta$, counting paths from that vertex decreases the bound to $1 + (\Delta - 1)^2/2 \leq (\Delta^2 - 1)/2$. So, what about $(\Delta^2 - 1)/2$? The toy graph of figure 1 shows that this bound $(\Delta^2 - 1)/2$ can be obtained for $\Delta = 3$. For the next odd degree $\Delta = 5$, the cartesian sum of K_3 and K_4 has the wanted property and order, namely $12 = (5^2 - 1)/2$.

For even degrees, the bound is attained only if a *distance-regular graph* (see [1]) with intersection matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ \Delta & 1 & 2 \\ 0 & \Delta - 2 & \Delta - 2 \end{bmatrix}$$

The techniques of distance-regular graphs lead to the computation of the eigenvalues: they are Δ and the two roots of $X^2 + X - +2 - \Delta$. Such an eigenvalue $\lambda \neq \Delta$ has then multiplicity

$$\frac{1 + \Delta^2/2)}{1 + \frac{\lambda^2}{\Delta} + \frac{(\lambda+1)^2}{\Delta(\Delta-2)/2)}}$$

that is also

$$\frac{(2 + \Delta^2)\Delta(\Delta - 2)}{4(\Delta - 1)^2 + (4 - \Delta)\lambda}$$

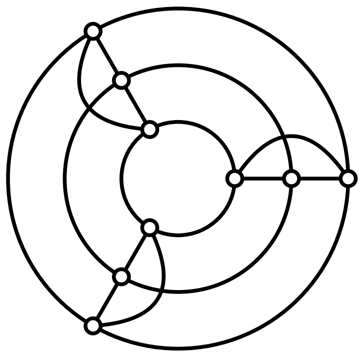


Figure 2: The optimal graph for degree 4 and diameter 2

Since this should be an integer, we should have either $\Delta = 4$ or λ integer, and in this case, $2(2\lambda + 1$ has to divide $(\lambda^2 + 2\lambda + 3)(\lambda^2 + \lambda + 2)\lambda$ and therefore $2(2\lambda + 1$ has to divide 63. This allows only the values 2, 4, 14, 22, 112, 994 for Δ.

The case $\Delta = 2$ is not interesting, the case $\Delta = 4$ gives a graph shown in figure 2 The case $\Delta = 22$, $n = 243$ is known: it is the Berlekamp, van Lint and Seidel graph, a Cayley graph on the group $(\mathbb{Z}/3\mathbb{Z})^5$ ([1, p. 360]).

The case $\Delta - 14$, $n = 99$ is unsolved, according to G. Exoo's list of unknown strongly regular graphs ([4])

3 Diameter 3, upper bound

The condition of edge-non-vulnerability is then: each edge lies in some cycle of length at most 4, each path of length 2 not already in a 4-cycle should be in a 5-cycle, unless each of its edges is in a 3-cycle, and at last, each path of length 3 should be in a cycle of length at most 6.

This provides the bound $n = 1 + \Delta + \Delta_2 + \Delta_3$, where $\Delta_2 \leq \Delta(\Delta - 1) - \lceil \Delta/2 \rceil$ and $\Delta_3 \leq \lfloor \Delta_2(\Delta - 2)/2 \rfloor$.

For $\Delta = 3$, this improved bound is 10, and the cartesian sum of a 5-cycle and K_2 is convenient.: figure 3

For $\Delta = 4$, the bound is 25. However, to attain this value it is necessary that the edge set is partitioned into 4-cycles, this is clearly not possible in a graph with $25 \cdot 4/2 = 50$ edges. The same obstruction occurs for all

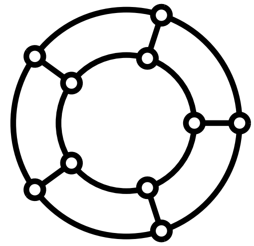


Figure 3: Optimal graph for $\Delta = 3$, and $D = 3$

degrees multiple of 4. For $\Delta \equiv 7$ or 9 (mod 8), the hand-shaking lemma also indicates that is bound is still too high!

4 Higher diameters, upper bound

It happens that the computation of improved upper bounds becomes more and more complicated. Just an example: for $D = 4$ and $\Delta = 3$, one has between vertices at distance 2 from a vertex v at least two edges. Thus at most 8 edges connect the sphere at distance 2 to the one at distance 3. Since the paths of length 2 that are not already in a 5-cycle have to be in a 6-cycle, the 7 sphere has at most 7 vertices, and the sphere at diatance 4 from v has at most 3. Thus a bound is 20. But the graph should then have two pentagons through each vertex, this makes at least 8 pentagons. If a vertex is on 3 pentagons, the bound becomes 19, and even 18 owing to the hand-shaking lemma. There are 30 edges. If an edge belong to 3 pentagons, its endvertices do. Otherwise there are 10 edges belonging to 2 pentagons, some pentagon has at least two such edges: if these edges are adjacent, their common vertex in on 3 pentagons, if the 10 edges are not adjacent they form a matching. The last vertex of a pentagon that has already 2 edges belonging to 2 pentagons is on 3 pentagons.

Thus the bound is now 18. It is easy to check that if the graph contains a cycle of length 3 or 4, the bound is only 16. On the other hand one can build convenient graphs on 16 vertices: figure 4.

Thus the optimal graph has 16 or 18 vertices.

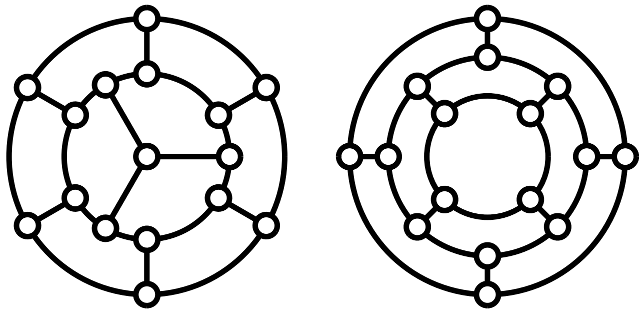


Figure 4: Graphs for degree 3 and diameter 4

5 Cartesian sums and categorical products

A first general construction is the *cartesian sum* of graphs,: the vertex set of $G_1 \square G_2$ has vertex set the product of the vertex sets, the edges of $GG_1 \square G_2$ are the pairs $\{(x,y),(x',y)\}$ where $\{x,x'\}$ is an edge of G_1 and y avertex of G_2 and the pairs $\{(x,y),(x,y')\}$ where x is a vertex of G_1 and $\{y,y'\}$ an edge of G_2.

The cartesian sum of G_1 (diameter D_1 , maximum degree Δ_1) and G_2 (diameter D_2 , maximum degree Δ_2) has maximum degree $\Delta_1 + \Delta_2$ and diameter $D_1 + D_2$, and is edge-non-vulnerable provided if $D_1 + D_2 > 2$,

For example, the cartesian sum of K_2 and Petersen graph has $n = 20$, $D = 3$, $\Delta = 4$, no so far from the (unaccessible) 25.

The *categorical product* of graphs (that may have loops) G_1 and G_2 has vertex set the product of the vertex sets, the edges of $G_1 \times G_2$ are the pairs (or loops) $\{(x,x'),(y,y')\}$ wher $\{x,x'\}$ and $\{y,y'\}$ are edges or loops of G_1 and G_2.

The maximum degree of $G_1 \times G_2$ is the product of the maximum degrees in G_1 and in G_2. The distance between (x,y) and (x',y') is the minimum between the lengths of paths (elementary or not) *of same parity* connecting x, x' and y, y'.

Figure 5 shows the product of the graph K_2^+ made from an edge with a loop at each end and first a 5-cycle with 3 loops and then with a triangle (this gives a graph isomorphic to the octahedron).

The graph of figure 2 is the categorical product of two triangles, that is $K_3 \times K_3$ or $K_3^{\times 2}$. It is also the cartesian sum of two triangles. The distance

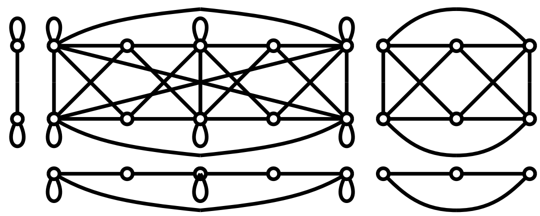


Figure 5: Examples od products with K_2^+

between (x, y) and (x, y') is 2 because there is a non-elementary path of length 2 from x to itself and a path of length 2 from y to y'; the distance between (x, y) and (x', y') is 1 because there is a path of length 1 from x to x' and a path of length 1 from y to y'; and so on.

6 Biplanes

A biplane is a bipartite distance-regular graph with intersection array $(d, d-1, d-2; 1, 2, d)$, thus of order $2n = d^2 - d + 2$ The Bruck-Ryser-Chowla theorem allows the existence of such graphs only if either n is even and $d-2$ is a square or n is odd and $x^2 = (d-2)y^2 + (-1)^{(n-1)/2}2z^2$ has integer non null solutions ([1, p. 698]).

If a biplane has a polarity, the quotient has degree d (with loops counting for 1), order $(d^2 + -d + 2)/2$ and diameter at most 2, and each edge either has a loop at its two endpoints, or lies in a triangle.

For $d = 2$, we have (with a bit cheating) the 4-cycle and its quotient K_2^+.

For $d = 3$,we have the usual cube, and the quotients K_4, $K_{2,1,1}$ (the toy graph of Figure 1 with loops on the vertices of degree 2, and C_4 with a loop at each vertex.

For degree 4, we have a graph with polarities, its quotient (that has always 4 loops) is shown in Figure 6.

For degree 5 the quotient also has edges with two ends occupied by loops (there are always 5 loops)

For degree 6, several quotients are possible, with 0 loops ($K_4 \square K_4$ or Shrikhande graph) or 16 loops (Clebsch graph), among other less symmetric graphs; Eigenvalue considerations impose that the number of loops is a multiple of 4.

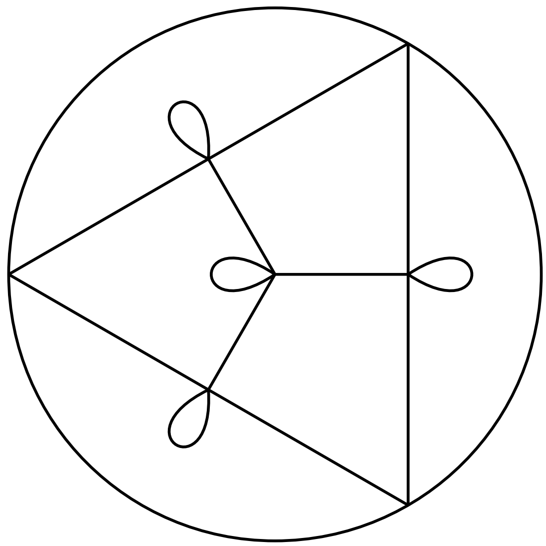


Figure 6: A quotient of a biplane of degree 4

For degree 9, one has a quotient with 37 vertices labeled with the elements of the field $\mathbb{Z}/37\mathbb{Z}$, and x, y are adjacent if their sum is one of the non-null 4-th powers in the field, that is 1, 7, 9, 10, 12, 16, 26, 33, 34. Here the vertices with loops are never adjacent.

For degree 11, there are several biplanes, with qoutients of order 56. One of them has 56 loops: the Gewirtz graph. Other quotients, have a number of loops congruent to 2 modulo 6.

The categorical products of these quotients with Erd***H***os-Rényi graphs are convenient and for some degrees and diameter 2.

7 Diameter 2: lower bounds

The product of K_2^+ with the Erd***H***os Rényi-graphs of degree d and order $d^2 - d + 1$ (with their loops) has diameter 2, degree $2d$, order $2(d^2 - d + 1)$, that is close to the upper bound $\sim d^2/2$.

For some degrees we have special constructions

- degree 6, some quotients of a biplane
- degree 8 the categorical product $K_3 \times K_3 \times K_3$

• degree 9, some quotients of a biplane.

Let us summarize our results for small degrees in table 1.

Table 1: Some results for diameter 2

Δ	3	4	5	6
n	$\underline{4}$	$\underline{9}$	$\underline{12}$	16
	$K_{1,1,2}$	$K_3 \times K_3$	$K_3 \square K_4$	$K_4 \square K_4$
Δ	7	8	9	10
n	20	27	37	42
	$K_5 \square K_4$	$(K_3)^{\times 3}$	quot. bipl.	$K_2^+ \times$ER(5)
Δ	12	14	16	
n	63	84	117	
	$K_3^{\times 2} \times$ ER(3)	$K_4 \times$ ER(5)	$K_3^{\times 2} \times$ ER(4)	
Δ	18	20	22	
n	146	189	$\underline{243}$	
	$K_2^+ \times$ER(9)	$(K_3)^{\times 2} \times$ ER(5)	BvLS	

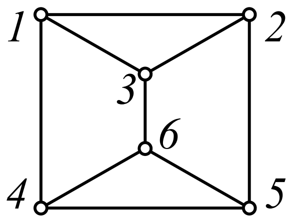
8 Twisted products

Since the cartesian sum clearly spills some edges with an excessive number of 4-cycles, we may improve things here and there.

• product $G \ltimes C_5$. The vertex set is the product of the vertex sets of G and C_5, the edges are the pairs $\{(g,a), g(,a')\}$ with g vertex of G and $\{a,a'\}$ an edge of C_5, and then G is endowed with an orientation, and C_5 with a permutation π exchanging the edges and non-edges of C_5, and we add the edges $\{(g,a),(g',\pi(a))\}$ (in other words, each edge of G is replaced by a Petersen graph). This gives a graph with diameter $D(G)+1$, maximum degree $\Delta(G)+2$, that is edge-non-vulnerable provided that $D(G) \geq 3$ and vertices at distance D in G are connected by two internally disjoint paths of length D.

• product $G \ltimes P(4t+1)$, where $P(4t+1)$ is the Paley graph on $4t+1$ vertices. The diameter is $D(G)+1$, and the maximum degree $\Delta(G)+2t$, and the graph is edge-non-vulnerable provided that $D(G) \geq 3$ and vertices at distance D in G are connected by two internally disjoint paths of length D.

• product $G \times C_{13}$, a similar construction, $\Delta(G \times C_{13}) = \Delta(G)+2$, and



The edges of the graph and their images by the involutive vertex permutation (1)(2)(36)(45) are all the edges of K_6.

Figure 7: The graph A_6 to be used in twisted products

$D(G \times C_{13}) = D(G) + 2$ under the same condition. Endowing C_{13} with the labels in $\mathbb{Z}/13\mathbb{Z}$ so that edges are labeled $\{a, a+1\}$, the permutation π sends the vertex i to the vertex $5i$ (so that π^2 is an isomorphism of C_{13}.

In the same vein, $G \ltimes A_6$, a similar construction, $\Delta(G \times A_6) = \Delta(G)+2$, and $D(G \times A_6) = D(G) + 2$ under the same condition. Here A_6 and its permutation π are represented in figure 7.

9 Line graphs

The line-graph L of a bipartite graph of degree $d \geq 3$, order n and diameter D has diameter D, degree $2d-2$ and order $dn/2$; each edge of L is in a triangle, and each pair of vertices of L at distance D is connected by 2 paths of length D. Thus the graph is non-edge-vulnerable.

The well-known large cubic bipartite graphs give for diameters 2, 3, 4 and 6 graphs of order 9 (the one we have already seen, from $K_{3,3}$), 21 (from Heawood graph), 45 (from Tutte's 8-cage), 189 (from Tutte's 12-cage). Besides the cubic bipartite graph of diameter 5 and order 56 described by Bond and Delorme [2] provides an edge-non-vulnerable graph on 84 vertices. having degree 4 and diameter 5. Some of these graphs are represented on figure 8. For diameters 3, 4 and 6, the line graphs of bipartite Moore graphs give some results.

10 A small census

We collect some results in the table 2.

This graph on 56 vertices of Figure 10 is the graph 56.2 in the list of M. Conder. Hea denotes Heawood graph, and TC Tutte-Coxeter graph; the O_k's are the so-called odd graphs

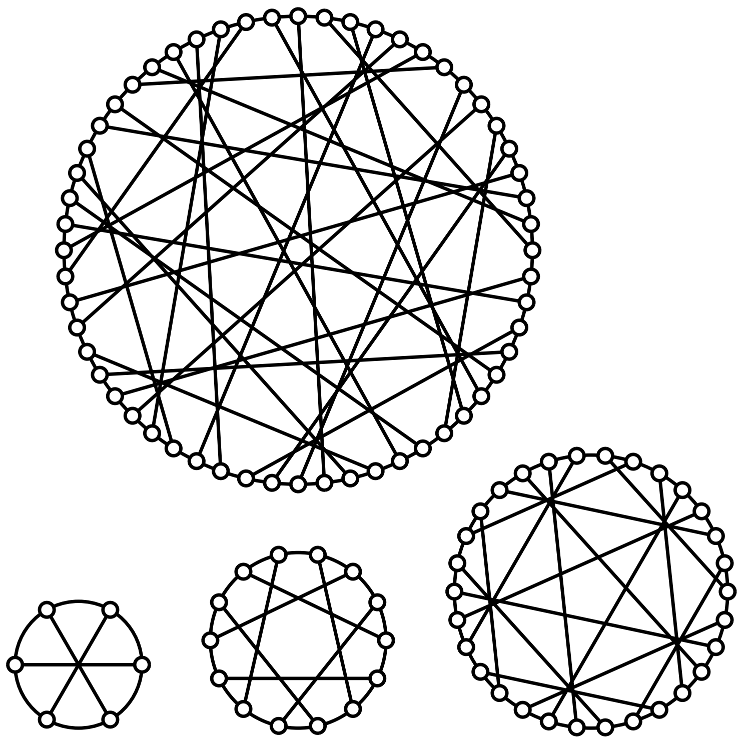


Figure 8: Large bipartite cubic graphs of diameters 2, 3, 4, 5

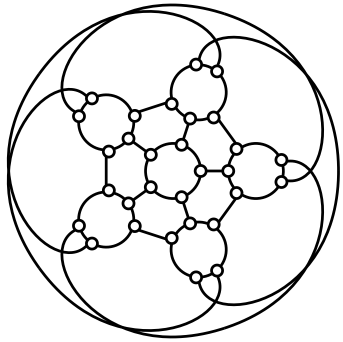


Figure 9: An edge- non-vulnerable graph on 30 vertices

Table 2: Some lower bounds

$\Delta\backslash D$	3	4	5	6
3	10	16	30	56
	$C_5 \,\square\, K_2$	fig. 4	fig.9	fig. 10
4	21	45	84	189
	LG	LG	LG	LG
5	30	70	182	390
	Pet.$\square K_3$	Hea. $\ltimes C_5$	Hea. $\ltimes C_{13}$	TC. $\ltimes C_{13}$.
6	52	175	462	1456
	LG	$O_4 \ltimes C_5$	O_6	LG
7	72	210	630	1716
	$24 \,\square\, K_3$	$O_4 \ltimes A_6$	$O_5 \ltimes C_5$	O_7
8	105	425	756	6825
	LG	LG	$O_5 \ltimes A_6$	LG

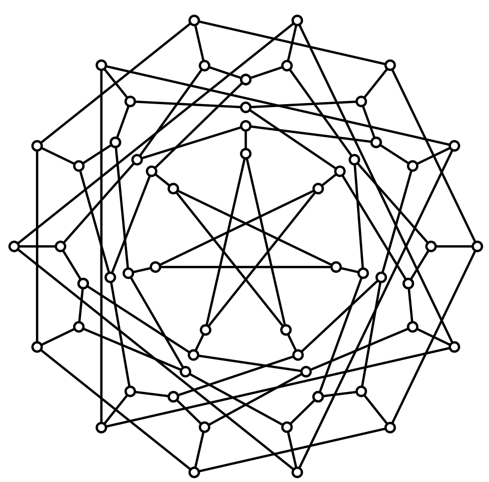


Figure 10: An edge- non-vulnerable graph on 56 vertices

11 Conclusion

We have given some indications on the large graphs with maximum degree and diameter whose diameter is unchanged after deletion of an edge. In the related problem with vertex deletion, the sufficient (but not necessary: see the line graphs of cubic graphs) condition that every path of length $\ell \geq 1$ should be in a cycle of length at most $\ell + D$ is replaced by the slightly weaker: every path of length $\ell \geq 2$ should be in a cycle of length at most $\ell + D$. Thus some of our graphs also provide solutions for the vertex-non-vulnerability, although not always as large as possible. See for example the survey paper by Fàbraga, Gómez and Yebra [5]

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