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# Propositional Calculus for Adjointness Lattices

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**Abstract**

Recently, Morsi has developed a complete syntax for the class of all adjointness algebras $(L, \leq, A, K, H)$. There, $(L, \leq)$ is a partially ordered set with top element 1, $K$ is a conjunction on $(L, \leq)$ for which 1 is a left identity element, and the two implication-like binary operations $A$ and $H$ on $L$ are adjoints of $K$.

In this paper, we extend that formal system to one for the class $ADJL$ of all 9-tuples $(L, \leq, 1, 0, A, K, H, \wedge, \vee)$, called *adjointness lattices*; in each of which $(L, \leq, 1, 0, \wedge, \vee)$ is a bounded lattice, and $(L, \leq, A, K, H)$ is an adjointness algebra. We call it *Propositional Calculus for Adjointness Lattices*, abbreviated $AdjLPC$. Our axiom scheme for $AdjLPC$ features four inference rules and thirteen axioms. We deduce enough theorems and inferences in $AdjLPC$ to establish its completeness for $ADJL$; by means of a quotient-algebra structure (a Lindenbaum type of algebra). We study two negation-like unary operations in an adjointness lattice, defined by means of 0 together with $A$ and $H$. We end by developing complete syntax for all adjointness lattices whose implications are $S$-type implications.

*Keywords:* Nonclassical logics; Syntax; Semantics; Adjointness; $S$-type implications

## 1 Propositional Calculus under Adjointness

In this section, we review the essentials of adjointness algebras, as well as the axioms, inference rules and main theorems of their complete syntax $AdjPC$ [16]. We show how a complete syntax (with fewer axioms and inference rules) has been developed in [16] for a syntax $EP$-$AdjPC$; with the smaller semantical domain of all adjointness algebras whose implications satisfy the exchange principle.

## 1.1 Adjointness Algebras

The logic *propositional calculus under adjointness*, denoted by $AdjPC$, is based on partially ordered sets (posets) $(L, \leq)$ whose elements are considered as *truth values*. Each poset is required to possess a top element 1; called *truth* or *validity*. The logic features three binary operations $A, K$ and $H$ on $(L, \leq)$. The operation $A$, called an *implication*, should be antitone in the left argument and monotone in the right argument, and should have 1 as a left identity element; that is $A(1, z) = z \quad \forall z \in L$. The operation $K$, called a *conjunction*, should be monotone in both arguments, and should also have 1 as a left identity element. ($K$ need neither be commutative nor be associative, and may have no right identity element.) The operation $H$, called a *forcing-implication*, should be antitone in the left argument and monotone in the right argument, and should satisfy: $\forall y, z \in L : \quad H(y, z) = 1 \quad iff \quad y \leq z$. ($H$ need not have a left identity element.) The logic *AdjPC* offers complete syntax (a formal system for deriving theorems) for the semantical domain consisting of all the quintuples of the following definition.

**Definition 1.1** *[16] An* adjointness algebra *is a quintuple* $(L, \leq, A, K, H)$*, in which* $(L, \leq)$ *is a poset with a top element,* $A$ *is an implication on* $(L, \leq)$*,* $K$ *is a conjunction on* $(L, \leq)$ *and* $H$ *is a forcing-implication on* $(L, \leq)$*, subject to the condition that* $A, K$ *and* $H$ *are mutually related, for all* $x, y, z$ *in* $L$*, by*

*(Adjointness):* $\forall x, y, z \in L : \quad y \leq A(x, z) \quad iff \quad K(x, y) \leq z \quad iff \quad x \leq H(y, z)$.

*We denote the class of all adjointness algebras by* $ADJ$.

The subject of a possibly noncommutative, nonassociative conjunction $K$ with two implication-like adjoints is an old one. See [2],[3],[4],[5],[6],[11],[19]. This idea lies also at the basis of that general trend in nonclassical logics collectively termed since 1990 *substructural logics*. Those are surveyed in the book [7], where one finds a detailed algebraic study of adjointness structures under the name *residuated partially ordered groupoids*, and a representation theorem for them is given in page 77 of [7]. See also *Galois connections* in [14]. The new contribution of [16] is the development of a complete syntax for those structures with weakest inference rules, in the general setting that 1 is a left identity, but not necessarily a right identity, for $K$.

The following are six basic inequalities in $(L, \leq, A, K, H)$:

$x \leq H(A(x, z), z) \quad , \quad x \leq H(y, K(x, y)) \quad , \quad y \leq A(H(y, z), z)$,

$y \leq A(x, K(x, y)) \quad , \quad K(x, A(x, z)) \leq z \quad , \quad K(H(y, z), y) \leq z$.

**Lemma 1.1** *[1] Let* $\{x_j\}$ *and* $\{y_s\}$ *be two subfamilies in an adjointness algebra* $(L, \leq, A, K, H)$ *that have suprema in* $L$*, and let* $\{z_t\}$ *be a subfamily of* $L$ *that has an infimum in* $L$*. Then*

$$A\left(\sup_j x_j, \inf_t z_t\right) = \inf_{j,t} A(x_j, z_t), \tag{1.1}$$

$$K\left(\sup_j x_j, \sup_s y_s\right) = \sup_{j,s} K(x_j, y_s), \tag{1.2}$$

$$H\left(\sup_s y_s, \inf_t z_t\right) = \inf_{s,t} A(y_s, z_t). \quad (1.3)$$

*Also, if* $(L, \leq)$ *has a bottom element* $0$*, then*
$A(0,0) = 1$ *and* $K(0,1) = K(1,0) = 0.$

**Lemma 1.2** *[18] Let* $(L, \leq)$ *be a complete lattice. If an implication* $A$ *on* $(L, \leq)$ *satisfies (1.1), then there exist unique* $K$ *and* $H$ *such that* $(L, \leq, A, K, H)$ *is an adjointness algebra. These are given by:*
$K(x,y) = \inf\{z \in L : \quad y \leq A(x,z)\}, \qquad x, y \in L,$
$H(y,z) = \sup\{x \in L : \quad y \leq A(x,z)\}, \qquad y, z \in L.$
*Similarly, a unique adjointness algebra will be obtained once a* $K$ *on* $(L, \leq)$ *that satisfies (1.2), or an* $H$ *on* $(L, \leq)$ *that satisfies (1.3), is given.*

Lemma 1.2 demonstrates that adjointness algebras constitute a readily available framework for the study of implications and conjunctions related by adjointness. The special case that $K$ is a supremum-preserving commutative triangular norm, and $A = H$ is its residuation implication, is well known.

## 1.2 Syntax: Axioms and Basic Theorems

The language of the propositional calculus under adjointness, *AdjPC*, features three logical connectives (binary operations) on the set $WF$ of formulae: *implication* $\Rightarrow$, *conjunction* & and *forcing-implication* $\supset$. In an interpretation of *AdjPC*, the three logical connectives $\Rightarrow$, & and $\supset$ will translate to the three operations $A$, $K$ and $H$ of some adjointness algebra, respectively. Lowercase Greek letters are used as metavariables running on formulae in $WF$.

Axioms of *AdjPC* [16]:

**P1:** $\gamma \supset (\alpha \Rightarrow \gamma)$.
**P2:** $\alpha \supset ((\alpha \Rightarrow \gamma) \supset \gamma)$.
**P3:** $(((\alpha \supset \beta) \Rightarrow \beta) \supset \gamma) \supset (\alpha \supset \gamma)$.
**P4:** $\beta \supset (\alpha \Rightarrow \alpha \& \beta)$.
**P5:** $(\alpha \Rightarrow (\beta \supset \gamma) \& \beta) \supset (\alpha \Rightarrow \gamma)$.
**P6:** $((\beta \supset \gamma) \& \beta) \& \delta \supset \gamma \& \delta$.
**P7:** $\alpha \& \beta \supset \beta$.

An inference $\alpha_1, \cdots, \alpha_n \vdash \beta$ is understood as usual; and is carried out by means of the four inference rules listed below. When $\emptyset \vdash \beta$ (that is, $\beta$ is derived from axioms alone), we write $\vdash \beta$, and we call $\beta$ a *theorem*.

A considerable simplification of notation is achieved [16] by using a new symbol "$\subset\supset$". We write $\alpha \subset\supset \beta$ to abbreviate the writing of two formulae $\alpha \supset \beta$ and $\beta \supset \alpha$. Thus, $\alpha \subset\supset \beta$ is a set of two formulae, and not one formula composed from two subformulae. So, an inference $\Gamma \vdash \alpha \subset\supset \beta$ is, in fact, two inferences $\Gamma \vdash \alpha \supset \beta$ and $\Gamma \vdash \beta \supset \alpha$. Likewise, a theorem $\vdash \alpha \subset\supset \beta$ is an abbreviated writing of two theorems. The meta-predicate $\vdash \alpha \subset\supset \beta$ is an equivalence relation on $WF$, called *equivalidity* [16]. Another equivalence relation $\equiv$ on $WF$ is defined by:

$\alpha \equiv \beta$ iff ($\alpha \vdash \beta$ and $\beta \vdash \alpha$).

It follows from modus ponens, below, that if $\vdash \alpha \subset\supset \beta$ then $\alpha \equiv \beta$, but not vice-versa.

This logic is too general. It may be that no finite set of axioms can complete *AdjPC* if inference uses modus ponens (**MP**) alone! However, by adopting **MP** and three bits of the substitution theorem as inference rules, the seven axioms **P1-P7** become complete for *AdjPC*.

Inference Rules of *AdjPC* [16]:

**I1= MP:** $\alpha,\ \alpha \supset \beta \vdash \beta$ (Modus Ponens for forcing-implication).

**I2:** $\alpha \subset\supset \beta \vdash (\alpha \supset \gamma) \subset\supset (\beta \supset \gamma)$ (substitution in left argument of $\supset$).

**I3:** $\alpha \subset\supset \beta \vdash \alpha\&\gamma \subset\supset \beta\&\gamma$ (substitution in left argument of &).

**I4:** $\beta \subset\supset \gamma \vdash (\alpha \Rightarrow \beta) \subset\supset (\alpha \Rightarrow \gamma)$ (substitution in right argument of $\Rightarrow$).

**Proposition 1.1** *[16] (transitivity of forcing-implication).* $\alpha \supset \beta, \beta \supset \gamma \vdash \alpha \supset \gamma$.

**Theorem 1.1** *[16] (reflexivity of forcing-implication).* $\vdash \alpha \supset \alpha$.

The above two results establish that the meta-predicate $\vdash \alpha \supset \gamma$ is a pre-order. In consequence, equivalidity is an equivalence relation on $WF$.

**Proposition 1.2** *[16] The following are correct inferences in AdjPC:*

$\alpha \supset \beta \vdash (\beta \supset \gamma) \supset (\alpha \supset \gamma)$.
$\beta \supset \gamma \vdash (\alpha \supset \beta) \supset (\alpha \supset \gamma)$.
$\alpha \supset \beta \vdash (\beta \Rightarrow \gamma) \supset (\alpha \Rightarrow \gamma)$.
$\beta \supset \gamma \vdash (\alpha \Rightarrow \beta) \supset (\alpha \Rightarrow \gamma)$.
$\alpha \supset \beta \vdash \alpha\&\gamma \supset \beta\&\gamma$.
$\beta \supset \gamma \vdash \alpha\&\beta \supset \alpha\&\gamma$.
$\alpha \vdash (\alpha \Rightarrow \gamma) \subset\supset \gamma$.
$\gamma \vdash \alpha \Rightarrow \gamma$.
$\alpha, \alpha \Rightarrow \gamma \vdash \gamma$.
$\alpha \vdash \beta \subset\supset \alpha\&\beta$.
$\alpha, \beta \vdash \alpha\&\beta$.
$\alpha\&\beta \vdash \beta$.
$\beta \supset \gamma \vdash \alpha \supset (\beta \supset \gamma)$.

**Theorem 1.2** *[16] The following are theorems in AdjPC:*

$\vdash \alpha \supset (\beta \supset \alpha\&\beta)$.
$\vdash (\beta \supset \gamma)\&\beta \supset \gamma$.
$\vdash \alpha\&(\alpha \Rightarrow \gamma) \supset \gamma$.
$\vdash \beta \supset ((\beta \supset \gamma) \Rightarrow \gamma)$.

**Proposition 1.3** *[16] (Adjointness):* $\beta \supset (\alpha \Rightarrow \gamma) \equiv \alpha\&\beta \supset \gamma \equiv \alpha \supset (\beta \supset \gamma)$.

This corresponds to the condition (*Adjointness*) in the definition of adjointness algebras (Definition 1.1).

*AdjPC* does not have a deduction theorem as strong as that of classical logic. For instance, the inference $\beta \supset \gamma \vdash (\alpha \Rightarrow \beta) \supset (\alpha \Rightarrow \gamma)$ is correct, but the formula $(\beta \supset \gamma) \supset ((\alpha \Rightarrow \beta) \supset (\alpha \Rightarrow \gamma))$ is not a theorem in *AdjPC*. It is equivalent to the exchange principle for the implication $\Rightarrow$ (Lemma 1.4, below). However, the following important fact holds in *AdjPC*.

**Lemma 1.3** *[16]. Let $\alpha$ be a theorem, and let $\lambda$ be any formula in $WF$. Then $\vdash \lambda \supset \alpha$. In particular, $\alpha$ and $\lambda$ will be equivalid if and only if $\lambda$ is also a theorem.*

## 1.3 Semantics

An *interpretation* of *AdjPC* is [16] a pair $\mathcal{A} = (\mathcal{L}, \pi)$, in which $\mathcal{L} = (L, \leq, A, K, H)$ is an adjointness algebra, and $\pi : WF \to L$ is called the *valuation function* of the interpretation; subject to the condition that the following three identities hold for all formulae $\alpha, \beta, \gamma$:

$\pi(\alpha \Rightarrow \gamma) = A(\pi(\alpha), \pi(\gamma))$,

$\pi(\alpha \& \beta) = K(\pi(\alpha), \pi(\beta))$,

$\pi(\beta \supset \gamma) = H(\pi(\beta), \pi(\gamma))$.

If $\pi(\alpha) = 1$, we say that $\alpha$ is *true* in $\mathcal{A}$, and we write $\mathcal{A} \vDash \alpha$. Given a set $\Gamma$ in $WF$, if $\mathcal{A} \vDash \lambda$ for all $\lambda \in \Gamma$, we write $\mathcal{A} \vDash \Gamma$. If $\mathcal{A} \vDash \delta$ for every interpretation $\mathcal{A}$ such that $\mathcal{A} \vDash \Gamma$, we write $\Gamma \vDash \delta$. If $\mathcal{A} \vDash \alpha$ for all interpretations $\mathcal{A}$, we say $\alpha$ is *universally valid* (or, a *tautology*), and we write $\vDash \alpha$.

**Semantics-Theorem 1.1** *[16] Suppose $\Gamma \vdash \alpha$ for some set of formulae $\Gamma \cup \{\alpha\}$. Then $\Gamma \vDash \alpha$. Consequently, AdjPC is sound for its semantics, in the sense that if $\vdash \alpha$ then $\vDash \alpha$ .*

**Semantics-Theorem 1.2** *[16] AdjPC is complete for ADJ; in the sense that its theorems are its universally valid formulae.*

**Corollary 1.1** *[16] Two formulae $\alpha, \beta$ will be equivalid if and only if $\pi(\alpha) = \pi(\beta)$ in all interpretations $(\mathcal{L}, \pi)$.*

The converse of Semantics-Theorem 1.1 fails. For instance, we have $\gamma \vDash \beta \supset \gamma$, but the inference $\gamma \vdash \beta \supset \gamma$ is incorrect [16]. However, that converse holds if all formulae in $\Gamma$ are equational; that is, they take the form $\beta \supset \gamma$ [16].

## 1.4 The Exchange Principle

An implication $A$ is said to satisfy the *exchange principle* [22] if it satisfies:

**EP:** $\forall x, y, z \in L : \quad A(x, A(y, z)) = A(y, A(x, z))$.

Morsi [16] has developed a complete syntax for the smaller semantical domain *EP-ADJ* of all adjointness algebras whose implications satisfy **EP**. He called it *propositional calculus under adjointness and exchange principle*, denoted *EP-AdjPC*. Its language is the same as that of *AdjPC*. A preliminary choice of the axioms of *EP-AdjPC* would be to augment the seven axioms of *AdjPC* with:

**EP:** $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \supset (\beta \Rightarrow (\alpha \Rightarrow \gamma))$.

And we may retain **I1-I4** as four inference rules for *EP-AdjPC*. Then *EP-AdjPC* would become sound and complete for *EP-ADJ*. However, it is possible to extract a smaller axiom scheme for *EP-AdjPC*; in the manner shown below.

**Lemma 1.4** *[16]. The following eight schema of equational formulae are equivalent in AdjPC:*

***E1=EP:*** $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \supset (\beta \Rightarrow (\alpha \Rightarrow \gamma))$ *(exchange principle for $\Rightarrow$).*

***E2:*** $\lambda \& (\alpha \Rightarrow \gamma) \supset (\alpha \Rightarrow \lambda \& \gamma)$.

***E3:*** $\alpha\& (\lambda\&\gamma) \supset \lambda\& (\alpha\&\gamma)$ *(exchange principle for* $\&$*).*
***E4:*** $(\beta \supset \gamma) \supset (\alpha\&\beta \supset \alpha\&\gamma)$ .
***E5:*** $(\beta \supset (\alpha \Rightarrow \gamma)) \supset (\alpha\&\beta \supset \gamma)$ .
***E6:*** $(\delta \supset \gamma)\&\beta \supset ((\beta \supset \delta) \Rightarrow \gamma)$ .
***E7:*** $(\alpha\&\beta \supset \gamma) \supset (\beta \supset (\alpha \Rightarrow \gamma))$ .
***E8:*** $(\beta \supset \gamma) \supset ((\alpha \Rightarrow \beta) \supset (\alpha \Rightarrow \gamma))$ .

It follows from this lemma that **P1-P7** and **E1-E8** are theorems of *EP-AdjPC*. A new axiom scheme for *EP-AdjPC* has been chosen from among these fifteen theorems.

Axioms of EP-AdjPC [16]:
**P1:** $\gamma \supset (\alpha \Rightarrow \gamma)$ .
**P2:** $\alpha \supset ((\alpha \Rightarrow \gamma) \supset \gamma)$ .
**P3:** $(((\alpha \supset \beta) \Rightarrow \beta) \supset \gamma) \supset (\alpha \supset \gamma)$ .
**P6:** $((\beta \supset \gamma)\&\beta)\&\delta \supset \gamma\&\delta$.
**E5:** $(\beta \supset (\alpha \Rightarrow \gamma)) \supset (\alpha\&\beta \supset \gamma)$ .
**E7:** $(\alpha\&\beta \supset \gamma) \supset (\beta \supset (\alpha \Rightarrow \gamma))$ .

Inference Rules of EP-AdjPC [16]: **MP**, **I2** and **I3**.

The formal system built upon these six axioms and three inference rules is sound and complete for *EP-ADJ* [16].

## 2 Adjointness Lattices

**Definition 2.1** *An* adjointness lattice *is a 9-tuple* $(L, \leq, 1, 0, A, K, H, \wedge, \vee)$*, in which* $(L, \leq, 1, 0, \wedge, \vee)$ *is a bounded lattice, and* $(L, \leq, A, K, H)$ *is an adjointness algebra. We denote the class of all adjointness lattices by* $ADJL$*.*

We aim to develop a complete syntax for the semantical domain $ADJL$. We call it propositional calculus for adjointness lattices, and denote it by *AdjLPC*. We select the axioms for *AdjLPC* from among the many inequalities derived algebraically in $ADJL$. Since the logic *AdjLPC* is an extension of *AdjPC*, the seven axioms of *AdjPC* can be adopted, and we choose six new axioms, namely, the following *universally valid* inequalities in $ADJL$:
**M8:** $x \wedge y \leq x \vee z$.
**M9:** $x \vee x \leq x$.
**M10:** $H(y, z) \leq H(x \vee y, z \vee x)$.
**M11:** $x \leq x \wedge x$.
**M12:** $H(y, z) \leq H(x \wedge y, z \wedge x)$.
**M13:** $0 \leq x$.

In forms free from $\leq$, these relations become: for all $x, y, z$ in $L$:
**N8:** $H(x \wedge y, x \vee z) = 1$.
**N9:** $H(x \vee x, x) = 1$.
**N10:** $H(H(y, z), H(x \vee y, z \vee x)) = 1$.
**N11:** $H(x, x \wedge x) = 1$.
**N12:** $H(H(y, z), H(x \wedge y, z \wedge x)) = 1$.
**N13:** $H(0, x) = 1$.

## 3 Syntax: Language, Axioms and Inference Rules

The language of the *Propositional Calculus for Adjointness Lattices*, $AdjLPC$, consists of a denumerable set $WF$ of *formulae* and five logical connectives (binary operations) on $WF$: *implication* $\Rightarrow$, *conjunction* &, *forcing-implication* $\supset$, *weak conjunction* $\wedge$ and *disjunction* $\vee$. The set $WF$ is constructed from a denumerable subset $WF_0$ of *atomic formulae* by means of repeated application of the logical connectives. We also add to $WF_0$ a special element $\perp$ called *Falsum.* We denote **P1** by $\top$ (*Truth*). As usual, brackets and comma are secondary symbols in the language.

To reduce the number of brackets appearing in complex formulae, we maintain a convention of priority among the eight operation symbols $\Rightarrow, \&, \supset, \subset\supset, \vdash, \equiv, \wedge, \vee$. We give $\&, \wedge, \vee$ the highest priority; whereas we give $\vdash, \equiv$ lower priority than the other symbols.

In Section 2, we identified six identities **N8-N13** (equivalently, six inequalities **M8-M13**) valid in all adjointness lattices. Their corresponding statements on formulae, together with the seven axioms **P1-P7** of $AdjPC$, are what follows:

Axioms of $AdjLPC$: The following are theorems:

**P1:** $\gamma \supset (\alpha \Rightarrow \gamma)$.
**P2:** $\alpha \supset ((\alpha \Rightarrow \gamma) \supset \gamma)$.
**P3:** $(((\alpha \supset \beta) \Rightarrow \beta) \supset \gamma) \supset (\alpha \supset \gamma)$.
**P4:** $\beta \supset (\alpha \Rightarrow \alpha \& \beta)$.
**P5:** $(\alpha \Rightarrow (\beta \supset \gamma) \& \beta) \supset (\alpha \Rightarrow \gamma)$.
**P6:** $((\beta \supset \gamma) \& \beta) \& \delta \supset \gamma \& \delta$.
**P7:** $\alpha \& \beta \supset \beta$.
**P8:** $\alpha \wedge \beta \supset \alpha \vee \gamma$.
**P9:** $\gamma \vee \gamma \supset \gamma$.
**P10:** $(\beta \supset \gamma) \supset (\alpha \vee \beta \supset \gamma \vee \alpha)$.
**P11:** $\beta \supset \beta \wedge \beta$.
**P12:** $(\beta \supset \gamma) \supset (\alpha \wedge \beta \supset \gamma \wedge \alpha)$.
**P13:** $\perp \supset \gamma$.

Inference Rules for $AdjLPC$ are those of $AdjPC$: **MP, I2, I3** and **I4**.

In an interpretation of $AdjLPC$, the five logical connectives $\Rightarrow, \&, \supset, \wedge$ and $\vee$ will translate onto the five operations $A, K, H, \wedge$ and $\vee$ of some adjointness lattice, respectively, whereas Falsum $\perp$ will translate onto 0. Also, formulae will translate onto functions on truth values; built up as composites of $A, K, H, \wedge, \vee$ and 0. $AdjLPC$ will be sound for these semantics, in the sense that all theorems will translate onto functions that are identically equal to 1.

## 4 Syntax: Essential Theorems

We derive enough theorems and inferences (called propositions) in $AdjLPC$ to establish, in Section 5, its completeness for the semantical domain $ADJL$ of adjointness lattices. In most proofs we shall use, as matters of course, both **MP** and the reflexivity and transitivity of the binary relation $\vdash \beta \supset \gamma$.

**Theorem 4.1** $\vdash \alpha \supset \alpha \vee \gamma$ *and* $\vdash \alpha \wedge \beta \supset \alpha$.

**Proof.** Use **P8** with $\frac{\alpha}{\beta}$, then **P11** to derive $\vdash \alpha \supset \alpha \vee \gamma$. The other part follows similarly. ■

**Theorem 4.2** *(idempotent laws for disjunction and for weak conjunction).*
$\vdash \gamma \vee \gamma \subset\supset \gamma$ *and* $\vdash \beta \wedge \beta \subset\supset \beta$.

**Proof.** Apply Theorem 4.1, **P9** and **P11**. ■

**Theorem 4.3** *(commutative laws for disjunction and for weak conjunction).*
$\vdash \alpha \vee \beta \subset\supset \beta \vee \alpha$ *and* $\vdash \alpha \wedge \beta \subset\supset \beta \wedge \alpha$.

**Proof.** By **P10**, $\vdash (\beta \supset \beta) \supset (\alpha \vee \beta \supset \beta \vee \alpha)$. So, by $\vdash \beta \supset \beta$ and **MP** we infer the first part. The second part follows similarly. ■

**Theorem 4.4** $\vdash (\beta \supset \gamma) \supset (\alpha \vee \beta \supset \alpha \vee \gamma)$ , $\vdash (\beta \supset \gamma) \supset (\beta \vee \alpha \supset \gamma \vee \alpha)$ , $\vdash (\beta \supset \gamma) \supset (\alpha \vee \beta \supset \alpha \vee \gamma)$ , $\vdash (\beta \supset \gamma) \supset (\beta \vee \alpha \supset \gamma \vee \alpha)$ .

**Proof.** These follow clearly from **P10, P12** and commutivity (Theorem 4.3). ■

Applying **MP** on the preceding theorem, we obtain

**Proposition 4.1** *(monotonicity).* $\beta \supset \gamma \vdash$
$\{\alpha \vee \beta \supset \alpha \vee \gamma \ , \ \beta \vee \alpha \supset \gamma \vee \alpha \ , \ \alpha \vee \beta \supset \alpha \vee \gamma \ , \ \beta \vee \alpha \supset \gamma \vee \alpha\}$ .

**Proposition 4.2** *(Substitution Theorem).* $\alpha \subset\supset \beta \vdash \Psi(\alpha) \subset\supset \Psi(\beta|\alpha)$ .
*Where* $\Psi(\alpha)$ *is a formula in which* $\alpha$ *occurs as a subformula, and* $\Psi(\beta|\alpha)$ *is a formula obtained from* $\Psi(\alpha)$ *by substituting* $\beta$ *for* $\alpha$*, in one or more of the occurrences of* $\alpha$ *of in* $\Psi(\alpha)$*. In particular, substitution preserves equivalidity.*

**Proof.** This follows clearly from all the monotonicity propositions of the five logical connectives $\Rightarrow, \&, \supset, \wedge$ and $\vee$. ■

**Theorem 4.5** $\vdash (\beta \supset \gamma) \subset\supset (\beta \vee \gamma \supset \gamma)$ .

**Proof.** (1) $(\beta \supset \gamma) \supset (\beta \vee \gamma \supset \gamma \vee \gamma)$ (Theorem 4.4)
(2) $(\beta \supset \gamma) \supset (\beta \vee \gamma \supset \gamma)$ ((1), Theorem 4.2, Substitution Theorem )
(3) $\beta \supset \beta \vee \gamma$ (Theorem 4.1)
(4) $(\beta \vee \gamma \supset \gamma) \supset (\beta \supset \gamma)$ ((3), Proposition 1.2). ■

**Theorem 4.6** $\vdash (\beta \supset \gamma) \subset\supset (\beta \supset \beta \wedge \gamma)$ .

**Proof.** Similar. ■

**Proposition 4.3** $\alpha \subset\supset \alpha \wedge \beta \equiv \alpha \vee \beta \subset\supset \beta \equiv \alpha \supset \beta$.

**Proof.** Follows by Theorem 4.5 and by Theorem 4.1. ■

**Proposition 4.4** $\{\gamma \supset \alpha, \gamma \supset \beta\} \equiv \gamma \supset \alpha \wedge \beta$.

**Proof.** (1) $\gamma \wedge \beta \supset \alpha \wedge \beta$ (first hypothesis, Proposition 4.1)
(2) $\gamma \supset \gamma \wedge \beta$ (second hypothesis, Proposition 4.3)
(3) $\gamma \supset \alpha \wedge \beta$ ((1), (2)).
The opposite inference follows from Theorem 4.1. ■

**Proposition 4.5** $\{\alpha \supset \gamma, \beta \supset \gamma\} \equiv \alpha \vee \beta \supset \gamma$.

**Proof.** Similar. ■

**Theorem 4.7** $\vdash \alpha \wedge (\alpha \vee \beta) \subset\supset \alpha$ *and* $\vdash \alpha \vee (\alpha \wedge \beta) \subset\supset \alpha$.

**Proof.** Use Theorem 4.1 together with Proposition 4.3. ■

# 5 Semantics

We explain how $ADJL$ (cf. Section 2) constitutes a semantical domain for *AdjLPC*. We prove that the syntax of *AdjLPC* is sound for $ADJL$. We then show that the quotient of the tuple $(WF, \vdash \cdot \supset \cdot, \top, \bot, \Rightarrow, \&, \supset, \wedge, \vee)$, with respect to the relation equivalidity, is a model of *AdjLPC*. We use it to prove completeness.

An *interpretation* of *AdjLPC* is a pair $\mathcal{T} = (\mathcal{L}, \pi)$, in which $\mathcal{L} = (L, \leq, 1, 0, A, K, H, \wedge, \vee)$ is an adjointness lattice, and $\pi$ is a function from the set $WF$ of formulae into $L$, called the *valuation function* (or *truth function*) of the interpretation, subject to the condition that the following six identities hold for all formulae $\alpha, \beta, \gamma$:

$$\pi(\alpha \Rightarrow \gamma) = A(\pi(\alpha), \pi(\gamma)), \tag{5.1}$$
$$\pi(\alpha \& \beta) = K(\pi(\alpha), \pi(\beta)), \tag{5.2}$$
$$\pi(\beta \supset \gamma) = H(\pi(\beta), \pi(\gamma)), \tag{5.3}$$
$$\pi(\beta \wedge \gamma) = \pi(\beta) \wedge \pi(\gamma), \tag{5.4}$$
$$\pi(\beta \vee \gamma) = \pi(\beta) \vee \pi(\gamma), \tag{5.5}$$
$$\pi(\bot) = 0. \tag{5.6}$$

$\pi(\alpha) \in L$ (also denoted by $\bar{\alpha}$) is called the *validity* (or, *truth*) of $\alpha$ in this interpretation. The symbol $\vDash$ is understood as in *AdjPC* (Subsection 1.3).

**Semantics-Theorem 5.1.** *AdjLPC is sound for its semantics, in the sense that if* $\vdash \alpha$ *then* $\vDash \alpha$; *that is, all theorems are universally valid.*

**Proof.** By the identities **N8-N13**, we know that the axioms **P8-P13** are universally valid in $ADJL$. Also, *AdjLPC* has the same inference rules as *AdjPC*. We can therefore imitate the proof in [16] of Semantics-Theorem 1.1, and deduce that *AdjLPC* is sound for $ADJL$. ■

We next address the question of the completeness of *AdjLPC* for $ADJL$. We follow a standard procedure due to Lindenbaum and Tarski. We begin by constructing the natural interpretation of *AdjLPC*. Denote the equivalence relation equivalidity (on $WF$) simply by $\tilde{}$. Let $p : WF \to WF/\tilde{} : \alpha \mapsto \bar{\alpha}$ be the quotient map. Then a partial order $\leq$ is well-defined on $WF/\tilde{}$ by: $\bar{\alpha} \leq \bar{\beta}$ iff

$\vdash \alpha \supset \beta$. By imitating for *AdjLPC* the proof in [16] of Lemma 1.3, we find that the poset $(WF/\tilde{}, \leq)$ has a top element which is precisely the set of all theorems in *AdjLPC*. We denote this top element by 1. Also by **P13**, the poset $(WF/\tilde{}, \leq)$ has a bottom element which is precisely the equivalence class of Falsum $\perp$. We denote this bottom element by 0. Moreover, the Substitution Theorem guarantees that the five logical connectives $\Rightarrow, \&, \supset, \wedge$ and $\vee$ possess the substitution property for $\tilde{}$. In consequence, the following five binary operations $\tilde{A}, \tilde{K}, \tilde{H}, \tilde{\wedge}, \tilde{\vee}$ are well defined on $WF/\tilde{}$. For all $\bar{\alpha}, \bar{\beta}, \bar{\gamma}$ in $WF/\tilde{}$:

$\tilde{A}(\bar{\alpha}, \bar{\gamma}) = p(\alpha \Rightarrow \gamma)$,
$\tilde{K}(\bar{\alpha}, \bar{\beta}) = p(\alpha \& \beta)$,
$\tilde{H}(\bar{\beta}, \bar{\gamma}) = p(\beta \supset \gamma)$,
$\bar{\beta} \tilde{\wedge} \bar{\gamma} = p(\beta \wedge \gamma)$,
$\bar{\beta} \tilde{\vee} \bar{\gamma} = p(\beta \vee \gamma)$.

From [16], we know that $(WF/\tilde{}, \leq, \tilde{A}, \tilde{K}, \tilde{H})$ is an adjointness algebra. Hence, we need only prove that $(WF/\tilde{}, \leq, 1, 0, \tilde{\wedge}, \tilde{\vee})$ is a bounded lattice. But, this follows in a routine way from the axioms and from the theorems and propositions of Section 4. This completes the proof that $\mathcal{L} = (WF/\tilde{}, \leq, 1, 0, \tilde{A}, \tilde{K}, \tilde{H}, \tilde{\wedge}, \tilde{\vee})$ is an adjointness lattice. Finally, by their construction, $\tilde{A}, \tilde{K}, \tilde{H}, \tilde{\wedge}, \tilde{\vee}, 0$ and $p$ satisfy the conditions (5.1)-(5.6) for $p$ to become a valuation function. This demonstrates that the pair $(\mathcal{L}, p)$ is an interpretation of *AdjLPC*. It is called the *natural interpretation* of *AdjLPC*.

Since, for any formula $\alpha$, we have $(\mathcal{L}, p) \vDash \alpha$ (that is, $\bar{\alpha}$ is the top element of $(WF/\tilde{}, \leq)$) if and only if $\alpha$ is a theorem, then in the light of Semantics-Theorem 5.1 we obtain:

**Semantics-Theorem 5.2.** *AdjLPC is complete for ADLJ; in the sense that its theorems are its universally valid formulae (that is, $\vdash \alpha$ if and only if $\vDash \alpha$, for all formulae $\alpha$).*

**Semantics-Theorem 5.3.** *Let $\Gamma$ be a nonempty set of equational formulae. Then for any formula $\alpha$ in $WF$, $\Gamma \vdash \alpha$ if and only if $\Gamma \vDash \alpha$.*

**Proof.** Adjoin $\Gamma$ to the set of axioms, then repeat all the arguments above. ■

Along the same lines of Subsection 1.4, we also possess a complete syntax for the semantical domain *EP-ADJL* of all adjointness lattices whose implications satisfy **EP**. We call it *propositional calculus for adjointness lattices and exchange principle*, and we denote it by *EP-AdjLPC*. The language of *EP-AdjLPC* is the same as that of *AdjLPC*. Our axiom scheme for *EP-AdjLPC* features three inference rules and twelve axioms. The inference rules and the first six axioms are those of *EP-AdjPC*, whereas the last six axioms **P8-P13** are as in *AdjLPC*. From Subsection 1.4 and this section, *EP-AdjLPC* is sound and complete for *EP-ADJL*.

# 6 Syntax: Additional Theorems

In this section we prove further useful theorems and inferences in *AdjLPC*.

**Theorem 6.1** $\vdash \alpha \& \perp \subset\supset \perp$.

**Proof.** By **P7** and **P13**. ■

**Theorem 6.2** $\vdash \bot \& \alpha \subset\supset \bot$.

**Proof.** By **P13**, $\bot \supset (\alpha \supset \bot)$, which gives by (Adjointness), $\bot\&\alpha \supset \bot$. This and **P13** yield the stated equivalidity. ■

**Theorem 6.3** $\vdash \bot \wedge \alpha \subset\supset \bot$ , $\vdash \bot \vee \alpha \subset\supset \alpha$ , $\vdash \top \wedge \alpha \subset\supset \alpha$ , $\top \vee \alpha \subset\supset \top$.

**Proof.** These follow clearly from **P13** and Proposition 4.3. ■

**Theorem 6.4** $\vdash \bot \Rightarrow \alpha$. *In particular,* $\vdash \bot \Rightarrow \bot$.

**Proof.** By **P13**, $\bot \supset (\top \supset \bot)$, from which we get by (Adjointness), $\top \supset (\bot \Rightarrow \alpha)$. So by **MP**, $\bot \Rightarrow \alpha$. ■

**Theorem 6.5** $\vdash (\alpha \Rightarrow \beta \wedge \gamma) \subset\supset (\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma)$ .

**Proof.** That $\vdash (\alpha \Rightarrow \beta \wedge \gamma) \supset (\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma)$ follows from Theorem 4.1 and Propositions 1.2, 4.4. The other half is proved as follows:

(1) $(\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma) \supset (\alpha \Rightarrow \beta)$ (Theorem 4.1)
(2) $(\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma) \supset (\alpha \Rightarrow \gamma)$ (Theorem 4.1)
(3) $\alpha\& ((\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma)) \supset \beta$ ((1), Adjointness)
(4) $\alpha\& ((\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma)) \supset \gamma$ ((2), Adjointness)
(5) $\alpha\& ((\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma)) \supset \beta \wedge \gamma$ ((3), (4), Proposition 4.4)
(6) $(\alpha \Rightarrow \beta) \wedge (\alpha \Rightarrow \gamma) \supset (\alpha \Rightarrow \beta \wedge \gamma)$ ((5), Adjointness). ■

**Theorem 6.6** $\vdash (\alpha \vee \beta \Rightarrow \gamma) \subset\supset (\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)$ .

**Proof.** That $\vdash (\alpha \vee \beta \Rightarrow \gamma) \supset (\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)$ follows from Theorem 4.1 and Propositions 1.2, 4.4. The other half is proved as follows:

(1) $(\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma) \supset (\alpha \Rightarrow \gamma)$ (Theorem 4.1)
(2) $(\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma) \supset (\beta \Rightarrow \gamma)$ (Theorem 4.1)
(3) $\alpha \supset ((\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma) \supset \gamma)$ ((1), Adjointness)
(4) $\beta \supset ((\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma) \supset \gamma)$ ((2), Adjointness)
(5) $\alpha \vee \beta \supset ((\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma) \supset \gamma)$ ((3), (4), Proposition 4.5)
(6) $(\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma) \supset (\alpha \vee \beta \Rightarrow \gamma)$ ((5), Adjointness). ■

The next two theorems have proofs along lines similar to the above two.

**Theorem 6.7** $\vdash (\alpha \supset \beta \wedge \gamma) \subset\supset (\alpha \supset \beta) \wedge (\alpha \supset \gamma)$ .

**Theorem 6.8** $\vdash (\alpha \vee \beta \supset \gamma) \subset\supset (\alpha \supset \gamma) \wedge (\beta \supset \gamma)$ .

**Theorem 6.9** $\vdash \alpha\& (\beta \vee \gamma) \subset\supset (\alpha\&\beta) \vee (\alpha\&\gamma)$ .

**Proof.** That $(\alpha\&\beta) \vee (\alpha\&\gamma) \supset \alpha\& (\beta \vee \gamma)$ follows from Theorem 4.1 and Propositions 1.2, 4.5. The other half is proved as follows:

(1) $\alpha\&\beta \supset (\alpha\&\beta) \vee (\alpha\&\gamma)$ (Theorem 4.1)
(2) $\alpha\&\gamma \supset (\alpha\&\beta) \vee (\alpha\&\gamma)$ (Theorem 4.1)
(3) $\beta \supset (\alpha \Rightarrow (\alpha\&\beta) \vee (\alpha\&\gamma))$ ((1), Adjointness)
(4) $\gamma \supset (\alpha \Rightarrow (\alpha\&\beta) \vee (\alpha\&\gamma))$ ((2), Adjointness)
(5) $\beta \vee \gamma \supset (\alpha \Rightarrow (\alpha\&\beta) \vee (\alpha\&\gamma))$ ((3), (4), Proposition 4.5)
(6) $\alpha\& (\beta \vee \gamma) \supset (\alpha\&\beta) \vee (\alpha\&\gamma)$ ((5), Adjointness). ■

**Theorem 6.10** $\vdash (\alpha \vee \beta) \& \gamma \subset\supset (\alpha \& \gamma) \vee (\beta \& \gamma)$.

**Proof.** Similar. ■

**Proposition 6.1** $(\alpha \vee \beta) \vee \lambda \supset \gamma \equiv \{\alpha \supset \gamma, \beta \supset \gamma, \lambda \supset \gamma\} \equiv \alpha \vee (\beta \vee \lambda) \supset \gamma$.

**Proof.** These equivalences follow easily from Proposition 4.5. ■

**Proposition 6.2** $\gamma \supset (\alpha \wedge \beta) \wedge \lambda \equiv \{\gamma \supset \alpha, \gamma \supset \beta, \gamma \supset \lambda\} \equiv \gamma \supset \alpha \wedge (\beta \wedge \lambda)$.

**Proof.** These equivalences follow easily from Proposition 4.4. ■

**Theorem 6.11** $\vdash (\alpha \vee \beta) \vee \lambda \subset\supset \alpha \vee (\beta \vee \lambda)$.

**Proof.** Apply Proposition 6.1 twice, with $\frac{\alpha \vee (\beta \vee \lambda)}{\gamma}$ and with $\frac{(\alpha \vee \beta) \vee \lambda}{\gamma}$. ■

**Theorem 6.12** $\vdash (\alpha \wedge \beta) \wedge \lambda \subset\supset \alpha \wedge (\beta \wedge \lambda)$.

**Proof.** Similar, using Proposition 6.2. ■

Using the monotoicity properties of $\Rightarrow, \supset, \&$ (Proposition 1.2), it is easy to conclude

**Theorem 6.13** $\vdash (\alpha \Rightarrow \beta) \vee (\alpha \Rightarrow \gamma) \supset (\alpha \Rightarrow \beta \vee \gamma)$,
$\vdash (\alpha \Rightarrow \gamma) \vee (\beta \Rightarrow \gamma) \supset (\alpha \wedge \beta \Rightarrow \gamma)$,
$\vdash (\alpha \supset \beta) \vee (\alpha \supset \gamma) \supset (\alpha \supset \beta \vee \gamma)$,
$\vdash (\alpha \supset \gamma) \vee (\beta \supset \gamma) \supset (\alpha \wedge \beta \supset \gamma)$,
$\vdash (\alpha \wedge \beta) \& \gamma \supset (\alpha \& \gamma) \wedge (\beta \& \gamma)$.
$\vdash \alpha \& (\beta \wedge \gamma) \supset (\alpha \& \beta) \wedge (\alpha \& \gamma)$.

**Theorem 6.14** $\vdash (\top \Rightarrow \alpha) \subset\supset \alpha$ *and* $\vdash \alpha \Rightarrow \top$.

**Proof.** Use Proposition 1.2, $\top$ and **MP**. ■

# 7 Negations from Implications

A *negation* $n$ on $(L, \leq 1, 0)$ is an order-reversing map that satisfies, $n(0) = 1$ and $n(1) = 0$, and it is a *strong negation* if it is also an involution; that is, $n(n(x)) = x$ for all $x$ [21].

In an adjointness lattice, we define two functions $n, m : L \to L$ by:

$$n(x) = A(x, 0), \quad (7.1)$$
$$m(y) = H(y, 0). \quad (7.2)$$

It is easy to see that $n$ is a negation on $(L, \leq)$, whereas $m$ may lack the property $m(1) = 0$. In the syntax of $AdjLPC$, the corresponding two unary operations $\neg, \#$ on $WF$ are defined on a formula $\alpha$ by:
$\neg\alpha = \alpha \Rightarrow \bot$ , $\#\beta = \beta \supset \bot$.

We have the following properties for $\neg$ and $\#$.

**Proposition 7.1** $\alpha \supset \beta \vdash \neg\beta \supset \neg\alpha \quad \textit{and} \quad \alpha \supset \beta \vdash \#\beta \supset \#\alpha$.

**Proof.** Use Proposition 1.2. ■

It follows clearly from the preceding proposition that the Substitution Theorem remains valid for complex formulae that may feature one or both of the two unary operations $\neg, \#$.

**Proposition 7.2** $\beta \supset \neg\alpha \equiv \alpha \& \beta \subset\supset \bot \equiv \alpha \supset \#\beta$.

**Proof.** Use (Adjointness) and **P13**. ■

**Theorem 7.1** $\vdash \neg\top \subset\supset \bot \quad , \quad \vdash \neg\bot \subset\supset \top \quad \textit{and} \quad \vdash \neg\neg\top \subset\supset \top$.

**Proof.** The first equivalidity follows from Theorem 6.14, and the second one from Theorem 6.4. The third one is a consequence of the first and the second. ■

We see from Proposition 7.1 and Theorem 7.1 that $\neg$ is a negation function.

**Theorem 7.2** $\vdash \#\bot \subset\supset \top$.

**Proof.** Direct from Theorem 1.1 and Lemma 1.3. ■

**Theorem 7.3** $\vdash \neg(\alpha \vee \beta) \subset\supset \neg\alpha \wedge \neg\beta \quad \textit{and} \quad \vdash \#(\alpha \vee \beta) \subset\supset \#\alpha \wedge \#\beta$.

**Proof.** Direct from Theorems 6.6, 6.8. ■

**Theorem 7.4** $\vdash \alpha \supset \#\neg\alpha \quad \textit{and} \quad \vdash \beta \supset \neg\#\beta$.

**Proof.** Direct from **P2** and Theorem 1.2. ■

**Theorem 7.5** $\vdash \alpha \& \neg\alpha \subset\supset \bot \quad \textit{and} \quad \vdash \#\beta \& \beta \subset\supset \bot$.

**Proof.** Direct from Theorem 1.2 and **P13**. ■

**Theorem 7.6** $\vdash \#\neg\#\beta \subset\supset \#\beta \quad \textit{and} \quad \vdash \neg\#\neg\alpha \subset\supset \neg\alpha$.

**Proof.** By Theorem 6 of [16], $(((\beta \supset \bot) \Rightarrow \bot) \supset \bot) \subset\supset (\beta \supset \bot)$, which is the first equivalidity. We get the second equivalidity from Theorem 7 of [16]. ■

It follows from Theorem 7.6 that the two unary operations $\neg\#$ and $\#\neg$ are idempotent, up to equivalidity.

**Theorem 7.7** $\vdash \#\beta \supset (\beta \supset \gamma) \quad , \quad \vdash \neg\alpha \supset (\alpha \Rightarrow \gamma) \quad , \quad \vdash \alpha \supset (\neg\alpha \supset \gamma) \quad \textit{and}$ $\vdash \beta \supset (\#\beta \Rightarrow \gamma)$.

**Proof.** Direct from Proposition 1.2 and **P13**. ■

**Proposition 7.3** $\alpha \supset \neg\neg\alpha \equiv \neg\alpha \supset \#\alpha \quad \textit{and} \quad \alpha \supset \#\#\alpha \equiv \#\alpha \supset \neg\alpha$.

**Proof.** By Adjointness. ■

**Proposition 7.4** $\neg\neg\alpha \subset\supset \alpha \vdash \neg\#\alpha \subset\supset \alpha$.

**Proof.** (1) $\neg\#\neg\neg\alpha \subset\supset \neg\#\alpha$ (hypothesis, Substitution Theorem)
(2) $\neg\#\neg\neg\alpha \subset\supset \neg\neg\alpha$ (Theorem 7.6)
(3) $\neg\#\alpha \subset\supset \neg\neg\alpha \subset\supset \alpha$ (hypothesis, (1), (2)). ■

**Proposition 7.5** $\#\#\alpha \subset\supset \alpha \vdash \#\neg\alpha \subset\supset \alpha.$

**Proof.** Similar. ■

**Proposition 7.6** $\#\beta \vdash (\alpha \Rightarrow \beta) \supset \neg\alpha$ *and* $\#\beta \vdash (\alpha \supset \beta) \supset \#\alpha.$

**Proof.** Direct from Proposition 1.2. ■

With the help of the preceding proposition, it is easy to deduce the following *Modus Tollens* schema.

**Proposition 7.7** $\alpha \Rightarrow \beta, \#\beta \vdash \neg\alpha,$
$\alpha \supset \beta, \#\beta \vdash \#\alpha,$
$\alpha \supset \beta, \neg\beta \vdash \neg\alpha,$
$\alpha \supset \#\beta, \beta \vdash \neg\alpha,$
$\alpha \supset \neg\beta, \beta \vdash \#\alpha.$

**Lemma 7.1** *The following five schema of formulae are equivalent in AdjLPC:*
***N1:*** $\neg\alpha \supset \#\alpha.$
***N2:*** $\alpha \supset \neg\neg\alpha.$
***N3:*** $\#\alpha \supset \neg\alpha.$
***N4:*** $\alpha \supset \#\#\alpha.$
***N5:*** $\neg\alpha \subset\supset \#\alpha.$

**Proof.** The equivalences **N1** $\equiv$ **N2** and **N3** $\equiv$ **N4** follow from Proposition 7.3.

**N2** entails **N3**: By Theorem 7.4, $\alpha \supset \neg\#\alpha$, and so by Proposition 7.1, $\neg\neg\#\alpha \supset \neg\alpha$. But by **N2** with $\frac{\#\alpha}{\alpha}$, $\#\alpha \supset \neg\neg\#\alpha$. Consequently, $\#\alpha \supset \neg\alpha$.

**N4** entails **N1**: Similar.

Finally, **N5** is the conjunction of **N1** and **N3**. ■

It is clear from the above lemma that $n$ need not equal $m$ (see the next example), and equality will hold if and only if for all $x$, $x \leq n(n(x))$.

**Example 7.1.** Define a conjunction $K$ on $[0, 1]$ by:

$$K(x,y) = \begin{cases} 0, & 2x + y \leq 1 \\ \min\{x,y\}, & 2x + y > 1 \end{cases}.$$

This $K$ is an associative conjunction with two-sided identity, but it is neither commutative nor continuous. It is direct to see that its implication triple is completed as follows:

$$A(x,z) = \begin{cases} 1, & x \leq z \\ \max\{1-2x, z\}, & x > z \end{cases},$$

$$H(y,z) = \begin{cases} 1, & y \leq z \\ \max\{(1-y)/2, z\}, & y > z \end{cases},$$

which are not comparable. For this adjointness lattice, we find:

$$n(x) = A(x,0) = \begin{cases} 1, & x = 0 \\ \max\{1-2x, 0\}, & x > 0 \end{cases},$$

$$m(y) = H(y,0) = \begin{cases} 1, & y \leq z \\ (1-y)/2, & y > z \end{cases}.$$

So, $n \neq m$. Also, we note that each of the two inequalities $x \leq n(n(x))$ and $x \leq m(m(x))$ fails for some $x$.

## 8 *S*-type Implications

In this final section, we consider a type of implications that has seen sufficient interest in the literature. Given a strong negation $n$ and a triangular norm $T$ on $(L, \leq)$, the *S-type implication* of $T$ and $n$ is defined on $(L, \leq)$ by:

$$A(x,y) = n(T(x, n(y))), \tag{8.1}$$

For simplicity of terminology, we shall say that an adjointness lattice is of the $S$-type if so its implication $A$ is. (N.B. It is direct to verify that if $A$ is given by (8.1), then $H$ will be the $n$-contrapositive of the residuated implication $J_T$ of $T$, whereas $K$ will be given by $K(x,y) = n(J_T(x, n(y)))$, $x, y \in L$.)

Our aim is to prove that adjoining to *EP-AdjLPC* one extra "involution" axiom (for the negation $\neg$) renders the implication $\Rightarrow$ an $S$-type implication. We denote the ensuing syntax by *S-AdjLPC*. Its language is that of *AdjLPC*. It is well known that in an adjointness lattice of the $S$-type, A satisfies **EP** and $n$ is involutive (see [18]). Therefore, the following axioms and inference rules are sound for those lattices:

Axioms of *S-AdjLPC*:

**P1:** $\gamma \supset (\alpha \Rightarrow \gamma)$.
**P2:** $\alpha \supset ((\alpha \Rightarrow \gamma) \supset \gamma)$.
**P3:** $(((\alpha \supset \beta) \Rightarrow \beta) \supset \gamma) \supset (\alpha \supset \gamma)$.
**P6:** $((\beta \supset \gamma)\,\&\beta)\,\&\delta \supset \gamma\&\delta$.
**E5:** $(\beta \supset (\alpha \Rightarrow \gamma)) \supset (\alpha\&\beta \supset \gamma)$.
**E7:** $(\alpha\&\beta \supset \gamma) \supset (\beta \supset (\alpha \Rightarrow \gamma))$.
**P8:** $\alpha \wedge \beta \supset \alpha \vee \gamma$.
**P9:** $\gamma \vee \gamma \supset \gamma$.
**P10:** $(\beta \supset \gamma) \supset (\alpha \vee \beta \supset \gamma \vee \alpha)$.
**P11:** $\beta \supset \beta \wedge \beta$.
**P12:** $(\beta \supset \gamma) \supset (\alpha \wedge \beta \supset \gamma \wedge \alpha)$.
**P13:** $\perp \supset \gamma$.
**IN:** $\alpha \subset\supset \neg\neg\alpha$.

Inference Rules of *S-AdjLPC*: **MP, I2** and **I3**.

The remaining arguments of this section are carried out within *S-AdjLPC*, whereby inferencing from its axioms will be denoted by $\vdash_S$. Recall that *S-AdjLPC* is just *EP-AdjLPC* with the involution axiom **IN** added. Accordingly, we are entitled to use all theorems and established inferences of *EP-AdjLPC*.

In terms of $\neg$ and $\Rightarrow$, we define the following, new logical connective $*$ :

$$\alpha * \beta = \neg(\alpha \Rightarrow \neg\beta). \tag{8.2}$$

We call it in *S-AdjLPC* the *tie conjunction* on $WF$.

**Proposition 8.1** *(monotonicity of* $*$*).*
$\alpha \supset \beta \vdash_S \{\alpha * \gamma \supset \beta * \gamma \ , \gamma * \alpha \supset \gamma * \beta\}$ .

**Proof.** Clear, from the monotonicity properties of $\neg$ and $\Rightarrow$. ■

**Theorem 8.1** *(commutivity of* $*$*).* $\vdash_S \alpha * \beta \subset\supset \beta * \alpha$.

**Proof.** This is just the following equivalidity from **EP** and Substitution Theorem:
$\neg(\alpha \Rightarrow (\beta \supset \bot)) \subset\supset \neg(\beta \Rightarrow (\alpha \supset \bot))$ . ■

**Theorem 8.2** *(exchange principle for* $*$*).* $\vdash_S \alpha * (\beta * \gamma) \subset\supset \beta * (\alpha * \gamma)$ .

**Proof.** $\alpha * (\beta * \gamma) = \neg(\alpha \Rightarrow \neg\neg(\beta \Rightarrow \neg\gamma)) \subset\supset \neg(\alpha \Rightarrow (\beta \Rightarrow \neg\gamma))$ (by (**IN**)) $\subset\supset$ $\neg(\beta \Rightarrow (\alpha \Rightarrow \neg\gamma))$ (by (**EP**)) $\subset\supset \neg(\beta \Rightarrow \neg\neg(\alpha \Rightarrow \neg\gamma))$ (by (**IN**)) $= \beta * (\alpha * \gamma)$ .
■

**Theorem 8.3** *(associativity of* $*$*).* $\vdash_S \alpha * (\beta * \gamma) \subset\supset (\alpha * \beta) * \gamma$.

**Proof.** This is a routine consequence of the preceding two theorems. ■

**Theorem 8.4** *(identity element).* $\vdash_S \alpha * \top \subset\supset \alpha$ *and* $\vdash_S \top * \alpha \subset\supset \alpha$

**Proof.** $\alpha * \top = \neg(\alpha \Rightarrow \neg\top) \subset\supset \neg(\alpha \Rightarrow \bot) = \neg\neg\alpha \subset\supset \alpha$ (by (**IN**)).
Also, $\top * \alpha = \neg(\top \Rightarrow \neg\alpha) \subset\supset \neg\neg\alpha \subset\supset \alpha$, by (**IN**). ■

It follows from Proposition 8.1 and Theorems 8.1-8.4 that the tie conjunction $*$ is a triangular norm. Also we know that $\neg$ is a negation, and so by **IN**, $\neg$ is a strong negation.

Now, from **IN** we have
$(\alpha \Rightarrow \beta) \subset\supset \neg\neg(\alpha \Rightarrow \neg\neg\beta) = \neg(\alpha * \neg\beta)$;
that is, $\Rightarrow$ is the $S$-type implication of these $*$ and $\neg$. This completes the proof that *S-AdjLPC* is a sound and complete syntax for the semantical domain of all adjointness lattices of the $S$-type.

We next study some essential features of $S$-type implications. The next theorem states that they satisfy self-contraposition.

**Theorem 8.5** $\vdash_S (\alpha \Rightarrow \neg\gamma) \subset\supset (\gamma \Rightarrow \neg\alpha)$ ,
$\vdash_S (\alpha \Rightarrow \gamma) \subset\supset (\neg\gamma \Rightarrow \neg\alpha)$,
$\vdash_S (\beta \supset \neg\alpha) \subset\supset \neg(\alpha \& \beta)$,
$\vdash_S \neg\alpha \subset\supset \#\alpha$.

**Proof.** The first equivalidity holds by **EP**. The second equivalidity follows from the first one and **IN**. The third equivalidity is just a restatement of axioms **E5** and **E7** with $\gamma = \bot$. The fourth holds by **IN** and Lemma 7.1. ■

**Proposition 8.2** $\beta \supset \neg\alpha \equiv_S \alpha \supset \neg\beta$ *and* $\alpha \& \beta \subset\supset \bot \equiv_S \beta \& \alpha \subset\supset \bot$.

**Proof.** The first equivalence follows from Proposition 7.2 and $\vdash_S \neg\alpha \subset\supset \#\alpha$ (Theorem 8.5). The second equivalence follows from the first one and Proposition 7.2. ■

**Proposition 8.3** $\alpha \subset\supset \beta \equiv_S \neg\alpha \subset\supset \neg\beta$.

**Proof.** This follows directly from Proposition 7.1 and **IN**. ■

The next theorem justifies the terminology "tie conjunction" for $*$. For a general study of such conjunctions in adjointness algebras, see [1].

**Theorem 8.6**

$$\vdash_S \ (\alpha \Rightarrow (\beta \Rightarrow \gamma)) \subset\supset (\alpha * \beta \Rightarrow \gamma)\,, \tag{8.3}$$

$$\vdash_S \ \alpha\& (\beta\&\gamma) \subset\supset (\alpha * \beta)\,\&\gamma. \tag{8.4}$$

**Proof.** We have the following equivalidities; by **IN** and the associativity of $*$:

$(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \subset\supset \neg(\alpha * \neg(\beta \Rightarrow \gamma)) \subset\supset \neg(\alpha * \neg\neg(\beta * \neg\gamma)) \subset\supset$ $\neg(\alpha * (\beta * \neg\gamma)) \subset\supset \neg((\alpha * \beta) * \neg\gamma) \subset\supset (\alpha * \beta \Rightarrow \gamma)$.

This proves (8.3).

By repeated application of (Adjointness), we obtain the following equivalences: $\alpha\&(\beta\&\gamma) \supset \delta \equiv_S \beta\&\gamma \supset (\alpha \Rightarrow \delta) \equiv_S \gamma \supset (\beta \Rightarrow (\alpha \Rightarrow \delta)) \equiv_S \gamma \supset (\beta * \alpha \Rightarrow \delta) \equiv_S$ $(\beta * \alpha)\,\&\gamma \supset \delta$ (by (8.3)) .

So by commutivity of $*$ (Theorem 8.1) we get the equivalence $\alpha\&(\beta\&\gamma) \supset \delta \equiv_S (\alpha * \beta)\,\&\gamma \supset \delta$.

Now, (8.4) ensues from applying this equivalence twice; once with $\frac{\alpha\&(\beta\&\gamma)}{\delta}$, and again with $\frac{(\alpha*\beta)\&\gamma}{\delta}$. ■

We next study the effects of adjoining to *S-AdjLPC* the following *commutivity axiom* for &:

**COM:** $\alpha\&\beta \supset \beta\&\alpha$.

**Proposition 8.4**

$$\textbf{COM} \vdash_S \ (\alpha \Rightarrow \gamma) \subset\supset (\alpha \supset \gamma)\,, \tag{8.5}$$

$$\textbf{COM} \vdash_S \ \alpha\&\beta \subset\supset \alpha * \beta. \tag{8.6}$$

**Proof.** We have the equivalences:

$\beta \supset (\alpha \Rightarrow \gamma) \equiv_S \alpha\&\beta \supset \gamma$ (Adjointness) $\equiv_S \beta\&\alpha \supset \gamma$ (**COM**).

So by (Adjointness), $\beta \supset (\alpha \Rightarrow \gamma) \equiv_S \beta \supset (\alpha \supset \gamma)$.

By applying this last equivalence, once with $\frac{\alpha\Rightarrow\gamma}{\beta}$ and again with $\frac{\alpha\supset\gamma}{\beta}$, we get (8.5).

Next, assuming **COM**, we get the following equivalidities in *S-AdjLPC*:

$\alpha * \beta \subset\supset \beta * \alpha$ (Theorem 8.1) $= \neg(\beta \Rightarrow \neg\alpha) \subset\supset \neg(\beta \supset \neg\alpha)$ (by(8.5)) $\subset\supset$ $\neg\neg(\alpha\&\beta)$ (Theorem 8.5) $\subset\supset \alpha\&\beta$ (**IN**). This renders (8.6). ■

The preceding proposition means that in *S-AdjLPC* enriched by **COM**, the implication $\Rightarrow$ is indistinguishable from the forcing implication $\supset$, and the conjunction & is indistinguishable from the triangular norm $*$. Thus, **COM** provides a complete characterization of an $S$-type implication $\Rightarrow$, of some triangular norm $*$, that is simultaneously the residuated implication of that $*$.

We remark that in residuated logic we have another complete characterization of such implications. They are those residuated implications (of triangular norms) that satisfy **IN**. For an algebraic proof, see [18].

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