EULER'S ANALYTICAL PROGRAM

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1.-Introduction.

"Read Euler, read Euler. He is the master of all us". This famous exhortation, which Libri[1] ascribes to Laplace, expresses Euler's influence on eighteenth-century mathematics very well. During his long and profitable activity Euler obtained an astonishing number of results, which were crucial in the development of mathematics and fill over 70 volumes of his *Opera omnia*[2]: they concern all parts of eighteenth-century mathematics – both pure and applied –, ranging from number theory, infinite series, the theory of equations, combinatorics and probability, the differential and integral calculus, elliptic integrals, the calculus of variations, musical harmony, mechanics, theory of machines, optics, astronomy, naval science, and much else besides. Thus, today, the name of Euler can found in the history of almost all the branches of mathematics, even in the history of those branches that did not yet exist in the eighteenth century but boast of Euler as an ancestor, such as graph theory.

In this paper, however, my intention is not to deal with one or more of the many Eulerian results but to call attention on what I term as *Euler's analytical program*, namely Euler's attempts of transforming analysis into an autonomous discipline and reorganizing the whole of mathematics around analysis. The main aspects of this program can be summarized as follows:

1 LIBRI, Gugliemo (1846) Review of FUSS, Paul Heinrich *Correspondance mathématique et physique de quelque célèbres géomètres du XVIII ème siècle, Journal des Savants*, Janvier 1846, 50–62; in particular 51.

2 *Leonhardi Euleri Opera omnia* Berlin, Leipzig, Heidelberg, Zurich, and Basel: 1911– (afterwards: *Opera*). The plan of Euler's *Opera omnia* originally involved three series, containing the works that Euler personally prepared for publication. Between 1911 and 2006 seventy volumes of series 1, 2, and 3 were published. Volumes 26 and 27 of series 2 are expected to be published in 2010. The publication of series 4 began in 1985. It is devoted to Euler's correspondence (series 4A) and manuscripts (series 4B). Series 4A is planned to consist of ten volumes (four volumes were published by 2006). Series B will contain Euler's hitherto unpublished manuscripts, notebooks, and diaries. An online resource for Leonhard Euler's original works and modern Euler scholarship is *The Euler Archive,* available from *http://math.dartmouth.edu/~euler*.

(a) the elimination of geometric and empirical evidence in the derivation of analytical propositions and the rise of analysis as an abstract, conceptual, and merely discursive theory;

(b) the construction of a set of abstract and general notions and propositions, which was thought to be the heart of mathematics (pure analysis);

(c) the application of pure analysis to geometry and mechanic so to transform these branches of mathematics in a sort of applied analysis (later termed analytical geometry and analytical mechanics).

In section 2, I will discuss the conceptual background of Euler's program. In sections 3 and 4, I will illustrate how Euler tried to carry out his program and its consequences on mechanics. Finally, in section 5, I give a glimpse on the reasons that led mathematicians, first, to share this program and, then, to reject it; I will also clarify why Euler's analytical program can be considered his main legacy.

2. -Analysis and geometry.

In the Leibnizian and Bernoullian conception, analysis was not an autonomous and self-founding mathematical discipline: it was an instrument for solving geometric problems and investigated the relations between geometric quantities (such as ordinate, abscissa, arc length, subtangent, normal, areas between curves and axes). In his first papers, written under Johann Bernoulli's influence, Euler followed this conception and dealt with typical problems of the Bernoullian school, such as isochronous curves[3], tautochrone curves[4], reciprocal trajectories[5]. In various cases Johann Bernoulli himself[6] suggested the topic of Euler's research. For instance, in his *De linea brevis-*

3 EULER, Leonhard (1726) "Constructio linearum isochronarum in medio quocunque resistente", *Acta Eruditorum*, 1726, 361-363 or *Opera,* ser. 2, vol. 6, 1-3.

4 EULER, Leonhard (1727a) "Dissertatio de novo quodam curvarum tautochronarum genere", *Commentarii academiae scientiarum Petropolitanae* (afterwards: *Comm.*), 2, 126-138 or *Opera*, ser. 2, vol. 6, 4 - 14.

5 EULER, Leonhard (1727b) "Methodus inveniendi traiectorias reciprocas algebraicas", *Acta Eruditorum*, 1727, 408-412 or *Opera*, ser. 1, vol. 27, 1-5 and EULER, Leonhard (1727c) "Problematis traiectoriarum reciprocarum solutio", *Comm.* 2, 90-111 or *Opera*, ser. 1, vol. 27, 6-23.

6 In his [1727b, 408], Euler stated that Johann Bernoulli was "the most renowned of masters" and that "not only was my teacher, greatly fostering my inquiries into such matters, but also looked after me as a patron".

sima[7], Euler stated: "The Celebrated Johann Bernoulli proposed this question[8] to me and urged me to write up my solution and to investigate these three kinds of surfaces ..."[9] However, already at the end of the 1720s, Euler played strong attention to the analytical instruments and attempted to improve them, since – he thought – analysis facilitated the understanding and solution of geometric and physical problems[10]. This was the starting-point of an evolutionary process that led him to a new idea of analysis, later expounded in a systematic way in *Introductio in analysin infinitorum*[11], *Institutiones calculi differentialis* [12], and *Institutionum calculi integralis*[13].

The crucial aspect of Euler's new conception was the separation of analysis from geometry; in practice, this meant that diagrammatic representations were eliminated from the derivation of analytical propositions and that analysis changed into an abstract, conceptual, and merely discursive theory. To clarify this point, I briefly examine some remarkable passages from two of Euler's articles, *Methodus universalis*[14] and *Inventio summae*[15]. In both these papers, Euler tackled the problem of determining and approximating evaluation of the *n*-partial sum of a series $\sum a_n$, but – he said – the method used in the former was geometric, whereas the method used in the latter was analytical:

> *"When I gave more precise consideration to the mode of summing which I had dealt with by using by the geometric method in the above dissertation [Methodus universalis] and investigated it analytically, I*

7 EULER, Leonhard (1728a) "De linea brevissima in superficie quacunque duo quaelibet puncta iungente", *Comm.* 3, 110-124 or *Opera*, ser. 1, vol. 25, 1-12.

8 The problem of finding "the shortest line between two points on a surface".

9 EULER (1728:§. 2).

10 For instance, he usually considered curves and surfaces as long as their nature can be expressed by equations (cf. EULER (1728b, 112).

11 EULER, Leonhard (1748) *Introductio in analysin infinitorum*, Lausannae, M. M. Bousquet et Soc., or *Opera*, ser. 1, vols. 8-9.

12 EULER, Leonhard (1755a), *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, Petropoli, Impensis Academiae Imperialis Scientiarum, or *Opera*, ser. 1, vol. 10.

13 EULER, Leonhard (1768-70) *Institutionum calculi integralis,* Petropoli, Impensis Academiae Imperialis Scientiarum or *Opera*, ser. 1, vols. 11-13.

14 EULER, Leonhard (1736a) "Methodus universalis serierum convergentium summas quam proxime inveniendi", *Comm.* 8, 3-9 or *Opera* ser. 1, vol. 14, 101-107.

15 EULER, Leonhard (1736b) "Inventio summae cuiusque seriei ex dato termino generali", *Comm.* 8, 9-22 or *Opera*, ser. 1, vol. 14, 108-123.

> *discovered that what I had derived geometrically could be deduced from a peculiar method for summing that I mentioned three years before in a paper*[16] *on the sum of series"*[17].

The method of *Inventio summae* was analytical because it was only based on the manipulation of formulas. Indeed, Euler assumed that the *n*-th term a_n of a given series $\sum a_i$ was a function *X=a(n)* of *n*[18] and, through a long sequel of calculations and formal manipulations, derived the so-called Euler-Maclaurin sum formula[19].

$$S(n)=\int Xdn+\frac{X}{2!}+\frac{dX}{3!2dn}-\frac{d^3X}{5!6dn^3}+\frac{d^5X}{7!6dn^5}-\frac{3d^7X}{9!10dn^7}+\frac{5d^9X}{11!6dn^9}-\frac{691d^{11}X}{13!210dn^{11}}+\frac{35d^{13}X}{15!2dx^{13}}-\frac{3617d^{15}X}{17!30dn^{15}}+\ldots$$ [20]

This formula allowed him to evaluate the *n*-partial sum $S(n)=\sum_{i=1}^{n} a_i$.

On the contrary, Euler defined the method of *Methodus universalis* as geometric because it was based on a geometric representation of a decreasing series $\sum a_n$. Euler considered the diagram shown in Fig. 1, where aA=a_1, bB=a_2, cC=a_3, dD=a_4, …, pP=a_n, qQ=a_{n+1} and AB=BC=CD=DE=…=PQ=1. An inspection of the diagram shows that

$$\sum_{i=1}^{n} a_i > \int_1^{n+1} a(n)dn.$$ [21]

16 EULER, Leonhard (1732-33) "Methodus generalis summandi progressiones", *Comm.* 6, 68-97 or *Opera,* ser. 1, vol. 14, 42-72.

17 EULER (1736b: 9).

18 On Euler's derivation of the Euler-Maclaurin sum formula, see FERRARO, Giovanni (1998) "Some Aspects of Euler's Theory of Series. Inexplicable functions and the Euler-Maclaurin summation formula", *Historia mathematica,*, 25, 290-317.

19 EULER (1736b: §. 19).

20 By a similar diagram, Euler derived $s_n=\sum_{i=1}^{n} a_i < \int_1^{n} a(n)dn$

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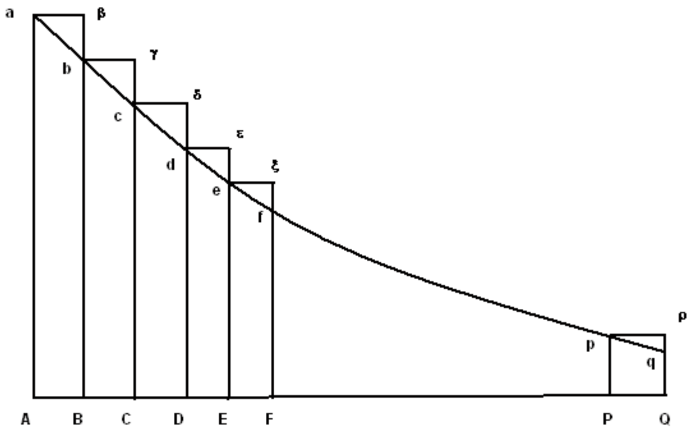


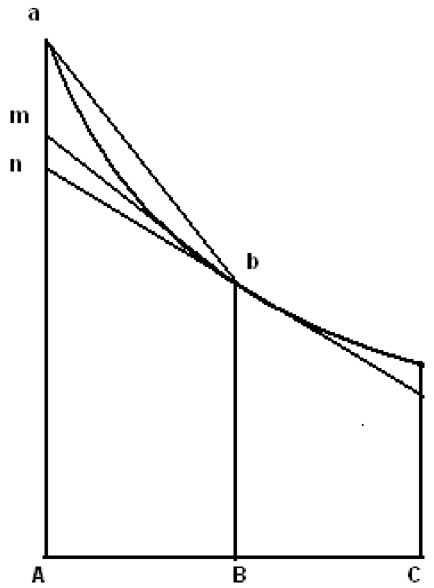
Euler noted that the integral $\int_1^{n+1} a(n)dn$ approximated the sum of the given series and that this approximation could be improved by observing that the curvilinear triangles $ab\beta$, $bc\gamma$, ..., $pq\rho$ were greater than the rectilinear triangles $ab\beta$, $bc\gamma$, ..., $pq\rho$ (the curved line aq is convex, at least for large enough n). Since the sum of the areas of the rectilinear triangles $ab\beta$, $bc\gamma$, ..., $pq\rho$ is $\frac{(Aa - Qq)AB}{2}$, he obtained

$$\sum_{i=1}^{n} a_i > \int_1^{n+1} a(t)dt + \frac{a_1}{2} - \frac{a_{n+1}}{2}.$$ [22]

Then, Euler considered the secant *bc* (Fig. 2) and approximated the arc *ab* by an appropriate arc of a parabola. By a series of geometric considerations based upon Fig. 2 and implicitly assuming the convexity of the curve, he obtained the following approximating formula for the sum of the series

$$\sum_{i=1}^{n} a_i = \int_1^{n+1} a(t)dt + \frac{a_1}{2} - \frac{a_{n+1}}{2} + \frac{a_1 - a_2}{12} - \frac{a_{n+1} - a_{n+2}}{12}.$$

22 Analogously, he obtained $\sum_{i=1}^{n} a_i < \int_1^{n} a(t)dt - \frac{a_1}{2} + \frac{a_{n+1}}{2}$.



According to Euler, the geometric method of *Methodus universalis* hinged on using appropriate geometric figures and some steps of the deduction were inferred by scrutinizing these figures; instead, the analytical method of *Inventio summae* dispensed with the diagrammatic representation. If, however, we look carefully at *Methodus universalis* and *Inventio summae*, we note that both papers are based upon similar concepts and, in effect, it would be easy to translate *Methodus universalis* into analytical symbols: the crucial difference between the analytical and geometric methods was merely the presence of the diagrams.

This is puzzling to modern eyes; nowadays, the diagrammatic representation of *Methodus universalis* might seem a dispensable tool for facilitating the comprehension of the proof, since in modern geometry figures improve the understanding of reasoning but are unessential. Modern proofs are merely linguistic deductions derived from explicit axioms and inference rules. This is not true for the classical conception of Euclidean geometry, where the reference to figures plays a crucial role[23]. Euler shared this concept of geometry, and it is just this concept that makes clearer the meaning of certain Euler's statements, such as the following from the preface of *Institutiones calculi differentialis*:

23 On the use of diagrams in Greek geometry, see NETZ, Reviel (1999) *The Shaping of Deduction in Greek Mathematics*, Cambridge, Cambridge University Press.

> *"I mention nothing of the use of this calculus in the geometry of curved lines, because its absence will be least felt, since it has been investigated so comprehensively that even the first principles of differential calculus are, so to speak, derived from geometry and, as soon as they had been sufficiently developed, were applied with extreme care to this science. Here, instead, everything is contained within the limits of pure analysis so that no figure is necessary to explain the rules of this calculus"*[24].

When Euler claimed the absence of geometric figures in his analytic treatises, he asserted the absence of inference derived from the mere inspection of a figure (*inspectio figurae*), which was crucial in classical geometric proofs. In Euler's opinion, analysis was as a system of merely conceptual and mediated notions; it functioned in a discursive way along abstract ideas. Geometry instead was a line of reasoning applied to figures that were shown in the concrete form of a diagram: geometry was entrusted, to a certain extent, to the intuitive immediacy of an inspection of the figure and the perception of the relationships shown in the diagram.

In conclusion, the crucial difference between analysis and geometry did not consist of the fact that analysis used symbols but depended on the fact symbols were the instruments by which analysis developed as an abstract conceptual, and merely discursive theory, which did not rely on the material, whereas geometry was concrete (or more concrete) and relied on the aid of figures. For this reason, the analytical method can be described as a *non-figural* method whereas the geometric one was *figural*[25].

3.- Analysis as algebraic analysis.

In this section, I will examine how Euler carried out his program. The key-word is abstract quantity: in Euler's opinion, analysis was the science of general or universal quantities. Indeed, following a traditional point of view, Euler defined mathematics as "the science of quantity or, the science which investigates the means of measuring quantity"[26] and gave the name of

24 EULER, Leonhard (1755a: 9).
25 See FERRARO, Giovanni (2001) "Analytical symbols and geometrical figures in eighteenth-century calculus", *Studies in History and Philosophy of Science* Part A, vol. 32, 535-555.
26 EULER, Leonhard (1770) *Vollständige Anleitung zur Algebra* , *Opera,* ser. 1, vol. 1, §. 2.

magnitude or quantity to "whatever is capable of increase or diminution"[27]. However, according to Euler, there existed many different kinds of quantity and this was "the origin of the different branches of mathematics, each being employed on a particular kind of quantity"[28]. In his opinion, a quantity was investigated in mathematics insofar as it could be measured and the operation of measuring quantity consisted:

> (a) in fixing at pleasure upon any one known magnitude of the same species with that which was to be determined, and considering it as the measure or unity;
> (b) in determining the relation of the given magnitude to this measure.

Euler stated that this relation was always expressed by numbers[29]; so he seems to reduce quantity to number. It, however, is to be emphasized that he did not possess a notion a number independent of that of quantity. Euler did not have the concept of the set of real numbers; he merely gave the name "number" to the result of the process of measuring any quantity – a process that expresses the relation between the quantity and the unity –, without explaining what this process is and how measurement can be performed and, therefore, without explaining what the relation between the quantity and the unity is.

Euler's concept of number remained at very intuitive level[30]; however, the number was the most important tool for treating quantity and, therefore, "the foundation of all the mathematical sciences must be laid in a complete treatise on the science of numbers, and in an accurate examination of the different possible methods of calculation"[31]. According to Euler, this science of numbers was analysis or algebra[32].

27 EULER (1770: §. 1).
28 EULER (1770: §. 2).
29 EULER (1770: §. 4).
30 In effect, Euler gave the name of number to any symbolic entity which can be manipulated in a similar way to natural numbers. On Euler's concept of number, see FERRARO, Giovanni (2004) "Differentials and differential coefficients in the Eulerian foundations of the calculus", *Historia Mathematica,* vol. 31, 34-61.
31 EULER (1770: §. 5).
32 "Analysis or Algebra is the fundamental part of mathematics that investigates numbers and methods of calculation" (EULER, 1770, §. 5). In his (1770) Euler did not make a distinction between analysis and algebra. Instead, according to d'Alembert, analysis "is properly the method for solving mathematical problems by reducing them to equations …. In order to

However, since all quantities could be expressed by numbers, all parts of mathematics dealt with numbers and it was necessary to provide a characterization of algebra or analysis with respect to the other parts of mathematics. Thus, in his *Algebra*, Euler explained that numbers are taken into account in analysis as they represented quantities considered *in general*, without regard to the differences between the special types of quantities[33]. The notion of quantity in general or universal quantity was discussed by Euler in the *Introductio*, where he stated that the idea of a general or universal quantity was generated from particular geometric quantities by means of a process of abstraction: a general quantity was what was common to all quantities, just as "redness" was possessed by all red particular objects. Using an explicitly philosophical language, Euler stated that "*in the same way as the ideas of species and genera are formed from the ideas of individuals, so a variable quantity is the genus, within which all determinate quantities are included*"[34].

In Euler's opinion, analysis was the science that investigated the pure concept of quantity (general or universal quantity) without no concrete determination; whereas the other branches of mathematics concerned the specific types of quantities[35]; more precisely, geometry and mechanics dealt with geometric and physical quantities (which could be represented by a diagram and had empirical evidence). Consequently, geometry and mechanics were more particular and specific disciplines with respect to analysis, which was the most general and abstract part of mathematics.

This approach had important consequences on the internal relationships among the different branches of mathematics. While a result valid for abstract quantities was also valid for geometric and mechanical quantities, the inverse was not true: the less general and more concrete notions of geometry and mechanics (lines, areas, times, forces,…) could not be used in analytical demonstrations. Mathematics was conceived of as a building whose heart was analysis, which investigated mathematical objects in all their generality and abstractness and, in this way, provided the method for other parts of mathematics. Thus, the other branches of mathematics, such as geometry

solve problems, Analysis resorts to Algebra, or the calculus of magnitudes in general: thus, these two words, Analysis and Algebra, are often regarded as synonyms" (d'ALEMBERT, Jean Baptiste Le Rond, "Analyse", *Encyclopédie, ou dictionnaire raisonné des sciences, des arts et de métiers*, Paris, Briasson, David l'aîné, le Breton, Durand, 1751-80, vol. 1, 400b.

33 EULER (1770: §. 6).

34 EULER (1748: 17).

35 EULER (1770: §. 6).

and mechanics, became fields of application of pure analysis: a sort of applied analysis. As a result of this conception, Euler's analytical program originated two subprograms aiming to do geometry and mechanics in an analytical way so to transform them into two new branches of analysis: these subprograms were of fundamental importance in the history of mathematics and, later, led to the rise of analytical geometry and analytical mechanics (see section 4).

As concerns arithmetic, Euler stated that it investigated numbers in the proper sense of the term and dealt with the common way of calculating with numbers, namely it investigated the operations between *specific* numbers that were not represented by means of letters. Analysis or algebra was more general than arithmetic since it comprehended all the cases that existed "in the doctrine and calculation of numbers"[36]; therefore, algebra was a generalization of arithmetic, where the usual arithmetic operations were generalized and applied to all kinds of numbers and to symbols, such as *a, b,* …, that represented indeterminate (unspecified) numbers or variable quantities.

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Analysis or algebra was subdivided into different parts. The first and more elementary subdivision was between "ordinary algebra" or "analysis of finite quantities" and "analysis of infinite quantities". Not only did these two branches of analysis differ for the use of infinity but also for meaning of the symbols. In ordinary algebra the symbols *a, b,* … stood for indeterminate numbers or unknown numbers; they were not variables. Indeed, the concept of quantity was essential to define the number, but once that numbers were introduced they were sufficient to develop ordinary algebra: ordinary algebra merely treated numbers in their indeterminate form *a, b*, ... Instead, the analysis of infinite quantities required the use of variables: according to Euler, a variable was merely a general quantity[37]. In other words, analysis of infinite quantities investigated the capability of quantity of being increasing or decreasing, while ordinary algebra investigated fixed determinations of quantity (substantially numbers even if these numbers could be treated in indeterminate form).

36 EULER, L. (1770:§. 7).
37 See FERRARO, Giovanni (2000a) "Functions, Functional Relations and the Laws of Continuity in Euler", *Historia mathematica*, 27, 107-132.

Despite these differences, Euler conceived analysis as a unitary theory based upon the step-by-step extensions of arithmetical rules[38]. He assumed that the methods of calculation used to natural and rational numbers could be applied to numbers of any kind, to indeterminate numbers represented by letters and even to abstract quantities[39]. For this reason, Euler's analytical program was a reductionist program, according to which the rules of analysis were conceived of as a generalization of the rules of arithmetic except for some necessary adjustments.

Of course, after Leibniz and Newton, the heart of analytical methods was constituted by the algorithm of the calculus and, therefore, Euler's program had to reduce the algorithm of the calculus to an appropriate extension of algebra of finite quantities. In other terms, Euler had to reduce some crucial notions of the calculus – namely, the notions of power series, differentiation, and integration – to algebraic notions (here the adjective "algebraic" is to be understood in the restrict sense of analysis of finite quantities or ordinary algebra[40]).

Following Newton and Leibniz, Euler thought that the notion of power series could easily be reduced to algebraic concepts; indeed, a power series was conceived of as an infinite polynomial, upon which one could apply the rules that were valid for finite polynomials. For instance, the common rule for the division of polynomials was used to expand a function into a power series. Thus, to expand $c/(b+x)$, one divided c by $b+x$ and obtained the quotient c/b and the remainder $-\frac{c}{b}x$. By dividing the remainder by $b+x$, one had the quotient $\frac{c}{b^2}x$ and the remainder $-\frac{c}{b^2}x^2$. By continuing *ad infinitum* one derived the series $\frac{c}{b}-\frac{c}{b^2}x+\frac{c}{b^3}x^2-\ldots$, which was the development of $\frac{c}{b+x}$.[41]

38 See PANZA, Marco (1992) *La forma della quantità. Analisi algebrica e analisi superiore: il problema dell'unità della matematica nel secolo dell'illuminismo*, vols. 38–39 of the *Cahiers d'Historie et de Philosophie des Sciences*, pp. 701-702 and JAHNKE, Hans Niels (1993) "Algebraic analysis in Germany, 1780–1840. Some mathematical and philosophical issues", *Historia Mathematica*, 20, 265–284 (in particular, 281).

39 See FERRARO (2004).

40 It is appropriate to make clear that this use of the term "algebraic" is not due to Euler. As we saw (footnote n. 31), Euler, in principle, considered algebra and analysis as synonyms; however, the fact he entitled his 1770 treatise as *Vollständige Anleitung zur Algebra* (*Complete instruction in algebra*) seems to be the sign of a trend to use the term "algebra" in place of "ordinary algebra".

41 For a detailed investigation of Euler's theory of series, I refer to FERRARO, Giovanni (2000b) "The value of an infinite sum. Some observations on the Eulerian theory of series", *Sciences et techniques en perspective*, 4, 73–113 and FERRARO, Giovanni (2008a), *The rise and development of the theory of series up to the early 1820s*, New York, Springer (in particular Part 2).

More generally, one can state that the procedures applied to power series were based upon the following principle, which was one of the cornerstones of Euler's reductionist program:

> **-P1.** *if a rule R was valid for finite expressions or if a procedure P depended on a finite number n of steps* S_1, S_2, S_3, …, S_n, *then it was legitimate to apply the rule R and the procedure P to infinite expressions and in an unending number of steps* S_1, S_2, S_3, ….[42]

While series were not problematic, differentials were very difficult to be treated coherently to the principles of algebraic analysis: the notion of differentials was felt obscure and not algebraic[43]. For this reason, Euler developed a strategy aiming to reduce the use of differentials to a minimum. This strategy can be summarized as follows.

1) Analysis of infinite quantities was subdivided into two parts:
- the introduction of the analysis of infinities or algebraic analysis[44],
- the calculus.

The introduction of the analysis of infinities was a corpus of knowledge that avoided the notion of differentials and could be treated using only the infinite extension of the rules valid in the algebra of finities: it investigated functions, their transformations and their expansions into series. Instead, the calculus was the part of analysis of infinities where the operations of differentiation and integration were investigated[45].

2) Euler attempted to make the notion of differentials as more arithmetical as possible and to reduce their use to the definition and determination of the differential ratios. Indeed, he defined differentials as evanescent quantities or

42 For more details, see FERRARO, Giovanni (2007a) "The foundational aspects of Gauss's work on the hypergeometric, factorial and digamma functions", *Archive for History of Exact Sciences* 61, 457-518 (in particular 464).

43 The same difficulty occurred for the theory of limits, since the notion of limits was also considered non-algebraic. In Euler's opinion, limits could only provide an intuitive justification for the rules of the calculus.

44 The expression "algebraic analysis" is due to Lacroix (cf. LACROIX, Silvestre François (1797-1798) *Traité du calcul différentiel et du calcul intégral*, 2 vols., Paris, Duprat).

45 On the different parts of analysis, see FERRARO, Giovanni (2007b) "Euler's treatises on infinitesimal analysis: *Introductio in analysin infinitorum, Institutiones calculi differentialis, Institutionum calculi integralis*", in BAKER, Roger (ed.) *Euler Reconsidered. Tercentenary Essays*, Heber City, UT, Kendrick Press, 39-101 (in particular, 44-45).

zero or nothing: a differential *dx* was only a way of denoting that a variable *x* vanished (namely, it tended to zero) and the numerical value of *dx* was zero. By means of a calculus of these evanescent quantities or zeroes, he attempted to reduce infinities and infinitesimals to a new kind of numbers[46]. This was not sufficient to transform differentials into an algebraic notion; thus, Euler stated that the differential calculus did not concern with investigating differentials – which are equal to 0 – but regarded with defining their mutual ratio – which had a determinate value –[47]. He claimed that functions were the genuine subject-matter of the differential calculus and that differentials were mere tools for dealing with functions. For example, given the function $y=x^2$ whose differential is $dy=2xdx$, the calculus studied the differential coefficient $\frac{dy}{dx}=2x$ and not the differential $2xdx$[48]. Therefore, the operation of differentiation involved finite quantities since it transformed a finite quantity y into the finite quantity dy/dx. However, differentials remained essential in defining this operation and Euler was fully aware of this[49].

3) As concerns the operation of integration, Euler thought that the use of differentials could be avoided by defining integration as the inverse operation of differentiation, namely, the integral $\int f(x)dx$ of the function $f(x)$ was a function $F(x)$ such that $dF=f(x)dx$. By integration, one returned from the differential to the function generating the differential[50].

In Euler's opinion, once the differential ratios were introduced, both the differential and integral calculus were not understood as calculi concerning differentials but as calculi concerning functions and their differential ratios – which were finite quantities –: the differential calculus was the direct calculus that led from functions to differential ratios; the integral calculus was the inverse calculus that led from the differential ratios to the functions generating that differential ratio.

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46 See FERRARO, G. (2004).
47 EULER (1755: 5).
48 EULER (1768-1770, vol.1: 6).
49 FERRARO (2004).
50 EULER (1768-70, vol. 1, 5). On Euler's concept of integration, see FERRARO, G. (2008b) "The integral as an anti-differential. An aspect of Euler's attempt to transform the calculus into an algebraic calculus", *Quaderns d'història de l'enginyeria*, 9, 25-58.

Functions were the main technical instruments used by Euler to carry out his analytical program; however, the meaning Euler gave to the term "function" is very different from the modern one[51]. Indeed, in Euler's works, when the word "function" was used in its proper sense, it has to be understood in this way: a function is given by one only analytical expression constructed from variables in a finite number of steps using exponential, logarithmic and trigonometric functions, algebraic operations, and composition of functions[52]. In other words, a function, in the proper sense of the term, was an elementary function[53].

This notion of a function was closely connected with:

(a) the above-described concept of analysis as a theory based upon the step-by-step extensions of arithmetical rules,

(b) the idea that functions could be considered as the basic and proper objects of analysis only if they were known objects.

As concerns point (a), I observe that elementary functions were thought to result from the generalization and symbolic representation of arithmetical operations[54], whereas other quantities, such as transcendental quantities expressed by means of integrals, were not felt as the direct generalisation of arithmetical operations.

As regards (b), I note that, according to Euler, analysis had to offer a methodology by which one could solve problems concerning any branch of mathematics. However, a problem is solved when one exhibited a known object; therefore a function, in the proper sense of the term, had to be a known object, so that it could be exhibited as the solution to a problem[55]. Only elementary functions seemed to satisfy these two conditions. I refer to my (2000a) for a detailed treatment of Euler's concept of a function; now I limit myself to men-

51 See FRASER, Craig (1989) "The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century", *Archive for History of Exact Sciences*, 39, 317-335 and FERRARO (2000a).

52 On the use of equations, such as $f(x,y)=0$, to define functions, see FERRARO (2000a).

53 On the use of functions different from the elementary ones in Euler's mathematics, see FERRARO (2000a) and FERRARO (2007a).

54 This idea even made the introduction of trigonometric functions problematic. Indeed, they were introduced later, when their link with the exponential function had been established and it had been highlighted that they occurred as solutions to certain differential equations (see KATZ, V. J. (1987) "The Calculus of the Trigonometric Functions", *Historia Mathematica*, vol. 14, 311-324).

55 See FERRARO (2000a: 115).

tion two aspects of this concept that were of great importance in establishing the causes of the crisis of Euler's analysis.

First, Eulerian functions were characterised in an essential way by the use of a formal methodology that it made possible to operate upon analytical expressions, independently of their meaning and was grounded upon principle **P1** and the following principle of the generality of algebra:

> **P2.** *If an analytical formula was derived by using the rules of algebra, then it was thought to be valid in general*[56]

For this reason, if a certain property of the function *f(x)* was proved to be valid for a certain interval of the values of the variable *x*, then it was generalized beyond the bounds of its original validity and was assumed to hold for any value of the function *x* – real, complex and also infinite or infinitesimal[57].

Second, in Euler's conception, a power series was viewed as the expansion of *generating* functions and always presupposed a function that generated them[58]. For this reason, a series could not be used to define a function. In the same way, since an integral was anti-differential, it could be used to define a function[59]. In FERRARO (2007a) I showed that this conception was a huge obstacle to the growth of analysis in the second part of eighteenth century[60].

4.- Analysis and mechanics.

The third point of Euler's analytical program was the reconstruction of the edifice of mathematics around analysis. Euler carried out this objective transforming geometry and mechanics into a sort of *applied analysis*: his treatment of geometry and mechanics can be considered as the starting point of

56 See FERRARO (2007a: 563).

57 On the principle of the generality of algebra, see FERRARO (2000a : 121-123).

58 For instance, the sum of $\sum_{i=0}^{\infty}(-1)^i x^i$ was 1/(1+x) since the function 1/(1+x) could be expanded into $\sum_{i=0}^{\infty}(-1)^i x^i$. On this question, see FERRARO (2000b) and (2008b: chapter 19).

59 On the nature of transcendental quantities $\int f(x)dx$ or $\int_a^b f(x)dx$ in Euler's calculus, see FERRARO (2008b).

60 FERRARO (2007a, 467 ff).

the process that led to the rise of analytical geometry (in a modern sense of the term[61]) and analytical mechanics.

Euler's main contribution to analytical geometry is the second book of the *Introductio*, which was just written to give a proof of the power and efficacy of the application of the analytical method. For the sake of brevity, I do not dwell upon this treatise[62] but prefer to deal with some of several papers Euler devoted to the attempt of making mechanics an analytical discipline.

In 1736 Euler published a treatise on kinematics and dynamics of a point-mass, *Mechanica sive motus scientia analytice exposita*[63], where analytic methods were, for the first time, applied to mechanics in a systematic way. He explained the reasons of his approach as follows:

> *"[W]hat distracts the reader the most, is the fact that everything is carried out synthetically, with the demonstrations presented in the manner of the old geometry, and the analysis hidden, and recognition of which is given only at the end of the work. Hermann's work is not a great deal different also, from the manner of the composition of Newton's Principia Mathematica Philosophiae, from which the science of motion has benefited the most. But what pertains to all the works composed without analysis, is particularly true for mechanics. In fact, the reader, even though he is persuaded about the truth of the things that are demonstrated, nonetheless cannot understand them clearly and distinctly. So he is hardly able to solve with his own strengths the same problems, when they are changed just a little, if he does not inspect them with the help of analysis and if he does not develop the propositions into the analytical methods. This is exactly what happened to me, when I began to study in detail Newton's Principia and Hermann's Phoronomia. In fact, even though I thought that I could understand the solution to numerous problems well enough; I could not solve problems that were slightly different. Therefore I strove, as much as I could, to get at the analysis behind those synthetic methods in order, for my purposes, to deal with those propositions in terms of analysis. Thanks to this procedure I perceived a remarkable improvement of knowledge. Thus I have endeavoured or a long time now, to use the old synthetic*

61 FERRARO (2007b: 50 ff).

62 For a discussion of the second book of the *Introductio*, see FERRARO (2007b : 50-55).

63 EULER, Leonhard (1736c) *Mechanica sive motus scientia analytice exposita. Tomus I.* Petropoli: ex typ. Academiae Scientiarum, 1736, in *Opera*: Ser. 2, vol.1.

> *method to elicit the same propositions that are more readily handled by my own analytical method, and so by working with this latter method I have gained a perceptible increase in my understanding. Then in like manner also, everything regarding the writings about this science that I have pursued, is scattered everywhere, whereas I have set out my own method in a plain and well-ordered manner, and with everything arranged in a suitable order"*[64].

The *Mechanica* originated from Leibniz's program to reformulate Newton's *Principia* in terms of the Leibnizian calculus[65] and can be considered "as a work of systematization of results achieved mainly in the Bernoullian school"[66]. However, Euler went beyond the intention of previous mathematicians, none of them had showed awareness of the possibility of transforming analytical methods into a new and autonomous field of mathematics. In the two books of the *Mechanica* analytical methods were conceived as the peculiar methodology of a general and abstract discipline (analysis), which, just because it was abstract and general, could be applied to a more specific and concrete discipline (mechanics). In this sense the *Mechanica* can be considered the starting point of the process that led, first, to the transformation of mechanics in applied analysis and, then, to what is now called analytic mechanics.

In practice, the application of analytical methods consisted of a wide use of differential and integral calculus – above all differential equations –. For example, Euler expressed Newton's second law[67] in the form[68]:

64 EULER (1736, 1: 38-39).

65 On this topic, see GUICCIARDINI, Niccolò (1999) *Reading the Principia: the debate on Newton's mathematical methods for natural philosophy from 1687 to 1736*, Cambridge, Cambridge University Press.

66 GUICCIARDINI (1999: 248).

67 At the beginning of Book I of his *Principia*, Newton formulated the second law of motion as follows: "the change of motion is proportional to the motive force impressed, and it takes place along the right line in which that force is impressed" (cf. NEWTON, Isaac (1687) *Philosophiae Naturalis Principia Mathematica*, Londini, Josephi Streater, p. 12). In modern terms, this definition corresponds to $F=\Delta(mv)$, where mv is the motion (momentum). The expression "change of motion" (*mutatio motus*) is not univocal in Newton and, elsewhere, Newton states that a centripetal force is proportional to the motion that it generates it in a given time: this sounds as $F=ma$ (see MALTESE, Giulio (2002) "On the Changing Fortune of the Newtonian Tradition in Mechanics", in Kim Williams (ed.) *Two Cultures Essays in Honour of David Speiser*, Basel, Birkhäuser, 97-113). Some mathematicians use Newton's law in Cartesian form; however, it was Euler who based the mechanics of rigid bodies and fluid mechanics on this principle.

68 EULER (1736, I : §. 154).

$$(1) \qquad dc = \frac{npdt}{A},$$

(where A is the body and its mass, p is the force, c the velocity, t the time, n is a constant of proportionality depending on the unity of measure)[69]. From equation (1) Euler was able to derive all differential equations necessary to describe the motion of a point-mass.

In *Mechanica*, however, Euler used an intrinsic coordinates system. He decomposed speeds and forces according to directions that depended upon the intrinsic nature of the problem[70]: this limited the generality of procedures. Some years later, a new, and more general approach appeared, first, in *Recherches sur le mouvement des corps célestes en général*[71] and, then, in *Découverte d'un nouveau principe de Mécanique*[72]. In these papers, Euler used an extrinsic references frames (a system of three orthogonal Cartesian axes) and formulated the second law of motion in this way:

$$(2) \qquad 2Mddx=Pdt^2,\ 2Mddy=Qdt^2,\ 2Mddz=Rdt^2,$$

where M is the mass and P, Q, and R the components of the force on the axis (the coefficient 2 depended on the unity of measure). In particular, in his (1750), Euler applied (2) to continuum mechanics: he stated that a physical continuum could be subdivided into elementary particles and one could apply differential equations (2) to these elementary particles (the mass M

69 The use of the differential calculus in mechanics was criticized in England. For instance, Benjamin Robins wrote: "I have no design to charge this author [Euler] with haste or negligence on account of these errors; but I consider them solely, as the effect of that inaccuracy in conception, to which the differential calculus is disposed to betray its admirers ... In the beginning of the third chapter, which treats of right-lined motion, Mister Euler has given Galileo's theory of falling bodies, in its own nature no difficult subject; but it is here so compounded with differential computations, that this subject may be much better learned from what has been writ in a more simple manner by others" (ROBINS, Benjamin, "Remarks on Mr. Euler's Treatise of Motion", in *Mathematical Tracts of the late Robins, Benjamin …* , edited by J. Wilson, London, J. Nourse: 1761,, 197-221, on 203 and 205.)

70 MARONNE, Sébastien and PANZA, Marco (2010) "Newton and Euler", in Mandelbrot, Scott and PULTE, Helmut (eds.) *The reception of Isaac Newton in the European Enlightenment*, (2 vols.), London, Continuum, to appear.

71 EULER, Leonhard (1747) *Recherches sur le mouvement des corps célestes en général, Mémoires de l'académie des sciences de Berlin* 3, 93-143 or *Opera,* ser. 2, vol. 25, 1 – 44.

72 EULER, Leonhard (1750) "Découverte d'un nouveau principe de Mécanique", *Mém.*, 6, 185-217 or *Opera*, ser. 2, vol. 5, 81 - 108

could also be an infinitesimal quantity). In his opinion, this fact constituted a new and fundamental principle of mechanics and any other principles or law of mechanics could be derived from it: mechanical problems could be formulated in a general, analytical way by means of an appropriate application of (2).

Later, in 1765, Euler introduced the concept of moment of inertia of a rigid body and decomposed the motion into the rectilinear motion of the centre of mass and the rotational motion about the centre of mass[73]; in 1775, he completed the construction of general equations of dynamics by formulating a system of six equations determining the motion of any body, which (except for an additional coefficient) he wrote in this way[74]:

$$\int dM\,\frac{d^2x}{dt^2}=P,\quad \int dM\,\frac{d^2y}{dt^2}=Q,\quad \int dM\,\frac{d^2z}{dt^2}=P,$$

$$\int zdM\,\frac{d^2y}{dt^2}-\int ydM\,\frac{d^2z}{dt^2}=S,\quad \int xdM\,\frac{d^2z}{dt^2}-\int zdM\,\frac{d^2y}{dt^2}=T,\quad \int ydM\,\frac{d^2x}{dt^2}-\int xdM\,\frac{d^2y}{dt^2}=U.$$

In his attempt to transform mechanics into a field of analysis Euler followed different paths[75] and dealt with all branches of mechanics. For

73 EULER, Leonhard (1765) *Theoria motus corporum solidorum seu rigidorum ex primis nostrae cognitionis principiis stabilita et ad omnes motus, qui in hujusmodi corpora cadere possunt, accommodata*, Rostochii et Gryphiswaldiae litteris et impensis A. F. Röse.

74 EULER, Leonhard (1775) "Nova methodus motum corporum solidorum rigidorum determinandi", *Novi Commentarii academiae scientiarum Petropolitanae* (afterwards: *Novi Comm.*) 20, 208-238 or *Opera*, ser. 2; vol. 9, 99 – 125.

75 I limit myself to mentioning the use of variational principles, which allowed Euler to pass from a geometric-based study of concrete particular system to an analytical treatment of any sort of systems based on a unique and general equation. In 1744 Euler published a fundamental book on the calculus of variation, where he faced "the method of finding curved lines that enjoy some maximum and minimum property, or solution of isoperimetric problems in the broadest accepted sense" and obtained the so-called Euler-Lagrange equation (see EULER, Leonhard (1744), *Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive solutio problematis isoperimetrici latissimo sensu accepti,* Lausannae, M. M. Bousquet et Soc., or *Opera*, ser. 1, vol. 24). In *Methodus inveniendi* Euler approached the question in an geometric way; however he stated that the problem of finding the curved line which enjoys some maximum and minimum property can be formulated analytically and so transformed into the problem of finding the function between *x* and *y* which enjoys some maximum and minimum property. In chapter 4 of the treatise, he even showed how the basic variational problem and its solution could be interpreted analytically. Euler (1744: 14) also called for the development of a simple method or an algorithm to obtain variational equations. This algorithm was developed by Lagrange, who recognized the dual usage of the symbol *dy* in *Methodus inveniendi*, where Euler denoted both the differential *dy* of *y* with respect to *x* and the variation of the curve *y(x)*. Euler immediately accepted Lagrange's presentation of the

instance, as concerns fluid mechanics[76], Euler considered the mass of the fluid as composed by three-dimensional infinitesimal parallelepipeds: so, he could to express the components of the force acting on the element of volume *dxdydz* as *Pqdxdydz*, *Qqdxdydz* and *Rqdxdydz*, where *R, Q, P* are the components of forces acting on the elementary parallelepiped with one corner at the point *Z* of coordinates *x, y, z* and with dimensions *dx*, *dy* , *dz*, and *q* is the body density[77]. By assuming that, during the time *dt*, the element of fluid at the point *Z* is carried to a point *Z′* of coordinates *x+udt*, *y+vdt*, *z+wdt* (*w, v, u* are the components of the velocity of the fluid element that is at point *Z*) and that the pressure is *p*, Euler succeeded in finding the differential equations of fluid motion and continuity[78]:

$$P-\frac{1}{q}\frac{\partial p}{\partial x}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

$$Q-\frac{1}{q}\frac{\partial p}{\partial y}=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}$$

$$R-\frac{1}{q}\frac{\partial p}{\partial z}=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}$$

$$\frac{\partial q}{\partial t}+\frac{\partial qu}{\partial x}+\frac{\partial qv}{\partial y}+\frac{\partial qw}{\partial z}=0.$$

These equations, as well as equations (2), show an important aspect of the process towards an analytical mechanics: the rise of the calculus of functions of several variables. This was one of novelty of Euler's analysis. However, Euler was unable to insert the calculus of the function of several variables in his program in organic way and this was one of the limits of his analytical program.

5.-The legacy of Euler's program.

In conclusion of this paper I discuss the legacy of Euler's analytical program on nineteenth and twentieth-century mathematics.

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calculus of variation and following it in the appendix of the *Institutionum calculi integralis.*

76 On p. 274 of a paper presented to the Berlin Academy in 1755 and published in 1757, Euler stated: "I hope to emerge successful at the end, so that if difficulties remain they will not be in the field of mechanics, but entirely in the field of analysis" (EULER, Leonhard (1755b) "Principes généraux du mouvement des fluides", *Mémoires de l'académie des sciences de Berlin*, (afterwards: *Mém.*) vol. 11, 274-315 or *Opera*, ser. 2, vol. 12, 54 – 91.

77 See, e. g., EULER, Leonhard (1755c) "Principes généraux de l'état d'équilibre des fluides", *Mém.*, vol. 11, 217-273 or *Opera*, ser. 2, vol. 12, 2 – 53 (in particular. 227 ff.).

78 EULER, Leonhard (1756-57) "Principia motus fluidorum", *Novi Comm.* 6, 271-311 or *Opera*, ser. 2, vol. 12, 133 - 168.

In section 1, I pointed out that the starting-point of Euler's analytical program was the elimination of diagrammatic and empirical evidence in the derivation of analytical propositions and the idea that analysis could be developed as an abstract, conceptual, and merely discursive theory, which did not rely on the material. In Euler's opinion, analysis was methodologically different from geometry and other parts of mathematics. Today, all mathematical theories are thought to be sets of propositions derived from explicit and arbitrary axioms by means of accepted rules of derivations: intuitive considerations, linked to the observation of diagram or other material objects, cannot be used to derive a theorem. Nowadays all mathematical theories are analytical in the sense Euler used this term, namely all mathematical theories (even modern synthetic geometry) developed in an abstract, conceptual, and merely discursive way without relying on a diagrammatic representation: a theory that unloads a part of reasoning on diagrams (as ancient geometry) is merely considered as a non-mathematical theory. For this reason, one can state that the whole modern mathematics presupposes Euler's analytical program and all parts of modern mathematics are analytical in Euler's sense: the whole modern mathematics (not only modern analysis) has to be considered as the heir of Euler's analytical program.

At the same time, it also clear that the foundations of modern mathematics and, in particular, of modern real analysis are entirely different from the foundations of Euler's analysis. Thus, the idea of analysis as a step-by-step construction based upon a mere generalization of arithmetic rules is no longer accepted and, in any case, the tools Euler used to carry out his program (for instance, the concept of abstract quantity and his notion of a function) have been rejected or have deeply been modified. The decline of Euler's foundations of analysis has often attributed to the lack of rigour (*e.g.*, the question of divergent series) and the consequent rise of Cauchy's rigorous mathematics; but this opinion did not grasp the core of the problem. Indeed, the crisis of Euler's foundations of analysis was mainly due to the fact these foundations failed in reaching their fundamental objective, the realization of the analytical program, namely Euler's foundations of analysis did not make possible the reorganization of mathematics around analysis, the latter being intended as a theory that did not rely upon geometric concepts[79].

Already around 1750, Euler's concepts of functions, series, differentia-

79 For a discussion of the decline of eighteenth formal methodology, see FERRARO (2008a, Part IV).

tion, and integration showed strong insufficiencies. For instance, the concept of functions was too restricted and not apt to the analytical investigation of natural phenomena[80]. In effect, if one followed Euler's foundations, it was impossible to provide a coherent theory that went beyond the restricted domain of elementary functions and their power series (see section 3). At the eighteenth century and the beginnings of the nineteenth century, Euler's concept of analysis was no longer sufficient for the needs of astronomy, probability, physics, etc. These sciences required the mathematical investigations of new kind of transcendental functions and the introduction of new methods that allowed giving a better interpretation to certain new objects, such as trigonometric series and partial differential equations. In other terms, the basic notions Euler used to carry out his analytical program led to a substantial failure of the program itself.

Euler was aware of the fact that analysis was not sufficiently developed theory at his time; he felt the need to introduce new mathematical objects in analysis so to obtain a complete analytical theory apt to mathematize the physical sciences. However, he believed that he had already determined essential parts of such a complete analytical theory: it should be sufficient to continue to develop his program according to the path he had shown to reach the final objective. In other terms, new chapters of the analytical theory could be added to the original theory without changing the foundations of analysis. Euler made various attempts to enlarge the domain of analysis adding new topics, such as trigonometric series or new kinds of functions. Unfortunately, these additions to old theory were not well integrated and often assumed the aspect of *ad hoc* arguments. For instance, this occurred in the case of the introduction of discontinuous functions (in Euler's sense, namely quantities that did not have an analytical expression or that were analytically expressed by means of more than one analytical expression). While D'Alembert rejected the use of discontinuous functions, since they were not coherent with the notion of a function as a single closed analytical expression, Euler accepted them since their rejection would have impeded the growth of mathematics and its capacity to interpret physical phenomena. However, Euler did not changed the old concept of a function as a single analytical expression, rather he merely added the new discontinuous functions to old functions without making the old and new functions compatible each others. As a result, the

80 See FERRARO (2000a) and (2007a).

discontinuous functions never entered in Euler's analysis as function in the proper sense of the term and Euler was never able to use them adequately[81]. The same thing occurred as concerns the gamma and beta functions[82] and elliptic integrals[83].

At the end of the eighteenth century, the process of construction of analysis, accordingly Euler's foundations, led to a poorly coherent construction. The structure of analysis retained the theory of elementary functions and its formal principles while the various attempts to extent its subject-matter lacked any adequate systematisation and were often supported by extra-analytical considerations[84]. Even the starting-point of Euler's program, the separation of analysis from geometry, did not reach to its natural conclusion and geometric principles were implicitly (and sometimes explicitly) used in analysis[85]. Thus, at the start of the nineteenth century, Bolzano was forced to observe that there were demonstrations of analytical theorems that depend "on a truth borrowed from geometry"; he even claimed that it was "an intolerable offence against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry"[86].

The last statement almost seems an Eulerian one: in effect, it expresses basic features of Euler's program very well. Thus Bolzano's innovative proof of the intermediate value theorem can be viewed as an attempt to carry out Euler's program, rather an attempt to reject it. Bolzano's proof does imply new foundations for mathematics but these new foundations seem to be a

81 See FERRARO (2000a).

82 See DELSHAMS, Amadeu and MASSA ESTEVE, Mª Rosa (2008) "Consideracions al voltant de la Funció Beta a l'obra de Leonhard Euler (1707-1783) *Quaderns d'Història de l'Enginyeria*, 9, 59-82; MASSA ESTEVE, Mª Rosa and DELSHAMS, Amadeu (2009) "Euler's beta integral in Pietro Mengoli's works", *Archive for History of Exact Sciences*, 63, 325-356.

83 See FERRARO (2008b : §. 3).

84 For instance, in FERRARO (2008b), I showed that in investigating gamma and beta functions, Euler used definite integration, but when he was forced to give the properties of definite integration he resorted to the geometric interpretation of the integral (namely, he conceived the integral as an area and referred to a diagram).

85 This also occurred because, as shown in my (2000a, 120-121), some Eulerian notions had an intrinsic geometric nature and the use of the symbolic representation (namely, by means of algebraic symbols) instead of the diagrammatic one hid but did not eliminate.

86 BOLZANO, Bernard (1817) *Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegengesetzes Resultat gewaehren, wenigstens eine reele Wurzel der Gleichung liege*, Prague, Enders. English translation in RUSS, Steve (1980) "A Translation of Bolzano's Paper on the Intermediate Value Theorem", *Historia Mathematica* 7, 156–185, in particular 157.

tool to save the core of Euler's analytical program.

The same thing is true for Gauss[87] and Cauchy[88]. These two mathematicians changed the foundation of analysis in a radical way; however, both Gauss and Cauchy saved the concept of analysis as an abstract, conceptual, and merely discursive theory and the idea that analysis had to be the main instrument to investigate geometric and mechanical objects. In this sense (but only in this sense), Gauss and Cauchy were Eulerian.

87 On Gauss's re-thinking of the fabric of eighteenth-century analysis in order to move beyond the restricted domain of Eulerian functions, see FERRARO (2007a).

88 On Cauchy's rejection of eighteenth-century theory of series, see FERRARO (2008a, chapter 33).