Large digraphs of given diameter and degree from coverings

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Abstract

We show that a construction of Comellas and Fiol for large vertex-transitive digraphs of given degree and diameter from small digraphs preserves the properties of being a Cayley digraph and being a regular covering.

1 Introduction

There age numerous ways to construct 'large' digraphs of a given in- and out-degree and a given diameter; we refer to [5] for a fairly recent survey. The two most prominent contribution for digraphs that, in addition, are *vertex-transitive*, are the constructions by Comellas and Fiol [1], Gómez [3], and Faber, Moore and Chen [2].

Our interest in these constructions has been motivated by the remark in [5] to the extent that *graph coverings* appear to be an appropriate language for presenting a number of constructions in the degree-diameter problem. Two constructions of Comellas and Fiol have already been studied from this point of view in [6]. For completeness we note that digraphs and graphs of Faber, Moore and Chen have been studied in depth in [4].

The underlying idea of all constructions of [1] is to take a 'small' digraph as input and by a kind of 'composition', as the operation is called in [1], produce a 'large' output digraph that depends on various parameters. Here we examine a construction of the paper [1] that has not been considered in [6], with the aim to show that the construction preserves coverings and the property of being Cayley in the sense that if the input digraph has the property, then so does the output digraph.

2 Constructions

Throughout, we assume that the reader is familiar with the concepts of a Cayley digraph as well as of regular coverings of digraphs and their description in the language of voltage assignments; we recommend [5] for details especially in connection with the directed version of the degree-diameter problem.

We begin with a description of the construction of Comellas and Fiol [1] which will be considered from now on. Let G be a digraph with vertex set V and dart set D, the *input digraph* for the construction. Let m and $n \geq 2$ be positive integers such that m is divisible by n. Further, let ℓ be a fixed element of the cyclic group Z_m such that $\ell \neq 0, 1$. The *output digraph* G' will have vertex set V' and dart set D' defined as follows. The vertex set V' consists of all ordered $(n+1)$-tuples $(p_0p_1 \dots p_{n-1}|j)$ such that $j \in Z_m$ and $p_i \in V$ for all i, $0 \leq i \leq n-1$. The dart set D' consists precisely of the darts of the form

$$(p_0p_1 \dots p_{j-1}\, u\, p_{j+1} \dots p_{n-1}|j) \rightarrow \begin{cases} (p_0p_1 \dots p_{j-1}\ v\ p_{j+1} \dots p_{n-1}|(j+1)) \\ (p_0p_1 \dots p_{j-1}\ u\ p_{j+1} \dots p_{n-1}|(j+\ell)), \end{cases}$$

whenever v is adjacent from u in G.

The role of this construction in the directed version of the degree-diameter problem is clear from the results of [1] which imply that if the input digraph G of order, say, c, is vertex-transitive, regular of degree d, and k-reachable, meaning that any ordered pair of vertices is connected by a directed walk of length precisely k, then the output digraph G' is vertex transitive, regular of degree $d+1$, of order mc^n and diameter at most $kn+b$ where b is the diameter of the Cayley digraph $Cay(Z_m, \{1, \ell\})$.

For our purposes it will be more convenient to work with an isomorphic image of G', defined by means of the bijection $\varphi : V' \rightarrow V'$ given by

$$(p_0p_1 \dots p_{j-1}\ u\ p_{j+1} \dots p_{n-1}|j) \mapsto (u\ p_{j+1} \dots p_{n-1}p_0p_1 \dots p_{j-1}|j)\ .$$

The isomorphic copy $G^* = \varphi(G')$ of G' thus has the same vertex set $V^* = V'$ but all the darts in its dart set D^* have the form

$$(p_0p_1 \dots p_{n-1}|j) \rightarrow \begin{cases} (p_1 \dots p_{n-1}q_0|(j+1)) \\ (p_\ell p_{\ell+1}p_{\ell+2} \dots p_{\ell-1}|(j+\ell)) \end{cases}$$

where q_0 is adjacent from p_0 in G and, as before, ℓ is a fixed element of Z_m such that $\ell \neq 0, 1$.

In order to have a more explicit notation, we denote the digraph G^* by writing $G^* = CF(G, n, m, \ell)$, where CF stands for Comellas-Fiol and G, n, m and ℓ are the parameters upon which the construction of G^* depends.

We note that a similar amendment of the description of this construction of Comellas and Fiol was suggested by Gómez [3] but the isomorphism proposed in [3] appears to be inconsistent with the actual graph description given therein.

3 Preservation of Cayley digraphs

We are now in position to show that the construction described in the previous section preserves the property of being Cayley in the sense outlined in the Introduction. We keep to all the notation introduced earlier.

Theorem 1 *If G is a Cayley digraph, then $G^* = CF(G, n, m, \ell)$ is a Cayley digraphs as well.*

Proof: Let $G = Cay(H, X)$ be a Cayley digraph for a group H and a generating set X. Let H^* be the semidirect product of $H^n = H \times H \times \ldots \times H$ (n times) by Z_m, with elements of the form $(a; j)$ where $a = (a_0, \ldots, a_{n-1}) \in H^n$ and $j \in Z_m$ and with the action j of Z_m on H^n defined by

$$j(a) = (a_j, a_{j+1}, \ldots, a_{n-1}, a_0, \ldots, a_{j-1})$$

for any $a = (a_0, \ldots, a_{n-1}) \in H^n$; at this point we recall that m is a multiple of n and hence the action is well defined. Multiplication in the semidirect product $H^* = H^n \rtimes Z_m$ is given by

$$(a; j)(a'; j') = (j'(a).a'; j + j') \quad . \tag{1}$$

Now, for any $x \in X$ we let $x^* = (e, e, \ldots, e, x) \in H^n$ where e is the unit element of A; also, let $e^* = (e, \ldots, e) \in H^n$. Finally, define $X^* = \{(x^*; 1), x \in X\} \cup \{(e^*, \ell)\}$ where $\ell \in Z_m$ is the fixed element different from 0 and 1. Note that the action of $j = 1 \in Z_m$ on H^n is given by $j(a) = 1(j_0, \ldots, j_{n-1}) = (j_1, \ldots, j_{n-1}, j_0)$.

We show that the Cayley digraph $Cay(H^*, X^*)$ is isomorphic to the digraph G^* for $G = Cay(H, X)$. The key is to observe that, for any $x \in X$,

$$\begin{aligned} 1(a).x^* &= 1(a_0, \ldots, a_{n-1})(e, \ldots, e, x) \\ &= (a_1, \ldots, a_{n-1}, a_0 x) \end{aligned}$$

On the other hand, right multiplication by $(x^*, 1) \in X^*$ gives:

$$\begin{aligned}(a;j)(x^*;1) &= (1(a).x^*; j+1)\\ &= ((a_1,\ldots,a_{n-1},a_0).(e,\ldots,e,x); j+1)\\ &= (a_1,\ldots,a_{n-1},a_0x; j+1).\end{aligned}$$

It follows that for any $(x^*, 1) \in X^*$ the vertex $(a; j)$ is adjacent to the vertex $(a; j)(x^*; 1) = (1(a).x^*; j + 1)$ in the Cayley digraph $Cay(H^*, X^*)$. This, however, is precisely the first adjacency rule in the definition of the output digraph G^*. Similarly, $(a; j)$ is in the Cayley digraph adjacent to the vertex $(a; j)(e^*; \ell) = (\ell(a).e^*; j + \ell)$, which gives the second adjacency rule for G^*. Consequently, G^* is isomorphic to the Cayley digraph $Cay(H^*, X^*)$ if the input digraph G is a Cayley digraph $Cay(H, X)$, as claimed. □

4 Preservation of coverings

We continue with the result regarding preservation of regular coverings.

Theorem 2 *If G regularly covers a digraph of a smaller order, then so does $G^* = CF(G, n, m, \ell)$.*

Proof: Let G be a regular covering space of a digraph of order smaller than the order of G. Equivalently, we assume that G is a lift of a base digraph $J = (V_J, D_J)$ by a voltage assignment α in some non-trivial group A.

We first introduce a new base digraph $L = (V_L, D_L)$. Its vertex set V_L will be the set $\{(r_0, \ldots, r_{n-1}); r_i \in V_J)\}$, where $i \in \{0, \ldots, n-1\}$. If there exists an arc b from a vertex $r' \in V_J$ to a vertex $s' \in V_J$ in the digraph J, then incidence in the digraph L is defined by the following rule. For every arc $b \in D_J$ from r' to s' in J and for every $(n-1)$-tuple $(r_1, \ldots, r_{n-1})$ of vertices in J there will be a dart labeled $\tilde{b} \in D_L$ from the vertex $r = (r', r_1, \ldots, r_{n-1}) \in V_L$ to the vertex $s = (r_1, \ldots, r_{n-1}, s') \in V_L$. Furthermore, for each vertex $r = (r_0, \ldots, r_{n-1}) \in V_L$ we include a dart $\tilde{c}_r$ from r to the vertex $r^* = (r_\ell, r_{\ell+1}, \ldots, r_{\ell-1})$.

Let us now introduce a voltage assignment β on L in the group $A^* = A^n \rtimes Z_m$, with the cyclic group Z_m acting on the n-fold direct product $A^n = A \times A \times \ldots \times A$ in the same way as it acted on H in the previous section.

Consider a dart $\tilde{b}: \ r \longrightarrow s$ of L that has originally come from a dart $b: \ r' \longrightarrow s'$ in J carrying voltage $\alpha(b)$. The new voltage of $\tilde{b}$ will be $\beta(\tilde{b}) = (e, \ldots, e, \alpha(b); 1)$; moreover, the darts $\tilde{c}_r$ will receive the voltage $\beta(\tilde{c}_r) = (e, \ldots, e|\ell)$.

Let us denote the lift of L with respect to this voltage assignment by L^β. By definition of a lift, vertices and darts of L^β have the form (r, a) and $(\tilde{b}, a)$, $(\tilde{c}_r, a)$ where $r \in V_L$, $\tilde{b} \in D_L$ and $a \in A^*$. A dart $(\tilde{b}, a)$ emanates from (r, a) and terminates at $(s, a.\beta(\tilde{b}))$ while a dart $(\tilde{c}_r, a)$ emanates from (r, a) and terminates at $(r^*, a.\beta(\tilde{c}))$. Note that the last two products evaluate as

$$\begin{aligned} a.\beta(\tilde{b}) &= (a_0, a_1, \ldots, a_{n-1}|j).(e, \ldots, e, \alpha(b)|1) \\ &= ((a_1, \ldots, a_{n-1}, a_0), (e, \ldots, e, \alpha(b))|j+1)) \\ &= (a_1, \ldots, a_{n-1}, a_0.\alpha(b)|j+1). \end{aligned}$$

and

$$\begin{aligned} a.\beta(\tilde{c}) &= (a_0, a_1, \ldots, a_{n-1})\beta(\tilde{c}) \\ &= (a_0, a_1, \ldots, a_{n-1})(e, \ldots, e|\ell) \\ &= (a_\ell, a_{\ell+1}, \ldots, a_{\ell-1}). \end{aligned}$$

We prove that the digraphs G^* and L^β are isomorphic. Having assumed that G is a lift of J by the voltage assignment α in the group A, it follows that the first type of incidence in the definition of the Comellas-Fiol digraph G^* can be described in the form

$$((r_0, a_0) \ldots (r_{n-1}, a_{n-1})|j) \longrightarrow ((r_1, a_1) \ldots (r_{n-1}, a_{n-1})(s_0, a_0.\alpha(b))|j+1)$$

for any $a_i \in A$, $0 \le i \le n-1$, whenever b is a dart from r_0 to s_0 in J. It is now a matter of routine to check that the mapping

$$((r_0, a_0)(r_1, a_1) \ldots (r_{n-1}, a_{n-1})|j) \mapsto ((r_0, \ldots, r_{n-1})(a_0, \ldots, a_{n-1})|j)$$

is an isomorphism from G^* onto L^β. □

In conclusion, let us note that although it is true that every Cayley digraph is a regular covering of a one-vertex digraph, our Theorem 2 does *not* imply Theorem 1 because, in the proof of Theorem 2, the digraph L^* need not be a one-vertex digraph even when J is.

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