Infinite families of 3-numerical semigroups with arithmetic-like links

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Abstract

Let $S = \langle a, b, N \rangle$ be a numerical semigroup generated by $a, b, N \in \mathbb{N}$ with $1 < a < b < N$ and $\gcd(a, b, N) = 1$. The *conductor* of S, denoted by $\mathrm{c}(S)$ or $\mathrm{c}(a, b, N)$, is the minimum element of S such that $\mathrm{c}(S) + m \in S$ for all $m \in \mathbb{N} \cup \{0\}$. Some arithmetic-like links between 3-numerical semigroups were remarked by V. Arnold. For instance he gave links of the form

$$\frac{\mathrm{c}(13,32,52)}{\mathrm{c}(13,33,51)} = \frac{\mathrm{c}(9,43,45)}{\mathrm{c}(9,42,46)} = \frac{\mathrm{c}(5,35,37)}{\mathrm{c}(5,34,38)} = 2 \quad \text{or} \quad \frac{\mathrm{c}(4,20,73)}{\mathrm{c}(4,19,74)} = 4.$$

In this work several infinite families of 3-numerical semigroups with similar properties are given. These families have been found using a plane geometrical approach, known as L-shaped tile, that can be related to a 3-numerical semigroup. This tile defines a plane tessellation that gives information on the related semigroup.

1 Introduction and known results

A 3-*semigroup* $S = \langle a, b, N \rangle$ with $a, b, N \in \mathbb{N}$ and $1 < a < b < N$, is defined as $\langle a, b, N \rangle = \{m \in \mathbb{N} \mid m = xa + yb + zN;\ x, y, z \in \mathbb{N}\}$. The values a, b and N are called the generators of S. The set $\overline{S} = \mathbb{N} \setminus S$ is called the set of *gaps* of S. If the cardinality of $\overline{S}$ is finite, then S is a 3-*numerical semigroup*. It is well known that S is a 3-numerical semigroup if and only if $\gcd(a, b, N) = 1$. The *Frobenius Number* of S is the value $\mathfrak{f}(S) = \max \overline{S}$. The *conductor* of S is the value $\mathrm{c}(S) = \mathfrak{f}(S) + 1$. Given $m \in S \setminus \{0\}$, the *Apéry set* of S with respect to m, $\mathrm{Ap}(S,m) = \{s \in S \mid s - m \notin S\}$, contains significant information of S. In particular, it is well known that $\mathfrak{f}(S) = \max \mathrm{Ap}(S,m) - m$. A 3-numerical semigroup $S = \langle a, b, N\rangle$ is *minimally generated* if the semigroups $\langle a, b\rangle$, $\langle a, N\rangle$ or $\langle b, N\rangle$ are proper subsets of S. You can find recent results on numerical semigroups in the book of Rosales and García-Sánchez [6]. Recent results mainly related on the Frobenius number can be found in the book of Ramírez Alfonsín [4].

The equivalence class of m modulo N will be denoted by $[m]_N$. A *weighted double-loop* digraph $G(N; a, b; \mathfrak{a}, \mathfrak{b})$ is a directed graph with set of vertices $V(G) = \{[0]_N, ..., [N-1]_N\}$ and set of weighted arcs $A(G) = \{[v]_N \xrightarrow{\mathfrak{a}} [v+a]_N, [v]_N \xrightarrow{\mathfrak{b}} [v+b]_N \mid [v]_N \in V(G)\}$. The idea of using weighted double-loop digraphs as a tool in the study of the Frobenius number of 3-numerical semigroups was already used by Selmer [8] in 1977 and Rødseth [5] in 1978.

Each weighted double-loop digraph G has related several *minimum distance diagrams* (MDD for short) that periodically tessellates the squared plane. Each vertex $[ia + jb]_N$ of G is associated with the unit square of the plane $(i, j) \in \mathbb{N}^2$, that is the interval $[i, i+1] \times [j, j+1] \in \mathbb{R}^2$. An MDD is composed by N unit squares and has a geometrical shape like the (capital) letter 'L' or it is a rectangle (that is considered a degenerated L-shape), see [5, 3] for more details. Sabariego and Santos [7] gave an algebraic characterization of these diagrams in any dimension. Here we include this characterization in two dimensions.

Definition 1 [Sabariego and Santos, [7]] A minimum distance diagram is any map $D : \mathbb{Z}_N \to \mathbb{N}^2$ with the following two properties:

(a) For every $[m]_N \in \mathbb{Z}_N$, $D([m]_N) = (i, j)$ satisfies $ia + jb \equiv m \pmod N$ and $\|D([m]_N)\|$ is minimum among all the vectors in $\mathbb{N}^2$ with that property ($\|(s, t)\| = s\mathfrak{a} + t\mathfrak{b}$).

(b) For every $[m]_N$ and for every $(s, t) \in \mathbb{N}^2$ that is coordinate-wise smaller than $D([m]_N)$, we have $(s, t) = D([n]_N)$ for some $[n]_N$ (with $n \equiv sa + tb \pmod N$).

An MDD $\mathcal{H}$ is denoted by the lengths of his sides, $\mathcal{H} = L(l, h, w, y)$, with $0 \le w < l$, $0 \le y < h$, $\gcd(l, h, w, y) = 1$ and $lh - wy = N$, as it is depicted in the Figure 1. The vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ define the tessellation of the plane by the L-shaped tile $\mathcal{H}$. These lengths fulfill the *compatibility*

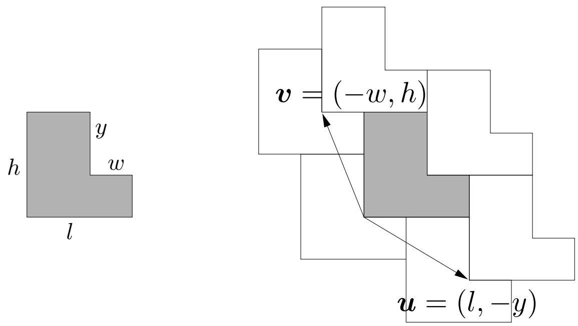

Figure 1: Generic MDD tessellating the plane

equations, stated by Fiol, Yebra, Alegre and Valero [3] in 1987, related to the tessellation

$$la - yb \equiv 0 \pmod N, \quad -wa + hb \equiv 0 \pmod N. \tag{1}$$

Definition 2 [Tessellation related to S] Let $S = \langle a, b, N\rangle$ be a 3-numerical semigroup. A tessellation related to S is a tessellation of the plane generated by an L-shaped MDD of the weighted double-loop digraph $G(N; a, b; a, b)$.

Let D be the map that appears in Definition 1 associated with $G = G(N; a, b; a, b)$, that is $\mathfrak{a} = a$ and $\mathfrak{b} = b$. Then

$$\mathrm{Ap}(S, N) = \{D([0]_N), ..., D([N-1]_N)\}$$

and $D([m]_N)$ can be though as the length of a minimum path from $[0]_N$ to $[m]_N$ in G. Definition 2 gives a metrical view of some properties of S. A geometrical characterization of MDD related to S is needed for practical reasons. This characterization is given in the following result.

Theorem 3 (A., Miralles and Zaragozá, [1]) *The L-shaped tile $\mathcal{H} = L(l, h, w, y)$ satisfying (1) with $lh - wy = N$ and $\gcd(l, h, w, y) = 1$ is related to $S = \langle a, b, N\rangle$ iff $la \geq yb$ and $hb \geq wa$ and both equalities are not satisfied.*

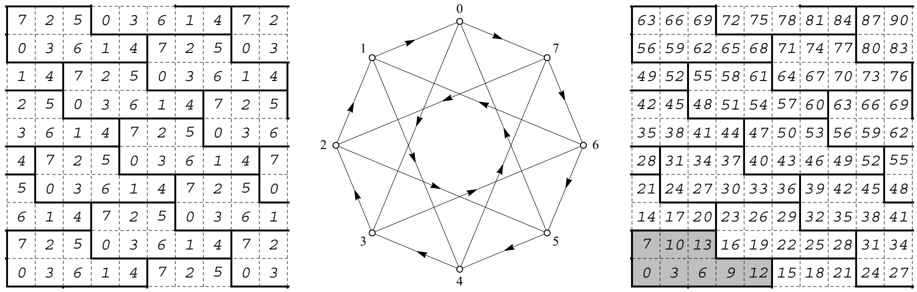

Figure 2: Minimum distance diagram related to $G(8;3,7;3,7)$

Example 4 Consider the weighted double-loop digraph $G=G(8;3,7;3,7)$ that is depicted in the Figure 2. An L-shaped MDD related to G is $\mathcal{H}=L(5,2,2,1)$. Note that the lengths of $\mathcal{H}$, $(l,h,w,y)=(5,2,2,1)$, fulfill the conditions $\gcd(l,h,w,y)=1$ and $lh-wy=N$, the compatibility equations (1) and Theorem 3. The left-hand side of Figure 2 shows a piece of the first quadrant of the squared plane and how $\mathcal{H}$ tessellates the plane. It also shows the periodic distribution of the equivalence classes modulo 8, where each unit square (i,j) is labelled by the class $[3i+7j]_8$. The right-hand side of this figure shows the same piece of the first quadrant, however each unit square (i,j) is labelled now by $\|D([3i+7j]_8)\|=3i+7j$ (D is the map of Definition 1). Note that the labels inside the grey L-shape (the one that contains the unit square $(0,0)$) form the set $\mathrm{Ap}(\langle 3,7,8\rangle,8)$. In particular, we have $\mathfrak{f}(\langle 3,7,8\rangle)=13-8=5$.

V. Arnold [2] in 2009 commented that his *1999 calculations of Frobenius numbers* provided hundreds of empirical properties. He remarked some *strange arithmetical facts* like

$$\frac{\mathrm{c}(13,32,52)}{\mathrm{c}(13,33,51)}=\frac{\mathrm{c}(9,43,45)}{\mathrm{c}(9,42,46)}=\frac{\mathrm{c}(5,35,37)}{\mathrm{c}(5,34,38)}=2,\quad \frac{\mathrm{c}(4,20,73)}{\mathrm{c}(4,19,74)}=4. \tag{2}$$

It was shown in [1] that if $\mathcal{H}=L(l,h,w,y)$ is related to $S=\langle a,b,N\rangle$, then the Frobenius number is

$$\mathfrak{f}(\langle a,b,N\rangle)=\max\{(l-1)a+(h-y-1)b,(l-w-1)a+(h-1)b\}-N. \tag{3}$$

Therefore, from the identities $\mathrm{c}(S)=\mathfrak{f}(S)+1$ and (3), arithmetic-like links between conductors as those appearing in (2) can be though as geometrical-like relations between related L-shaped MDD tiles.

When the semigroup is 2-minimally generated, that is $S=\langle a,b\rangle$ with $\gcd(a,b)=1$, it is well known that his Frobenius number is

$$\mathfrak{f}(\langle a,b\rangle)=ab-a-b. \tag{4}$$

Although this result was published by Sylvester [9] in 1884, it seems to be true that (4) was given first by Frobenius in his lectures. Therefore, the conductor is given by the expression $\mathrm{c}(a,b)=\mathfrak{f}(\langle a,b\rangle)+1=(a-1)(b-1)$.

In this work, several infinite families of pairs of 3-numerical semigroups are given such that each pair fulfills a (2)-like relation.

2 Computer assisted numerical remarks

Properties in (2) suggest looking for semigroups like

$$\frac{\mathrm{c}(\alpha,n,m)}{\mathrm{c}(\alpha,n-1,m+1)}=k, \tag{5}$$

where $\langle\alpha,n,m\rangle$ and $\langle\alpha,n-1,m+1\rangle$ are 2 and 3 minimally generated numerical semigroups respectively, for different natural numbers n and m and fixed values of α and k.

α \ k	1	2	3
4	11	0	0
5	0	109	0
6	4	0	1
7	0	55	6
8	4	0	1
9	5	13	3
10	2	0	1

Table 1: Cardinalities of some sets $P(\alpha,k,100)$

Let us consider the set

$$P(\alpha,k,\ell)=\{\langle\alpha,n,m\rangle \mid \frac{\mathrm{c}(\alpha,n,m)}{\mathrm{c}(\alpha,n-1,m+1)}=k,\;\; m\leq\ell\}$$

where $\langle \alpha, n, m\rangle$ and $\langle \alpha, n-1, m+1\rangle$ are 2 and 3 minimally generated. A computer search reveals the cardinality of some sets $P(\alpha, k, 100)$. These cardinalities are included in Table 1.

Let us consider now the set $Q(\alpha, k, \ell)$, defined as $P(\alpha, k, \ell)$ but now both semigroups $\langle \alpha, n, m\rangle$ and $\langle \alpha, n-1, m+1\rangle$ are 3-minimally generated. The cardinalities of $Q(\alpha, 1, 100)$, with $\alpha = 4, \ldots, 10$, are $276, 5, 0, 15, 0, 218$ and 4, respectively. We have now $Q(\alpha, k, 100) = \emptyset$ for $(\alpha, k) \in \{4, \ldots, 10\} \times \{2, 3\}$. Let us denote the sets

$$P(\alpha, k) = \bigcup_{\ell \geq \alpha+2} P(\alpha, k, \ell) \text{ and } Q(\alpha, k) = \bigcup_{\ell \geq \alpha+2} Q(\alpha, k, \ell).$$

We use the numerical data of this section to search infinite families of pairs of semigroups belonging to $P(\alpha, k)$ or $Q(\alpha, k)$, for some values of α and k.

3 Infinite families

In this section we use the L-shaped tile technique included in Section 1 for finding infinite families of 3-numerical semigroups that belong to $P(4, 1)$, $P(7, 3)$ and $Q(9, 1)$.

Theorem 5 *Let us consider the* 3*-numerical semigroups* $S_t = \langle 4,\, 4t+3,\, 8t+6\rangle$ *for* $t \geq 1$*. Then* $\{S_t\}_{t\geq 1} \subset P(4, 1)$*.*

Proof: Let us consider S_t and $T_t = \langle 4, 4t+2, 8t+7\rangle$. First, we check that S_t and T_t are numerical semigroups for $t \geq 1$, that is $\gcd(4, 4t+3, 8t+6) = \gcd(4, 4t+2, 8t+7) = 1$,

$$\begin{aligned} \gcd(4, 4t+3, 8t+6) &= \gcd(4, 3, 6) = \gcd(3, 2) = 1, \\ \gcd(4, 4t+2, 8t+7) &= \gcd(4, 2, 7) = \gcd(2, 7) = 1. \end{aligned}$$

Second, we have to see that S_t and T_t are 2 and 3 minimally generated, respectively. To this end, note that $8t+6 = 2 \times (4t+3)$ and so $S_t = \langle 4, 4t+3, 8t+6\rangle = \langle 4, 4t+3\rangle$, that is a 2-minimally generated semigroup because $4t+3$ can not be a multiple of 4. Consider now $T_t = \langle 4, 4t+2, 8t+7\rangle$, we have that neither $4t+2$ nor $4t+7$ are multiples of 4; also $8t+7$ is not a multiple of $4t+2$. Let us see also that $8t+7 \notin \langle 4, 4t+2\rangle$, that is $8t+7 \neq c_t \times 4 + d_t \times (4t+2)$ with $c_t, d_t \in \mathbb{N}$, for $t \geq 1$; if so, the even number $c_t \times 4 + d_t \times (4t+2)$ would equalize the odd one $8t+7$, a contradiction.

Third, we have to see the identity $\mathrm{c}(S_t) = \mathrm{c}(T_t)$, for all $t \geq 1$. The conductor $\mathrm{c}(S_t)$ is easy to compute because S_t is 2-generated and we can apply (4), that is $\mathrm{c}(a,b) = \mathfrak{f}(a,b)+1 = (a-1)(b-1)$. So, $\mathrm{c}(S_t) = (4-1)(4t+3-1) = 12t+6$. To compute the conductor $\mathrm{c}(T_t)$, we use the expression (3). To this end, we have to find the related sequence of L-shaped minimum distance diagrams.

Let us see that T_t has related the L-shaped MDD $\mathcal{H}_t = L(5t+4, 2, 2t+1, 1)$, for all $t \geq 1$. Obviously $\gcd(5t+4, 2, 2t+1, 1) = 1$. Set $N_t = 8t+7$, $a_t = 4$, $b_t = 4t+2$, $l_t = 5t+4$, $h_t = 2$, $w_t = 2t+1$ and $y_t = 1$. It is easily checked that $l_t h_t - w_t y_t = (5t+4) \times 2 - (2t+1) = N_t$ and the compatibility equations (1)

$$l_t a_t - y_t b_t \equiv 0 \pmod{N_t} \Leftrightarrow 20t+16-4t-2 = 16t+14 \equiv 0 \pmod{N_t},$$
$$h_t b_t - w_t a_t \equiv 0 \pmod{N_t} \Leftrightarrow 8t+4-8t-4 = 0 \equiv 0 \pmod{N_t}.$$

$\mathcal{H}_t$ is also an MDD because Theorem 3 is fulfilled, that is $l_t a_t > y_t b_t$ and $h_t b_t = w_t a_t$, for all $t \geq 1$. Therefore $\mathcal{H}_t$ is related to T_t and we can use the expression (3) to compute the conductor $\mathrm{c}(T_t)$

$$\mathrm{c}(T_t) = \mathfrak{f}(T_t)+1 = \max\{(5t+3)\times 4+0, (3t+2)\times 4+4t+3\} - 8t-7+1 = 12t+6.$$

Hence, $\mathrm{c}(S_t) = \mathrm{c}(T_t)$ as it is stated. □

Theorem 6 *Consider the* 3*-numerical semigroups* $S_t = \langle 7, 7t+7, 14t+9 \rangle$ *for* $t \geq 1$. *Then* $\{S_t\}_{t\geq 1} \subset P(7,3)$.

Proof: Consider S_t and $T_t = \langle 7, 7t+6, 14t+10 \rangle$. We have $\gcd(7, 7t+7, 14t+9) = \gcd(7, 7t+6, 14t+10) = 1$, so S_t and T_t are numerical semigroups. The semigroup S_t is minimally 2-generated and $S_t = \langle 7, 14t+9 \rangle$, so his conductor is $\mathrm{c}(S_t) = (7-1)(14t+9-1) = 84t+48$.

Let us see that T_t is 3-minimally generated. We have $7 \nmid 7t+6$, $7 \nmid 14t+10$ and $7t+6 \nmid 14t+10$, for all $t \geq 1$. We have to see now $14t+10 \notin \langle 7, 7t+6 \rangle$. If $7 \times m_t + (7t+6) \times n_t = 14t+10$ with $m_t, n_t \in \mathbb{N}$, then $0 \leq n_t \leq 1$ (if $n_t \geq 2$ then $n_t \times (7t+6) > 14t+10$). If $n_t = 0$, the identity can not be satisfied, hence $n_t = 1$. So the equality turns to be $7m_t = 7t+4$ that has no solution for $m_t \in \mathbb{N}$ because $7m_t \equiv 0 \pmod 7$ and $7t+4 \equiv 4 \pmod 7$. Therefore, the semigroup T_t is 3-minimally generated.

The semigroup T_t has related the L-shaped MDD $\mathcal{H}_t = L(5t+4, 4, 2t+2, 3)$, that is $\gcd(5t+4, 4, 2t+2, 3) = 1$, his area is $14t+10$ and $\mathcal{H}_t$ fulfills the compatibility equations (1) and Theorem 3. Therefore, by using (3), his conductor is

$$\mathrm{c}(T_t) = \max\{(5t+3)\times 7+0, (3t+1)\times 7+3\times(7t+6)\} - 14t - 10 + 1 = 28t+16.$$

So $\mathrm{c}(S_t) = 3\mathrm{c}(T_t)$ as it is stated. $\square$

Theorem 7 *Consider the* 3*-numerical semigroups* $S_t = \langle 9, 9t+7, 9t+12\rangle$ *for* $t \geq 1$*. Then* $\{S_t\}_{t\geq 1} \subset Q(9,1)$*.*

Proof: Consider S_t ad $T_t = \langle 9,\ 9t+6,\ 9t+13\rangle$. From the identities $\gcd(9,\ 9t+7,\ 9t+12) = \gcd(9, 9t+6, 9t+13) = 1$, the semigroups S_t and T_t are numerical semigroups. Let us see that both semigroups are 3-minimally generated.

From $9 \not| 9t+7, 9t+6, 9t+12, 9t+13$ and $9t+7 \not| 9t+12$ and $9t+6 \not| 9t+13$, we have to see $9t+12 \notin \langle 9, 9t+7\rangle$ and $9t+13 \notin \langle 9, 9t+6\rangle$. Let us assume that $9\times m_t + (9t+7)\times n_t = 9t+12$ with $m_t, n_t \in \mathbb{N}$ and $0 \leq n_t \leq 1$ (if $n_t \geq 2$ then $n_t \times (9t+7) > 9t+12$). Then $n_t = 1$ because $9 \not| 9t+12$ and so we have the identity $9m_t = 5$ for $m_t \in \mathbb{N}$, that is a contradiction. A similar argument proves that $9t+13 \notin \langle 9, 9t+6\rangle$.

It can be checked that S_t and T_t have related the L-shaped minimum distance diagrams $L(3t+4, 3, 2t+1, 0)$ and $L(4t+5, 3, 3t+2, 1)$, respectively. Therefore, from (3), we have $\mathrm{c}(S_t) = \mathrm{c}(T_t) = 36t+30$. $\square$

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