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Abstract

In this paper we characterize symmetric L-graphs, which are either Kronecker products of two cycles or Gaussian graphs. Vertex symmetric networks have the property that the communication load is uniformly distributed on all the vertices so that there is no point of congestion. A stronger notion of symmetry, edge symmetry, requires that every edge in the graph looks the same. Such property ensures that the communication load is uniformly distributed over all the communication links, so that there is no congestion at any link.

1 Introduction

Many interconnection networks have been based on vertex-transitive (or vertex-symmetric) graphs. This is the case of current parallel computers from Cray, HP and IBM, among others, that are built around torus networks. Tori have displaced meshes that do not use wraparound edges which simplifies planar design at the price of losing vertex-transitivity. Less work has been devoted to edge-transitive (or edge-symmetric) networks. Square tori would be the network of choice as it is symmetric (vertex and edge symmetric). However, for practical reasons such as packaging, modularity, cost and scalability, the number of nodes per dimension might be different. These topologies, are denoted as mixed-radix networks in [3]. Mixed-radix tori have the drawback of being non-edge-transitive which leads to an imbalanced utilization of network links and buffers. For many traffic patterns, the load on the longer dimension is higher than on the shorter one and, hence, links in longer dimension become network bottlenecks, [1].

The present work is devoted to find and characterize the symmetric members of the two-dimensional family of undirected *multidimensional circulant graphs*, defined in [5]. These graphs can be informally denoted as *L-graphs*, since their set of vertices is a subset of a two-dimensional lattice that can be represented as an L-shaped mesh with wrap-around edges. These two-dimensional networks can be considered as a generalization of torus graphs as they have a planar representation when laying them out on a torus surface. L-shaped graphs have been widely considered, for example in [2] and [6], with applications to interconnection networks.

We recall now some definitions and results appeared in [5]. Let $M \in \mathcal{M}_{n \times n}(\mathbb{Z})$ be a $n \times n$ matrix of integers. Given two vectors $a, b \in \mathbb{Z}^n$ we say that a is congruent to b modulo M , which we denote as $a \equiv b \pmod{M}$, if $a - b \in M\mathbb{Z}^n$, where $M\mathbb{Z}^n$ stands for the additive group of column n -vectors with integral coordinates. We will also denote as $\mathbb{Z}^n/M\mathbb{Z}^n$, the group of integral vectors modulo M , which has $|\det(M)|$ elements when M is nonsingular.

Definition 1 Let $M \in \mathcal{M}_{2 \times 2}(\mathbb{Z})$ be a non-singular matrix. Let $A = \{\pm a_1, \pm a_2\}$ be a subset of vectors of \mathbb{Z}^2 such that $\{a_1, a_2\}$ are linearly independent. The L-graph generated by M and adjacency A , $L(M; A)$, is defined as the graph whose vertex set is formed by the elements of $\mathbb{Z}^2/M\mathbb{Z}^2$ and every vertex u is adjacent to $u + A \pmod{M}$.

In particular, we are interested in those cases whose generating set of jumps is $A = \{e_1, e_2\}$, where $e_1 = (1, 0)^t$ and $e_2 = (0, 1)^t$ form the orthonormal two-dimensional basis. In general we will assume $A = \{e_1, e_2\}$ and therefore we will just denote $L(M; A) = L(M)$.

As stated above, in this paper we study the symmetry of L-graphs. We will prove that the only symmetric L-graphs are either Kronecker products of two cycles or Gaussian graphs. Perfect codes over L-graphs were characterized in [7]. It was shown that L-graphs include Gaussian graphs [8], torus, twisted torus, Kronecker products of two cycles, etc. It is straightforward that any Gaussian graph G_{a+bi} is isomorphic to $L\left(\begin{pmatrix} a & -b \\ b & a \end{pmatrix}\right)$. Kronecker products of cycles have been proposed for interconnection networks in [11], [10], and codes over them were characterized in [12].

Definition 2 Given a graph $G = (V, E)$, $Aut(G)$ denotes its automorphisms group. G is said to be vertex-transitive if, for any pair of vertices

$v_1, v_2 \in V$ there exists $f \in \text{Aut}(G)$ such that $f(v_1) = v_2$. Similarly, G is said to be edge-transitive if for any pair of edges $e_1 = (v_1, v_2), e_2 \in E$ there exists $f \in \text{Aut}(G)$ such that $f(e_1) = (f(v_1), f(v_2)) = e_2$. Then, if G is both vertex and edge transitive, then it is called symmetric.

It is easy to see that all L-graphs are vertex-transitive. Therefore, in order to characterize symmetric L-graphs we will study those which are edge-transitive. With this aim, in Section 2 we will consider some properties of isomorphisms between L-graphs. In Section 3, we determine those L-graphs which are symmetric. In Subsection 3.1 we will focus on linear automorphisms (group automorphisms of $\mathbb{Z}^2/M\mathbb{Z}^2$) of L-graphs which act transitively on the set of edges in order to characterize those which are edge-transitive (and therefore symmetric) by means of their generator matrix. Finally, in Subsection 3.2 we analyze some marginal cases of symmetric L-graphs with non-linear automorphisms.

2 Linear Automorphisms of L-graphs

Two multidimensional circulants are *Ádam isomorphic* if there exists an isomorphism between their groups of vertices such that sends the set of jumps of one into the other one. It is clear that any Ádam isomorphic multidimensional circulants are isomorphic, but the opposite it is not true. In [4] it was proved that:

Theorem 3 [4] *Any two finite isomorphic connected undirected Cayley multigraphs of degree 4 coming from abelian groups are Ádam isomorphic, unless they are obtained with the groups and families $\mathbb{Z}_{4n}, (1, -1, 2n + 1, 2n - 1)$ and $\mathbb{Z}_{2n} \times \mathbb{Z}_2, ((1, 0), (-1, 0), (1, 1), (-1, 1))$.*

In this Section we obtain a similar result in a different form. We address the method here since it can be extended to higher dimensions, which is considered as future work.

Definition 4 We define the *neighborhood* of a vertex v in the graph $G = (V, E)$ as $N(v) = \{w : (v, w) \in E\}$. Then, the *common neighborhood* of a list of vertices denoted as v_1, \dots, v_n as $N(v_1, \dots, v_n) = \bigcap_{i=1}^n N(v_i)$.

Theorem 5 *The neighborhood is preserved in graph isomorphisms. That is, if f is a graph isomorphism, then*

$$N(f(v_1), \dots, f(v_n)) = \{f(w) : w \in N(v_1, \dots, v_n)\}.$$

Proof: Let f be a graph isomorphism from $G = (V, E)$ into $G' = (V', E')$. We have that $f(w) \in N(f(v_1), \dots, f(v_n))$ if and only if $\forall i, f(w) \in N(f(v_i))$, that is $\forall i, (f(w), f(v_i)) \in E'$. Since f is an isomorphism we have that this is equivalent to $\forall i, (w, v_i) \in E$ so $w \in N(v_1, \dots, v_n)$. \square

Next, we analyze which multidimensional circulant graphs isomorphisms are linear mappings. We need the following:

Definition 6 We say that $a, b, c, d \in A$ form a 4-cycle in $G(M; A)$ if $0 = a + b + c + d$ ¹. Then, we say that $G(M; A)$ has not nontrivial 4-cycles if $a, b, c, d \in A$ such that $0 = a + b + c + d$ implies $a = -b$ or $a = -c$ or $a = -d$.

Theorem 7 *If $G(M; A)$ is a multidimensional circulant graph without nontrivial 4-cycles, then for all $a, b \in A$ with $a \neq b$*

$$N(a, b) = \{0, a + b\}.$$

Proof: If $v \in N(a, b)$ then $\exists x, y \in A$ such that $v = a + x = b + y$. Since we have $a - b + x - y = 0$ and $G(M; A)$ has not nontrivial 4-cycles, it must be fulfilled one of the following expressions:

- $a = b$ contradicting the hypothesis,
- $a = -x$ and thus $v = a - a = 0$,
- $a = y$ and thus $v = b + y = a + b$. \square

Lemma 8 *Let $G(M; A)$ and $G(M'; A')$ be two isomorphic multidimensional circulant graphs without nontrivial 4-cycles. Then any isomorphism f between $G(M; A)$ and $G(M'; A')$ with $f(0) = 0$ is such that $f(a + b) = f(a) + f(b)$ for any $a, b \in A$ with $a \neq b$.*

¹each of $\{(v, v + a, v + a + b, v + a + b + c, v + a + b + c + d) : v \in V\}$ is a cycle of length 4

Proof: Let $a, b \in A$ with $a \neq b$. From the previous theorem we get that $N(a, b) = \{0, a + b\}$, hence $N(f(a), f(b)) = \{f(0), f(a + b)\} = \{0, f(a) + f(b)\}$. As $f(0) = 0$ we have that $f(a + b) = f(a) + f(b)$. \square

Lemma 9 *In multidimensional circulant graphs translations are automorphisms.*

Corollary 10 *If f is an automorphism of $G(M; A)$, then there exists another automorphism f' with $f'(0) = 0$.*

Lemma 11 *Let $f \in \text{Aut}(G(M; A))$ such that $f(0) = 0$ and $G(M; A)$ has not nontrivial 4-cycles. Then, we have that*

$$\forall t \in G(M; A), \forall a, b \in A, f(t + a + b) = f(t + a) + f(t + b) - f(t).$$

Proof: Let $t \in G(M; A)$ and $a, b \in A$. We can define $f_t(v) = f(t + v) - f(t)$, which is an automorphism with $f_t(0) = f(t) - f(t) = 0$. Now if $a \neq b$, by Lemma 8 $\forall t \in G(M; A)$, $f_t(a + b) = f_t(a) + f_t(b)$, which implies $\forall t \in G(M; A)$, $f(t + a + b) - f(t) = f(t + a) - f(t) + f(t + b) - f(t)$.

If $a = b$ and $a \neq -a$ then we have $f(t + a - a) + f(t) = f(t + a) + f(t - a)$ and taking $t = t' + a$, $f(t' + a) + f(t' + a) = f(t' + a + a) + f(t')$.

Finally if $a = b = -a$ then let $c = f_t(a)$ and suppose $c \neq -c$. We have $f_t^{-1}(c) = a$, and because of Lemma 8, $0 = f_t^{-1}(c - c) = f_t^{-1}(c) + f_t^{-1}(-c)$. Hence $f_t^{-1}(c) = -f_t^{-1}(-c) = a = -a = -f_t^{-1}(c)$, so $f_t^{-1}(-c) = f_t^{-1}(c)$ and $c = -c$, which is a contradiction. Hence $f_t(a) = c$ implies $c = -c$ and as consequence we obtain $f_t(a) = -f_t(a)$. From this we get that $f(t + a) - f(t) = -f(t + a) + f(t)$, and taking $t = t' + a$ that $f(t' + a + a) = -f(t' + 2a) + f(t' + a) + f(t' + a)$ and by $2a = 0$ that $f(t' + a + a) = f(t' + a) + f(t' + a) - f(t')$. \square

Theorem 12 *If the connected multidimensional circulant graph $G(M; A)$ has not nontrivial 4-cycles then any graph automorphism with $f(0) = 0$ is a group automorphism of $\mathbb{Z}^n / M\mathbb{Z}^n$.*

Proof: We need to prove $\forall n_i \in \mathbb{N}, a_i \in A, f(\sum_i n_i a_i) = \sum_i n_i f(a_i)$, for which is enough to see that for any $v \in G(M; A)$, $a \in A$, $f(a + v) =$

$f(a) + f(v)$. We proceed by induction in the number of summands of v in terms of elements of A . We have already proved that for 1 summand it holds. Assuming that it holds for N summands: let v be expressed with $N + 1$ summands. There exist $b \in A$, $w \in G(M; A)$ such that $v = b + w$ with w expressed with only N summands. Now, because of Lemma 11, $f(a + v) = f(w + a + b) = f(w + a) + f(w + b) - f(w)$. And applying the induction hypothesis we have that $f(a + v) = f(w) + f(a) + f(w + b) - f(w) = f(a) + f(v)$. \square

3 Edge-transitivity of L-graphs

In this Section we determine those L-graphs which are edge-transitive. First, in Subsection 3.1 we consider the case where $L(M)$ is an L-graph without nontrivial 4-cycles, that is, we can consider that any automorphism of $L(M)$ is a linear mapping. Later, in Subsection 3.2 we analyze the special cases in which such cycles exist.

3.1 Edge-transitive of L-graphs by Linear Automorphisms

In this Subsection we will consider those L-graphs such that any automorphism is a linear mapping. Some of the following results will be presented not only for L-graphs but for any multidimensional circulant.

Definition 13 A signed permutation is a composition of a permutation with sign changing function.

Definition 14 A signed permutation matrix is a matrix with entries in $\{-1, 0, 1\}$ which has exactly one ± 1 in each row and column.

Note that in $\mathbb{Z}^{n \times n}$ the signed permutation matrices are exactly the unitary matrices, this is, the matrices U such $UU^t = I$. They are related with permutations in the way that for each signed permutation σ we can find a matrix P_σ such that

$$\begin{pmatrix} v_{\sigma(1)} \\ \vdots \\ v_{\sigma(n)} \end{pmatrix} = P_\sigma \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

Theorem 15 *Let $G(M)$ be a multidimensional circulant. Then, for each linear automorphism f there exists a signed permutation matrix P such that $\forall \alpha, f(\alpha) = P\alpha$.*

Proof: We define P as:

$$P_{i,j} = \begin{cases} 1 & f(e_j) = e_i \\ -1 & f(e_j) = -e_i \\ 0 & \text{otherwise} \end{cases}$$

having $f(e_i) = Pe_i$. Let $\alpha = \sum n_i e_i$.

$$f(\alpha) = \sum_i n_i f(e_i) = \sum_i n_i Pe_i = P \sum_i n_i e_i = P\alpha. \quad \square$$

Theorem 16 *Let $G(M)$ be a multidimensional circulant. Then $f(\alpha) = P\alpha$ is a linear automorphism in $G(M)$ if and only if there exists a matrix Q such that $PM = MQ$.*

Proof: We prove first the left to right implication. As f must be well-defined, for all $i, 0 = P0 \equiv_M PMe_i$. And then exists q_i such that $PMe_i = Mq_i$, gathering all i 's together

$$PM = [PMe_1, \dots, PMe_n] = M[q_1, \dots, q_n] = MQ$$

Now, we prove the right to left implication. We begin proving f is well-defined. Let $a \equiv_M b$, there is a γ such that $a - b = M\gamma$, so $Pa - Pb = PM\gamma = MQ\gamma = M\gamma'$, getting $Pa \equiv_M Pb$. Now we prove injectivity. Let $Pa - Pb = M\gamma$ then $a - b = P^{-1}M\gamma = MQ^{-1}\gamma = M\gamma'$. And as $|\det(M)| = |\det(PM)|$ the sizes are equal and so we have bijection. And finally we prove edge preservation. If a and b are connected in $G(M)$ then we have e_i and γ such $a - b = \pm e_i + M\gamma$, and so $Pa - Pb = \pm Pe_i + PM\gamma = \pm e_j + M\gamma'$ thus Pa is connected to Pb in $L(M)$. \square

To know if a multidimensional circulant $G(M)$ without nontrivial 4-cycles is edge-transitive we need to look to the multiplicative group of the signed permutation matrices P such $PM = MQ$. It is clear that if a matrix representing a cycle of length n (even if it changes signs) is in the group then by composing it with itself, we can map e_1 to every e_i making the

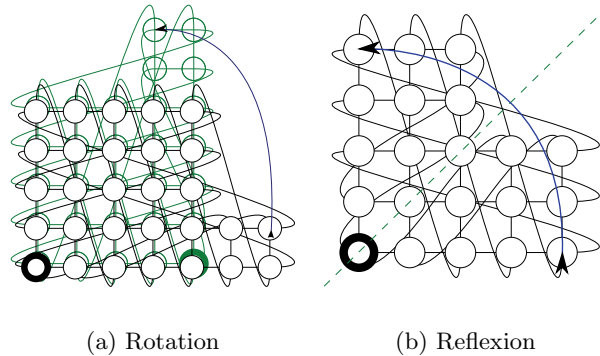


Figure 1: Linear Automorphisms of L-graphs

graph edge-transitive. However, there exist groups which have not such cycles but get the mapping, like the fourth alternating group.

In dimension 2 this is simply to see that if for M one of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is in its group.

Theorem 17 $L(M)$ is edge-transitive if and only if M is right equivalent to one of the following matrices:

- $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$, for $P = Q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.
- $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$, for $P = Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.
- $\begin{pmatrix} a & -b \\ a & b \end{pmatrix}$, for $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $Q = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Proof: The characteristic polynomial of $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is $\lambda^2 - 1$, and the one of $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is $\lambda^2 + 1$. By Theorem III.2 in [9] it must be the characteristic polynomial of both P and Q in $PM = MQ$. Therefore, we have two cases:

- $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$, being reducible over \mathbb{Q} , Q must be similar to a matrix $Q' = \begin{pmatrix} 1 & p \\ 0 & -1 \end{pmatrix}$, which is similar to either $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ or $\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix} \tilde{S} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- $P = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, $\lambda^2 + 1$ which is irreducible over \mathbb{Q} and the class number of $\mathbb{Z}[i]$ is 1, so Q must be similar to P (Theorem III.14, in [9]). \square

It was proved in [7] that the Kronecker product of two cycles is always an L-graph, as next Theorem shows.

Theorem 18 [7] *Let $a, b \in \mathbb{Z}$. Then, the Kronecker product of the cycles of lengths a and b , denoted as $C_a \times C_b$, is isomorphic to:*

- $L\left(\begin{pmatrix} \frac{a+b}{2} & \frac{a-b}{2} \\ \frac{a-b}{2} & \frac{a+b}{2} \end{pmatrix}\right)$ if a and b are odd integers such that $a \geq b$.
- 2 disjoint copies of $L\left(\begin{pmatrix} \frac{a}{2} & \frac{-b}{2} \\ \frac{a}{2} & \frac{b}{2} \end{pmatrix}\right)$ if a and b are both even integers such that $a \geq b$.
- $L\left(\begin{pmatrix} \frac{a}{2} & -b \\ \frac{a}{2} & b \end{pmatrix}\right)$, if a is an even integer and b is an odd integer.

Therefore, $L(M)$ in the first and third cases of Theorem 17 is a Kronecker product of two cycles and in the second one it is a Gaussian graph.

3.2 Edge-transitive L-graphs by Nonlinear Automorphisms

In this Subsection we focus on those L-graphs with nontrivial 4-cycles, that is, L-graphs which can be isomorphic but not Ádam isomorphic. Clearly, if there is a nontrivial 4-cycle then there exist $a, b \in A$ which fulfill:

1. $4a \equiv 0 \pmod{M}$
2. $3a + b \equiv 0 \pmod{M}$
3. $2a + 2b \equiv 0 \pmod{M}$

If we consider $ma + nb = 0 \pmod{M}$ it means that there exists $\gamma \in \mathbb{Z}^2$ such that $\kappa = \begin{pmatrix} m \\ n \end{pmatrix} = M\gamma$. Now, let $\gcd \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \gcd(a_1, a_2)$, $\gamma' = \frac{\gamma}{\gcd \gamma}$ and $\kappa' = \frac{\kappa}{\gcd \gamma}$, having $\kappa' = M\gamma'$. As $\gcd \gamma' = 1$ we can build a unit matrix U with γ' as one of its columns, and therefore $M' = MU$ has κ' as a column.

We'll begin with item (3). In this case we obtain the matrix $M = \begin{pmatrix} m & 2 \\ n & 2 \end{pmatrix}$. If $n = 2k$ we have that $\begin{pmatrix} m & 2 \\ 2k & 2 \end{pmatrix}$ is right equivalent to $\begin{pmatrix} m-n & -2 \\ 0 & 2 \end{pmatrix}$. On the other hand, if $n = 2k + 1$ then $\begin{pmatrix} m & 2 \\ 2k+1 & 2 \end{pmatrix}$ is right equivalent to $\begin{pmatrix} m-n-1 & -2 \\ 1 & 2 \end{pmatrix}$. Both matrices generate the same graph and the isomorphism is not linear in this case.

If we analyze (1) and (2) in detail, we can obtain an equivalent to Theorem 12 for this case. In fact, we can finally prove that in these cases, all automorphisms are linear.

Finally, there are a few marginal cases which correspond to those matrices M with both columns being nontrivial 4-cycles or with such a column and $2e_i$ in the other, that is, those which can be built by selecting two columns in the set:

$$C = \left\{ \begin{pmatrix} 4 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \begin{pmatrix} 0 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ -3 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \end{pmatrix} \right\}.$$

A complete study of the following cases, shows as that most of the combinations are edge-transitive. However, there are cases that lack of a non-linear automorphism, leading to non-edge-transitive graphs.

Up to isomorphism, the L-graphs with 2 different nontrivial solutions to the 4-cycles are:

- With nontrivial 4 cycles but without nonlinear automorphisms.

$$\begin{pmatrix} 4 & 3 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 4 & 1 \\ 0 & 3 \end{pmatrix}$$

- With nonlinear automorphism, which makes them edge-transitive,

$$\begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}, \begin{pmatrix} 3 & 3 \\ 1 & -1 \end{pmatrix} \simeq \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$$

with an example in Figure 2.

- With nonlinear automorphism, but its linear automorphisms already make them edge-transitive,

$$\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}, \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}, \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}$$

with torus as example in Figure 3.

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