# of some graph and digraph compositions

Romain Boulet
Centre national de la recherche scientifique
and Université de Toulouse
Toulouse

Abstract

Let G be a graph of order n the vertices of which are labeled from 1 to n and let $G_1, \cdots, G_n$ be n graphs. The graph composition $G[G_1, \cdots, G_n]$ is the graph obtained by replacing the vertex i of G by the graph G_i and there is an edge between $u \in G_i$ and $v \in G_j$ if and only if there is an edge between i and j in G. We first consider graph composition $G[K_k, \cdots, K_k]$ where G is regular and K_k is a complete graph and we establish some links between the spectral characterisation of G and the spectral characterisation of $G[K_k, \cdots, K_k]$. We then prove that two non isomorphic graphs $G[G_1, \cdots G_n]$ where G_i are complete graphs and G is a strict threshold graph or a star are not Laplacian-cospectral, giving rise to a spectral characterization of these graphs. We also consider directed graphs, especially the vertex-critical tournaments without non-trivial acyclic interval which are tournaments of the shape $t[\overrightarrow{C}_{k_1}, \cdots, \overrightarrow{C}_{k_m}]$, where t is a tournament and $\overrightarrow{C}_{k_i}$ is a circulant tournament. We give conditions to characterise these graphs by their spectrum.

1 Introduction

Some informations about the structure of the graph can be obtained from the spectrum of a matrix associated to the graph. The most used matrices are the adjacency matrix A and the Laplacian matrix $L = D - A$ where D

is the diagonal matrix of degrees. A graph G is *determined by its spectrum* (DS for short) if any other graph having the same spectrum as G is isomorphic to G; we shall specify the matrix only if there is a risk of confusion (we recall that a regular graph is DS with respect to A if and only if it is DS with respect to L). We can focus on a particular family $\mathcal{F}$ of graphs: a graph G is *characterised by its spectrum in* $\mathcal{F}$ if there are no other graphs in $\mathcal{F}$ cospectral non-isomorphic to G.

Let G be a graph of order n the vertices of which are labeled from 1 to n and let $G_1, \cdots, G_n$ be n graphs. The graph composition $G[G_1, \cdots, G_n]$ is the graph obtained by replacing the vertex i of G by the graph G_i and there is an edge between $u \in G_i$ and $v \in G_j$ if and only if there is an edge between i and j in G. If all the G_i's are isomorphic to a graph H then the graph composition $G[H, \cdots, H]$ is the lexicographic product of G and H and will be noted $G[H]$.

The vertex set of $G[H]$ is the cartesian product $V(G) \times V(H)$ and there is an edge between (u, u') and (v, v') if and only if there is an edge between u and v in G or $u = v$ and there is an edge between u' and v' in H. Throughout this paper, a vertex of a lexicographic product $G[H]$ will be denoted by (u, u') where $u \in V(G)$ and $u' \in V(H)$.

We first consider simple graphs and the lexicographic product of a graph with a complete graph (Section 2). Then in Section 3 we develop a specific example of graph composition $G[K_{k_1}, K_{k_2}, \cdots, K_{k_n}]$ with G a strict threshold graph or a star. Finally (Section 4), we deal with digraphs and in particular compositions of tournaments.

To fix notations, $Sp(M)$ denotes the spectrum of a matrix M; for a graph G, $Sp(G)$ denotes the spectrum of its adjacency matrix and $\mu_1^{(m_1)} \in Sp(M)$ means that μ_i is m_i times an eigenvalue of M (the multiplicity of μ_i is at least m_i, we may allow $\mu_i = \mu_j$ for $i \neq j$). The Laplacian spectrum of G is denoted by $Sp_L(G)$. For a vertex v of a graph G, $N(v)$ denotes the set of neighbours of v in G. A complete graph on n vertices is denoted by K_n. The neighbourhood of a vertex v is denoted by $N(v)$ and is the set of vertices adjacent to v.

2 Composition of simple graphs with complete graphs

We consider the graph composition $G[K_k, \cdots, K_k]$ where K_k stands for the complete graph with k vertices. This kind of graph composition $G[H, \cdots H]$, often denoted by $G[H]$ is also called the lexicographic product of G and H and denoted by $G.H$. Moreover we remark that when H is the complete graph then $G[K_k]$ is equal to the strong product of G and H: $G \boxtimes H$.

Proposition 1 *Let λ_i be the eigenvalues of a graph G on n vertices ($1 \leq i \leq n$). Then the nk eigenvalues of $G[K_k]$ are*

$$Sp(G[K_k]) = \{(-1)^{(nk-n)}\} \cup \{k\lambda_i + k - 1, 1 \leq i \leq n\}.$$

Proof: The proof of this proposition is conducted by writing the block matrix of $G[K_k]$; the sketch of the proof is the same as that of Theorem 16. Another to prove this proposition is to remark that the adjacency matrix of $G[K_k]$ can be written as a Kroneker product of matrices: $(A+I) \otimes J - I$, where A is the adjacency matrix of G, I is the identity matrix and J is the all-ones matrix. Then we use classical result of Kronecker products [7]. □

Lemma 2 [8, 9, 10] *Two regular graphs G and G' are isomorphic if and only if the graphs $G[K_k]$ and $G'[K_k]$ are isomorphic.*

The following lemma is a consequence of Proposition 1.

Lemma 3 *Let $\mathcal{C}_r = \{G[K_k], G$ regular $, k \in \mathbb{N}, k \geq 2\}$. If $H = G[K_k]$ is a graph cospectral with $H' = G'[K_k] \in \mathcal{C}_r$ then G is cospectral with G'.*

We can state the following theorem:

Theorem 4 *Let $\mathcal{C}_r = \{G[K_k], G$ regular $, k \in \mathbb{N}, k \geq 2\}$. If the graph $H = G[K_k] \in \mathcal{C}_r$ is characterised by its spectrum in $\mathcal{C}_r$ then G is determined by its spectrum.*

Proof: If G is not determined by its spectrum then there is a graph G' cospectral with G and non isomorphic to G. Then the graphs $G[K_k]$ and $G'[K_k]$ are cospectral (Proposition 1) and not isomorphic (Lemma 2) and therefore the graph H is not characterised by its spectrum in $\mathcal{C}$. □

Corollary 5 *If $G = \tilde{G}[K_k] \in \mathcal{C}_r$ is DS then $\tilde{G}$ is DS.*

The main problem to prove the converse of this theorem is to prove that if $G[K_k]$ (G DS) is cospectral with $G'[K_{k'}]$ then these graphs are isomorphic. Here we consider this problem for some sub-classes of $\mathcal{C}_r$.

Theorem 6 *Let $\mathcal{B}$ be the family of regular bipartite graph and let $\mathcal{C}_r^{\mathcal{B}} = \{G[K_k], G \in \mathcal{B}, k \in \mathbb{N}, k \geq 2\}$. If $G \in \mathcal{B}$ is determined by its spectrum then the graph $H = G[K_k] \in \mathcal{C}_r^{\mathcal{B}}$ is characterised by its spectrum in $\mathcal{C}_r^{\mathcal{B}}$.*

Proof: Let $G \in \mathcal{B}$ be a regular bipartite graph determined by its spectrum and let $H' = G'[K_{k'}] \in \mathcal{C}_r^{\mathcal{B}}$ be a graph cospectral with $H = G[K_k]$; we have to show that H and H' are isomorphic. Let μ (resp. μ') be the spectral radius of G (resp. G'); since G and G' are bipartite, the minimum eigenvalue of G (resp. G') is $-\mu$ (resp. $-\mu'$). The maximal eigenvalue of H is $k(\mu+1)$ and its minimal eigenvalue is $k(-\mu+1)$. The maximal eigenvalue of H' is $k'(\mu'+1)$ and its minimal eigenvalue is $k'(-\mu'+1)$. Since H and H' are cospectral, we have $k(\mu+1)+k(-\mu+1) = k'(\mu'+1)+k'(-\mu'+1)$ that is $k = k'$. Applying Lemma 3 we have that G' is cospectral with G and since G is DS, G' is isomorphic to G and so H' is isomorphic to H. □

Theorem 7 *Let $\mathcal{C}_r^{\mathcal{P}} = \{G[K_k], |G| prime, G \text{ regular }, k \in \mathbb{N}, k \geq 2\}$. If G is a regular DS graph on a prime number of vertices then $\forall k > 1$ the graph $G[K_k] \in \mathcal{C}_r^{\mathcal{P}}$ is characterised by its spectrum in $\mathcal{C}_r^{\mathcal{P}}$.*

Proof: Let G be a regular DS graph on a prime number of vertices determined by its spectrum and let $H' = G'[K_{k'}] \in \mathcal{C}_r^{\mathcal{P}}$ be a graph cospectral with $H = G[K_k]$; we have to show that H and H' are isomorphic. Let $d = \gcd(k, k')$ and let q, q' be such that $k = dq$ and $k' = dq'$ (q and q' are coprime). We have $H = (G[K_q])[K_d]$ cospectral with $H' = (G[K_{q'}])[K_d]$ and applying Lemma 3 we have that $G[K_q]$ is cospectral with $G'[K_{q'}]$. Let n (resp. n') and r (resp. r') be the number of vertices and the degree of G (resp. G'). We have $nq = n'q'$ and $(r+1)q = (r'+1)q'$. So q' divides nq but q and q' are coprime, thus q' divides n and q' is equal to 1 or n (n is prime).

- If $q' = 1$ then $q = 1$ (otherwise n' is not prime) and we have $n = n'$, $k = k'$ and $G'[K_k]$ is cospectral with $G[K_k]$, so (Lemma 3) G' is cospectral with (and therefore isomorphic to) G. So H' is isomorphic to H.

- If $q' = n$ then n divides $r+1$ but $n \geq r+1$ so $n = r+1$ and G is a complete graph. Then H is also a complete graph wich is DS so H' is isomorphic to H. □

To end this section, we compute the Laplacian spectrum of a graph $G[K_{k_1}, K_{k_2}, \cdots, K_{k_n}]$. The proof, using block matrices, is a classical way in this paper to compute eigenvalues of (di)graphs compositions, we describe it in details.

Theorem 8 *The Laplacian spectrum of $G[K_{k_1}, K_{k_2}, \cdots, K_{k_n}]$ is:*

$$\bigcup_{i=1..n} \left\{ \left(k_i + \sum_{j \in N(i)} k_j \right)^{(k_i - 1)} \right\} \cup Sp(-A(G)\hat{D} + \Delta),$$

where the vertices of G are labelled from 1 to n, $A(G)$ is the adjacency matrix of G, $\hat{D}$ is the diagonal matrix of the k_i's and Δ is the diagonal matrix whose i^{th} entry is $\sum_{j \in N(i)} k_j$.

Proof: The adjacency matrix of K_{k_i} will be also denoted by K_{k_i}, the adjacency matrix of $G[K_{k_1}, K_{k_2}, ..., K_{k_n}]$ will be denoted by A, D is the diagonal matrix of degrees of $G[K_{k_1}, ..., K_{k_n}]$ and $L = D - A$ is the Laplacian of $G[K_{k_1}, ..., K_{k_n}]$. The vector $(\underbrace{1, 1, \cdots, 1}_{p \text{ times}})^T$ is denoted by $\mathbf{1}_p$ or by $\mathbf{1}$ if no confusion can be made. Let u be an eigenvector of K_{k_i} associated to the eigenvalue -1, since the multiplicity of the eigenvalue -1 is $k_i - 1$, there is $k_i - 1$ independant eigenvectors u. As $\mathbf{1}$ is an eigenvector of K_{k_i} associated ti the eigenvalue $k_i - 1$, we have $< u, \mathbf{1} >= 0$ (where $<, >$ is the usual scalar product). Let $\tilde{u} = (\underbrace{0, ..., 0}_{k_1 + ... + k_{i-1} \text{ times}}, u^T, \underbrace{0, ..., 0}_{k_{i+1} + ... + k_n \text{ times}})^T$, we have $A\tilde{u} = -\tilde{u}$ and $D\tilde{u} = (k_i - 1 + \sum_{j \in N(i)} k_j)\tilde{u}$. So $L\tilde{u} = (k_i + \sum_{j \in N(i)} k_j)\tilde{u}$. As a result $k_i + \sum_{j \in N(i)} k_j$ is $k_i - 1$ times an eigenvalue of $G[K_{k_1}, K_{k_2}, \cdots, K_{k_n}]$.

There remains n eigenvalues to find (and n eigenvectors). Let $w = \begin{pmatrix} \alpha_1 \mathbf{1}_{k_1} \\ \alpha_2 \mathbf{1}_{k_2} \\ \vdots \\ \alpha_n \mathbf{1}_{k_n} \end{pmatrix}$ and $v = (\alpha_1, \alpha_2, ..., \alpha_n)^T$ where $\alpha_i \in \mathbb{R}$ for $1 \leq i \leq n$, we have:

$$Aw = \begin{pmatrix} (k_1-1)\alpha_1\mathbf{1}_{k_1} + (\sum_{j\in N(1)} k_j\alpha_j)\mathbf{1}_{k_1} \\ (k_2-1)\alpha_2\mathbf{1}_{k_2} + (\sum_{j\in N(2)} k_j\alpha_j)\mathbf{1}_{k_2} \\ (k_3-1)\alpha_3\mathbf{1}_{k_3} + (\sum_{j\in N(3)} k_j\alpha_j)\mathbf{1}_{k_3} \\ \vdots \\ (k_n-1)\alpha_n\mathbf{1}_{k_n} + (\sum_{j\in N(n)} k_j\alpha_j)\mathbf{1}_{k_n} \end{pmatrix},$$

$$Dw = \begin{pmatrix} (k_1-1+\sum_{j\in N(1)} k_j)\alpha_1\mathbf{1}_{k_1} \\ (k_2-1+\sum_{j\in N(2)} k_j)\alpha_2\mathbf{1}_{k_2} \\ (k_3-1+\sum_{j\in N(3)} k_j)\alpha_3\mathbf{1}_{k_3} \\ \vdots \\ (k_n-1+\sum_{j\in N(n)} k_j)\alpha_n\mathbf{1}_{k_n} \end{pmatrix}$$

so

$$Lw = \begin{pmatrix} ((\sum_{j\in N(1)} k_j)\alpha_1 - \sum_{j\in N(1)} k_j\alpha_j)\mathbf{1}_{k_1} \\ ((\sum_{j\in N(2)} k_j)\alpha_2 - \sum_{j\in N(2)} k_j\alpha_j)\mathbf{1}_{k_2} \\ ((\sum_{j\in N(3)} k_j)\alpha_3 - \sum_{j\in N(3)} k_j\alpha_j)\mathbf{1}_{k_3} \\ \vdots \\ ((\sum_{j\in N(n)} k_j)\alpha_n - \sum_{j\in N(n)} k_j\alpha_j)\mathbf{1}_{k_n} \end{pmatrix}.$$

As a consequence w is an eigenvector of L if and only if $\exists\lambda\in\mathbb{R}$, $\forall i = 1,\cdots,n$, $(\sum_{j\in N(i)} k_j)\alpha_i - \sum_{j\in N(i)} k_j\alpha_j = \lambda\alpha_i$, that is if and only if $v = (\alpha_1,\cdots,\alpha_n)^T$ is an eigenvector of $-A(G)\hat{D}+\Delta$ where $\hat{D}$ is the diagonal matrix of the k_i's and Δ is the diagonal matrix whose i^{th} diagonal entry is equal to $\sum_{j\in N(i)} k_j$. As a result $v = (\alpha_1,\cdots,\alpha_n)^T$ is an eigenvector of $-A\hat{D}+\Delta$ associated to the eigenvalue λ if and only if w is an eigenvector of L associated to the eigenvalue λ.
Moreover a vector w is not a linear combination of the vectors $\tilde{u}$ previously defined because the vectors u are orthogonal to $\mathbf{1}$ so there is no linear combination of vectors u equals to $\alpha\mathbf{1}$, $\alpha\in\mathbb{R}^*$. The w are the n missing eigenvectors and the n missing eigenvalues are the eigenvalues of $-A(G)\hat{D}+\Delta$. $\square$

3 Composition of a threshold graph with complete graphs

3.1 Starlike threshold graphs: definition and Laplacian spectrum

In this section we consider a special class of graphs, the characterization of which cannot be done with theorems stated in the previous section: the graph is not regular and the composition is made with complete graphs of various orders. Moreover, showing the characterisation by the spectrum of a special class of graph is quite frequent in spectral graph theory; indeed, more the considered family of graphs is large, more the risk to have a pair of cospectral non-isomorphic graphs within this family is important.

A *threshold graph* [2] is a graph that can be partitioned into a stable subgraph S and a maximal complete subgraph K such that $S = \{i_1, \cdots, i_p\}$ and $N(i_1) \subset N(i_2) \subset \cdots \subset N(i_p)$. If these inclusions are strict then the threshold graph is called *strict threshold graph*. A *(strict) starlike-threshold graph* is a graph $G[K_{k_1}, \cdots, K_{k_n}]$ where G is a (strict) threshold graph. We can give an alternative definition of a (strict) starlike-threshold graph [2]:

Definition 9 A *starlike-threshold graph* is a connected graph where vertices can be partitioned into $C, D_1, D_2, \cdots, D_p$ such that:

- C is a maximal complete subgraph;
- D_i is a complete subgraph and $\forall u, v \in D_i$, $N(u) \cup \{u\} = N(v) \cup \{v\}$;
- $C_1 \subset C_2 \subset \cdots \subset C_p \subset C$ where $C_i = (N(u) \cup \{u\}) \backslash D_i$ with $u \in D_i$.

If the latest inclusions are strict the starlike-threshold graph is called *strict starlike-threshold graph*

Notations: We set $d_i = |D_i|, c = |C|$, $c_i = |C_i|$, $c'_i = |C_i \backslash C_{i-1}|$ with $c'_1 = c_1$, $c' = |C \backslash C_p|$. The number of vertices of a starlike-threshold graph is denoted by n and we set $n_i = n - \sum_{k=1}^{i-1} d_k = c + \sum_{k=i}^{p} d_k$ for $2 \leq i \leq p$ (we have $n_1 = n$ and $n_p = c + d_p$). A starlike-threshold graph is determined by the parameters p, c, $(d_i)_{1 \leq i \leq p}$, $(c_i)_{1 \leq i \leq p}$. By analogy with a star, the parameter p is called the number of branches.

Before dealing with starlike-threshold graphs, we give some general results on the Laplacian spectrum.

Theorem 10 [6, 13] *Let G be a graph on n vertices whose Laplacian spectrum is $\mu_1 \geq \mu_2 \geq ... \geq \mu_{n-1} \geq \mu_n = 0$. Then:*

(i) $\mu_{n-1} \leq \frac{n}{n-1} \min\{d(v), v \in V(G)\}$.

(ii) *If G is not a complete graph then $\mu_{n-1} \leq \min\{d(v), v \in V(G)\}$.*

(iii) $\mu_1 \leq \max\{d(u) + d(v), uv \in E(G)\}$.

(iv) $\mu_1 \leq n$.

(v) $\sum_i \mu_i = 2|E(G)|$.

(vi) $\mu_1 \geq \frac{n}{n-1} \max\{d(v), v \in V(G)\} > \max\{d(v), v \in V(G)\}$.

Theorem 11 [6] *Let G be a non-complete graph, κ_0 its vertex connectivity, κ_1 its edge connectivity, μ_{n-1} its second smallest Laplacian eigenvalue (also called algebraic connectivity), d_m its minimum degree. Then $\mu_{n-1} \leq \kappa_0 \leq \kappa_1 \leq d_m$*

Definition 12 Let G be a simple graph on n vertices, a vertex of degree $n-1$ is an *universal vertex.*

The following lemma uses only basic properties on the Laplacian spectrum [13].

Lemma 13 *Let G be a graph on n vertices with k universal vertices, then $n^{(k)} \in Sp_L(G)$ and the Laplacian spectrum of $G \backslash \{$universal vertices$\}$ is $(Sp_L(G)\backslash\{n^{(k)},\ 0\} - k) \cup \{0\}$.*

Proposition 14 *Let G be a graph with only one non-zero Laplacian-eigenvalue a, then there is $r \in \mathbb{N}^*$, $p \in \mathbb{N}$ such that the Laplacian spectrum of G is $\{a^{(ra-r)}, 0^{(r+p)}\}$ and G is isomorphic to $rK_a \cup pK_1$.*

Proof: Let G be a graph with only one non-zero Laplacian-eigenvalue a and let H be a connected component of G which is not an isolated vertex; the graph H has only one non-zero eigenvalue a. If H is not complete, by Theorem 11 we have $a \leq \min\{d(v), v \in V(H)\}$, but Theorem 10 gives $a > \max\{d(v), v \in V(H)\}$, contradiction. So H is the complete graph K_a with Laplacian spectrum $\{a^{(a-1)}, 0\}$. $\square$

Theorem 15 [1] *Let G be a graph without isolated vertex. If the Laplacian spectrum of G is $\{k_1^{(k_1-1)}, k_2^{(k_2-1)}, ..., k_n^{(k_n-1)}, 0^{(n)}\}$ with $k_i \in \mathbb{N}\backslash\{0,1\}$ then G is a disjoint union of complete graphs of order $k_1, ..., k_n$.*

Theorem 16 *The Laplacian spectrum of a strict starlike-threshold graph with parameters p, c, $(d_i)_{1\leq i\leq p}$, $(c_i)_{1\leq i\leq p}$ is the multiset:*

$$\bigcup_{i=1}^{p}\left\{n_i^{(c'_i)}, (d_i + \sum_{j=1}^{i} c'_j)^{(d_i-1)}, c_i\right\} \cup \{c^{(c'-1)}, 0\}.$$

Proof: The proof is made by induction on p. Induction Hypothesis: the Laplacian spectrum of a threshold graph of completes with p branches is

$$\bigcup_{i=1}^{p}\left\{n_i^{(c'_i)}, (d_i + \sum_{j=1}^{i} c'_j)^{(d_i-1)}, c_i\right\} \cup \{c^{(c'-1)}, 0\}.$$

$p = 1$: Let $n = |G|$ and let $\mu_1 \geq \mu_2 \geq ... \geq \mu_{n-1} \geq \mu_n = 0$ be the Laplacian eigenvalues (counted with multiplicity). Since G has c_1 universal vertices we have (Lemma 13) $n^{(c_1)} \in Sp_L(G)$. The graph $G\backslash C_1$ is the disjoint union of two completes with d_1 and c' vertices so $Sp_L(G\backslash C_1) = \{d_1^{(d_1-1)}, c'^{(c'-1)}, 0^{(2)}\}$. According to Lemma 13,

$$Sp_L(G\backslash\{\text{universal vertices}\}) = (Sp_L(G)\backslash\{n^{(c_1)}, 0\} - c_1) \cup \{0\}$$

so

$$Sp_L(G\backslash C_1) = (Sp_L(G)\backslash\{n^{(c_1)},\ 0\} - c_1) \cup \{0\}$$

and

$$Sp_L(G)\backslash\{n^{(c_1)},\ 0\} = Sp_L(G\backslash C_1) \setminus \{0\} + c_1 = \{(d_1 + c_1)^{(d_1-1)}, c^{(c'-1)}\}.$$

Thus

$$Sp_L(G) = \{n^{(c_1)}, (d_1 + c_1)^{(d_1-1)}, c^{(c'-1)}, 0\}.$$

The induction hypothesis is true for $p = 1$.

Let us assume that the induction hypothesis is true at rank p and let G be a strict threshold graph of completes with $p + 1$ branches and let $\mu_1 \geq \mu_2 \geq ... \geq \mu_{n-1} \geq \mu_n = 0$ be its Laplacian eigenvalues (counted with multiplicity).

According to Lemma 13, the spectrum of $G\backslash C_1$ is $\{\mu_{c_1+1}-c_1, ..., \mu_{n-1}-c_1, 0\}$. The graph $G\backslash C_1$ has two connected components: the complete graph D_1 and a strict threshold graph of completes denoted by G_1. The Laplacian of $G\backslash C_1$ has twice the eigenvalue 0, so $\exists i$ such that $\mu_i - c_1 = 0$ *i.e.* $\mu_i = c_1 = c'_1$. We also have that $Sp_L(G\backslash C_1) = Sp_L(D_1) \cup Sp_L(G_1)$. Since the spectrum of a complete graph on k vertices is k with multiplicity $k-1$ and 0 with multiplicity 1 we have that d_1 is an eigenvalue of $G\backslash C_1$ with multiplicity $d_1 - 1$. The graph G_1 is a strict threshold graph of completes whose the partitioning of vertices is $C\backslash C_1$, D_2,...,D_{p+1}. Moreover, for $u \in D_i$ we have $(N(u)\cup\{u\})\backslash D_i = C_i\backslash C_1$ and so $|(N(u)\cup\{u\})\backslash D_i| = c_i - c_1$. We also have $|G_1| = n - c_1$.

We apply the induction hypothesis to G_1 in order to obtain its spectrum:

$$\bigcup_{i=1}^{p}\left\{(n_{i+1}-c_1)^{(c'_{i+1})}, (d_{i+1}+\sum_{j=1}^{i} c'_{j+1})^{(d_{i+1}-1)}, c_{i+1}-c_1\right\}\cup\{(c-c_1)^{(c'-1)}, 0\}$$

i.e.

$$\bigcup_{i=2}^{p+1}\left\{(n_i-c_1)^{(c'_i)}, (d_i+\sum_{j=2}^{i} c'_j)^{(d_i-1)}, c_i-c_1\right\}\cup\{(c-c_1)^{(c'-1)}, 0\}$$

so the spectrum of $G\backslash C_1$ is

$$\{d_1^{(d_1-1)}, 0\}\cup\bigcup_{i=2}^{p+1}\left\{(n_i-c_1)^{(c'_i)}, (d_i+\sum_{j=2}^{i} c'_j)^{(d_i-1)}, c_i-c_1\right\}\cup\{(c-c_1)^{(c'-1)}, 0\}$$

As $Sp_L(G) = \{n^{(c_1)}, 0\} \cup (Sp_L(G\backslash C_1) \setminus \{0\} + c_1)$ we have

$$Sp_L(G) = \bigcup_{i=1}^{p+1}\left\{n_i^{(c'_i)}, (d_i+\sum_{j=1}^{i} c'_j)^{(d_i-1)}, c_i\right\}\cup\{c^{(c'-1)}, 0\}.$$

As a conclusion the induction hyptohesis is true for $p+1$. $\square$

3.2 There are no cospectral non-isomorphic strict starlike-threshold graphs.

Lemma 17 *For a threshold graph of completes with parameters* p, c, $(d_i)_{1\leq i\leq p}$, $(c_i)_{1\leq i\leq p}$, *we have the following inequalities:*

$$n_1 > n_2 > n_3 > ... > n_p$$

$$\forall j \geq i,\ n_i > d_j + \sum_{k=1}^{i} c'_k$$

$$\forall i, j,\ n_i > c_j$$

$$\forall i,\ n_i > c$$

$$c_1 \leq c_2 \leq c_3 \leq ... \leq c_{p-1} < c$$

Lemma 18 *For* $p \geq 2$, *if* $d_1 + c_1 > n_2$ *and* $c'_2 \neq 0$ *then the multiplicity of* $d_1 + c_1$ *is* $d_1 - 1$.

Proof: We already know that the multiplicity of $d_1 + c_1$ is greater than or equal to $d_1 - 1$; it remains to show that the other eigenvalues are not equal to $d_1 + c_1$. These eigenvalues are $n_i^{(c'_i)}, (d_i + \sum_{k=1}^{i} c'_k)^{(d_i-1)}, c_i, c^{(c'-1)}, 0$ for $i = 1, ..., p$.

• With the first inequalities of the previous lemma and with $d_1 + c_1 > n_2$ we have $d_1 + c_1 > n_i$ for $i \geq 2$. Obviously we have $n_1 > d_1 + c_1$.

• $d_1 + c_1 > n_2 \Rightarrow d_1 + c_1 > d_j + \sum_{k=1}^{i} c'_k$ for $j \geq 2$ (second inequality of the previous lemma).

• $d_1 + c_1 > n_2 \Rightarrow d_1 + c_1 > c_j$ for all j (third inequality of the previous lemma).

• $d_1 + c_1 > n_2 \Rightarrow d_1 + c_1 > c$ (fourth inquality of the previous lemma).

As a result the remaining eigenvalues are not equal to $d_1 + c_1$, thus the multiplicity of $d_1 + c_1$ is $d_1 - 1$. □

Lemma 19 *For* $p \geq 2$, *if* $d_1 + c_1 < n_2$ *and* $c'_2 \neq 0$ *then the multiplicity of* n_2 *is* c'_2.

Proof: We already know that the multiplicity of n_2 is greater than or equal to c'_2; it remains to show that the other eigenvalues are not equal to n_2. These eigenvalues are $n_i^{(c'_i)}, (d_i + \sum_{k=1}^{i} c'_k)^{(d_i-1)}, c_i, c^{(c'-1)}, 0$ for $i = 1, ..., p$.

- $n_i \neq n_2,\ \forall i \neq 2$ (Lemma 17).
- $n_2 > d_1 + c_1$ by hypothesis.
- $n_2 > d_j + \sum_{k=1}^{i} c'_k$ pour $j \geq 2$ (second inequality of Lemma 17).
- $n_2 > c_j$ pour tout j (third inequality of Lemma 17).
- $n_2 > c$ (fourth inequality of Lemma 17).

As a result the remaining eigenvalues are not equal to n_2, thus the multiplicity of n_2 is c'_2. □

Lemma 20 *If $d_1 + c_1 = n_2$ and $c'_2 \neq 0$ then the multiplicity of n_2 (i.e. the multiplicity of $d_1 + c_1$) is $c'_2 + d_1 - 1$.*

Proof: We already know that the multiplicity of n_2 is greater than or equal to c'_2 and that the multiplicity of d_1+c_1 is greater than or equal to d_1-1, so if $d_1+c_1 = n_2$ then the multiplicity if n_2 (*i.e.* that of d_1+c_1) is greater than or equal to c'_2+d_1-1. It remains to show that the other eigenvalues are not equal to n_2. These eigenvalues are $n_i^{(c'_i)}, (d_i+\sum_{k=1}^{i} c'_k)^{(d_i-1)}, c_i, c^{(c'-1)}, 0$ for $i = 1, ..., p$.

- $n_i \neq n_2,\ \forall i \neq 2$ (Lemma 17).
- $n_2 > d_j + \sum_{k=1}^{i} c'_k$ pour $j \geq 2$ (second inequality of Lemma 17).
- $n_2 > c_j$ pour tout j (third inequality of Lemma 17).
- $n_2 > c$ (fourth inequality of Lemma 17).

As a result the remaining eigenvalues are not equal to n_2, thus the multiplicity of n_2 is $c'_2 + d_1 - 1$. □

Lemma 21 *Let G be a strict starlike threshold graph with $p \geq 2$ and $c'_2 \neq 0$, the spectrum of which is $\mu_1 > \mu_2 > ... > \mu_q > 0$, and let $m_1, m_2, ..., m_q$ be the multiplicities of these eigenvalues. If $m_2 = \mu_2 - \mu_q - 1$ then $d_1 = \mu_2 - \mu_q$ otherwise $d_1 = \mu_1 - \mu_2$.*

Proof: The spectrum of G is

$$\bigcup_{i=1}^{p} \left\{ n_i^{(c'_i)}, (d_i + \sum_{k=1}^{i} c'_k)^{(d_i-1)}, c_i \right\} \cup \{c^{(c'-1)}, 0\}$$

(Theorem 16). According to Lemma 17, the greatest eigenvalue μ_1 is equal to n and the smallest eigenvalue μ_q is equal to c_1. According to Lemma 17, there are two possible values for the second largest eigenvalue μ_2: n_2

or $d_1 + c_1$. Indeed for $j > 2$ we have $n_j < n_2$, $d_j + \sum_{k=1}^{i} c'_k < n_2$ and for $j \leq 1$ we have $c_j < c < n_2$; except n_1 and $d_1 + c_1$ all the eigenvalues are strictly loweer than n_2.

• If $\mu_2 = n_2 = d_1 + c_1$ the we have $d_1 = \mu_2 - \mu_q$ and $d_1 = \mu_1 - \mu_2$. The lemma is true in this case.

• If $\mu_2 = d_1 + c_1 > n_2$ then, by Lemma 18, the multiplicity of $d_1 + c_1$ is $d_1 - 1$ and we have $m_2 = \mu_2 - \mu_q - 1$ and $d_1 = \mu_2 - \mu_q$. The lemma is true in this case.

• If $\mu_2 = n_2 > d_1 + c_1$ then, by Lemma 19, the multiplicity of n_2 is c'_2, we have $c'_2 < n - d_1 - c_1 - 1 = n_2 - \mu_q - 1$ thus $m_2 \neq \mu_2 - \mu_q - 1$ and $d_1 = \mu_1 - \mu_2$. The lemma is true in this case. □

Theorem 22 *Let G be a strict starlike threshold graph cospectral with a strict starlike threshold graph F with $p = 1$. Then G and F are isomorphic.*

Proof: Let F be a strict starlike threshold graph with $p = 1$ and with parameters d_1, c_1, c'. As G is cospectral with F, the spectrum of G is $\{n^{(c_1)}, (d_1 + c_1)^{(d_1-1)}, c_1, c^{(c'-1)}, 0\}$ and G has c_1 universal vertices. Let G_1 be the graph $G_1 = G \backslash \{\text{universal vertices}\}$, the spectrum of G_1 is $\{d_1^{(d_1-1)}, c'^{(c'-1)}, 0, 0\}$. The graph G_1 has two connected components: a complete and a complete or a strict starlike threshold graph.

• If $Sp_L(G_1) = \{0, 0\}$ then G_1 consists in two isolated vertices and G is completely determined.

• If $Sp_L(G_1) = \{d_1^{(d_1-1)}, 0, 0\}$ then G_1 (and consequently G) is completely determined (Proposition 14).

• If $Sp_L(G_1) = \{c'^{(c'-1)}, 0, 0\}$ then G_1 (and consequently G) is completely determined (Proposition 14).

• Let us assume that $Sp_L(G_1) = \{d_1^{(d_1-1)}, c'^{(c'-1)}, 0, 0\}$. If G_1 has an isolated vertex, then the spectrum of a connected component of G_1 is $\{d_1^{(d_1-1)}, c'^{(c'-1)}, 0\}$. This is not the spectrum of a complete nor the spectrum of a strict starlike threshold graph because the greatest eigenvalue is not equal to the number of vertices. As a result G_1 does not have an isolated vertex and (Theorem 15) G_1 is the union of two completes with d_1 and c' vertices. □

Theorem 23 *There are no cospectral non-isomorphic strict starlike threshold graphs.*

Proof: Let G be a strict starlike-threshold graph cospectral with another strict starlike-threshold graph G'; we have to show that G is isomorphic to G'. The proof is made by induction on the number of branches of G' denoted by p; the induction hypothesis is 'If G is a strict starlike threshold graph cospectral with a starlike threshold graph on p branches then these two graphs are isomorphic'.
• $p = 1$. It is Theorem 22.
• Let us assume the hyothesis true at rank $p-1$ and let G be a strict starlike threshold graph cospectral with a strict starlike threshold graph with p branches. We denote by m_i the multiplicity of the eigenvalue μ_i. We have:

- n is given by the number of eigenvalues or by μ_1.
- $c_1 = m_1$.
- d_1 is given by Lemma 21.

The graph $G \backslash C_1$ is the disjoint union of a complete with d_1 vertices and a strict starlike threshold graph G_1. As we know c_1 and n, we know the spectrum of $G \backslash C_1$ (Lemma13); and as we know d_1, we know the spectrum of G_1:

$$\bigcup_{i=2}^{p} \left\{ (n_i - c_1)^{(c'_i)}, (d_i + \sum_{j=2}^{i} c'_j)^{(d_i - 1)}, c_i - c_1 \right\} \cup \{(c - c_1)^{(c'-1)}\} \cup \{0\}.$$

But (Theorem 16) this is the spectrum of a strict starlike threshold graph with $p-1$ branches, $n_2 - c_1$ vertices, so the graph G_1 is a strict starlike threshold cospectral with a strict starlike threshold graph with $p-1$ branches and therefore isomorphic to this graph by the induction hypothesis. As a result G is isomorphic with G' and the induction hypothesis is true at rank p. □

3.3 Star of completes

A *star of completes* is the graph $S_n[K_{k_0}, K_{k_1}, \cdots, K_{k_n}]$ where S_n is a star with $n+1$ vertices labeled from 0 to n such that the vertex with degree greater than 1 is labeled 0. A star can be seen as a particular case of a threshold graph, a star of completes can be seen as a particular case of a starlike threshold graph.

Theorem 24 *The Laplacian spectrum of a star $S_n[K_{k_0}, \cdots, K_{k_n}]$ of completes is*

$$\{(k_0 + \cdots + k_n)^{(k_0)}\} \cup \bigcup_{i=1}^{n} \{(k_0 + k_i)^{(k_i-1)}\} \cup \{k_0^{(n-1)}\} \cup \{0\}.$$

We can now state two theorems of characterizations of stars of completes:

Theorem 25 *There are no Laplacian-cospectral non-isomorphic stars of completes.*

Proof: Two cospectral stars of completes have the same number of universal vertices, the deletion of which gives a union of complete graphs. Since there are not two disjoint unions of complete graphs cospectral and non-isomorphic, it ensues that there are no Laplacian-cospectral non-isomorphic stars of completes. □

Theorem 26 *Let $\mathcal{H}$ be the set of graphs with minimum degree strictly greater than the minimum non-zero eigenvalue of its Laplacian matrix. A star of completes belonging to $\mathcal{H}$ is characterised by its Laplacian spectrum in $\mathcal{H}$.*

Proof: Let G be a graph on N vertices with spectrum $\{(k_0+...+k_n)^{(k_0)}\} \cup \bigcup_{i=1}^{n}\{(k_0 + k_i)^{(k_i-1)}\} \cup \{k_0^{(n-1)}\} \cup \{0^{(1)}\}$ and such that $d_{\min} > k_0$. We have $N = k_0 + ... + k_n$ and the spectrum of $\overline{G}$ is $\bigcup_{i=1}^{n}\{(N - k_0 + k_i)^{(k_i-1)}\} \cup \{(N - k_0)^{(n-1)}\} \cup \{0^{(k_0+1)}\}$ so $\overline{G}$ has $k_0 + 1$ connected components. One of these components which has the eigenvalue $N - k_0$, has more than $N - k_0 - 1$ vertices (Theorem 10). Then the k_0 other connected components are isolated vertices. Let H be the connected component of size $N - k_0$, The spectrum of H is $\bigcup_{i=1}^{n}\{(N - k_0 + k_i)^{(k_i-1)}\} \cup \{(N - k_0)^{(n-1)}\} \cup \{0^{(1)}\}$ so the spectrum of $\overline{H}$ is $\bigcup_{i=1}^{n}\{(k_i)^{(k_i-1)}\} \cup \{(0)^{(n)}\}$.

The graph $\overline{H}$ have no isolated vertex; indeed the maximum degree of H, denoted by $d^H_{\max}$, is the maximum degree of $\overline{G}$ that is $d^H_{\max} = N - d_{\min} - 1$. The minimum degree of $\overline{H}$ is $|H| - d^H_{\max} - 1 = (N - k_0) - (N - d_{\min} - 1) - 1 = d_{\min} - k_0 > 0$ so $\overline{H}$ does not have isolated vertices.

By Theorem 15, $\overline{H}$ is a union of complete graphs of size $k_1, ..., k_n$, thus H is the complete multipartite graph $K_{k_1,...,k_n}$ and $\overline{G}$ is the disjoint union of the complete multipartite graph $K_{k_1,...,k_n}$ and k_0 isolated vertices. As a result G is a star of completes $S_n[K_{k_0},...,K_{k_n}]$. □

Remark 27 For a graph G, let κ_0 be its vertex connectivity and κ_1 be its edge connectivity. We have (Theorem 11) $\mu_{n-1} \leq \kappa_0 \leq \kappa_1 \leq d_{\min}$ (μ_{n-1} is the second smallest Laplacian eigenvalue, that is k_0). Thus we can obtain corollaries of the previous theorem by replacing the condition $d_{\min} > k_0$ by the condition $d_{\min} > \kappa_0$ or $\kappa_0 < \kappa_1$.

Remark 28 There exists non-DS star of completes, for instance the star of completes $S_6[K_{k_0}, K_5, K_2, K_2, K_2, K_2, K_2]$ is Laplacian-cospectral with and non-isomorphic to $S_6[K_{k_0}, P, K_1, K_1, K_1, K_1, K_1]$ where P is the Petersen graph. Of course, owing to Theorem 26, the cospectral mate is such that its minimum non-zero eigenvalue equals its minimum degree.

4 Composition of tournaments

In this section we deal with the adjacency spectrum of tournaments (that is a digraph in which each pair of nodes is joined by an arc). Compared with simple graphs, few is done to characterise digraphs by their spectrum.

A *circulant matrix* [5] is a matrix whose k^{th} column is a circulant shift of the $(k-1)^{\text{th}}$ column. A *circulant tournament* is a tournament whose adjacency matrix is circulant. We denote by $\overrightarrow{C}_k$ (k odd) the circulant tournament, the vertices of which are labeled from 0 to k such that $N(0) = \{1, 2, \cdots, \frac{k-1}{2}\}$ and $N(i) = (N(0) + i) \bmod[k]$.

Proposition 29 [5] *The eigenvalues of a circulant matrix are*

$$\lambda_r = \sum_{j=0}^{n-1} a_j e^{\frac{2i\pi j}{n} r}, \quad r = 0, ..., n-1,$$

where $(a_0, a_1, ...a_{n-1})'$ is the first column of the matrix. In particular, the eigenvalues of a circulant tournament the vertices of which are labeled from 0 to n are

$$\lambda_r = \sum_{j \in N(0)} e^{\frac{2i\pi j}{n} r}, \quad r = 0, ..., n-1$$

and the eigenvalues of $\overrightarrow{C}_k$ (k odd) are

$$\lambda_r = \sum_{j=1}^{\frac{k-1}{2}} e^{\frac{2i\pi j}{k} r}, \quad r = 0, ..., k-1.$$

This following well-known and straightfoward result is useful to characterise circulant tournament.

Proposition 30 *A tournament is a circulant tournament if and only if its automorphism group contains a full-length cycle.*

This section is motivated by obtaining an algebraic characterization (mainly spectral characterization) of *vertex-critical tournament without non-trivial acyclic interval* (see the definition hereafter). Culus and Jouve [3] recently found a characterization of these graphs through a combinatorial and graph-theoretic approach: these graphs are compositions $t[\overrightarrow{C}_{k_1}, \cdots, \overrightarrow{C}_{k_m}]$.

Definition 31 A subset X of a tournament T is an *interval* (also called convex subset) if for all v in $V(T) \setminus V(X)$ then for all $x \in X$ there is a link from v to x or for all $x \in X$ there is a link from x to v. An acyclic interval is an interval without any cycle, that is a transitive interval. A *non-trivial acyclic interval* is an acyclic interval with at least two vertices.

A *vertex-critical tournament without non-trivial acyclic interval* is a tournament T such that T is without non-trivial acyclic intervals and, for every vertex u of T, $T \setminus u$ has a non-trivial acyclic interval.

Theorem 32 [3] *Every vertex-critical tournament without non-trivial acyclic interval is isomorphic to $t[\overrightarrow{C}_{k_1}, \cdots, \overrightarrow{C}_{k_m}]$ where t is a tournament of order m and where $k_i \in \mathbb{N} \setminus \{0, 1, 2\}$.*

Proposition 33 [5, 11] *The eigenvalues of $\overrightarrow{C}_k$ are:*

$$\lambda_j = \sum_{s=1}^{\frac{k-1}{2}} e^{\frac{2sj\pi}{k} i} = \begin{cases} -\frac{1}{2} + \frac{i}{2}\cot\left(\frac{j\pi}{2k}\right) & \text{if } j \text{ odd} \\ -\frac{1}{2} + \frac{i}{2}\cot\left(\frac{(k+j)\pi}{2k}\right) & \text{if } j \text{ even} \end{cases}, j < k \text{ and } \lambda_k = \frac{k-1}{2}.$$

Theorem 34 *The spectrum of $T = t[\overrightarrow{C}_{k_1}, ..., \overrightarrow{C}_{k_n}]$ is the multiset*

$$\bigcup_{j=1}^{n} \left(Sp(\overrightarrow{C}_{k_j}) \setminus \left\{ \frac{k_j - 1}{2} \right\} \right) \cup Sp(A\hat{D} + \Delta),$$

where A is the adjacency matrix of t, $\hat{D}$ is the diagonal matrix of the k_j's and Δ is the diagonal matrix whose j^{th} entry is $\frac{k_j-1}{2}$.

Proof: The proof is conducted in the same manner as that of Theorem 16. □

We define $\mathcal{T}_r = \{t[\overrightarrow{C}_k]$, t regular, $k \geq 3\}$; it is a subset of the regular tournaments of $\mathcal{T}$ and a subset of the vertex-critical tournaments without non-trivial acyclic interval.

Theorem 35 *The tournament $T = t[\overrightarrow{C}_k] \in \mathcal{T}_r$ is characterised by its spectrum in $\mathcal{T}_r$ if and only if t is determined by its spectrum.*

Proof: We show the first implication ($\Rightarrow$). Let t be a r-regular tournament such that $t[\overrightarrow{C}_k]$ is characterised by its spectrum in $\mathcal{T}_r$ and let t' a tournament cospectral with t. Then tournament t' is r-regular and $t'[\overrightarrow{C}_k]$ is cospectral with $t[\overrightarrow{C}_k]$ and so there is an isomorphism $\varphi : t[\overrightarrow{C}_k] \to t'[\overrightarrow{C}_k]$. Now let assume that u_1 and u_2 are two distinct vertices of t such that there exists a vertex u' of t' and a, a', b' vertices of $\overrightarrow{C}_k$ with $\varphi(u_1, a) = (u', a')$ and $\varphi(u_2, a) = (u', b')$. Since φ is an isomorphism, we have that

$$\{\varphi(v, x), v \in N(u_1), x \in V(\overrightarrow{C}_k)\} \subset N(\varphi(u_1, a)) = N(u', a')$$

and

$$\{\varphi(v, x), v \in N(u_2), x \in V(\overrightarrow{C}_k)\} \subset N(\varphi(u_2, a)) = N(u', b').$$

Consequently:

$$\{\varphi(v, x), v \in N(u_1) \cup N(u_2), x \in V(\overrightarrow{C}_k)\} \subset N(u', a') \cup N(u', b')$$

Since $u_1 \neq u_2$ we have $|N(u_1) \cup N(u_2)| \geq r + 1$ and $|N(u', a') \cup N(u', b')| \geq (r+1)k$ But $N(u', a') \cup N(u', b') = \{(v', x), v' \in N(u'), x \in V(\overrightarrow{C}_k)\} \cup \{(u', y'), y' \in N(a') \cup N(b')\}$ and $|N(u', a') \cup N(u', b')| \leq rk + 2\frac{k-1}{2} < (r+1)k$ involving a contradiction. As a result if $\varphi(u_1, a) = (u', a')$ and $\varphi(u_2, a) = (u', b')$ then $u_1 = u_2$. If we define the following surjective homomorphism $\pi : t'[\overrightarrow{C}_k] \to t', \pi(u', x) = u'$, we have that $u_1 \neq u_2$ implies $\pi(\varphi(u_1, a)) \neq \pi(\varphi(u_2, a))$. Now, for a given $a \in V(\overrightarrow{C}_k)$ we define the injection $i_a : t \to t[\overrightarrow{C}_k], i(u) = (u, a)$ and $\psi_a = \pi \circ \varphi \circ i_a$ is an isomorphism from t to t'. (it is easy to see that $u_1 \sim u_2 \Rightarrow \psi(u_1) \sim \psi(u2))$. As a result t is isomorphic to t' and t is DS; the first implication of the theorem is proved.

Now we show the converse ($\Leftarrow$): we assume t DS. Let $T = t[\overrightarrow{C}_k]$ cospectral with $T' = t'[\overrightarrow{C}_{k'}]$, that is :

$$\begin{aligned} & \underbrace{n\left(\mathrm{Sp}(\overrightarrow{C}_k) \setminus \left\{\frac{k-1}{2}\right\}\right) \cup \mathrm{Sp}\left(kA_t + \frac{k-1}{2}I\right)}_{\mathrm{Sp}(T)} \\ = \ & \underbrace{n'\left(\mathrm{Sp}(\overrightarrow{C}_{k'}) \setminus \left\{\frac{k'-1}{2}\right\}\right) \cup \mathrm{Sp}\left(kA_{t'} + \frac{k'-1}{2}I\right)}_{\mathrm{Sp}(T')} \end{aligned}$$

If $n = |t| > |t'| = n'$ then $k < k'$ (because $nk = n'k'$).
Let $\lambda \in \mathrm{Sp}(\overrightarrow{C}_k) \setminus \{\frac{k-1}{2}\}$ such that $\lambda \notin \mathrm{Sp}(\overrightarrow{C}_{k'}) \setminus \left\{\frac{k'-1}{2}\right\}$, so $\lambda \in \mathrm{Sp}\left(kA_{t'} + \frac{k'-1}{2}I\right)$ which is impossible according to the multiplicity of λ in $\mathrm{Sp}(T)$. So:

$$\mathrm{Sp}(\overrightarrow{C}_k) \setminus \left\{\frac{k-1}{2}\right\} \subset \mathrm{Sp}(\overrightarrow{C}_{k'}) \setminus \left\{\frac{k'-1}{2}\right\}$$

As $\mathrm{Sp}(\overrightarrow{C}_r) \setminus \{\frac{r-1}{2}\} = \{-\frac{1}{2} + \frac{i}{2}\cot\left(\pi\frac{2j+1}{2r}\right), j \in \{0...r-1\} \setminus \{\frac{r-1}{2}\}\}$ we have $\exists j' > 0 : \frac{1}{2k} = \frac{2j'+1}{2k'}$ wich implies $k' \geq 3k$ and as a consequence $n \geq 3n'$.

As the eigenvalues of $\mathrm{Sp}(\overrightarrow{C}_{k'})$ are simple, we have that each copy of $\mathrm{Sp}(\overrightarrow{C}_{k'}) \setminus \left\{\frac{k'-1}{2}\right\}$ can only contain one copy of $\mathrm{Sp}(\overrightarrow{C}_k) \setminus \{\frac{k-1}{2}\}$; for that reason $\mathrm{Sp}\left(kA_{t'} + \frac{k'-1}{2}I\right)$ contains $(n-n')$ copies of $\mathrm{Sp}(\overrightarrow{C}_k) \setminus \{\frac{k-1}{2}\}$, so $n' \geq (n-n')(k-1) > n$, contradiction !

As a result $n = n'$, $k = k'$ and therefore $\mathrm{Sp}(t) = \mathrm{Sp}(t')$. Since t is DS, t and t' are isomorpic and $t[\overrightarrow{C}_k]$ is isomorphic to $t'[\overrightarrow{C}_{k'}]$. □

Corollary 37 is an exemple of application of Theorem 35:

Theorem 36 [11] *The circulant tournament $\overrightarrow{C}_p$, p odd, is DS.*

Corollary 37 *There are no graphs in $\mathcal{T}_r$ cospectral non isomorphic to $\overrightarrow{C}_p[\overrightarrow{C}_k]$.*

We end this section by giving an algebraic characterization of some vertex-critical tournament without non-trivial acyclic interval.

Lemma 38 *Let p be a prime number, ζ_p be a primitive root of* 1 *and let $\lambda = \sum_{j=1}^{\frac{p-1}{2}} e^{\frac{2i\pi j}{p}}$. Then we have that the field extentions $\mathbb{Q}(\zeta_p)$ and $\mathbb{Q}(\lambda)$ are equal. As a consequence, the degree of the minimal polynomial in $\mathbb{Q}$ of λ is $p-1$ and The conjugates of λ in $\mathbb{Q}$ are $\lambda_r = \sum_{j=1}^{\frac{p-1}{2}} e^{\frac{2i\pi j}{p}r}$, $r = 1, ..., p-1$.*

Proof: We have $\lambda \in \mathbb{Q}(\zeta_p)$ and so $\mathbb{Q}(\lambda) \subset \mathbb{Q}(\zeta_p)$. Let $\zeta_p = e^{\frac{2i\pi}{p}}$ and $k = \frac{p-1}{2}$, then $\lambda = \zeta_p(1 + \zeta_p + ... + \zeta_p^{k-1})$ and

$$\begin{aligned} \bar{\lambda} &= \zeta_p^{p-1}(1 + \zeta_p^{p-1} + ... + \zeta_p^{p-k+1}) \\ &= \zeta_p^{k+1}\zeta_p^{k-1}(1 + \zeta_p^{p-1} + ... + \zeta_p^{p-k+1}) \\ &= \zeta_p^{k+1}(\zeta_p^{k-1} + \zeta_p^{k-2} + ... + 1). \end{aligned}$$

So $\zeta_p^k = \lambda^{-1}\bar{\lambda}$ and $\mathbb{Q}(\zeta_p) \subset \mathbb{Q}(\lambda)$.

Let $\chi(X)$ be the characteristic polynomial of $\overrightarrow{C}_p$, then there is a polynomial $P(X)$ of degree $p-1$ and with coefficients in $\mathbb{Q}$ such that $\chi(X) = (X - \frac{p-1}{2})P(X)$. Since λ is an eigenvalues of $\overrightarrow{C}_p$, then it is a root of P and therefore P is the minimal polynomial of λ. The conjugates of λ in $\mathbb{Q}$ are then the eigenvalues of $\overrightarrow{C}_p$ different from $\frac{p-1}{2}$ and are described in Proposition 29. □

Theorem 39 *Let T be a tournament on n vertices. If the three following conditions are satisfied*
(i) *there is an integer m and prime number $p_1, p_2, ...p_m$ such that $p_1 + .. + p_m = n$ and $m < \min\{p_i\}$;*
(ii) *the automorphism group of T contains the cycles $(0\ 1\ ...\ p_1-1), (p_1\ p_1+1\ ...\ p_2-1), ..., (p_1 + ... + p_{m-1}\ \ p_1 + ... + p_{m-1} + 1\ \ ...\ \ p_1 + ... + p_m - 1)$;*
(iii) *the adjacency spectrum of T contains the following eigenvalues*

$$\lambda^{(p_s)} = \sum_{j=1}^{\frac{p_s-1}{2}} e^{\frac{2i\pi j}{p_s}}, \quad s = 1, ..., m;$$

then T is a vertex-critical tournament without non-trivial acyclic interval and there exists a tournament t such that $T = t[\overrightarrow{C}_{p_1}, \cdots, \overrightarrow{C}_{p_m}]$.

Proof: Condition (ii) implies that there exists tournaments T_1, ..., T_m with respectively p_1, ..., p_m vertices and a tournament t on m vertices such that $T = t[T_1, T_2, ..., T_m]$. Moreover the automorphism group of T_i contains a full-length cycle, so T_i are circulant tournaments and consequeltly are regular of degree $\frac{p_i-1}{2}$. It remains to show that these tournaments T_i are isomorphic to $\overrightarrow{C}_{p_i}$.

Using block-matrices, as done in Theorem 16, or by analogy with Theorem 34, the spectrum of $T = t[T_1, T_2, ..., T_m]$ is

$$\bigcup_{j=1}^{n} \left(Sp(T_{p_j}) \setminus \left\{ \frac{p_j - 1}{2} \right\} \right) \cup Sp(A\hat{D} + \Delta)$$

where A is the adjacency matrix of t, $\hat{D}$ is the diagonal matrix of the p_j's and Δ is the diagonal matrix whose j^{th} entry is $\frac{p_j-1}{2}$.

Fact 1: an eigenvalue $\lambda^{(p_s)}$ described in condition (iii) cannot be an eigenvalue of $A\hat{D} + \Delta$. Indeed, the characteristic polynomial of $A\hat{D} + \Delta$ has its coefficients in $\mathbb{Z}$ and is of degree m and $\lambda^{(p_s)}$ is complex and its minimal polynomial in $\mathbb{Q}$ is of degree $p_s - 1$ (Lemma 38). But $m < p_s$ by condition (i) so $m = p_s - 1$ and the eigenvalues of $A\hat{D} + \Delta$ are $\lambda^{(p_s)}$ and its conjugates described in Lemma 38. On one hand the trace of $A\hat{D} + \Delta$ is the trace of Δ (which is positive) and on the other hand the trace of $A\hat{D} + \Delta$ is the sum of its eigenvalues that is (according to Proposition 33) $-\frac{1}{2}(p_s - 1) < 0$, involving a contradiction.

Fact 2: an eigenvalue $\lambda^{(p_s)}$ described in condition (iii) cannot be an eigenvalue of T_r with $|T_r| = p_r \neq p_s$. Indeed, if $\lambda^{(p_s)}$ is an eigenvalue of T_r then according to Proposition 29 we have $\lambda^{(p_s)} \in \mathbb{Q}(\zeta_{p_r})$, but $\lambda^{(p_s)} \in \mathbb{Q}(\zeta_{p_s})$, so $\lambda^{(p_s)} \in \mathbb{Q}(\zeta_{p_r}) \cap \mathbb{Q}(\zeta_{p_s}) = \mathbb{Q}$, a contradiction.

According to Facts 1 and 2, $\lambda^{(p_s)}$ is an eigenvalue of T_s with $|T_s| = p_s$. Therefore the p_s eigenvalues of $|T_s|$ are $\lambda_r^{(p_s)} = \sum_{j=1}^{\frac{p_s-1}{2}} e^{\frac{2i\pi j}{p_s} r}$, $r = 1, ..., p_s - 1$ and $\frac{p_s-1}{2}$ (T_s is $\frac{p_s-1}{2}$-regular). The tournament T_s has the same eigenvalues than $\overrightarrow{C}_{p_s}$, so T_s is isomorphic to $\overrightarrow{C}_{p_s}$ (Theorem 36).

This ends the proof of this theorem. □

5 Conclusion

As a conclusion, if we focus on the well-known question (but far from being solved) *Which graphs are determined by their spectrum?*, we realise that the spectrum is not sufficient to (easily) determine graphs. An easier problem, which is often a first step in proving DS graph, consists in showing that some given graphs are *characterised by their spectrum* within a smaller family of graphs. This is what we have done in this paper by giving characterisations of strict starlike threshold graphs or stars of completes. In this paper we also established some links between graphs determined by their spectrum and graph compositions characterised by their spectrum.

Another difficult point is to deal with directed graphs. Indeed the eigenvalues are complex (and as a consequence more difficult to handle) and few digraphs are proved to be DS; moreover if we compare to undirected graph, there are less properties linking the spectrum to the structure for digraphs than for graphs. As we have done with some vertex-critical tournament without non-trivial acyclic interval (which can be written as a digraph composition), another way to extend the problem of finding DS graphs is to consider other algebraic objects related to the graphs: we do not only consider the spectrum but also the automorphism group for instance. The new question arising is *Which (di)graphs are determined by their spectrum and their automorphism group?* in the sense that if we have these two informations (the spectrum and the automorphism group), we wonder if we could associate one and only one graph (up to isomorphism).

References

[1] R. Boulet. Disjoint union of complete graphs characterized by their Laplacian spectrum *Electronic Journal of Linear Algebra*, 18:773–783, 2009

[2] M.R. Cerioli and J.L. Szwarcfiter. Edge clique graphs and some classes of chordal graphs. *Discrete Mathematics*, 242:31–39, 2002.

[3] J.F. Culus and B. Jouve. Convex circuit-free coloration of an oriented graph. *European Journal of Combinatorics*, 30(1):43–52, 2009.

[4] E.R. van Dam and W.H. Haemers. Which graphs are determined by

their spectrum?. *Linear Algebra and its Applications*, 373:241–272, 2003.

[5] P.J. Davis. *Circulant Matrices.* Wiley, New-York, 1979.

[6] M. Fiedler. Algebraic connectivity of graphs. *Czechoslovak Mathematical Journal*, 23(98):298–305, 1973.

[7] R. A. Horn, C. R. Johnson. *Topics in matrix analysis.* Cambridge University Press, 1994.

[8] W. Imrich. Über das lexikographische Produkt von Graphen. *Archiv der Mathematik*, 20(3):228–234, 1969.

[9] L. Lovász. Operations with structures. *Acta Mathematica Academiae Scientiarum Hungaricae*, 18:321–328, 1967.

[10] L. Lovász. On the cancellation law among finite relational structures. *Periodica Mathematica Hungarica*, 1(2):145–156, 1971.

[11] D.A. Gregory, S.J. Kirkland and B.L. Shader. Pick's inequality and tournaments. *Linear algebra and its applications*, 186:15–36, 1993.

[12] S.J. Kirkland and B.L. Shader. Tournament Matrices with Extremal Spectral Properties. *Linear algebra and its applications*, 196:1–17, 1994.

[13] B. Mohar. The Laplacian spectrum of graphs. *Graph Theory, Combinatorics, and Applications*, 2:871–898, 1991.