# of matched sum graphs

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Abstract

A *matched sum graph* G_1MG_2 of two graphs G_1 and G_2 of the same order n is obtained by adding to the union (or sum) of G_1 and G_2 a set M of n independent edges which join vertices in $V(G_1)$ to vertices in $V(G_2)$. When G_1 and G_2 are isomorphic, G_1MG_2 is just a *permutation graph*. In this work we derive bounds for the k-restricted edge connectivity $\lambda_{(k)}$ of matched sum graphs G_1MG_2 for $2 \leq k \leq 5$, and present some sufficient conditions for the optimality of $\lambda_{(k)}(G_1MG_2)$.

1 Introduction

Georges and Mauro introduced in [11] the concept of matched sum graphs as follows. Given two graphs G_1, G_2 of the same order $|V(G_1)| = |V(G_2)| = n$ and a set M of n independent edges with one endvertex in $V(G_1)$ and the other one in $V(G_2)$ (a matching between $V(G_1)$ and $V(G_2)$), the *matched sum graph of* G_1 *and* G_2 is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2) \cup M$. Even though these authors denoted such a graph by $G_1M^+G_2$, we will simplify this writing to G_1MG_2 heretofore for the sake of simplicity. Matched sum graphs are in fact *permutation graphs* —as they were introduced by Chartrand and Harary in [6]— when G_1 and G_2 are isomorphic; hence, matched sum graphs generalize the concept of permutation graphs. Examples of permutation graphs include hypercubes, prisms and some generalized Petersen graphs; see [12, 15, 17, 18] for results on permutation graphs.

This work is devoted to study a particular measure of the connectivity of matched sum graphs, extending (and somehow improving) some other related known results. This measure —which can be seen within the framework of *conditional connectivities*, introduced by Harary in [13]— is the so-called *k-restricted edge connectivity* of a graph G, denoted $\lambda_{(k)}(G)$, which corresponds to the minimum cardinality of a set of edges of G whose deletion results in a disconnected graph with all its components of cardinality at least k. We first derive bounds for the k-restricted edge connectivity of matched sum graphs $G = G_1MG_2$ for $2 \leq k \leq 5$. As a consequence of this, we can present some sufficient conditions to guarantee optimality for $\lambda_{(k)}(G)$, G being a matched sum graph. These new results extend and improve those obtained in [2, 3] in some senses.

From now on, every graph will be assumed to be simple; that is, with neither loops nor multiple edges.

1.1 Notation and terminology

Unless otherwise stated we follow [7] for additional terminology and definitions.

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. For every subset X of $V(G)$, $G[X]$ denotes the subgraph of G *induced* by X. For every vertex $x \in V(G)$, the *neighborhood of x* denoted by $N(x) = N_G(x)$ is the set of vertices that are adjacent to x. The *degree* of a vertex x is $d(x) = d_G(x) = |N(x)|$, whereas $\delta = \delta(G)$ is the *minimum degree* over all vertices of G. For every two given proper subsets X, Y of $V(G)$ we denote by $[X, Y]$ the set of edges with one end in X and the other end in Y; when $X = \{x\}$, we write $[x, Y]$ instead of $[\{x\}, Y]$. If X is a proper subset of $V(G)$, let us denote by $w(X) = w_G(X)$ to the set $[X, V(G) \setminus X]$. If the graph G is connected and $1 \leq k \leq |V(G)|$ is an integer, the *minimum k-edge degree of G* is defined as

$$\xi_{(k)}(G) = \min\{|w(X)| : |X| = k,\ G[X] \text{ is connected}\}.$$

Clearly $\xi_{(1)}(G) = \delta(G)$ and $\xi_{(2)}(G) = \min\{d(u)+d(v)-2 : uv \in E(G)\}$, the latter being usually denoted as $\xi(G)$ and called the *minimum edge-degree* of G.

Inspired by the definition of conditional connectivity introduced by Harary [13], Fàbrega and Fiol [9, 10] proposed the concept of k-restricted edge connectivity as follows. For an integer $k \geq 1$ an edge cut W is called a *k-restricted edge cut* if every component of $G-W$ has at least k vertices, where $k \geq 1$ (in the former version due to Fàbrega and Fiol all components obtained by deleting a k-restricted edge cut W from G should have at least $k+1$ vertices, hence $k \geq 0$ was taken; nevertheless, in view of recent related literature we consider in this work cardinality at least k for the components of $G-W$). Assuming that G has k-restricted edge cuts (then G is said to be *$\lambda_{(k)}$-connected*), the *k-restricted edge connectivity* of G, denoted by $\lambda_{(k)}(G)$, is defined as the minimum cardinality over all k-restricted edge cuts of G. From the definition, we immediately have that if $\lambda_{(k)}(G)$ exists, then $\lambda_{(i)}(G)$ exists for any $i < k$ and $\lambda_{(i)}(G) \leq \lambda_{(k)}(G)$. Observe that any edge cut of G is a 1-restricted edge cut and $\lambda_{(1)}(G)$ is just the standard connectivity $\lambda(G)$. Furthermore, the restricted edge connectivity $\lambda'(G)$ defined in [8] is $\lambda'(G) = \lambda_{(2)}(G)$.

As far as the existence of k-restricted edge cuts is concerned, it was shown in [8] that $\lambda_{(2)}(G)$ exists and $\lambda_{(2)}(G) \leq \xi(G)$ if G is not a star and its order is at least 4. For $k=3$, it was shown [5, 16] that except for a special class of graphs named *flowers*, 3-restricted edge cuts exist and $\lambda_{(3)}(G) \leq \xi_{(3)}(G)$ for any connected graph G with order at least 7. Following Ou [16], a graph F of order $n \geq 2k$ is called a *flower* if it contains a cut-vertex s such that every component of $F-s$ has order at most $k-1$. The following result was given by Zhang and Yuan in [21].

Theorem 1 [21] *Let G be a connected graph of minimum degree δ and order $n \geq 2(\delta+1)$ that is not isomorphic to any $G^*_{m,\delta}$ (where $G^*_{m,\delta}$ consists of m disjoint copies of K_δ and a new vertex u adjacent to all the vertices in those copies). For all $k \leq \delta+1$, G is $\lambda_{(k)}$-connected with $\lambda_{(k)}(G) \leq \xi_{(k)}(G)$.*

A graph G is said to be *$\lambda_{(k)}$-optimal* if $\lambda_{(k)}(G) = \xi_{(k)}(G)$. For other interesting results on the k-restricted edge connectivity of graphs see [1, 3, 4, 14, 19, 20, 22], among others.

2 Main results

Given a matched sum graph G_1MG_2, it is clear that if $B \subset V(G_i)$ is a set of cardinality k that induces a connected subgraph of G_i then

$$\xi_{(k)}(G_1MG_2) \leq |w_{G_1MG_2}(B)| = |w_{G_i}(B)| + k,$$

which in particular yields to the following remark.

Remark 2 Let $k \geq 1$ and let G_1, G_2 be two graphs of minimum k-edge degrees $\xi_{(k)}(G_1)$, $\xi_{(k)}(G_2)$, respectively. Then for every matched sum graph G_1MG_2 it follows that

$$\xi_{(k)}(G_1MG_2) \leq \min\{\xi_{(k)}(G_1), \xi_{(k)}(G_2)\} + k.$$

A useful result obtained in [3] is recalled next.

Lemma 3 [3] *Let G be a connected graph with minimum degree δ and minimum k-edge-degree $\xi_{(k)}(G)$ with $k \leq \delta + 1$. Then for every $k \geq 2$ and for every $j \in \{0, \ldots, k\}$ it follows that*

$$\xi_{(k)}(G) \geq \xi_{(k-j)}(G) + j\delta - 2jk + j(j+1).$$

The following theorem constitutes the main result of this work.

Theorem 4 *Let $2 \leq k \leq 5$ be an integer and let G_1, G_2 be two connected $\lambda_{(k)}$-connected graphs of the same order n and minimum degrees $\delta(G_1) \geq k$, $\delta(G_2) \geq k$, respectively. Then every matched sum graph G_1MG_2 is $\lambda_{(k)}$-connected and*

$$\begin{aligned} &\min\{n, \lambda_{(k)}(G_1) + \lambda_{(k)}(G_2), \lambda_{(k)}(G_1) + \delta(G_1) - k + 3, \\ &\quad \lambda_{(k)}(G_2) + \delta(G_2) - k + 3, \xi_{(k)}(G_1MG_2)\} \\ &\leq \lambda_{(k)}(G_1MG_2) \leq \xi_{(k)}(G_1MG_2). \end{aligned}$$

Proof: Set $\mathcal{M} = G_1MG_2$ from now on. Observe that $n \geq 2k$ because both G_1 and G_2 are $\lambda_{(k)}$-connected. Notice also that $\mathcal{M}$ has no cutvertex, because G_1 and G_2 are connected.

Consider first $G_1 \simeq G_2 \simeq K_n$. In this case, $\mathcal{M}$ is isomorphic to $K_2 \times K_n$, and it is easily seen that this graph is $\lambda_{(k)}$-connected with

$$\lambda_{(k)}(K_2 \times K_n) = n < k(n-k+1) = \xi_{(k)}(K_2 \times K_n).$$

Suppose now that G_1 is a noncomplete graph, then $n = |V(G_1)| \geq \delta(G_1) + 2$. First, when $G_2 \simeq K_n$ we get $\delta(G_2) = n - 1 \geq \delta(G_1) + 1$, hence $\delta(\mathcal{M}) = \delta(G_1) + 1 \leq n - 1$. As a consequence,

$$|V(\mathcal{M})| = 2n \geq 2(\delta(\mathcal{M}) + 1),$$

and $\mathcal{M}$ is $\lambda_{(k)}$-connected with $\lambda_{(k)}(\mathcal{M}) \leq \xi_{(k)}(\mathcal{M})$ following Theorem 1 as $\mathcal{M}$ has no cutvertex. Second, suppose that G_2 is also a noncomplete graph, $n = |V(G_2)| \geq \delta(G_2) + 2$. Then $\delta(\mathcal{M}) = \min\{\delta(G_1), \delta(G_2)\} + 1 \leq n - 1$ and $|V(\mathcal{M})| = 2n \geq 2(\delta(\mathcal{M}) + 1)$ holds. Again from Theorem 1 it follows that $\mathcal{M}$ is $\lambda_{(k)}$-connected with $\lambda_{(k)}(\mathcal{M}) \leq \xi_{(k)}(\mathcal{M})$.

The rest of the proof concerns with the lower bound for $\lambda_{(k)}(\mathcal{M})$. Let $W \subset E(\mathcal{M})$ be a minimum k-restricted edge cut of $\mathcal{M}$, $|W| = \lambda_{(k)}(\mathcal{M})$. Hence $\mathcal{M} - W$ consists of exactly two connected components, H, H^* such that $|V(H)| \geq k$ and $|V(H^*)| \geq k$. Observe that $w(V(H)) = w(V(H^*)) = W = [V(H), V(H^*)]$. If $|V(H)| = k$, then $\lambda_{(k)}(\mathcal{M}) = |W| \geq \xi_{(k)}(\mathcal{M})$ and the result holds. If $W = M$ the result is also true since $\lambda_{(k)}(\mathcal{M}) = |M| = n$. Let us next prove the following claim.

Claim A. *The inequality $\lambda_{(k)}(\mathcal{M}) \geq \xi_{(k)}(\mathcal{M})$ holds provided that any of the following situations occurs:*

(i) *There exist two sets $S_1 \subset V(G_1)$, $S_2 \subset V(G_2)$, $2 \leq |S_1| = k - 2, |S_2| = k - 1$, such that the following conditions hold altogether: $S_1 \cup S_2 \subseteq V(H)$; the subgraphs $\mathcal{M}[S_i]$ are connected, $i = 1, 2$; $\mathcal{M} - W$ contains no edge cd with $c \in S_i$ and $d \in (V(G_i) \setminus S_i) \cap V(H)$, $i = 1, 2$; there exist two vertices $u \in S_1$, $u' \in S_2$ such that $uu' \in E(\mathcal{M} - W)$; $\mathcal{M} - W$ contains no edge $ab' \in M$ where $a \in (V(G_1) \setminus S_1) \cap V(H)$ and $b' \in S_2 - u'$.*

(ii) *There exist two sets $S_1 \subset V(G_1)$, $S_2 \subset V(G_2)$, $|S_1| = |S_2| = k - 1$ for $3 \leq k \leq 4$, and $|S_1| = |S_2| \in \{k - 2, k - 1\}$ for $k = 5$, such that the following conditions hold altogether: $S_1 \cup S_2 = V(H)$; the subgraphs $\mathcal{M}[S_i]$ are connected, $i = 1, 2$; there exist two vertices $u \in S_1$, $u' \in S_2$ such that $uu' \in E(\mathcal{M} - W)$.*

(iii) *$k = 5$ and there exist $S_1 = \{u, w\} \subset V(G_1)$, $S_2 = \{u', v', t'\} \subset V(G_2)$, $S_3 = \{w', z'\} \subset V(G_2)$ ($S_2 \cap S_3 = \emptyset$), $|S_1| = |S_3| = 2$, $|S_2| = 3$, such that the following conditions hold altogether: $S_1 \cup S_2 \cup S_3 \subseteq V(H)$; the subgraphs $\mathcal{M}[S_i]$ are connected, $i = 1, 2, 3$; $\mathcal{M} - W$ contains no edge cd with $c \in S_i$ and $d \in (V(G_i) \setminus S_i) \cap V(H)$, $i = 1, 2, 3$; $uu', ww' \in E(\mathcal{M} - W)$; $\mathcal{M} - W$ contains no edge $ab' \in M$ where $a \in (V(G_1) \setminus S_1) \cap V(H)$ and $b' \in S_2 - u'$.*

(iv) *$k = 5$ and there exist $S_1 = \{u, w\} \subset V(G_1)$, $S_2 = \{u', v', t', z'\} \subset V(G_2)$, $|S_1| = 2$, $|S_2| = 4$, such that the following conditions hold* *altogether:* $S_1 \cup S_2 \subseteq V(H)$*; the subgraphs* $\mathcal{M}[S_i]$ *are connected,* $i = 1,2$*;* $\mathcal{M} - W$ *contains no edge cd with* $c \in S_i$ *and* $d \in (V(G_i) \setminus S_i) \cap V(H)$*,* $i = 1,2$*;* $uu' \in E(\mathcal{M} - W)$*;* $\mathcal{M} - W$ *contains no edge* $ab' \in M$ *where* $a \in (V(G_1) \setminus S_1) \cap V(H)$ *and* $b' \in S_2 - u'$*.*

(v) $k = 5$ *and there exist* $S_1 = \{u, w\} \subset V(G_1)$*,* $S_2 = \{u', v'\} \subset V(G_2)$*,* $S_3 = \{v, t\} \subset V(G_1)$ $(S_1 \cap S_3 = \emptyset)$*,* $|S_1| = |S_2| = |S_3| = 2$*, such that the following conditions hold altogether:* $S_1 \cup S_2 \cup S_3 \subseteq V(H)$*; the subgraphs* $\mathcal{M}[S_i]$ *are connected,* $i = 1,2,3$*;* $\mathcal{M}-W$ *contains no edge cd with* $c \in S_i$*,* $d \in (V(G_i)\setminus S_i)\cap V(H)$*,* $i = 1,2,3$*;* $uu', vv' \in E(\mathcal{M}-W)$*.*

Proof of Claim A. We give the proof for items (i), (ii) and (iii), since (iv) and (v) are proved similarly.

(i) Considering the set $\Omega = \{u\} \cup S_2$ of cardinality k it is clear that the subgraph of $\mathcal{M}$ induced by Ω is connected. Observe that, for every vertex $v \in S_1 - u$, it may exist an edge in $M \setminus W$ which connects v and some vertex in $(V(G_2) \setminus S_2) \cap V(H)$. Then,

$$\begin{aligned}
\lambda_{(k)}(\mathcal{M}) &= |w_{\mathcal{M}}(V(H))| \\
&\geq |w_{\mathcal{M}}(\Omega)| + \textstyle\sum_{v\in S_1-u}(d_{\mathcal{M}}(v) - 2|[v,\Omega]| - 1) - (|S_1|-1)(|S_1|-2) \\
&\geq \xi_{(k)}(\mathcal{M}) + \textstyle\sum_{v\in S_1-u}(k+1-2\cdot 2-1) - (k-3)(k-4) \\
&\geq \xi_{(k)}(\mathcal{M}) + (k-3)(k-4) - (k-3)(k-4) = \xi_{(k)}(\mathcal{M}),
\end{aligned}$$

after taking into account that $|[v,\Omega]| \leq 2$ for every $v \in S_1 - u$.

(ii) When $|S_1| = |S_2| = k-1$ consider again the set $\Omega = \{u\} \cup S_2$, which induces a connected subgraph of $\mathcal{M}$. It follows that:

$$\begin{aligned}
\lambda_{(k)}(\mathcal{M}) &= |w_{\mathcal{M}}(V(H))| \\
&\geq |w_{\mathcal{M}}(\Omega)| + \textstyle\sum_{v\in S_1-u}(d_{\mathcal{M}}(v) - 2|[v,\Omega]|) - (|S_1|-1)(|S_1|-2) \\
&\geq \xi_{(k)}(\mathcal{M}) + \textstyle\sum_{v\in S_1-u}(k+1-2\cdot 2) - (k-2)(k-3) \\
&\geq \xi_{(k)}(\mathcal{M}) + (k-2)(k-3) - (k-2)(k-3) = \xi_{(k)}(\mathcal{M}).
\end{aligned}$$

And when $|S_1| = |S_2| = k-2 = 3$ $(k = 5)$, take the set $L = \{u, w\} \cup S_2$ with $uw \in E(G_1)$, $w \in S_1$. This set has cardinality $k = 5$ and clearly induces a connected subgraph of $\mathcal{M}$. In this case, if $S_1 \setminus \{u, w\} = \{z\}$:

$$\begin{aligned}
\lambda_{(5)}(\mathcal{M}) &= |w_{\mathcal{M}}(V(H))| \geq |w_{\mathcal{M}}(L)| + d_{\mathcal{M}}(z) - 2|[z, L]| \\
&\geq \xi_{(5)}(\mathcal{M}) + (6 - 2\cdot 3) \geq \xi_{(5)}(\mathcal{M}),
\end{aligned}$$

noticing that $|[z, L]| \leq 3$.

(iii) Take the set of cardinality five $\Omega = S_1 \cup \{u'\} \cup S_3$, which induces a connected subgraph of $\mathcal{M}$. Then:

$$\begin{aligned}\lambda_{(5)}(\mathcal{M}) &= |w_{\mathcal{M}}(V(H))| \\ &\geq |w_{\mathcal{M}}(\Omega)| + d_{\mathcal{M}}(v') + d_{\mathcal{M}}(t') - 2|[\{v',t'\},\Omega]| - 2|[v',t']| - 1 \\ &\geq \xi_{(5)}(\mathcal{M}) + 6 + 6 - 2\cdot 2 - 2 - 1 = \xi_{(5)}(\mathcal{M}) + 5 > \xi_{(5)}(\mathcal{M}),\end{aligned}$$

because vertices v', t' cannot be adjacent in $\mathcal{M}$ to any vertex of S_1 and since it may exist one edge in $M \setminus W$ which connects z' to some vertex in $(V(G_1) \setminus S_1) \cap V(H)$. $\square$

We continue the proof of the theorem by assuming $|V(H)| \geq k+1$, $|V(H^*)| \geq k+1$, $W \neq M$, and that none of the aforementioned five situations (i) to (v) of Claim A (or the corresponding ones obtained by interchanging the roles of either G_1, G_2, or H, H^*) occurs. We write heretofore $W = W_1 \cup W_M \cup W_2$, with $W_1 \subset E(G_1)$, $W_M \subset M$, $W_2 \subset E(G_2)$. Notice that if $W_i \neq \emptyset$ then W_i is an edge cut of G_i due to the minimality of W. The following claim needs to be proved at this point.

Claim B. *If $W_i \neq \emptyset$, every component of $G_i - W_i$ has at least k vertices.*

Proof of Claim B. We use proof by contradiction. Assume that some component of $G_i - W_i$ has at most $k-1$ vertices. Let C be such a component of $(G_1 - W_1) \cup (G_2 - W_2)$ on at most $k-1$ vertices, chosen so that no other component of $(G_1 - W_1) \cup (G_2 - W_2)$ has fewer vertices than C, and (in case two or more components have this minimum order) with the minimum possible number of components of $(G_1 - W_1) \cup (G_2 - W_2)$ to which these components are linked by means of an edge (of M) in $\mathcal{M} - W$. Assume without loss of generality that $W_1 \neq \emptyset$ and that C is a component of $G_1 - W_1$, with $V(C) \subset V(H)$. As $\mathcal{M}$ is $\lambda_{(k)}$-connected it follows that there exist two adjacent vertices $u \in V(C) \subset V(G_1 - W_1) \cap V(H)$ and $u' \in V(G_2 - W_2) \cap V(H)$ such that the edge $uu' \in M$ does not belong to W. Let us prove now the following assertion:

$$\textit{All components of } H - V(C) \textit{ have at least } k \textit{ vertices.} \tag{1}$$

To this end, let C^* be a component of $G_2 - W_2$ to which C is linked by means of an edge of $M \setminus W$, and assume that $|V(C)| \leq |V(C^*)| \leq k-1$ (otherwise the component of $H - V(C)$ containing C^* has cardinality at least k).

Suppose first that $|V(C)| = 1$, $V(C) = \{u\}$. Then $H - u$ is connected as vertex u is only adjacent in H to vertex $u' \in V(C^*)$, and $|V(H - u)| = |V(H)| - 1 \geq k$. Thus, assertion (1) is proved when $k = 2$.

Now, suppose that $2 \leq |V(C)| \in \{k-2, k-1\}$, hence $3 \leq k \leq 5$. Observe that C^* must be linked in $\mathcal{M} - W$ (by means of an edge of $M \setminus W$) to some component $\tilde{C} \neq C$ of $G_1 - W_1$. Indeed, let us see that supposing otherwise that the only component of $G_1 - W_1$ to which C^* is linked is C yields to one of the five situations of Claim A, against our assumptions. When $|V(C^*)| > |V(C)|$ it must be $|V(C^*)| = k - 1$ and $|V(C)| = k - 2$, which corresponds to situation (i) of Claim A; and when $|V(C^*)| = |V(C)|$, it follows that the only component of $G_2 - W_2$ to which C is linked is C^* (by the way C has been chosen), that is to say, $V(H) = V(C) \cup V(C^*)$ and then $|V(C^*)| = |V(C)| = k - 1$ for $3 \leq k \leq 4$ or $|V(C^*)| = |V(C)| \in \{k-2, k-1\} = \{3, 4\}$ for $k = 5$, because $|V(H)| \geq k + 1$; this is situation (ii) of Claim A.

Hence when $2 \leq |V(C)| \in \{k-2, k-1\}$ it follows that C^* is linked in $\mathcal{M} - W$ (by means of an edge of $M \setminus W$) to some component $\tilde{C} \neq C$ of $G_1 - W_1$. In this case, the component of $H - V(C)$ containing C^* has cardinality at least

$$\begin{aligned} &|V(C^*)| + |V(\tilde{C})| \geq 2 \cdot 2 = 4, && \text{if } k = 3, \\ &|V(C^*)| + |V(\tilde{C})| \geq 2(k-2) \geq k, && \text{if } k = 4, 5. \end{aligned}$$

Observe that assertion (1) is then proved when $k = 3, 4$. Hence, to complete the proof of (1) it must be assumed next that $k = 5$ and $|V(C)| = 2$, $V(C) = \{u, w\}$.

First, if C^* is not linked in $\mathcal{M} - W$ (by means of an edge of $M \setminus W$) to any component $\tilde{C} \neq C$ of $G_1 - W_1$ ($H - V(C^*)$ is connected), it turns out that $|V(C^*)| \in \{3, 4\}$; otherwise $|V(C^*)| = 2$ and so $V(H) = V(C) \cup V(C^*)$ according to the way C has been chosen, which is an absurdity because $|V(H)| \geq 6$ by assumption. When $|V(C^*)| = 3$, C is necessarily linked in $\mathcal{M} - W$ (by means of an edge of $M \setminus W$) to some component $\hat{C} \neq C^*$ of $G_2 - W_2$, because $|V(H)| \geq 6$. If $|V(\hat{C})| \geq 3$ then $|V(H) \setminus V(C^*)| \geq |V(C)| + |V(\hat{C})| \geq 5$; hence the set of edges

$$W' = (W \cup \{uu'\}) \setminus w_{G_2}(V(C^*))$$

is a 5-restricted edge cut of $\mathcal{M}$, of cardinality

$$|W'| \leq |W| + 1 - |V(C^*)|(\delta(G_2) - 2) \leq |W| - 8 < |W|,$$

an absurdity. As a consequence $|V(\hat{C})| = 2$, situation (iii) of Claim A. The case $|V(C^*)| = 4$ corresponds to situation (iv) of Claim A.

Second, suppose that C^* is linked in $\mathcal{M} - W$ by means of an edge of $M \setminus W$ to some component $\tilde{C} \neq C$ of $G_1 - W_1$. When $|V(C^*)| \geq 3$ assertion (1) holds, as $|V(C^*)| + |V(\tilde{C})| \geq 3 + 2 = 5$. Hence, consider the case $|V(C^*)| = 2$. Again, if $|V(\tilde{C})| \geq 3$ we are done, then assume $|V(\tilde{C})| = 2$, which corresponds to situation (v) of Claim A. At this point, assertion (1) has been shown to be true for all $2 \leq k \leq 5$.

Once we have seen that every component of $H - V(C)$ has order at least k, it follows that the set of edges

$$W^* = (W \cup \{ww' : w \in V(C), w' \in V(G_2), ww' \in E(H) \setminus W_M\}) \setminus w_{G_1}(V(C))$$

is a k-restricted edge cut of $\mathcal{M}$. But W^* has cardinality

$$|W^*| \leq |W| + |V(C)| - |w_{G_1}(V(C))| \leq |W| - |V(C)| \leq |W| - 1$$

(because $|w_{G_1}(V(C))| \geq 2|V(C)|$ since $\delta(G_1) \geq k$ and $|V(C)| \leq k - 1$), an absurdity. Then the claim has been proved. $\square$

As a consequence of Claim B, if $W_i \neq \emptyset$ then W_i is indeed a k-restricted edge cut of G_i, hence $|W_i| \geq \lambda_{(k)}(G_i)$.

Therefore, when both $W_1, W_2 \neq \emptyset$, then $\lambda_{(k)}(\mathcal{M}) = |W| \geq |W_1| + |W_2| \geq \lambda_{(k)}(G_1) + \lambda_{(k)}(G_2)$, and the theorem holds. Hence we may assume $W_1 \neq \emptyset$ and $W_2 = \emptyset$, and in this case $V(H) \subset V(G_1)$ and $k + 1 \leq |V(H)| = |W_M|$. It follows that

$$\lambda_{(k)}(\mathcal{M}) = |W| = |W_1| + |W_M| = |W_1| + |V(H)|. \tag{2}$$

Set $r = |V(H)| \geq k + 1$. First observe that if $r \geq \delta(G_1) - k + 3$, then from (2) and from the fact that $|W_1| \geq \lambda_{(k)}(G_1)$ (because W_1 is a k-restricted edge cut of G_1) it follows

$$\lambda_{(k)}(\mathcal{M}) \geq \lambda_{(k)}(G_1) + \delta(G_1) - k + 3,$$

and the theorem holds. Therefore we assume $k + 1 \leq r \leq \delta(G_1) - k + 2$. By Lemma 3 we have

$$|W_1| \geq \xi_{(r)}(G_1) \geq \xi_{(k)}(G_1) + (r - k)(\delta(G_1) - r - k + 1). \tag{3}$$

If $r \leq \delta(G_1) - k + 1$, then $(r-k)(\delta(G_1) - r - k + 1) \geq 0$, hence from (2), (3), and from Remark 2 it follows that

$$\lambda_{(k)}(\mathcal{M}) \geq \xi_{(k)}(G_1) + r \geq \xi_{(k)}(G_1) + k + 1 > \xi_{(k)}(\mathcal{M}).$$

Suppose finally that $r = |V(H)| = \delta(G_1) - k + 2$. Taking into account Remark 2 and expressions (2) and (3) yields

$$\lambda_{(k)}(\mathcal{M}) \geq \xi_{(k)}(G_1)+(2k-\delta(G_1)-2)+(\delta(G_1)-k+2) = \xi_{(k)}(G_1)+k \geq \xi_{(k)}(\mathcal{M}).$$

Similarly, under the alternative assumption $W_2 \neq \emptyset$ and $W_1 = \emptyset$ we obtain either

$$\lambda_{(k)}(\mathcal{M}) \geq \xi_{(k)}(\mathcal{M})$$

or

$$\lambda_{(k)}(\mathcal{M}) \geq \lambda_{(k)}(G_2) + \delta(G_2) - k + 3,$$

and the proof of the theorem is now complete. □

A very similar expression to that in Theorem 4 was obtained in [2] for matched sum graphs when $k = 2$. In fact, the only difference lies on the terms $\lambda_{(k)}(G_i)+\delta(G_i)-k+3 = \lambda_{(2)}(G_i)+\delta(G_i)+1$ for $i = 1, 2$ (in the lower bound for $\xi_{(2)}(G_1MG_2)$ in Theorem 4), which are one unit larger than the corresponding terms in the mentioned result in [2]; in this sense, Theorem 4 (slightly) improves the result in [2] for the case $k = 2$. When $k = 3$ and $G_1 \simeq G_2$ (then G_1MG_2 is a permutation graph), Theorem 4 recovers the main result in [3]. Hence the case $k = 3$ of Theorem 4 is a natural generalization for matched sum graphs of the corresponding known result for permutation graphs. As far as we know, cases $k = 4, 5$ of Theorem 4 must be considered as new contributions for the k-restricted edge connectivity of matched sum graphs (thus, also for permutation graphs).

The following results —consequences of Theorem 4— provide conditions on G_1, G_2 to guarantee $\lambda_{(k)}$-optimality for matched sum graphs G_1MG_2 ($\lambda_{(k)}(G_1MG_2) = \xi_{(k)}(G_1MG_2)$) when $2 \leq k \leq 5$.

Corollary 5 *Let $3 \leq k \leq 5$ be an integer and let G_1, G_2 be two connected $\lambda_{(k)}$-connected graphs of minimum degrees $\delta(G_1) \geq 2k-3$, $\delta(G_2) \geq 2k-3$ and order $|V(G_1)| = |V(G_2)| \geq \min\{\xi_{(k)}(G_1), \xi_{(k)}(G_2)\} + k$, and such that $\lambda_{(k)}(G_i) \geq \xi_{(k)}(G_i) - \delta(G_i) + 2k - 3$ for both $i = 1, 2$. Then every matched sum graph G_1MG_2 is $\lambda_{(k)}$-optimal.*

Corollary 6 *Let $3 \leq k \leq 5$ be an integer and let G_1, G_2 be two connected $\lambda_{(k)}$-connected graphs such that $\lambda_{(k)}(G_1) \leq \lambda_{(k)}(G_2)$. Suppose that G_1 and G_2 are $\lambda_{(k)}$-optimal, with minimum degrees $\delta(G_1) \geq 2k-3$, $\delta(G_2) \geq k+2$ and order $|V(G_1)| = |V(G_2)| \geq \xi_{(k)}(G_1) + k$. Then every matched sum graph G_1MG_2 is $\lambda_{(k)}$-optimal.*

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